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THE EPSILON EXPANSION OF TWO-DIMENSIONAL QUANTUM GRAVITY

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Abstract

Motivated by recent suggestions that renormalisation constants evaluated at the fixed point of the ϵ -expansion for two-dimensional quantum gravity might reproduce Polyakov's exact two-dimensional results, we discuss the perturbative calculation of the effective action for quantum gravity in $(2 - \epsilon)$ dimensions. We show that the theory is non-renormalizable at two-loop order.

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1. Introduction

The theory of two-dimensional quantum gravity is of crucial importance for the study of string theories away from criticality. Recently interest in this area has been stimulated by a series of important advances. Polyakov[1] derived an $SL(2, R)$ formulation of induced 2-dimensional gravity which yielded results for the central charge and also for the anomalous dimensions of matter fields interacting with the induced gravity[2]. Subsequently these results were re-obtained using the theory formulated in the conformal gauge, and extended to 2-dimensional surfaces of arbitrary genus[3].

A parallel development has been the matrix-model approach[4] to discretized two-dimensional quantum gravity, extending previous work on dynamically triangulated random lattices[5]. The main new ingredient is the “double scaling limit” which enables one in principle to obtain the partition function exactly in the continuum limit, while retaining information from 2-dimensional surfaces of every genus, hence including non-perturbative effects.

In Polyakov’s approach[1] the gravitational action is the conformal anomaly induced by the coupling of gravity to matter; the Einstein Lagrangian in two dimensions is a total derivative, whose integral gives a topological quantity, the Euler number. Hence the Einstein action has no dynamics in two dimensions. Recently, however, an intriguing hypothesis was advanced by Kawai and Ninomiya[6] (henceforth abbreviated to KN), which held out the promise of assigning a rôle to the Einstein action and hence potentially emphasizing the relation of the induced gravity action to the conventional action used in four dimensions. They considered the ϵ -expansion for the theory of gravity coupled to matter near two dimensions, taking as their action the sum of the Einstein action and the action for scalars coupled to gravity, defined in $(2 - \epsilon)$ dimensions (where the Einstein action is no longer topological). They conjectured that at the fixed point of the β -function for G , the gravitational coupling constant, the anomalous dimensions of matter fields interacting with gravity would reproduce the exact results of ref.[1] as $\epsilon \rightarrow 0$. This conjecture was confirmed to first order in $\frac{1}{\epsilon}$, where ϵ is the number of scalar fields, by a one-loop calculation.(n.b. Our ϵ has the opposite sign to that used by KN.)

In this paper we shall discuss the issues involved in pursuing this hypothesis beyond leading order. Regrettably, we shall show that the conjecture fails at the two-loop level. For simplicity and clarity of presentation we shall concentrate on the computation of the two-loop β -function for G , but the problems which we shall find would arise in any two-loop calculation, for instance of the anomalous dimension of some operator—and in any case knowledge of the β -function is a prerequisite for the programme envisaged in Ref.[6]. In fact, we find that there are non-local divergences which cannot be removed by renormalisation and which prevent the unambiguous determinations of the two-loop β -function. Their presence is due to a peculiar feature of calculations for quantum gravity in $(2 - \epsilon)$ dimensions: namely the graviton propagator contains a pole in ϵ [6,7], reflecting the topological nature of the Einstein action at $\epsilon = 0$. A consequence is that quantities such as β -functions and anomalous dimensions, which are usually finite, may acquire poles in ϵ . This is not necessarily a problem provided such poles are well-defined. Indeed in KN’s calculation, such poles are essential in ensuring non-zero results for the anomalous

dimension at the fixed point as $\epsilon \rightarrow 0$. However, we shall find that at two loops the effective action contains pole terms involving non-local quantities arising, for example, from the finite parts of momentum integrals, which are ambiguous since they depend on global properties of the spacetime. These pole terms cannot be removed by adjusting the coupling constants in the usual fashion, and they prevent the determination of the β -function. We restrict ourselves to consideration of pure gravity in $(2 - \epsilon)$ dimensions, but of course our results will apply with at least as much force to the theory coupled to matter.

2. The One-Loop Calculation

In this section we establish notation and recapitulate the one-loop calculation which dates back to the pioneering work of Refs.[8,9]. The action for gravity in $d = 2 - \epsilon$ dimensions, incorporating a cosmological constant term, is

$$S[g] = \int d^d x \sqrt{g} \left(\frac{Z_G}{G} R + Z_A \Lambda \right) \quad (2.1)$$

where $g = \det g_{\mu\nu}$, G is the gravitational constant and Z_G , Z_A are renormalisation constants chosen to render the quantum theory finite, order by order in perturbation theory. We use the background field formalism, expanding the quantum field $g_{\mu\nu}^q$ around a classical background configuration, viz.

$$g_{\mu\nu}^q = Z_g (g_{\mu\nu} + Z_h G^{\frac{1}{2}} h_{\mu\nu}). \quad (2.2)$$

where Z_h is an additional renormalisation constant necessary for finiteness, and Z_g sets the scale for $g_{\mu\nu}$ and may be adjusted at will. The background field generating functional is then defined by

$$Z[g] = \int d[h, c, c^*] e^{-S[g^*] - S_{gf} - S_{gh} - S_{gh}(g^*, c, c^*)} \quad (2.3)$$

where S_{gf} is a gauge-fixing term and S_{gh} the corresponding ghost action. A suitable gauge-fixing term is [8,9]

$$S_{gf} = -\frac{1}{2} \int d^d x (\nabla_\mu h^\mu{}_\nu - \frac{1}{2} \nabla_\nu h^\mu{}_\mu)^2, \quad (2.4)$$

where ∇_μ is the covariant derivative with respect to the background metric $g_{\mu\nu}$. The ghost action is then

$$S_{gh}[g, c, c^*] = \int d^d x \sqrt{g} Z_c c^{*\mu} (\nabla^2 g_{\mu\nu} + Z_d R_{\mu\nu}) c^\nu \quad (2.5)$$

where Z_c and Z_d are further renormalisation constants required for finiteness. In accord with the usual BRST analysis for gauge theories, we expect that the gauge-fixing term will be unrenormalised.

