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THE TURAEV-VIRO STATE SUM MODEL AND 3-DIMENSIONAL QUANTUM GRAVITY

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Abstract

We discuss the relationship between the Turaev-Viro state sum model and 3-dimensional quantum gravity. In particular we study the semiclassical limit of trivalent graph invariants in the state sum model. These are shown to behave like parallel transport operators along the graph links.

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1. Introduction

In 1968 Regge and Ponzano pointed out a curious relationship between the semiclassical (large spin) limit of 6j-symbols of angular momentum theory and a Euclidean path integral expression for simplicial 3-dimensional quantum gravity with an Einstein-Regge action [1]. Recent work by Turaev and Viro [2], has gone a long way to help understand the Regge-Ponzano observation.

Turaev and Viro's state sum model shows one how to construct a 3-dimensional topological quantum field theory whose basic ingredients are the quantum 6j-symbols associated with the quantum group $U_q(sl_2C')$, where q is a root of unity. Turaev later extended these results to include 3-dimensional cobordisms with coloured trivalent framed graphs sitting inside the manifold, hence producing invariants of such graphs in 3-manifolds. In the case of S^3 these invariants are generalizations of the Jones polynomial. This model is described here in a form suitable to examine its semiclassical limit. This form is also suitable for describing the natural extension of the model which includes framed graphs with free ends lying on the manifold boundary and associates vector spaces with marked (or punctured) 2-dimensional surface.

There are two lines of argument to relate this TQFT to 3-dimensional quantum gravity. Firstly it has recently been proven that it is closely related to Witten's TQFT [3], based on a Chern-Simons action with group $SU(2)$ at level k , with $q = e^{2\pi i/(k+2)}$. The manifold invariants of the T.V. theory are the modulus squared of the invariants of Witten's theory, which can be regarded as an Euclidean functional integral for quantum gravity in the following way.

$$\begin{aligned} Z &= \left| \int [dA] \exp \left(\frac{ik}{4\pi} \int (AdA + \frac{2}{3}A^3) \right) \right|^2 \\ &= \int [dA, dB] \exp \left(\frac{ik}{4\pi} \int (AdA + \frac{2}{3}A^3) - (BdB + \frac{2}{3}B^3) \right) \\ &= \int [de, d\omega] \exp \left(i \int e \wedge R + \frac{\Lambda}{3} e \wedge e \wedge e \right) \end{aligned} \quad (1)$$

where e is a dreibein, ω is a spin connection on the manifold, $R = d\omega + \omega \wedge \omega$, $\omega = \frac{1}{2}(A+B)$, $e = \frac{k}{8\pi}(A-B)$, and cosmological constant $\Lambda = (\frac{4\pi}{k})^2$.

The Regge-Ponzano result about the large spin limit of 6j-symbols gives the other connection of the Turaev-Viro state sum model to 3-dimensional quantum gravity. This is discussed in the section 3. and we extend their ideas to the semi-classical limit of graph expectation values (or graph invariants) in the state sum model. We find that these resemble a Regge calculus, Wilson graph expectation value, path integral.

2. Turaev-Viro Theory

Here we describe the basic ingredients and procedure for constructing the invariants of trivalent graphs in 3-manifolds, and the associated TQFT. The reader is referred to [2] for details of proofs and generalisations. We work in the dual picture to Turaev's work on graph invariants, this being more suitable for studying the geometry of the model and its connection with Regge calculus.

A. Initial Data

- Fix an integer $r \geq 3$
- Denote by I the set $\{0, \frac{1}{2}, 1, \dots, \frac{r-2}{2}\}$
- Set $q = \exp(\frac{2\pi\sqrt{-1}}{r})$
- for integers $n \geq 1$ set $[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$ and $[n]! = [n][n-1] \dots [2][1]$
- The triple $(i, j, k) \in I^3$, is said to be admissible if $i+j+k$ is a positive integer, $i+j+k \leq r-2$ and $i \leq j+k$, $j \leq i+k$, $k \leq i+j$
- The 6-tuple $(i, j, k, l, m, n) \in I^6$ is said to be admissible if each of the triples (i, j, k) , (k, l, m) , (m, n, i) , and (j, l, n) are admissible.

