COMPOSITENESS, ANOMALOUS DIMENSIONS AND RENORMALIZABILITY IN FOUR - FERMI THEORIES

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Abstract

We study the interplay of compositeness, non-triviality and hyperscaling in a four-fermi theory with interaction $(\bar{\psi}\psi)^2$ in spacetime dimension d , $2 < d \leq 4$. We establish that nonvanishing anomalous dimensions, compositeness and hyperscaling lead to renormalizability and a nontrivial continuum limit. We argue that the theory possesses a renormalizable $1/N$ expansion, where N is the number of fermion species, about a strongly-coupled fixed point in all dimensions $2 < d \leq 4$. Explicit calculations of the critical indices to $O(1/N)$ illustrate these points. For $d = 4$ we confirm that the theory's cutoff can be removed only at the expense of making the continuum theory trivial and that this result prevails beyond leading order. This model illustrates that strongly-coupled theories can be renormalizable in a continuum of dimensions, rather than at just one upper critical dimension as occurs in weakly coupled theories.

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1. Introduction

Although quantum field theory is necessarily defined with the help of some ultraviolet cutoff procedure, in practice we expect to be able to make physical predictions independent of this, by absorbing all cutoff-dependence into redefinitions of parameters such as particle masses and coupling constants. If such a manipulation is possible, the theory is said to be renormalizable. The “physical” or “renormalized” theory is defined at a point in the bare parameter space at which the cutoff theory undergoes a second order phase transition. At this point, known as the continuum limit, an infinite correlation length is generated. The system is scale invariant and the cutoff-dependence disappears [1,2]. Close to the phase transition point the relevant scale is the divergent correlation length, and all physical quantities scale with respect to it.

A renormalizable theory should have practically unlimited powers of prediction: e.g. by fitting a small number of bare parameters like coupling constant and mass, we hope to get as output a consistent mass spectrum, effective couplings, matrix elements etc. Such a system is overdetermined (a large number of equations in only two unknowns) and, in order to have a solution, the theory must exhibit a high degree of self-consistency. One way to test this self-consistency is through hyperscaling [2]. The program consists of computing the critical indices which characterize the scaling behavior of various thermodynamic quantities in the critical region. Since there are only two parameters the system responds to (eg. temperature and pressure or mass and coupling constant), only two indices can be independent. The relations between the exponents present the consistency conditions and reflect the existence of only one scale.

In this paper we study the interplay between renormalizability and hyperscaling in four-fermi theories. First we introduce the model we will be working with. The Lagrangian reads [3]:

$$\mathcal{L} = \bar{\psi}_i(\not{\partial} + m)\psi_i - \frac{g^2}{2N}(\bar{\psi}_i\psi_i)^2, \quad (1.1)$$

where $\psi, \bar{\psi}$ are four component fermion fields, summation is implied on the repeated index i which runs over N flavors. Analysis is simplified by the introduction of an auxiliary scalar field σ :

$$\mathcal{L} = \bar{\psi}_i(\not{\partial} + m)\psi_i + \frac{g}{\sqrt{N}}\sigma\bar{\psi}_i\psi_i + \frac{1}{2}\sigma^2. \quad (1.2)$$

For $d = 4 - \epsilon$ the coupling g^2 has dimension $\epsilon - 2$. Using the gap equation, as described below, it can be shown that in the chiral limit $m \rightarrow 0$, for g^2 greater than some critical value g_c^2 , operators σ and $\bar{\psi}\psi$ both acquire non-vanishing expectation values signalling breakdown of the discrete chiral symmetry $\psi \mapsto \gamma_5\psi$, $\bar{\psi} \mapsto -\bar{\psi}\gamma_5$ present in (1.1). It is the behavior of the model in the vicinity of the fixed point $g = g_c$ that will interest us, and in particular, the claim [4-10] that the four-fermi model has a renormalizable expansion in $1/N$ about a strongly-coupled fixed point in three spacetime dimensions. We extend the result to all

dimensionalities d such that $2 < d \leq 4$. For $d = 4$ we find that the cutoff can be removed only at expense of making the continuum theory trivial; this result prevails beyond leading order.