For the one-loop calculation all we need are the expansions of S , S_{gf} and S_{gh} to quadratic order in $h_{\mu\nu}$, c and c^* and zeroth order in $\hbar$ (i.e. setting all Z 's equal to 1). We have

$$S[g^q] + S_{gf} = S[g] + \int d^d x \sqrt{g} j^{\mu\nu} h_{\mu\nu} + \frac{1}{2} \int d^d x \sqrt{g} h_{\mu\nu} \Delta^{\mu\nu,\rho\sigma} h_{\rho\sigma} + \dots \quad (2.6)$$

$$S_{gh}[g^q, c, c^*] = \int d^d x \sqrt{g} c^* \Delta_{\mu\nu}^{gh} c^\nu + \dots \quad (2.7)$$

where

$$\Delta^{\mu\nu,\rho\sigma} = -\nabla^2 p^{\mu\nu,\rho\sigma} + X^{\mu\nu,\rho\sigma} \quad (2.8a)$$

$$\Delta_{\mu\nu}^{gh} = \nabla^2 g_{\mu\nu} + R_{\mu\nu} \quad (2.8b)$$

$$J^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} \quad (2.8c)$$

with

$$p^{\mu\nu,\rho\sigma} = \frac{1}{4} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \quad (2.9)$$

and

$$\begin{aligned} X^{\mu\nu,\rho\sigma} &= \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} R - \frac{1}{4} (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) R \\ &\quad + \frac{1}{4} (g^{\mu\rho} R^{\nu\sigma} + g^{\mu\sigma} R^{\nu\rho} + g^{\nu\rho} R^{\mu\sigma} + g^{\nu\sigma} R^{\mu\rho}) \\ &\quad - \frac{1}{2} (g^{\mu\nu} R^{\rho\sigma} + g^{\rho\sigma} R^{\mu\nu}) + \frac{1}{2} (R^{\nu\rho\mu\sigma} + R^{\nu\sigma\mu\rho}) \\ &\quad + \frac{1}{4} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \Lambda. \end{aligned} \quad (2.10)$$

The simplicity of the derivative part of $\Delta^{\mu\nu,\rho\sigma}$ in Eq.(2.8a) is due to the particular choice of gauge-fixing action in Eq.(2.4). However, we intend to use configuration-space heat-kernel techniques which require elliptic operators. Although $\Delta^{\mu\nu,\rho\sigma}$ is not of this form, we may define a related elliptic operator $\tilde{\Delta}^{\mu\nu,\rho\sigma}$ [8,9] by

$$\tilde{\Delta}^{\mu\nu,\rho\sigma} = \Delta^{\mu\nu,\alpha\beta} P^{-1}_{\alpha\beta}{}^{\rho\sigma} \quad (2.11a)$$

$$\tilde{\Delta}^{\mu\nu,\rho\sigma} = -\nabla^2 I^{\mu\nu,\rho\sigma} + \tilde{X}^{\mu\nu,\rho\sigma} \quad (2.11b)$$

where I is the unit operator acting on h , given by

$$I^{\mu\nu,\rho\sigma} = \frac{1}{2} (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}), \quad (2.12)$$

$\tilde{X}$ is defined by

$$\tilde{X}^{\mu\nu,\rho\sigma} = (X P^{-1})^{\mu\nu,\rho\sigma} \quad (2.13)$$

and we have, from Eq.(2.9)

$$P^{-1}_{\mu\nu,\rho\sigma} = -2(I^{\mu\nu,\rho\sigma} + \frac{1}{\epsilon} g^{\mu\nu} g^{\rho\sigma}) \quad (2.14)$$

displaying the pole which, as mentioned previously, may be ascribed to the topological nature of the Einstein action in exactly two dimensions.

Writing $Z = e^W$ where W is the generating functional for connected graphs, we have the one-loop result

$$W^{(1)} = -\frac{1}{2} \ln \det \Delta + \ln \det \Delta^{gh} - S^{(1)} \quad (2.15)$$

For an operator $M = -\nabla^2 1 + X$ acting in two dimensions we have the well-known result, derived using heat-kernel techniques[10],

$$\ln \det M \sim -\frac{1}{2\pi\epsilon} \text{tr} \left[\frac{1}{6} R 1 - X \right] \quad (2.16)$$

and hence, since $\det \Delta \sim \det \tilde{\Delta}$, we find, combining Eqs.(2.8b), (2.11),(2.15) and (2.16)

$$W^{(1)} \sim \frac{1}{2\pi\epsilon} \left(\frac{1}{2} \text{tr} \left[\frac{1}{6} R 1 - \tilde{X} \right] - \text{tr} \left[\frac{1}{6} R g_{\mu\nu} + R_{\mu\nu} \right] - S^{(1)} [Z_g g] \right), \quad (2.17)$$

so

$$W^{(1)} \sim \frac{1}{2\pi\epsilon} \left[\frac{19}{12} R - 3G\Lambda \right] - S^{(1)} [Z_g g] \quad (2.18)$$

where we see that the potential “kinematic” pole due to the presence of P^{-1} has cancelled out. From Eqs. (2.1), (2.2) and (2.18), we see that $W^{(1)}$ can be made finite by taking

$$\begin{aligned} (Z_g Z_g^{\frac{1}{2}-1})^{(1)} &= \frac{19}{12} \frac{1}{2\pi\epsilon} G \\ (Z_\Lambda Z_g^{\frac{1}{2}})^{(1)} &= -\frac{3}{2\pi\epsilon} G \end{aligned} \quad (2.19)$$

It is clear from Eq.(2.19) that the combination $Z_g Z_\Lambda^{\frac{1}{2}-1}$ is independent of the arbitrary scale parameter Z_g and hence it is this combination which is of physical significance. Indeed it is demonstrated in Ref.[6] that if one calculates in a more general gauge than that given by Eq.(2.4), then the gauge dependence cancels in $Z_g Z_\Lambda^{\frac{1}{2}-1}$ at one loop. Hence it is more natural to compute the beta function for $G\Lambda^{\frac{1}{2}-\frac{1}{2}}$ than for G alone, and by a slight abuse of notation this is what we shall mean by β_G . (In fact at one loop in the gauge we are using, the β -functions for G and $G\Lambda^{\frac{1}{2}-\frac{1}{2}}$ are numerically the same.) We have, then,

$$\beta_G = -\epsilon G - \frac{19}{12} \frac{1}{2\pi} G^2 + \dots \quad (2.20)$$