-For an admissible 6-tuple (i, j, k, l, m, n) , set

$$\left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right| = (\sqrt{-1})^{2(i+j+k+l+m+n)} \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}_q^{RW}$$

where $\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}_q^{RW}$ is the Racah-Wigner 6j-symbol. (See [2] for explicit formula)

-Set $\left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right| = 0$, if (i, j, k, l, m, n) is not an admissible 6-tuple.

-For $j \in I$ set $q_j = \exp(\frac{2\pi\sqrt{-1}}{r}(j - \frac{j(j+1)}{r}))$, $\omega_j = (\sqrt{-1})^{2j}[2j+1]^{\frac{1}{2}}$, and $\omega = \frac{\sqrt{2r}}{|q|^{1/2} - q^{-1/2}}$

B. Trivalent graphs in 3-manifolds.

A trivalent graph G in a 3-manifold M is a 1-dimensional structure such that each point has a neighbourhood homeomorphic to a line, three half-lines meeting at their common ends, or a half-line, and embedded in M such that $\partial M \cap G$ are the points of G with neighbourhoods homeomorphic to half-lines. A colouring of G is a map which associates an element of I with each edge of G such that the three colours at any vertex form an admissible triple. A framing of a trivalent graph G is a map from the integers to each link of G (which can be thought of as the twist in each link of a fattened ribbon graph).

C. Constructing the state sum.

The state sum for the invariant of a coloured trivalent framed graph G in a 3-manifold M is constructed in the following way.

First choose a triangulation M_T of the 3-manifold with embedded, coloured graph G which has the following properties. The graph may not meet the 0-simplices or 1-simplices and must cross the 2-simplices transversally. The intersection of each 3-simplex (tetrahedron) with the graph must be either empty, a line element with ends in distinct 2-simplices, or a trivertex with ends in distinct 2-simplices. Let a be the number of 0-simplices of M_T . Suppose that e of them lie on the boundary ∂M . Let $E_1, \dots, E_k$ be the 1-simplices of M_T ,

FIG. 1

FIG. 3

FIG. 2

FIG. 4

and let $Y_1, \dots, Y_d$ be the tetrahedra of M_T . Let exactly the first f of the edges lie on ∂M . The twist associated with each link is now factored between the tetrahedra containing it so that the link elements in each tetrahedron now have integral twist indices. Denote by $N_j(Y_t)$ the twist of the link with colour j in the tetrahedra Y_t . Denote this construction (M_T, G) . A colouring of (M_T, G) is a map from I to the 1-simplices of the triangulation, together with a representation index δ_j associated with each point at which the graph meets a 2-simplex. j is the colour of the link at this point and $\delta_j \in \{-j, -j+1, \dots, j-1, j\}$

The graph is now shifted inside each tetrahedra to a neighbourhood of its boundary so that each non-empty tetrahedron now resembles either figure 1, 2 or 3. The tetrahedra may now be split into 4 disjoint sets, coloured tetrahedra of types 1, 2, 3 and 4. For each colouring η of (M_T, G) a complex number $[M, G]_\eta$ is associated with the construction, by first associating a complex number $Y_t^{\eta}(\Phi)$ with each tetrahedron $Y_t(\Phi)$, of type Φ for $\Phi = 1, 2, 3$ and 4.

For η corresponding to the colouring in fig. 1.

$$Y_t^\eta(1) = (-1)^{2(i+\delta_p)} \omega_{i+\delta_p} (-1)^{2(j+\delta_p)} \omega_{j+\delta_p} q_i q_j q_{i+\delta_p}^{-1} q_{j+\delta_p}^{-1} q_p^{-4N_p(Y_t)} \times \left| \begin{matrix} p & i+\delta_p & i & j+\delta_p & i+\delta_p & m \\ m & j & j+\delta_p & l & n & k \end{matrix} \right| \quad (2)$$

For η corresponding to the colouring in fig. 2.

