A METHOD FOR CALCULATING DISTURBANCES

IN CONSTANT GRADIENT MAGNETS AND ITS APPLICATION

TO THE VACUUM CHAMBER EFFECT.

Summary.

By using the conformal transformation \( w'' = w^2 \), the hyperbolical polefaces of constant gradient magnets are transformed into parallel planes. The design of a constant gradient magnet (synchrotron magnet or quadrupole lens) may be made easier by this transformation, because a constant gradient in the \( w \)-plane corresponds to a constant field on the \( w' \)-plane. The precise calculation of disturbances in the constant gradient, due e.g. to limited width of poles, eddy currents in the vacuum chamber, or currents in poleface windings, is simplified.

As an example of this method, the effect of eddy currents in the vacuum chamber (final design) are calculated and compared with measurements.

A similar transformation \( w^n = w^{2n} \) can be used to transform the field inside a multipole lens (2n poles) into a homogeneous field.

1. The Transformation \( w'' = w^2 \).

The magnetic potential of a two-dimensional field with a symmetry plane (median plane) satisfies the general relation (in polar coordinates)

\[
\psi = \sum_{n=1}^{\infty} a_n C^n \sin n\phi
\]  

(1)

We call \( C e^{i\phi} = w \), and apply the transformation

\[
w'' = w^2 = C^2 e^{i \cdot 2\phi}
\]  

(2)

The polar coordinates in the \( w' \)-plane are called \( C'' \) and \( \phi' \), giving
\[ \psi' = \psi \]

\[ \psi' = 2\psi \]

The potential distribution in the \( \psi' \)-plane is therefore

\[ V' = \sum_{n=1}^{\infty} \frac{a_n}{n} \cos \frac{n\pi}{2} \sin \frac{\psi}{2} \]

The \( \psi' \)-plane is the image of only half of the \( \psi \)-plane (e.g. the upper half).

The horizontal axis in the \( \psi \)-plane corresponds to half of the horizontal axis in the \( \psi' \)-plane, on the right hand side of the origin, because of the transformation

\[ \psi = 0 \quad \Rightarrow \quad \psi' = 0 \]

\[ \psi = \pi \quad \Rightarrow \quad \psi' = 2\pi \]

This half-axis in the \( \psi' \)-plane will therefore in general be a line of discontinuity in the field, unless a field in the \( \psi \)-plane is considered with symmetry about the origin, which corresponds to even values of \( n \) only in (1) and (5).

If also odd values occur, the axis \( \psi' = 0 \) will in any case be an equipotential line \( (W = 0) \), as long as a field with a median plane is considered.

2. Application to a Constant Gradient Field.

For a pure quadrupole field in the \( \psi \)-plane we have only to consider the second term of (1):

\[ V = a_2 \cos^2 \psi \]

giving in the \( \psi' \)-plane

\[ V' = a_2 \cos^2 \psi' \]

This potential produces a homogeneous field. The hyperbolic equipotentials in the \( \psi \)-plane become parallel planes in the \( \psi' \)-plane (fig. 1).

In this case, no discontinuity exists at the \( \psi' = 0 \) axis.

In fig. 2 and 3 the transformation of the CERN synchrotron magnet is shown (open and closed block, respectively). In this case, no symmetry about the origin.
exists (in other words: the term with \( n = 1 \) is present), and therefore the equipotential \( \phi'' = 0 \) at \( \phi = 0 \) is a separation between two regions with quite different fields.

The problem of shaping the edge of the pole profile so that the gradient is as constant as possible, is now reduced to the simpler problem of producing a field as constant as possible in the \( w' \)-plane. The deviation of the pole profile in the \( w' \)-plane from the ideal straight line, necessary to compensate for the limited pole width, might be computed using the analogue method described in the report CERN-PS/WW-36, but with a greater precision, as now fields instead of gradients must be measured on the analogue.

The influence on the gradient of currents flowing between the poles at a right angle with the \( w' \)-plane, can now be calculated accurately. In the next section an example is given.

1. Calculation of the Vacuum Chamber Effects.

Measurements of the effect on \( n \) from the eddy currents in the vacuum chamber were described in the reports CERN-PS/WW 15 and WW 25. Measurements outside the median plane are difficult because of the dimensions of the measuring coils. Therefore only limited data exist about this effect outside the median plane.

In principle, it is possible to calculate the effect for any value of the vertical coordinate \( z \), if it is known in the median plane. This calculation can be performed in the following way.