Clearly at one loop the fixed point where $\beta_G = 0$ occurs for $G = G^* \sim \epsilon$. When pure gravity is coupled to c scalar fields, KN[6] show that the fixed point is at $G^* \sim \frac{1}{\epsilon} 24\pi$,

while the anomalous dimension Δ of some primary conformal field at one loop is of the form

$$\Delta - \Delta_0 = \frac{1}{2\pi\epsilon}(\Delta_0(1 - \Delta_0)G + O(\epsilon)) + \dots \quad (2.21)$$

where Δ_0 is the dimension in the absence of gravity. Hence at the fixed point

$$\Delta - \Delta_0 \sim \frac{12}{-c}\Delta_0(1 - \Delta_0) + O(\epsilon) + \dots \quad (2.22)$$

and as $\epsilon \rightarrow 0$ this reproduces the first term in an expansion in powers of $\frac{1}{-c}$ of the exact result[1,2]:

$$\Delta - \Delta_0 = \frac{12\Delta_0(1 - \Delta_0)}{13 - c + \sqrt{(1 - c)(25 - c)}} \quad (2.23)$$

It is this tantalising result which motivated the present investigation of whether this correspondence might persist beyond leading order.

To complete the one-loop calculation we should also determine Z_h in Eq.(2.2) and Z_c, Z_d in Eq.(2.5) at one loop. However, since the ghost propagator (the inverse of Δ^{gh} in Eq.(2.8b)) contains no kinematic pole, Z_c and Z_d will not contribute to the two-loop calculation at the order to which we shall be working, so we shall not compute them. Also the calculation of Z_h is somewhat involved and relies on heat-kernel techniques which are described in Section 3, and hence is deferred to Appendix B. Here we shall simply state the result,

$$Z_h = 1 + \frac{2}{\epsilon} \frac{1}{2\pi\epsilon} G + O\left(\frac{1}{\epsilon}\right) + \dots \quad (2.24)$$

The leading one-loop contribution to Z_h is a double pole in ϵ , one factor of ϵ^{-1} arising from the “kinematic” pole in the graviton propagator (i.e. the inverse of Δ in Eq.(2.11a), and one factor from a standard ultra-violet singularity. This term is completely unambiguous and well-defined. However, the $O(\epsilon^{-1})$ term in Eq.(2.24) cannot be determined explicitly, as we show in Appendix B. This is the first indication of potential problems in computing Z_G at two loops, since clearly Z_h will be an ingredient in this calculation.

3. The Two-Loop Calculation

In this section we examine non-local terms in the effective action at two loops. We specialize to pure gravity, setting the cosmological constant Λ to zero in Eqs.(2.1), (2.10). We find that the leading non-local poles for each two-loop diagram are $O(\epsilon^{-3})$, and we show that these leading non-local terms cancel with appropriate counterterms included, up to terms proportional to the Einstein tensor, which vanishes in exactly two dimensions. However the next-to-leading $O(\epsilon^{-2})$ non-local poles are impossible to calculate for a general spacetime and there is no reason to expect any cancellation. We shall argue that for the

KN programme to be carried out at two loops, the non-local poles would have to cancel at least down to $O(\epsilon^{-2})$ inclusive. Hence the conjecture of KN fails at two loop order.

We shall adopt the formalism and notation developed for background field calculations in configuration space, exploiting heat-kernel techniques[11-13]. This machinery is particularly useful for calculations on curved spacetimes since it maintains explicit general covariance. We shall confine ourselves to a brief summary here, referring the reader to Refs.[11-13] for a fuller explanation.

The propagator G_M for an operator

$$M = -\nabla^2 + A, \quad (3.1)$$

(where A is some general matrix), acting on a curved 2-dimensional manifold is expanded as

$$G_M(x, x') = G_0(x, x')a_0^M(x, x') + G_1(x, x')a_1^M(x, x') + H_M(x, x') \quad (3.2)$$

where

$$G_0(x, x') = \Delta^{\frac{1}{2}}(x, x') \left(\frac{\Gamma(-\frac{1}{2}\epsilon)}{4\pi^{\frac{d}{2}}} (2\sigma(x, x'))^{\frac{1}{2}} + \frac{1}{2\pi\epsilon} \right) \quad (3.3)$$

$$G_1(x, x') = \Delta^{\frac{1}{2}}(x, x') \left(-\frac{\Gamma(-1 - \frac{1}{2}\epsilon)}{16\pi^{\frac{d}{2}}} (2\sigma(x, x'))^{1 + \frac{1}{2}} - \frac{1}{4\pi\epsilon} \sigma(x, x') \right)$$

with $2\sigma(x, x')$ the squared geodesic distance between x and x' , and $\Delta(x, x')$ the Van Vleck-Morette determinant[10]. The $\frac{1}{\epsilon}$ terms in Eq.(3.3) should contain a factor $\mu^{-\epsilon}$, where μ is the usual dimensional regularisation scale, but for our purposes we may take $\mu = 1$. The $a_i^M(x, x')$ are the Seeley-De Witt coefficients of the heat-kernel expansion for the operator M . For our purposes the most important properties are

$$\begin{aligned} a_0^M|_{x'=x} &= 1 \\ (\nabla_\mu \nabla_\nu a_0^M)|_{x'=x} &= \frac{1}{2} [\nabla_\mu, \nabla_\nu] \\ a_1^M|_{x'=x} &= \frac{1}{6} R1 - A \end{aligned} \quad (3.4)$$

where A is defined in Eq.(3.1). It is important to note that while G_0, G_1 depend only on the metric locally for $x' \sim x$, H_M is non-local and depends on global aspects of the spacetime. In a renormalizable theory all poles with residues involving H_M should cancel—this is vital in order that divergences be well-defined local quantities which may be cancelled by appropriate coupling-constant renormalizations. In the case at hand we shall find that this requirement is not met.