2. Anomalous Dimensions and Compositeness

As a prelude to a technical discussion of renormalizability, we first focus on the physical aspects of the model, and especially on the notion of compositeness. Consider a scattering process that involves the exchange of a scalar particle. The amplitude at the tree level is proportional to $1/(k^2 + M_0^2)$, which in coordinate space translates into a potential of the form $V(r) \sim \exp(-M_0 r)/r$. Due to quantum fluctuations, the scalar propagator is modified to $1/(k^2 + M_0^2 + \Pi(k^2))$. By power counting, we infer that the leading term in the vacuum polarization behaves as $\Pi(k^2) \sim \Lambda^2$. Thus in the continuum limit quantum corrections drive the range of the potential to zero and, for particles to interact, they have to sit on top of each other. There is no physical scale in the system and such a theory cannot have an interacting continuum limit. We call such a theory trivial. One way out of this problem is if the exchanged particle is composite: finiteness of its radius is then related to the strength of the effective coupling. A well known example of this is encapsulated in the Goldberger-Treiman relation [11] which relates the nucleon mass with the pion-nucleon coupling: $g_{\pi N} = M_N/f_\pi$. After recognizing that $R_\pi \sim 1/f_\pi$, we get that $g_{\pi N} \sim M_N R_\pi$. Hence $g_{\pi N}$ vanishes when pions become pointlike.

This behavior should be contrasted with the exchange of a vector particle which couples to a conserved current; this conservation prevents the appearance of the constant term in the vacuum polarization, and triviality does not follow. In the case of Yukawa couplings, it is possible to adjust the coupling so that the screening disappears. This requires that the attraction in the scalar channel becomes strong and results in the appearance of light scalar composites. The corresponding renormalization constants vanish and the three- and higher point functions satisfy homogeneous Bethe-Salpeter equations. The dynamics is dominated by strong fluctuations and the correlation functions have scale invariant form with anomalous dimensions [12]. Thus, the critical coupling for the onset of the composite scalars is a fixed point [8].

In four-fermi theories as represented by eq.(1.2), scalar degrees of freedom are not canonical, but auxiliary fields – they don't propagate and the asymptotics of such an exchange is hard. In coordinate space $D_\phi(x) = \delta^{(d)}(x)$. In general, the scalar propagator has the following short-distance form: $D_\phi(x) \sim 1/|x|^{2d_\phi} = 1/|x|^{d-2+\eta}$, where d_ϕ is the *scaling*, and $\eta/2$ the *anomalous* dimension of the field. In the large- N limit, the radiative corrections amount to summing fermion bubble diagrams. To understand the UV behavior, we

neglect fermion mass. In that case the expression for the σ -propagator becomes [4,13]

$$D_\sigma(k^2) = \frac{1}{1 + g^2(-C_1\Lambda^{2-\epsilon} - C_2k^2\Lambda^{-\epsilon} + C_3k^{2-\epsilon})}, \quad (2.1)$$

where the C_i 's are positive dimensionless constants dependent on d only. Clearly, the k^2/Λ^ϵ term can be neglected for $k^2 \ll \Lambda^2$. Of the three constants C_i , it turns out that only C_3 is regularization scheme-independent. In d dimensions it is equal to $4B(\frac{d}{2}, \frac{d}{2} - 1)\Gamma(\frac{\epsilon}{2})/(4\pi)^{\frac{d}{2}}$. We now adjust the bare coupling so that screening disappears. This defines the critical coupling,

$$g_c^2 = \frac{1}{C_1\Lambda^{2-\epsilon}}. \quad (2.2)$$

This is a UV-stable fixed point of the theory. With this notation, the scalar propagator can be rewritten as,

$$D_\sigma^{-1}(k^2) = \left(1 - \frac{g^2}{g_c^2}\right) + \frac{g^2}{g_c^2} \frac{C_3}{C_1} \left(\frac{k}{\Lambda}\right)^{2-\epsilon} \quad (2.3)$$

Denote $t = 1 - g^2/g_c^2$. Although in the critical region $t \rightarrow 0$, $g^2 \sim 1/\Lambda^{2-\epsilon}$ from eq.(2.2), the scattering amplitude is unsuppressed because we have

$$g^2 D_\sigma(k^2) \approx g_c^2 \frac{C_1}{C_3} \left(\frac{\Lambda}{k}\right)^{2-\epsilon} \approx \frac{1}{C_3} \frac{1}{k^{2-\epsilon}} \quad (2.4)$$