The gradient error in the median plane is expressed as a power series of \( z' \):

\[
G = \sum_{k=0}^{\infty} r_k \frac{z'}{k}
\]  

(8)

Outside the median plane:

\[
G = \sum_{k=0}^{\infty} p_k \frac{z'}{k}
\]  

(9)
If the \( m_k \)'s are known (from the measurements in the median plane), the \( p_k \)'s can be calculated from

\[
\begin{align*}
p_0 &= m_0 = 2 m_2 + 4 m_4 + 6 m_6 + \cdots \\
p_1 &= m_1 = 3 m_3 + 5 m_5 + \cdots \\
p_2 &= m_2 = 4 m_4 + 6 m_6 + \cdots \\
p_3 &= m_3 = 5 m_5 + 7 m_7 + \cdots \\
p_4 &= m_4 = \cdots 
\end{align*}
\]

in which \( ^{a \choose b} \) stands for

\[
\frac{(a - 1)(a - 2) \cdots (a - b + 1)}{b!}
\]

The formula (10) can be derived by writing (1) in rectangular coordinates, differentiating twice to find the gradient, and comparing this expression with (9) and, after putting \( z = 0 \), with (8).

It will appear from (10), as is found in practice, that for high values of \( z \) the series for \( p_{k} \) will converge very slowly. Therefore it is necessary to know the error curve in the median plane very accurately to predict the effect far from the median plane. The measurements are not precise enough for this purpose.

It is, however, possible to use the method described in section 2 above to compute the influence of the vacuum chamber.

First, the eddy current distribution is calculated in the following way.

The vacuum chamber wall is divided into \( a \) longitudinal slices of equal material cross section. The resistance of each slice per metre length is called \( R_a \).

The rate of change of flux \( (d\phi_k/dt) \) passing per metre length between the centre of slice \( k \) and the centre of the vacuum chamber is calculated.

Now the current through each slice is not determined by \( (d\phi_k/dt) \), but by \( (d\phi_k/dt) - (d\phi_o/dt) \), in which \( (d\phi_o/dt) \) is the rate of change of the flux passing between the centre of the chamber and the point of the chamber cross section corresponding to zero current. We can write:
\[ i_k = \frac{\sum_{k=1}^{n} \frac{d\phi_k}{dt}}{R_s} \]  

(11)

in which \( \frac{d\phi}{dt} \) is still an unknown quantity. We know, however that

\[ \sum_{k=1}^{n} i_k = 0 \]

so that

\[ \sum_{k=1}^{n} \frac{d\phi_k}{dt} = a \frac{d\phi}{dt} \]

\[ \frac{R_s}{R_s} = 0 \]

or

\[ \frac{d\phi}{dt} = \frac{\sum_{k=1}^{n} \frac{d\phi_k}{dt}}{n} \]  

(12)

From (11) and (12):

\[ i_k = \frac{1}{n} \left( \sum_{k=1}^{n} i_k \right) \]

\[ \frac{d\phi}{dt} = \frac{\sum_{k=1}^{n} \frac{d\phi_k}{dt}}{n} \]  

(13)

This expression enables us to calculate the current distribution. In fig. 4 the result is shown for the final vacuum chamber for the CERN proton synchrotron. The conductivity of the walls is 95 \( \mu \Omega \cdot m \), the rate of rise of the field on the equilibrium orbit is supposed to be 12 kg/sec.

Calculating the gradient disturbance caused by this current distribution is a more difficult task. A possible way is the approximation of the polefaces by a straight-line wedge (Reinemeyer, CERN-PS/ER 42), but the results in this case would probably not be precise enough to give a reasonable picture outside the median plane. With the aid of the transform (2), however, it is possible to calculate more exactly the influence of a current between hyperbolical polefaces.

For a wire with current \( i_k \) between two parallel plates of infinite width and permeability (fig. 5), the magnetic potential can be written as
\[ \nu'' = 0, 2 \, i \, \arctan \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \sinh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \sin \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \]

This formula can be derived either by "mirror" system, taking an infinite number of mirror images on both sides, or by the Schwarz–Christoffel transformation \(^1\).