The interactions which yield the three- and four-point vertices are given by the third and fourth order terms in the expansions of $S[g+h]$ and $S^{gh}[g+h]$, while the propagators are the Green functions for the operators $\Delta^{\mu\nu, \rho\sigma}$ and $\Delta_{\mu\nu}^{gh}$ in Eq.(2.8), satisfying

$$\begin{aligned} \Delta^{\mu\nu, \rho\sigma} G_{\alpha\beta, \rho\sigma} &= \delta_{(\rho}^{\mu} \delta_{\sigma)}^{\nu} \delta^d(x, x') \\ \Delta_{\mu\rho}^{gh} G^{gh, \rho\nu} &= \delta_{\mu}^{\nu} \delta^d(x, x'). \end{aligned} \quad (3.5)$$

There are eight two-loop graphs, shown in Figs. 1 and 2, where R denotes interactions involving a background Ricci tensor. We also need to consider counterterm graphs involving insertions of Z_G, Z_A, Z_c, Z_d and Z_g (in Eqs. (2.1), (2.2), and (2.5)) at one loop. These are depicted in Fig. 3. We shall concern ourselves only with the *leading* non-local pole terms. These are generated by Figs. 1 and 3(a); the ghost graphs have fewer graviton propagators and hence fewer kinematic poles. The relevant vertices for Fig. 1 may be read off from the expansion of $S[g+h]$ given in Appendix A.

The amplitude for Fig. 1(a) is of the form

$$W_{1a}^{(2)} = \int d^4x d^4x' \sqrt{g(x)} \sqrt{g(x')} T^{\alpha\beta\gamma\delta, \mu\nu} T^{\alpha'\beta'\gamma'\delta', \mu'\nu'} \cdot \\ [G_{\alpha\beta, \alpha'\beta'} (\nabla_\mu G_{\gamma\delta, \gamma'\delta'}) \bar{\nabla}_\mu (\nabla_\nu G_{\rho\sigma, \rho'\sigma'}) \bar{\nabla}_\nu + \\ (G_{\alpha\beta, \gamma'\delta'} \bar{\nabla}_\mu) (\nabla_\mu G_{\gamma\delta, \alpha'\beta'}) (\nabla_\nu G_{\rho\sigma, \rho'\sigma'}) \bar{\nabla}_\nu + \\ + \text{permutations of } \alpha-\nu, \alpha'-\nu'] \quad (3.6)$$

The propagator $G_{\mu\nu, \rho\sigma}$ in Eq.(3.6) is not appropriate for the application of heat-kernel techniques since it corresponds to the operator $\Delta^{\mu\nu, \rho\sigma}$ which is not elliptic. However, from Eq.(2.11a), $G_{\mu\nu, \rho\sigma}$ is related to $\tilde{G}_{\mu\nu, \rho\sigma}$, the Green function for the elliptic operator $\tilde{\Delta}$, by

$$G_{\mu\nu, \rho\sigma} = P^{-1}{}_{\mu\nu}{}^{\alpha\beta} \tilde{G}_{\alpha\beta, \rho\sigma}. \quad (3.7)$$

The procedure for extracting the divergent behaviour of the RHS of Eq.(3.6) is then to write $\tilde{G}$ in terms of G using Eq.(3.7), and then to insert the expansion for $\tilde{G}$ given by Eq.(3.2). We then use results given in Ref.[13] for the singular behaviour of products of G_0 's and G_1 's with various derivatives. We shall confine ourselves to terms which include one factor of the non-local part H . For our purposes the relevant results from Ref.[13] are the following:

$$(\nabla_\mu \partial_\nu G_0) \tilde{G}_0 \sim -\frac{1}{2} \frac{1}{2\pi\epsilon} g_{\mu\nu} \delta^d \quad (3.8a)$$

$$\partial_\mu G_0 \partial_\nu G_0 \sim \frac{1}{2} \frac{1}{2\pi\epsilon} g_{\mu\nu} \delta^d \quad (3.8b)$$

$$(\nabla_\mu \partial_\nu G_0) \partial_\rho G_0 \sim -\frac{1}{4} \frac{1}{2\pi\epsilon} (g_{\mu\nu} \partial_\rho - g_{\mu\rho} \partial_\nu - g_{\nu\rho} \partial_\mu) \delta^d \quad (3.8c)$$

$$(\nabla_\mu \partial_\nu G_0) (\nabla_\sigma \partial_\rho G_0) \sim \frac{1}{24} \frac{1}{2\pi\epsilon} \{ 2(g_{\mu\sigma} \nabla_\nu \partial_\rho + g_{\mu\rho} \nabla_\sigma \partial_\nu + g_{\nu\sigma} \nabla_\mu \partial_\rho + g_{\nu\rho} \nabla_\mu \partial_\sigma) \\ - 4(g_{\mu\nu} \nabla_\sigma \partial_\rho + g_{\rho\sigma} \nabla_\mu \partial_\nu) + (g_{\mu\sigma} g_{\nu\rho} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\nu} g_{\rho\sigma}) \nabla^2 \} \delta^d \\ + \frac{1}{12} \frac{1}{2\pi\epsilon} \{ 2(R_{\mu\nu\sigma\rho} + R_{\mu\rho\nu\sigma}) \\ - (R_{\mu\rho} g_{\nu\sigma} + R_{\nu\rho} g_{\mu\sigma} + R_{\mu\sigma} g_{\nu\rho} + R_{\nu\sigma} g_{\mu\rho}) \} \delta^d \quad (3.8d) \\ - \frac{1}{12} \frac{1}{2\pi\epsilon} (R_{\mu\nu} g_{\rho\sigma} + R_{\rho\sigma} g_{\mu\nu}) \delta^d \\ \nabla_\mu \partial_\nu G_0 \nabla_\rho \partial_\sigma G_1 \sim \frac{1}{8} \frac{1}{2\pi\epsilon} (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) \delta^d \quad (3.8e)$$

To deal with derivatives at x as well as at x' it is convenient to introduce $g_{\mu'}^{\nu'}$, the matrix giving parallel transport of vectors along the geodesic from x' to x . This has the properties

$$g_{\mu\rho'} g^{\nu\rho'} = \delta_{\mu}^{\nu}, \quad g_{\rho\mu'} g^{\rho\nu'} = \delta_{\mu'}^{\nu'}, \\ g^{\mu\nu'} \sigma_{\nu'} = -\sigma^{\mu}, \quad \sigma_{\mu} g^{\mu\nu'} = -\sigma^{\nu'}, \\ (\nabla_\mu \partial_\nu g_{\rho'}^{\sigma'})|_{x'=x} = -\frac{1}{2} R_{\sigma'}^{\rho'}{}_{\mu\nu}. \quad (3.9)$$