From here the anomalous dimension of the σ -field is identified as $\eta = \epsilon$. Near the fixed point, scattering processes have the same UV asymptotics as those in four dimensions with canonical $1/k^2$ exchange. For $d = 4 - \epsilon$ we have $\int d^d k/k^{2-\epsilon} \sim \int d^4 k/k^2$. In coordinate space $D_\sigma(x) \sim 1/|x|^2$ since $d - 2 + \epsilon = 2$. At this point $\sigma\bar{\psi}\psi$ is a marginal operator, and the theory is *renormalizable* (as opposed to superrenormalizable). Here, we see how nonvanishing anomalous dimensions and masslessness of the σ are tied together – the fact that $\eta = \epsilon$ ensures that the interaction is sufficiently strong at short distances to guarantee the appearance of light bound states. In superrenormalizable theories the UV asymptotics are dominated by the scalar kinetic energy term and would lead to $1/k^2$ behavior of the σ -propagator. Below four dimensions, this is too weak to produce light bound states.

In the limit $d \rightarrow 4$, $\eta = \epsilon \rightarrow 0$, we find the coefficients C_i of equation (2.1) are universal and behave as: $C_1 \simeq 1/4\pi^2$, $C_2 \simeq C_3 = 1/4\pi^2\epsilon$. The expression for the scalar propagator thus becomes

$$D^{-1}(k^2) = 1 + g^2 \left(-\frac{\Lambda^2}{4\pi^2} + \frac{k^2}{8\pi^2} \ln \frac{\Lambda^2}{k^2} \right). \quad (2.5)$$

At the critical point the constant term is absent, and the remaining term vanishes logarithmically as the cutoff is removed:

$$g^2 D(k^2) \simeq 8\pi^2 \frac{1}{k^2 \ln(\frac{\Lambda^2}{k^2})}. \quad (2.6)$$

This is the well known result that in four dimensions scalars and fermions decouple and the theory describes free pointlike bosons [14]. There are logarithmic violations of scaling that are typical for marginal operators in four dimensions. They make $\sigma\bar{\psi}\psi$ marginally irrelevant. Thus, we see that nonvanishing η becomes the statement of both compositeness and nontriviality in the continuum.

3. Leading order in the $1/N$ expansion

To leading order in $1/N$, the gap equation consists simply of a fermion tadpole insertion, and reads

$$\Sigma(k) = m - g^2 \langle \bar{\psi}\psi \rangle = m + g^2 \text{tr} \int_p \frac{1}{i\not{p} + \Sigma(p)}, \quad (3.1)$$

where $\Sigma(k)$ is the fermion self-energy and m is the bare mass. To this order there is no wavefunction renormalization and the physical fermion mass M is equal to its inverse susceptibility $\Sigma_0 = \Sigma(0)$. Clearly, the solution of the gap equation is $\Sigma(k) = \Sigma_0$, which in the chiral limit becomes

$$\frac{m}{\Sigma} + 4 \left(1 - \frac{m}{\Sigma}\right) \left(1 - \frac{g_c^2}{g^2}\right) = 4g_c^2 \int_q \frac{\Sigma^2}{q^2(q^2 + \Sigma^2)} \quad (3.2)$$

where $1/g_c^2 = 4 \int_q 1/q^2$. Equation (3.2) can be interpreted as a renormalization condition relating the bare parameters (g^2, m) to a physical mass scale $M = \Sigma_0$, from which we see that, as $M/\Lambda \rightarrow 0$, $(g^2, m) \rightarrow (g_c^2, 0)$, so that this is indeed a UV-stable fixed point. The critical exponents are easily calculated. From (3.1) and (3.2), we see that $\langle \bar{\psi}\psi \rangle \propto \Sigma_0 \propto t^{1/(2-\epsilon)}$ giving $\beta = \nu = 1/(d-2)$. Exponents γ and δ may be obtained by considering the effects of the bare fermion mass in the gap equation. For $g^2 = g_c^2$ we find from (3.2) $\Sigma_0 \propto m^{1/(3-\epsilon)}$ and hence $\delta = d-1$. Similarly, if we evaluate $\chi \equiv \partial \langle \bar{\psi}\psi \rangle / \partial m|_{m=0}$ we find $\chi \propto \Sigma_0^{-(2-\epsilon)} \propto t^{-1}$ and hence $\gamma = 1$ (this could also be deduced by considering equation (2.3) at $k^2 = 0$). As we saw in the previous section $\eta = 4-d$ and so we have:

$$\gamma = 1, \quad \eta = 4-d, \quad \beta = \nu = \frac{1}{d-2}, \quad \delta = d-1. \quad (3.3)$$

The exponents satisfy the hyperscaling relations [2]:

$$2\beta + \gamma = d\nu, \quad 2\beta\delta - \gamma = d\nu, \quad \beta = \frac{1}{2}\nu(d-2+\eta), \quad \gamma = \nu(2-\eta). \quad (3.4)$$

The fact that hyperscaling is obeyed justifies the identification of $\xi = 1/\Sigma_0$ as the sole physical length scale in the problem. From (3.3) we see that in the limit $d \rightarrow 4$ the exponents assume their mean field values: $\beta = \frac{1}{2}$; $\delta = 3$; $\gamma = 1$; $\nu = \frac{1}{2}$; $\eta = 0$.