On this formula we apply the transformation (2), written in dimensionless form:

\[ \frac{x'' + i \, y''}{R} = \left( \frac{x + i \, y}{R} \right)^2 \]

The result is

\[ \nu'' = 0, 2 \, i \, \arctan \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \sinh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \sin \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \]

The gradient in the median plane follows from this:

\[ \begin{vmatrix} \frac{\nu''}{\nu''} & 0, 2 \, 2 \, \pi \, i \, \nu'' \\ y = 0 & R^2 \\ \end{vmatrix} \]

\[ \begin{vmatrix} \nu''^2 = x^2 + y^2 \\ \sinh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \sin \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \end{vmatrix} \]

\[ \begin{vmatrix} 1 = \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cos \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \sin \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \quad \cosh \left( \frac{\pi \, \nu'' \, \nu''}{2 \, \nu' \, \nu'} \right) \end{vmatrix} \]

\[ (16) \]

Substituting into this formula the values of \( x_k, y_k \) and \( z_k \) corresponding to the vacuum chamber slices, and summing their effects, we find the gradient disturbance on the median plane.

In principle, it would also be possible to calculate the gradient outside the median plane with this method. However, the expressions become very complicated, and it is simpler to compute the gradient in a number of points in the median plane and calculate the values outside, using equations (10).

The gradient was computed for eleven points in the median plane. This gradient was then expressed as a power series of \( r \):

\[
\left( \frac{\Delta B}{\Delta r} \right) = \sum_{k=0}^{10} m_k \cdot \frac{r^k}{r^k}
\]

with

<table>
<thead>
<tr>
<th>( k )</th>
<th>( m_k ) (gauss/cm(^k))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-8,886 \cdot 10^{-2}</td>
</tr>
<tr>
<td>1</td>
<td>+3,667 \cdot 10^{-2}</td>
</tr>
<tr>
<td>2</td>
<td>+4,214 \cdot 10^{-3}</td>
</tr>
<tr>
<td>3</td>
<td>+7,399 \cdot 10^{-5}</td>
</tr>
<tr>
<td>4</td>
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</tr>
<tr>
<td>5</td>
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<td>6</td>
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<tr>
<td>7</td>
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</tr>
<tr>
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<td>+7,516 \cdot 10^{-9}</td>
</tr>
<tr>
<td>10</td>
<td>+1,567 \cdot 10^{-9}</td>
</tr>
</tbody>
</table>

For \( r = 2 \) cm the corresponding gradient curve was computed, using equations (10).

The values of \( \Delta p/a \) at injection (\( B_0 = 147 \) gauss), caused by this gradient, are shown in Fig. 6. In computing these curves the decrease of \( B_0 \) (magnitude 2 s 24 gauss), calculated in a similar way as the gradient, was taken into account.
Two measured curves of \((\Delta z/a)\) \((z = 0\) and \(z = 2\ cm)\) are also shown in this diagram. The small difference between measured and computed curves can be explained by the following facts:

a) The measurements were done on a vacuum chamber of dimensions 160 x 80 mm instead of 150 x 74 mm.

b) The gradient was measured by means of two coils, each about 1 cm wide, with a centre-to-centre distance of 1 cm.

c) The precision of the measurements was not very high.

The precision of the computed curve for \(z = 0\) should be sufficient for all practical purposes, since the only approximations made are the following:

a) In deriving (14), the permeability of the pole-pieces was supposed to be infinite. This should not introduce any appreciable error, as the real permeability at injection is at least 600. The effect of the \(n^{th}\) mirror image of the current would be reduced by a factor \((\frac{a_n - 1}{a_n + 1})^n\). As the effect of the mirror images is relatively small, and decreases rapidly with increasing \(n\), this can be neglected.

b) Also, the width of the pole-pieces was supposed to be infinite, and the magnet structure as well as the currents inside the gap were supposed to be symmetric about the origin (fig. 5). It is probable that the influence of this approximation is relatively small, as the field caused by the eddy currents decreases rapidly outside the chamber (see fig. 7).

c) Instead of considering a continuous current distribution, the vacuum chamber was divided into 40 slices for the purpose of this calculation. The contribution of each slice is small compared with the total effect. Therefore this procedure has little influence on the precision. Only for the two points to nearest the chamber wall \((r = \pm 6.25\ cm)\) an error of a few percent was made, as was found by making a further subdivision of the slice nearest to these points. For all other calculated points the errors are negligible.

The precision of the curve for \(z = 2\) is somewhat less, especially far away from the centre. This is due to the computing method. If a more precise knowledge of this effect would be necessary, it might be obtained by direct computation, with a digital computer, using equation (15).
4. Acknowledgement

Thanks are due to Miss A. Bead and Miss M. Hanney, who made the numerical computations for the vacuum chamber effect.

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3.4.58
\[ \rho = 96 \mu \Omega \text{cm} \]
\[ \frac{dB_o}{dt} = 12 \text{ kg/sec} \]

**Fig. 4**

CERN-PS/MM35
Fig. 6

CERN - PS/MM85