We then have

$$\partial_{\nu'} G_0 = -g^{\mu}{}_{\nu'} \partial_\mu G_0 \quad (3.10)$$

which, together with Eqs. (3.8),(3.9), is sufficient to obtain the singularities of products of G_0 's and G_1 's with derivatives at both x and x' , such as occur in the expansion of Eq.(3.6). It is useful to note that

$$\alpha_{0\mu\nu, \mu'\nu'} = \frac{1}{2} (g_{\mu\mu'} g_{\nu\nu'} + g_{\mu\nu'} g_{\nu\mu'}) \quad (3.11)$$

and hence, using either Eq.(3.4) or Eq.(3.9),

$$(\nabla_\rho \partial_\sigma \alpha_{0\mu\nu, \mu'\nu'})|_{x'=x} = \frac{1}{4} (R_{\mu'}^{\mu}{}_{\rho\sigma} g_{\nu\nu'} + R_{\mu'}^{\mu}{}_{\nu\sigma} g_{\rho\rho'} + R_{\nu'}^{\nu}{}_{\rho\sigma} g_{\mu\mu'} + R_{\nu'}^{\nu}{}_{\mu\sigma} g_{\rho\rho'}). \quad (3.12)$$

We also have, from Eqs.(3.4), (2.11),

$$\alpha_{1\mu\nu, \rho\sigma}|_{x'=x} = \frac{1}{6} R_{\mu\nu, \rho\sigma} - \tilde{X}_{\mu\nu, \rho\sigma}. \quad (3.13)$$

We find

$$W_{1a}^{(2)NL} = \frac{G}{2\pi\epsilon} \frac{1}{\epsilon} \int d^4x \sqrt{g} (-\frac{1}{2} \nabla_\alpha H_{\beta}{}^{\gamma} \bar{\nabla}^\alpha + \frac{1}{2} \nabla_\alpha H_{\beta}{}^{\gamma} \bar{\nabla}^\alpha \\ + \frac{1}{2} \nabla_\alpha H_{\beta}{}^{\gamma} \bar{\nabla}^\alpha \bar{\nabla}^\alpha + \frac{1}{2} \nabla_\alpha H^{\gamma\alpha}{}_{,\beta} \bar{\nabla}^\beta - \nabla_\sigma H^{\beta\gamma}{}_{,\alpha} \bar{\nabla}^\alpha \\ - \frac{23}{12} R H^{\alpha}{}_{\beta}{}_{,\gamma} + \frac{5}{3} R H^{\alpha\beta}{}_{,\gamma} + \frac{13}{6} R_{\alpha\beta} H^{\alpha\beta}{}_{,\gamma} \\ + \frac{13}{6} R_{\alpha\beta} H^{\gamma}{}_{\gamma}{}^{\alpha\beta} - \frac{13}{3} R_{\alpha\beta} H^{\gamma\alpha}{}_{,\gamma}{}^{\beta})|_{x'=x} + \dots \quad (3.14)$$

This result, together with those following, was computed with the aid of the algebraic manipulation package FORM. The leading non-local poles appear to be $O(\epsilon^{-2})$ but are in fact $O(\epsilon^{-3})$ since they are written in terms of H , where according to Eq.(3.7), $H = P^{-1} \tilde{H}$ where P^{-1} contains a pole in ϵ .

The amplitude for Fig. 1(b) is of the form

$$W_{1b}^{(2)} = \int d^4x d^4x' \sqrt{g(x)} \sqrt{g(x')} T^{\alpha\beta\gamma\delta, \mu\nu} U^{\alpha'\beta'\gamma'\delta', \mu'\nu'} (x') \\ [G_{\alpha\beta, \alpha'\beta'} (\nabla_\mu G_{\gamma\delta, \gamma'\delta'}) (\nabla_\nu G_{\rho\sigma, \rho'\sigma'}) + \text{permutations}] \quad (3.15)$$

where T is the same vertex as in Fig. 1(a) and U involves a factor of the Ricci tensor. We follow the same procedure as before and obtain

$$W_{1b}^{(2)NL} = \frac{G}{2\pi\epsilon} \frac{1}{\epsilon} \int d^4x \sqrt{g} (RH^\alpha{}_\beta{}^\gamma{}_\gamma - R_{\alpha\beta} H^\gamma{}_\gamma{}^\alpha{}_\beta - R_{\alpha\beta} H^\gamma{}_\gamma{}^\alpha{}_\beta) |_{x'=x} + \dots \quad (3.16)$$

The amplitude for Fig. 1(c) is of the form

$$W_{1c}^{(2)} = \int d^4x \sqrt{g} V^{\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\lambda\mu\nu} G_{\alpha\beta\gamma\delta} |_{x'=x} \nabla_\mu G_{\rho\sigma\kappa\lambda} \bar{\nabla}_\nu |_{x'=x} \quad (3.17)$$

The singular behaviour may again be extracted by inserting the expansion Eq.(3.2) and now using

$$\begin{aligned} G_0 &\sim \frac{1}{2\pi\epsilon} \\ \nabla_\mu G_1 \bar{\nabla}_\nu |_{x'=x} &\sim \frac{1}{4\pi\epsilon} g_{\mu\nu} \\ \nabla_\mu G_0 \bar{\nabla}_\nu |_{x'=x} &\sim -\frac{1}{2\pi\epsilon} \frac{1}{6} R_{\mu\nu} \end{aligned} \quad (3.18)$$