A careful calculation of the scalar propagator D_σ in the broken phase, this time including the effects of a fermion mass Σ_0 , yields

$$D_\sigma^{-1}(k^2) = \frac{2\Gamma(\frac{\epsilon}{2})}{(4\pi)^{\frac{\epsilon}{2}}} \frac{k^2 + 4\Sigma_0^2}{\Sigma_0^\epsilon} F(1, \frac{\epsilon}{2}; \frac{3}{2}; -\frac{k^2}{4\Sigma_0^2}), \quad (3.5)$$

where F is the hypergeometric function. Hence for small k/Σ_0 we see that the scalar propagates canonically with mass $2\Sigma_0$. For $k/\Sigma_0 \rightarrow \infty$, however, we recover the asymptotic form (2.4). This holds in the vicinity of the critical point as $\langle \bar{\psi}\psi \rangle, \Sigma_0 \rightarrow 0$. Since the anomalous dimension of the scalar field here is $\eta/2 = \epsilon/2$, its scaling dimension is

$$d_\sigma = \frac{1}{2}(d - 2 + \epsilon) = 1. \quad (3.6)$$

This holds for any d . At leading order nothing happens to the scaling dimension of $\bar{\psi}\psi$ because wavefunction renormalization and vertex corrections are $O(1/N)$ effects. Thus, $\bar{\psi}\psi$ retains its canonical dimension $d_{\bar{\psi}\psi} = d - 1$. Therefore, the scaling dimension of the interaction term is,

$$d_{\sigma\bar{\psi}\psi} = d - 1 + 1 = d. \quad (3.7)$$

Consequently, $\sigma\bar{\psi}\psi$ is marginal and the theory is renormalizable at this order for any d , as can be verified by power-counting arguments [4-10].

In general the identification $\bar{\psi}\psi \sim \sigma$ is incorrect, although the correlation functions $D_\sigma(x) = \langle \sigma(x)\sigma(0) \rangle$ and $D_{\bar{\psi}\psi}(x) = \langle \bar{\psi}\psi(x)\bar{\psi}\psi(0) \rangle$ have the same pole structure which in Euclidean space translates into identical long range behavior. As far as UV behavior is concerned, the two propagators are different. To exhibit this, let's look at the four-fermi theory without introducing the auxiliary field and consider the short distance behavior of the correlation function $D_{\bar{\psi}\psi}(x)$:

$$D_{\bar{\psi}\psi}(x) = \int_{x'} \text{tr} S(x - x') \Gamma(x') S(x' - x), \quad (3.8)$$

where $\Gamma(x)$ is the Fourier transform of $\Gamma(k) = 1/(1 - g^2\Pi(k^2))$ with $\Pi(k^2)$ being the fermion bubble. In fact, we recognize $\Gamma(k)$ as the σ propagator in the bosonized version of the theory. From this expression it is clear that the expressions for $D_\sigma(x)$ and $D_{\bar{\psi}\psi}(x)$ differ because of the two extra fermion propagators that contribute to the scaling dimension of the fermion composite (note that $D_{\bar{\psi}\psi}$ as defined here requires an additional regularization, such as normal ordering of $\bar{\psi}\psi$, to remove divergences from terms proportional to $\delta^d(x - x')$). At the critical point the constant term in $\Pi(k^2)$ is killed by the fixed point condition and the scalar propagator scales as $\Gamma(x) \sim 1/|x|^{d-2+\epsilon}$. Also, the fermion mass vanishes and its propagator scales as $S(x) \propto 1/|x|^{2d_\psi}$ with $d_\psi = (d - 1)/2$. Therefore, we have

$$D_{\bar{\psi}\psi}(\lambda x) = \lambda^d \int_{x'} \text{tr} S(\lambda(x - x')) \Gamma(\lambda x') S(\lambda(x' - x)) = \lambda^{d-2(d-1)-(d-2+\epsilon)} D_{\bar{\psi}\psi}(x), \quad (3.9)$$

where d comes from the integration over x' , $2(d-1) = 4d_\psi$ from two fermion propagators and $d-2+\epsilon$ from $\Gamma(x)$. Thus, the scaling dimension is $d_{\bar{\psi}\psi} = (2d-4+\epsilon)/2 = d/2$ and so $d_{(\bar{\psi}\psi)^2} = d$. The interaction is marginal for any d . This is the same conclusion we reached in eq.(3.7).