$$Y_t^\eta(2) = (-1)^{2(k+\delta_{p_2})} \omega_{k+\delta_{p_2}} (-1)^{2(k+\delta_{p_3})} \omega_{k+\delta_{p_3}} (-1)^{2(j+\delta_{p_1})} \omega_{j+\delta_{p_1}} q_j q_{k+\delta_{p_2}} q_{j+\delta_{p_1}}^{-1} q_{k+\delta_{p_3}}^{-1} \times q_{p_1}^{-4N_{p_1}(Y_t)} q_{p_2}^{-4N_{p_2}(Y_t)} q_{p_3}^{-4N_{p_3}(Y_t)} (-1)^{p_1+p_2+p_3} (\omega_{p_1} \omega_{p_2} \omega_{p_3})^{1/2} \times \left| \begin{matrix} p_1 & j+\delta_{p_1} & j & p_2 & p_3 & j+\delta_{p_1} & i & m \\ n & k+\delta_{p_3} & k+\delta_{p_2} & k & k+\delta_{p_3} & k+\delta_{p_2} & l & n & k+\delta_{p_2} \end{matrix} \right| \quad (3)$$

For η corresponding to the colouring in fig. 3.

$$Y_t^\eta(3) = \delta_{\delta_p, \delta_p} q_p^{-4N_p(Y_t)} \left| \begin{matrix} j & i+\delta_p & m \\ l & n & k \end{matrix} \right| \quad (4)$$

For η corresponding to the colouring in fig. 4.

$$Y_t^\eta(4) = \left| \begin{matrix} j & i & m \\ l & n & k \end{matrix} \right| \quad (5)$$

We then define:

$$|M, G|_\eta = \omega^{-2a+c} \prod_{r=1}^f \omega_{\eta(E_r)} \prod_{s=f+1}^b \omega_{\eta(E_s)}^2 \prod_{t=1}^d Y_t^\eta \quad (6)$$

A triangulation of a marked closed surface is a triangulation of the surface such that each marked point is associated with a 2-simplex and a 1-simplex in its boundary (see figs. 5, 6.) Each 2-simplex may have at most one marked point associated with it. A colouring of such a marked surface triangulation is a map from I to the links of the triangulation, together with a representation index δ_j associated to each marked point. Clearly in the above construction a colouring of (M, G) induces a colouring of the triangulated manifold surface. A colouring of a marked surface is said to be admissible if in every empty 2-simplex,

FIG. 5

FIG. 6

coloured as in fig. 5., (i, j, k) form an admissible triple, and in every 2-simplex containing a marked point, coloured as in fig. 6., $(i + \delta_p, j, k)$ and $(i, p, i + \delta_p)$ form admissible triples.

Denote by $adm(\partial M, G)$ the set of admissible colourings of the marked triangulated surface in the above construction. Then for $\alpha \in adm(\partial M, G)$, define the state sum;

$$\Omega_{M, \alpha, G} = \sum_{\eta_\alpha} |M, G|_\eta \quad (7)$$

the sum being over all colourings η_α of (M, G) , agreeing with α on ∂M .

Theorem Let (M, α, G) be a compact 3-manifold with embedded coloured, framed graph and a suitably triangulated marked surface with colouring α , as boundary. Then any construction of (M, G) extending this surface triangulation results in the same value for the state sum $\Omega_{M, \alpha, G}$ defined above.

D. Comments on the theorem.

(i) When there is no framed graph present, the proof of this theorem [2] involves showing that the state sum is invariant under certain moves which alter the triangulation and relate any two distinct triangulations of $Int M$. This is more easily done if one translates the state sum construction to one on the dual cell subdivision of the triangulated 3-manifold. Due to work of Alexander and Newman in the 30's there are known to be three local moves on the dual 2-skeleton, a finite sequence of which can relate any two triangulations, so the proof involves showing that the initial data satisfies three identities. When a framed graph

is present, $\Omega_{M,\alpha,G}$ is also an invariant of the graph in $IntM$. To prove this it is shown to be invariant under a set of generalised Reidemeister moves on the dual 2-skeleton.