We find

$$\begin{aligned} W_{1c}^{(2)NL} &= \frac{G}{2\pi\epsilon} \frac{1}{\epsilon} \int d^4x \sqrt{g} \left\{ \frac{3}{2} (\nabla_\alpha H_{\beta\gamma}^\beta{}^\gamma \bar{\nabla}^\alpha - \nabla_\alpha H_{\beta\gamma}^\beta{}^\gamma \bar{\nabla}^\alpha \right. \\ &\quad \left. - \nabla_\alpha H_{\beta\gamma}^\beta{}^\gamma \bar{\nabla}^\alpha - \nabla_\alpha H_{\beta\gamma}^\beta{}^\gamma \bar{\nabla}^\alpha + 2\nabla_\alpha H_{\beta\gamma}^\beta{}^\gamma \bar{\nabla}^\alpha \right. \\ &\quad \left. + \frac{11}{12} (RH^\alpha{}_\beta{}^\gamma{}_\gamma - RH_{\alpha\beta}^\alpha{}_\beta{}^\gamma{}_\gamma - R_{\alpha\beta} H^\gamma{}_\gamma{}^\alpha{}_\beta) \right. \\ &\quad \left. - R_{\alpha\beta} H^\alpha{}_\beta{}^\gamma{}_\gamma + 2R_{\alpha\beta} H^\alpha{}_\beta{}^\gamma{}_\gamma \right\} |_{x'=x} + \dots \end{aligned} \quad (3.19)$$

Finally, the amplitude for Fig. 1(d) is of the form

$$W_{1d}^{(2)} = \int d^4x \sqrt{g} W^{\alpha\beta\gamma\delta\epsilon\zeta\eta\theta\lambda\mu\nu} G_{\alpha\beta\gamma\delta} |_{x'=x} G_{\rho\sigma\kappa\lambda} |_{x'=x} \quad (3.20)$$

and we find, using Eq.(3.18),

$$\begin{aligned} W_{1d}^{(2)NL} &= \frac{G}{2\pi\epsilon} \frac{1}{\epsilon} \int d^4x \sqrt{g} \left\{ (RH^\alpha{}_\beta{}^\gamma{}_\gamma - 2RH_{\alpha\beta}^\alpha{}_\beta{}^\gamma{}_\gamma - 2RH_{\alpha\beta}^\alpha{}_\beta{}^\gamma{}_\gamma) \right. \\ &\quad \left. - 2R_{\alpha\beta} H^\gamma{}_\gamma{}^\alpha{}_\beta - 2R_{\alpha\beta} H^\alpha{}_\beta{}^\gamma{}_\gamma + 8R_{\alpha\beta} H^\alpha{}_\beta{}^\gamma{}_\gamma \right\} |_{x'=x} + \dots \end{aligned} \quad (3.21)$$

As mentioned earlier, the ghost graphs of Fig. 2 do not contribute at $O(\epsilon^{-2})$, so only the counterterm diagrams of Fig.3 remain to be considered. Z_c and Z_d in Eq.(2.5) contain no kinematic poles and hence we may disregard Fig. 3(b) to leading order as well. From Eqs.(2.1), (2.2), (A.4) and (A.5), the amplitude for Fig. 3(a) is

$$\begin{aligned} W_{3a}^{(2)} &= \frac{1}{4} \int d^4x \sqrt{g} (Z_G Z_h^2)^{(1)} (-\nabla_\alpha G_{\beta\gamma}^\beta{}^\gamma \bar{\nabla}^\alpha + \nabla_\alpha G_{\beta\gamma}^\beta{}^\gamma \bar{\nabla}^\alpha \\ &\quad + \nabla_\alpha G_{\gamma\gamma}^\gamma{}_\alpha \bar{\nabla}^\beta + \nabla_\alpha G_{\gamma\gamma}^\gamma{}_\alpha \bar{\nabla}^\beta - 2\nabla_\alpha G_{\gamma\gamma}^\beta{}_\beta \bar{\nabla}^\alpha \\ &\quad - \frac{1}{2} R G_{\alpha\beta}^\alpha{}_\beta{}^\gamma{}_\gamma + R G_{\alpha\beta}^\alpha{}_\beta{}^\gamma{}_\gamma + R_{\alpha\beta} G_{\gamma\gamma}^\gamma{}_\gamma{}^\alpha{}_\beta \\ &\quad + R_{\alpha\beta} G_{\gamma\gamma}^\gamma{}_\gamma{}^\alpha{}_\beta - 4R_{\alpha\beta} G_{\gamma\gamma}^\alpha{}_\beta{}^\beta{}_\gamma) \end{aligned} \quad (3.22)$$

The leading contribution to Eq.(3.22) comes solely from Z_h in Eq.(2.24) since Z_G in Eq.(2.19) (with $Z_g = 1$) has no kinematic pole. The non-local term $W_{3a}^{(2)NL}$ is obtained simply by replacing G by H in Eq.(3.22). Adding Eqs.(3.14), (3.16), (3.19), (3.21), and (3.22) with $G \rightarrow H$ and Z_h as in Eq.(2.24), we find

$$W^{(2)NL} = -\frac{G}{8\pi\epsilon} \frac{1}{\epsilon} \int \sqrt{g} (R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}) (H^\gamma{}_\gamma{}^\alpha{}_\beta + H^\alpha{}_\beta{}^\gamma{}_\gamma + 2H^\alpha{}_\gamma{}^\beta{}_\gamma) |_{x'=x} + \dots \quad (3.23)$$

All the terms involving derivatives of H have cancelled, leaving terms proportional to the Einstein tensor. The topological nature of the Einstein action in 2 dimensions implies that the Einstein tensor must vanish in exactly 2 dimensions, but it appears that one cannot on that account argue that it is $O(\epsilon)$ away from 2 dimensions[14]. Hence we cannot unequivocally state even that the leading $O(\epsilon^{-2}H)$ terms cancel. Moreover, although it is impossible to determine in general whether the non-local terms cancel at next-to-leading order, i.e. $O(\epsilon^{-1}H)$, it seems highly unlikely that they do. At this order there are contributions from the finite parts of the singular products in Eq.(3.8) which are themselves potentially non-local, and hence depend on knowledge of the global nature of the spacetime. For conventional renormalizable field theories, the non-local pole terms cancel identically, independently of the form of the non-local term H and of the global properties of the spacetime, and there is no reason to expect otherwise in this case. Moreover it is impossible to extract Z_h at this order from the calculation of Appendix B. If one tries to circumvent the first problem by adopting the prescription of ignoring finite parts in Eq.(3.8), one finds that there is no possible choice for the $O(\epsilon^{-1})$ part of Z_h which would make the resulting non-local $O(\epsilon^{-1}H)$ terms cancel. These divergences are really $O(\epsilon^{-2})$, since $H = P^{-1}\bar{H}$ with P^{-1} as in Eq.(2.14), and $\bar{H}$ is the true, finite non-local part of the propagator $\bar{G}$. These divergences cannot be removed by redefining the couplings of the theory. Hence the theory is non-renormalizable, and it is impossible to extract Z_G and obtain the β -function for G . We should emphasize that our findings are not simply an artefact of the background field method in configuration space. The cancellation of non-local terms H corresponds precisely to the cancellation of contributions from the finite parts of momentum integrals in momentum space calculations, involving $\ln p^2$ type terms.