4. Beyond Leading Order

At $O(1/N)$, we find in addition to terms proportional to $\Lambda^{2-\epsilon}$ there are cutoff dependencies of the form $\ln(\Lambda/\Sigma)$. It is thus necessary to introduce wavefunction and vertex renormalization factors to define a physical theory. In previous treatments of this question [9,15], using the form (1.2) of the model, it was argued using power counting that all divergent terms can be absorbed by suitable redefinition of the coupling g and the fields ψ and σ : three independent renormalization constants corresponding to the three terms in $\mathcal{L}$. However, since $\mathcal{L}$ can also be written in the form (1.1) there are really only two independent terms, and hence a full treatment of the divergences in the model will overdetermine the constants and yield a non-trivial consistency condition. We therefore feel that the previous “proofs” of renormalizability [9], though suggestive, are incomplete.

We begin our investigation of $O(1/N)$ effects by considering corrections to the gap equation in the broken phase (figure 1), and hence the critical exponents. Using a simple momentum cutoff, the gap equation in the chiral limit is found to be:

$$\frac{1}{g^2} = \frac{8\Lambda^{2-\epsilon}}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})(2-\epsilon)} \left(1 - \frac{(d-1)}{2N}\right) - \frac{8\Sigma_0^{2-\epsilon}}{(4\pi)^{\frac{d}{2}}(2-\epsilon)}\Gamma(\frac{\epsilon}{2}) \left(1 - \frac{2^{d-3}(d-1)}{N}\right), \quad (4.1)$$

where Σ_0 denotes the contribution to the fermion self-energy from the gap equation alone, ie. the mass appearing in the bare fermion propagator. We see that $1/N$ corrections lead to a larger critical coupling: $g_c^2 \rightarrow g_c^2(1 + (d-1)/2N)$. Now, from (3.1) we know Σ_0 is proportional to the order parameter $\langle \bar{\psi}\psi \rangle$, and hence can extract $\beta = 1/(d-2)$ as before. The extraction of δ and γ is straightforward and we merely give their values

$$\delta = (d-1) \left[1 + \frac{C_d}{N}\right], \quad \gamma = 1 + \frac{(d-1)}{(d-2)} \frac{C_d}{N}, \quad (4.3)$$

where $C_d = 1/\Gamma(\frac{\epsilon}{2})\Gamma(\frac{d}{2})B(\frac{d}{2}, \frac{d}{2}-1)$.

To extract ν we identify the inverse correlation length with the renormalized fermion mass M obtained by considering the logarithmic divergence of figure 2:

$$M = Z_M^{-1}\Sigma_0 = \Sigma_0 \left(1 - \frac{(d-1)}{d} \frac{C_d}{N} \ln \frac{\Lambda}{\Sigma_0}\right). \quad (4.4)$$

Using the result for β we deduce

$$\nu = \frac{1}{(d-2)} \left[1 + \frac{(d-1)}{d} \frac{C_d}{N} \right]. \quad (4.5)$$

To extract η we need to study the $1/N$ corrections to D_σ given by figure 3; after some work we find

$$D_\sigma^{-1}(k^2)|_{i=0} = g^2 C_3 k^{2-\epsilon} \left(1 - 2 \frac{(d-1)}{d} \frac{C_d}{N} \ln \frac{\Lambda}{k} \right) \propto k^{2-\eta}, \quad (4.6)$$

whence

$$\eta = 4 - d - 2 \frac{(d-1)}{d} \frac{C_d}{N}. \quad (4.7)$$

This completes the calculation of critical exponents to $O(1/N)$: hyperscaling relations (3.4) continue to be satisfied.