(ii) Turaev and Viro use the state sum to construct a topological quantum field theory, associating a finite dimensional vector space with each closed surface, and homomorphisms, associated with 3-manifolds, between the vector spaces of its boundary components. A straight-forward generalisation of this now produces a TQFT which associates a vector space with every marked closed surface and for every 3-manifold with boundary $S_1 \cup S_2$, $(S_1 \cap S_2 = \emptyset, \partial S_1 = \partial S_2 = \emptyset)$ and embedded coloured, framed, trivalent graph, a homomorphism between the vector space of S_1 and that of S_2 . The space generated by admissible colourings of a marked triangulated surface S must be factored out by the kernel of the identity cobordism (here this would be the homomorphism associated with manifold $S \times [0, 1]$ with trivial graph connecting markings on $S \times 0$ and $S \times 1$). This will be discussed in more detail elsewhere and used to study the Verlinde Algebra associated with this TQFT.

3. The semiclassical limit, and Regge calculus.

The relationship of T.V. theory with Witten's Chern-Simons theory tells us that the $r \rightarrow \infty$ limit corresponds to the semi-classical limit, $\hbar \rightarrow 0$. In this limit the quantum deformation parameter $q \rightarrow 1$ and $[n] \rightarrow n$, and hence quantum 6j-symbols $\rightarrow$ classical 6j-symbols.

If one now assumes that the state sum for a graph invariant is dominated by the large spin (colouring) terms in the sum one can use the results of Regge and Ponzano to get a geometric interpretation of the state sum. We proceed by examining the state sum expression when limited to large internal link colourings. For this we need asymptotic values of 6j-symbols with different numbers of small arguments (corresponding to finite values of the colourings on the graph links) and large arguments (corresponding to large valued colourings of the internal 1-simplices). From equations (2)...(5) it is clear that we need to consider the following four cases. These are all results from Regge and Ponzano's 1968 paper, which show that it is the association of link colourings in the triangulation

with link lengths that is the key to seeing the geometry in the model. Note that the admissibility relations translate as triangle inequalities.

(i) All arguments large:

$$\begin{vmatrix} j & i & m \\ l & n & k \end{vmatrix} \simeq \frac{q^{-1} (-1)^{i+j+k+l+m+n}}{(12\pi V(Y_t))^{1/2}} \cos(i\theta_i(t) + j\theta_j(t) + k\theta_k(t) + l\theta_l(t) + m\theta_m(t) + n\theta_n(t) + \frac{\pi}{4}) \quad (8)$$

where $V(Y_t)$ is the volume of the tetrahedra Y_t , with edge lengths $i + \frac{1}{2}, j + \frac{1}{2}, k + \frac{1}{2}, l + \frac{1}{2}, m + \frac{1}{2}, n + \frac{1}{2}$, and $\theta_i(t)$ is the angle between the outward normals to the two faces meeting at the edge coloured i (similarly for $\theta_j, \theta_k, \theta_l, \theta_m, \theta_n$).

(ii) All arguments large, but some differing by small amounts from case (i): $(\delta_p, \delta'_p, \bar{\delta}_p \ll i, j, k, l, m, n)$

$$\begin{vmatrix} j + \delta'_p & i + \delta_p & m \\ l & n & k + \bar{\delta}_p \end{vmatrix} \simeq \frac{q^{-1} (-1)^{\delta_p + \delta'_p + \bar{\delta}_p} \exp(\delta_p \frac{\partial}{\partial i} + \delta'_p \frac{\partial}{\partial j} + \bar{\delta}_p \frac{\partial}{\partial k})}{(2i+1)^{1/2} (2j+1)^{1/2}} d_{\delta_p, \delta'_p}^{(p)}(\Theta) \quad (9)$$

(iii) Five arguments large, one small: $(p, \delta_p, \delta'_p \ll i, j, m)$

$$\begin{vmatrix} p & i + \delta_p & i \\ m & j & j + \delta'_p \end{vmatrix} \simeq \frac{q^{-1} (-1)^{3i+3j+2m+2p+2\delta_p+\delta'_p}}{(2i+1)^{1/2} (2j+1)^{1/2}} d_{\delta_p, \delta'_p}^{(p)}(\Theta) \quad (10)$$

where $d_{\delta_p, \delta'_p}^{(p)}(\Theta)$ is a rotation matrix, in representation p , through Θ , which is the angle between i and j in the triangle with sides i, j and m .