Of course, in the presence of kinematic poles, it would perhaps be unreasonable to expect all non-local poles to cancel. KN show that the two-loop β_G is expected to be $O(\epsilon^{-1})$, rather than finite as in the usual case—in other words the 2-loop result is expected to contain one factor of ϵ^{-1} from a kinematic pole. Since for a conventional theory one certainly has finite non-local contributions, one might expect $O(\epsilon^{-1})$ non-local terms in the case at hand. One could then still evaluate the leading $O(\epsilon^{-1})$ term in β_G unambiguously, since this comes from the $O(\epsilon^{-2})$ term in Z_G . This would probably suffice for the computation of the anomalous dimension of some operator at the fixed point of β_G as $\epsilon \rightarrow 0$. However, our calculation shows that the situation is worse than one might have hoped a priori since the non-local terms come with two kinematic pole factors rather than only one, and hence there is no possibility of obtaining even the leading term in Z_G .

Conclusions

The ideas of KN appeared to offer access to the exact results of two-dimensional quantum gravity via a conventional perturbative calculation, using the standard Einstein action. By measuring the anomalous dimensions of operators at the fixed point of β_G , and then proceeding to the limit $\epsilon = 0$, they conjectured that one would regain the exact two-dimensional results. However, we have shown that there are insuperable obstacles to this programme, in the form of intractable non-local pole terms of $O(\epsilon^{-2})$ in the effective action, which prevent the calculation of β_G beyond one loop. This also presents a grave impediment to the “asymptotic safety” programme proposed by Weinberg[15].

Other authors [16,17] have discussed perturbative calculations of two-dimensional quantum gravity using Polyakov’s induced action[1]. In Ref.[16] the leading term in the expansion of Polyakov’s exact result in powers of $\frac{1}{\epsilon}$ is obtained, in this case by calculating with the induced action in the light-cone gauge using a momentum cut-off. In Ref.[17] dimensional regularisation is applied to the induced action. It would be interesting to see if these approaches may be consistently implemented at the two-loop level.

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Appendix A

In this appendix we record the expansion up to fourth order of the Einstein action $S = \int d^d x \sqrt{g} R$.

$$S[g_{\mu\nu} + h_{\mu\nu}] = \sum_i S^{(i)}[g_{\mu\nu}, h_{\mu\nu}] \quad (A.1)$$

with

$$S^{(0)} = S[g], \quad (A.2)$$

$$S^{(1)} = \int d^d x \sqrt{g} \left(\frac{1}{2} R h - R_{\mu\nu} h^{\mu\nu} - \nabla^2 h + \nabla_\mu \nabla_\nu h^{\mu\nu} \right), \quad (A.3)$$

defining $h \equiv h^\mu{}_\mu$,

$$S^{(2)} = \frac{1}{2} \int d^d x \sqrt{g} \left(X[h_{\mu\nu}] - \frac{1}{2} R h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} R h^2 - h h^{\mu\nu} R_{\mu\nu} + 2 h^{\mu\rho} h_\rho{}^\nu R_{\mu\nu} \right) \quad (A.4)$$

where

$$X[h_{\mu\nu}] = \frac{1}{2} h_{,\mu} h^{,\mu}{}_{,\nu} - h^{,\mu}{}_{,\nu} h^{,\nu}{}_{,\mu} - \frac{1}{2} h_{\mu\nu;\rho} h^{\mu\nu;\rho} + h_{\mu\nu;\rho} h^{\mu\rho;\nu}, \quad (A.5)$$

$$S^{(3)} = \int d^d x \sqrt{g} \left(\frac{1}{4} h X - \frac{1}{4} h^{\mu\nu} Y_{\mu\nu} + \frac{1}{4} h_{\mu\nu} h^{\mu\nu}{}_{;\alpha\beta} R^{\alpha\beta} + \frac{1}{2} h h_{\mu\rho} h^\rho{}_\nu R^{\mu\nu} - \frac{1}{8} h h_{\mu\nu} h^{\mu\nu} R - \frac{1}{8} h^2 h^{\mu\nu} R_{\mu\nu} + \frac{1}{48} h^3 R + \frac{1}{6} h_{\mu\nu} h^{\nu\rho} h_\rho{}^\mu R - h_{\mu\nu} h^{\nu\rho} h_{\rho\sigma} R^{\mu\sigma} \right) \quad (A.6)$$

where

$$Y_{\mu\nu} = -2 h_{\mu\nu;\rho} h^{\rho\sigma}{}_{;\sigma} + 2 h_{\mu\nu;\rho} h^{\rho\sigma}{}_{;\sigma} + h_{,\mu} h_{,\nu} - 2 h_{,\mu} h_{,\nu}{}^\rho - h_{\rho\sigma;\mu} h^{\rho\sigma}{}_{;\nu} + 4 h_{\rho\mu;\sigma} h^{\rho\sigma}{}_{;\nu} - 2 h_{\rho\mu;\sigma} h_{\nu}{}^{\rho\sigma} - 2 h_{,\rho} h^\rho{}_{;\mu\nu} + 2 h_{\mu\rho;\sigma} h_{\nu}{}^{\sigma\rho}, \quad (A.7)$$