We have already argued that the physical assumptions underlying hyperscaling and renormalizability are equivalent: moreover it is simple to repeat the calculation of scaling dimensions in the bosonized model of section 3 to find

$$d_\sigma = 1 - \frac{(d-1)}{d} \frac{C_d}{N}; \quad d_{\bar{\psi}\psi} = d - 1 + \frac{(d-1)}{d} \frac{C_d}{N}, \quad (4.8)$$

so $d_{\sigma\bar{\psi}\psi} = d$ as before, and the interaction continues to be marginal – this is perhaps the most direct way of seeing that the model is renormalizable. There is, however, another argument, which highlights the equivalence between hyperscaling and renormalizability, starting from an identity relating the n th order contributions to the fermion self-energy $\Sigma'(k^2) \equiv \Sigma(k^2) - \Sigma_0$ and the vertex $\Gamma_{\sigma\bar{\psi}\psi}(k^2)$ [9]:

$$\Sigma'^{(n)}(0) = -\Sigma_0 \frac{\sqrt{N}}{g} \Gamma_{\sigma\bar{\psi}\psi}^{(n)}(0). \quad (4.9)$$

Adding Σ_0 to each side we deduce

$$Z_\psi^{-1} Z_M^{-1} \Sigma_0 \propto -\frac{\Sigma_0}{g} Z_\sigma^{-\frac{1}{2}} Z_\psi^{-1}, \quad (4.10)$$

where Z_M is defined in (4.6) and Z_ψ, Z_σ are wavefunction renormalization constants. The constant of proportionality is finite. We deduce the consistency relation

$$Z_M^2 \propto Z_\sigma g^2. \quad (4.11)$$

Now, once g^2 is determined in terms of M and Λ by the gap equation, all subsequent divergences are logarithmic: writing $Z_M = 1 + a \ln(\Lambda/M)$ and assuming that higher order corrections exponentiate, we can relate α to the exponents β and ν using (4.4):

$$1 - a = \frac{\beta}{\nu}, \quad (4.12)$$

so that

$$Z_\sigma g^2 \propto 1 + 2 \left(1 - \frac{\beta}{\nu}\right) \ln \frac{\Lambda}{M}. \quad (4.13)$$

However, $Z_\sigma g^2$ is also constrained by the divergence structure of the inverse scalar propagator. At criticality we have

$$D_\sigma^{-1}(k^2) \propto k^{2-\eta} \sim k^{2-\epsilon} \left[1 - (\epsilon - \eta) \ln \frac{\Lambda}{k}\right], \quad (4.14)$$

and at $k^2 = 0$ we have

$$D_\sigma^{-1}(0) \equiv \chi \propto M^{\frac{\gamma}{\nu}} \sim M^{2-\epsilon} \left[1 - \left(\epsilon - 2 + \frac{\gamma}{\nu}\right) \ln \frac{\Lambda}{M}\right]. \quad (4.15)$$

Renormalizability requires that the interaction strength $Z_\sigma g^2 D_\sigma$ should be Λ -independent in both limits. Using (4.13) we obtain $2\beta = \nu(d - 2 + \eta)$ and $2\beta + \gamma = d\nu$, which are both hyperscaling relations. This puts the renormalizability of the model on a very physical footing. A further consequence is that $O(1/N)$ and higher corrections to the four-point scattering amplitude must vanish.

To conclude, we have seen how renormalizability, non-triviality and hyperscaling are intimately related in a strongly interacting model field theory. The underlying physical picture is one of interaction via exchange of composite scalar degrees of freedom signalled by non-vanishing anomalous dimensions at the fixed point. Although our excursion into non-integer d is rather formal, it is interesting that this theory is exactly renormalizable in a *continuum* of dimensionality, rather than at one (upper) critical value as is the case in weakly-coupled theories. At leading order this can be traced back to the fact that the anomalous dimensions are such that the UV asymptotics remain independent of dimensionality: in coordinate space $D_\sigma(x) \sim 1/|x|^2$ for $2 < d < 4$. The leading order results seem to be stable against $1/N$ corrections: although the critical indices are changed, the physics remains the same. We note that $C_{d=4} = 0$, so in four dimensions the critical indices retain their mean-field values and fermions decouple from scalars in the continuum. A more detailed discussion of the four-dimensional theory can be found in reference [7] and [16]. Numerical simulations of the realistic case $d = 3$ are in progress.

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Figure Captions

Figure 1: $O(1/N)$ correction to the gap equation. Solid lines denote fermion propagators, and dotted ones scalar.

Figure 2: $O(1/N)$ contribution to the fermion self-energy $\Sigma'(k^2)$.

Figure 3: $O(1/N)$ corrections to the scalar propagator.

Fig. 1

Fig. 2

Fig. 3