(iv) Three arguments large, three small: $(p_1, p_2, p_3, \delta_{p_1}, \delta_{p_2}, \delta_{p_3} \ll k)$

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ k & k + \delta_{p_3} & k + \delta_{p_2} \end{vmatrix} \simeq \frac{q^{-1} (-1)^{k+3\delta_{p_2}+3\delta_{p_3}} (2k)^{-1/2}}{\delta_{p_3} - \delta_{p_2} \delta_{p_3}} \begin{pmatrix} p_1 & p_2 & p_3 \\ \delta_{p_3} & \delta_{p_2} & -\delta_{p_3} \end{pmatrix} \quad (11)$$

where $\begin{pmatrix} p_1 & p_2 & p_3 \\ \delta_{p_3} & \delta_{p_2} & -\delta_{p_3} \end{pmatrix}$ is a 3j-symbol. (Clebsch-Gordan coefficient) [1].

In the case where there is no embedded graph ($G = 0$), using equation (8) the large spin

limit of $|M, 0\rangle_\eta$ can be written as

$$|M, 0\rangle_\eta \simeq \prod_{r=1}^{q-1} (2\eta(E_r) + 1) \exp\left(\pi\sqrt{-1}\eta(E_r)(2 - N(E_r))\right)^{-1/2} \prod_{i=1}^d (48\pi V(Y_i))^{-1/2} \left\{ \exp\left(\sqrt{-1}\sum_{s=t_1}^{t_6} \eta(E_s)\theta_s(t) + \frac{\pi}{4}\right) + \exp\left(-\sqrt{-1}\sum_{s=t_1}^{t_6} \eta(E_s)\theta_s(t) + \frac{\pi}{4}\right) \right\} \quad (12)$$

where $\eta(E_r)$ is the colour of 1-simplex E_r ; $E_{t_1}, \dots, E_{t_6}$ are the 1-simplexes in tetrahedron t , and $N(E_r)$ is the number of tetrahedra sharing the 1-simplex E_r . Regge and Ponzano noticed that the first term in the expansion (the term with positive frequency exponential from each tetrahedron) looks like $\exp(\sqrt{-1}S_R)$, where

$$S_R = \sum_{r=1}^b \eta(E_r) \epsilon_r \quad (13)$$

with $\epsilon_r = 2\pi - \sum_{i=1}^{N(E_r)} (\pi - \theta_i(t))$, the deficit angle at 1-simplex E_r . Hence the state sum $\sum_\eta |M, 0\rangle_\eta$ resembles a quantum gravity path integral. It was suggested, [1] that the contribution to Ω_M from the other $2^d - 1$ terms vanished in the continuum limit. Hasslacher and Perry claim in [4] that detailed analysis confirms this and that there is only a contribution to the WKB limit from the first of the 2^d terms.

We show that the semi-classical limit of $\Omega_{M, \alpha, G}$ similarly resembles a path integral for the expectation value of a certain function, W_G , of the graph G . From equations (8), ..., (11), the semiclassical limits of the Y_i^η 's are given by

$$\begin{aligned} Y_i^\eta(1) &\simeq (-1)^{2p} (\sqrt{-1})^{-\delta_p - \delta'_p} d_{\delta_p, \delta'_p}^{(p)}(\Theta) \exp(\delta_p \frac{\partial}{\partial \delta'_i} + \delta'_p \frac{\partial}{\partial \delta_i}) \left| \begin{matrix} j & i & m \\ l & n & k \end{matrix} \right| \\ Y_i^\eta(2) &\simeq (-1)^{p_1 + p_2 + p_3} (\omega_{p_1} \omega_{p_2} \omega_{p_3})^{1/2} (\sqrt{-1})^{-\delta_{p_1} - \delta_{p_2} - \delta_{p_3}} (-1)^{\delta_{p_2} + \delta_{p_3}} d_{\delta_{p_1}, \delta_{p_2}, \delta_{p_3}}^{(p_1)}(\Theta) \\ &\quad \times \left(\begin{matrix} p_1 & p_2 & p_3 \\ \delta_{p_2} - \delta_{p_3} & \delta_{p_2} & -\delta_{p_3} \end{matrix} \right) \exp(\delta_{p_1} \frac{\partial}{\partial \delta_j} + \delta_{p_2} \frac{\partial}{\partial \delta_k}) \left| \begin{matrix} j & i & m \\ l & n & k \end{matrix} \right| \\ Y_i^\eta(3) &\simeq \delta_{\delta_p, \delta'_p} (-1)^{2p} (-1)^{-\delta_p} \exp(\delta_p \frac{\partial}{\partial \delta'_i}) \left| \begin{matrix} j & i & m \\ l & n & k \end{matrix} \right| \end{aligned} \quad (14)$$