$$S^{(4)} = \int d^d x \sqrt{g} \left\{ \frac{1}{16} (h^2 - 2 h_{\mu\nu} h^{\mu\nu}) X - \frac{1}{8} (h h_{\mu\nu} - 2 h_{\mu\rho} h^\rho{}_\nu) Y^{\mu\nu} + h^{\mu\rho} h^{\nu\sigma} Z_{\mu\rho,\nu\sigma} - \frac{1}{6} h_{\mu\nu} h^{\nu\rho} h_\rho{}^\mu h^{\sigma\beta} R_{\alpha\beta} - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} h^{\sigma\gamma} h_\gamma{}^\beta R_{\alpha\beta} - \frac{1}{2} h h_{\alpha\gamma} h^{\gamma\delta} h_{\delta\beta} R^{\alpha\beta} + \frac{1}{8} h h_{\mu\nu} h^{\mu\nu} h^{\alpha\beta} R_{\alpha\beta} + \frac{1}{8} h^2 h_{\sigma\gamma} h^{\gamma\beta} R^{\alpha\beta} - \frac{1}{48} h^3 h^{\alpha\beta} R_{\alpha\beta} + h_{\alpha\gamma} h^{\gamma\delta} h_{\delta\rho} h^\rho{}_\beta R^{\sigma\beta} + R \left[\frac{1}{32} (h_{\mu\nu} h^{\mu\nu})^2 + \frac{1}{12} h_{\mu\nu} h^{\nu\rho} h_\rho{}^\mu - \frac{1}{32} h^2 h_{\mu\nu} h^{\mu\nu} - \frac{1}{8} h_{\mu\nu} h^{\nu\rho} h_{\rho\alpha} h^{\alpha\mu} + \frac{1}{384} h^4 \right] \right\} \quad (A.8)$$

where

$$Z_{\mu\rho,\nu\sigma} = h_{\mu\rho;\nu} h_{\sigma\alpha}{}^{;\alpha} - h_{\mu\rho;\sigma} h_{\nu}{}^{\alpha} + \frac{1}{2} h_{\mu\nu;\alpha} h_{\rho\sigma}{}^{;\alpha} - \frac{1}{2} h_{\mu\rho;\alpha} h_{\nu\sigma}{}^{;\alpha} + h_{\alpha\mu;\nu} h^{\alpha}{}_{\rho\sigma} - h_{\alpha\mu;\sigma} h^{\alpha}{}_{\nu\rho} + h_{\mu\nu;\sigma} h_{\rho}{}^{\alpha} - 2 h_{\alpha\mu;\nu} h_{\rho\sigma}{}^{;\alpha} + h_{\mu\rho;\alpha} h^{\alpha}{}_{\nu\sigma}. \quad (A.9)$$

The quantities X, Y and Z are not independent; in fact they are related by

$$Z_{\alpha}{}^{\alpha}{}_{,\mu\nu} + Z_{\mu\nu,\alpha}{}^{\alpha} = -Y_{\mu\nu} \\ Y_{\mu}{}^{\mu} = 6X. \quad (A.10)$$

Eqs.(A.4) and (A.5) agree with the results of Ref.[8]; Eqs.(A.6)-(A.9) agree with Ref.[18] apart from some discrepancies of sign.

Appendix B

In this Appendix we discuss the calculation of the quantum field renormalisation Z_h at one loop. This is determined by requiring that

$$\langle h_{\mu\nu}(x) \rangle = \int d^d x' \sqrt{g(x')} G_{\mu\nu,\rho\sigma}(x, x') \bar{h}_{\rho\sigma}(x') \quad (B.1)$$

is finite, where $\bar{h}_{\rho\sigma}$ is the h -to-vacuum amplitude. $\bar{h}_{\rho\sigma}$ is given at one loop by the diagrams in Fig. 4, where again vertices involving the Ricci tensor are marked R. Using Eqs.(3.10), (3.11) and (3.17), and the expansions in Appendix A, we find that the pole term in Fig. 4(a) is given by

$$\begin{aligned} \bar{h}_{\mu\nu}^{(a)} \sim & \frac{1}{16\pi\epsilon} ((P^{-1}a_1)_{\alpha}{}^{\alpha}{}_{,\beta} g_{\mu\nu} - (P^{-1}a_1)_{\alpha\beta}{}^{,\alpha\beta} g_{\mu\nu} \\ & + 2(P^{-1}a_1)_{\alpha\mu}{}_{,\nu}{}^{\alpha} - (P^{-1}a_1)_{\alpha}{}^{\alpha}{}_{,\mu\nu} - (P^{-1}a_1)_{\mu\nu,\alpha}{}^{\alpha}) \end{aligned} \quad (B.2)$$

while that in Fig. 4(b) is given by

$$\bar{h}_{\mu\nu}^{(b)} \sim -\frac{1}{2\pi\epsilon} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}). \quad (B.3)$$

Figs. 4(c) and (d) yield no leading $O(\epsilon^{-2})$ contribution, while Fig. 4(e) gives

$$\bar{h}_{\mu\nu}^{(e)} = Z_h Z_G (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}). \quad (B.4)$$

Using Eqs.(2.10), (2.13), (2.14) and (3.12), we find that Eq.(B.2) in fact has no leading pole, and hence from Eqs.(B.3) and (B.4) we see that $\langle h \rangle$ is made finite to leading order by taking

$$Z_h^{(1)} = \frac{1}{2\pi\epsilon} G + \dots \quad (B.5)$$

The non-leading $O(\epsilon^{-1})$ pole in $\bar{h}_{\mu\nu}$ is bedevilled by the same problems as we have encountered in the main text in the context of the two-loop calculation—namely $\bar{h}_{\mu\nu}$ can receive contributions from the finite parts of singular quantities such as those in Eq.(3.17). Even if we ignore such contributions, we find that the sum of the $O(\epsilon^{-1})$ contributions from Figs. 4(a)-(d) is not proportional to $(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu})$ and hence it is impossible to choose Z_h in Eq.(B.4) so as to cancel this pole.

In principle we could calculate Z_c and Z_d , at least to leading order, in a similar fashion by considering one-loop graphs with external ghost lines. Since Z_c and Z_d do not contain $O(\epsilon^{-2})$ poles we have omitted this calculation.

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Figure Captions

Fig. 1: Two-loop graphs with no ghost propagators (with full lines representing graviton propagators).

Fig. 2: Two-loop graphs with ghost propagators (represented by dashed lines).

Fig. 3: Counterterm graphs at two loops, with X representing one-loop counterterm insertion.

Fig. 4: One-loop graphs contributing to $\bar{h}_{\rho\sigma}$.

Figures

Fig. 1

Fig. 2

Fig. 3

Fig. 4