FIG. 7

FIG. 8

We again restrict ourselves to the positive frequency term from the contribution Y_i of each tetrahedron. This means that we have the correspondence $\frac{\partial}{\partial \delta'_i} \sim \sqrt{-1}\theta_i(t)$ in equation (14) showing that the graph elements contribute exactly the rotation matrices for parallel transport along the graph links. The colour of the graph link is the representation of $SU(2)$ that the rotation matrix is in, and at trivertices these are fused by the appropriate Clebsch-Gordan coefficients. This is made clear by considering cartesian coordinates at each point where the graph intersects 2-simplices as in figures 7 and 8. A basis for the $(2p+1)$ -dimensional representation of $SU(2)$ is chosen at this point to be the set of angular momentum states $|p, \delta_p\rangle$, (eigenstates of J_z , and J^2). Then recall [5] that the matrix element for a rotation through Euler angles $(\theta_1, \theta_2, \theta_3)$ is

$$\langle p, \delta'_p | R(\theta_1, \theta_2, \theta_3) | p, \delta_p \rangle = e^{-\sqrt{-1}\theta_1 \delta_p} d_{\delta_p, \delta'_p}^{(p)}(\theta_2) e^{-\sqrt{-1}\theta_3 \delta'_p} \quad (15)$$

Given a set of orthonormal vectors (x, y, z) this rotation is the following operation: a rotation about the z -axis through angle θ_3 , followed by a rotation about the resulting y -axis through angle θ_2 , followed by a rotation about the resulting z -axis through angle θ_1 . With the identifications $\theta_1 = -\theta_i(t)$, $\theta_2 = \Theta$, and $\theta_3 = -\theta_j(t)$, this is exactly the rotation between coordinate systems x, y, z and $\hat{x}, \hat{y}, \hat{z}$ at the end points of the graph line in figure 8 and the positive frequency part of the ratio $Y_i^\eta(2)/Y_i^\eta(0)$ is the above matrix element. Note that the $(-1)^{p-\frac{1}{2}}$ at each end of the graph line is needed for the π rotation, or orientation switch, in going between neighbouring tetrahedra.

Hence, in the semiclassical limit W_G is the Wilson graph operator corresponding to parallel

transport along the graph links. This is the expected result from examining the equivalences between the graph invariants of Turaev's state sum model and those of Witten's Chern-Simons theory. The exact relationship for closed 3-manifolds [6] is

$$|M, G| = \tau(M, G) \overline{\tau(M)} \quad (16)$$

where $\tau(M, G)$ is the Chern-Simons Wilson graph expectation value (or the equivalent graph invariant defined by Reshetikin and Turaev in [7]). Hence using the identifications shown in the introduction, the right-hand side of (16) can be written as $\int [dc, d\omega] \text{Tr}(e^{\int_{\mathcal{O}} (\omega + \frac{1}{2}e)}) e^{iS}$, where $\text{Tr}(e^{\int_{\mathcal{O}} (\omega + \frac{1}{2}e)})$ is schematic for the holonomy of $\omega + \frac{1}{2}e$ along the graph links and fused by appropriate Clebsch-Gordan coefficients at trivertices. Semiclassically, $k \rightarrow \infty$, and this reduces to the holonomy of the connection ω , or parallel transport, along the graph lines.

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