On Holomorphic Factorization for Free Conformal Fields II

M. Knecht ^a, S. Lazzarini ^b and R. Stora ^c

^a *IPN *, Division de Physique Théorique, F-91406 Orsay Cedex, France*

^b *International School for Advanced Studies, SISSA,
Strada Costiera 11, I-34014 Trieste, Italy*

^c *Laboratoire de Physique Théorique ENSLAPP [†],
Chemin de Bellevue, BP 110, F-74941 Annecy-le-Vieux Cedex, France
and
Theory Division, CERN, CH-1211 Geneva 23, Switzerland*

Abstract

The holomorphic factorization theorem proved in [1] (Phys. Lett. **B262** (1991) 25) is extended to the case of free fields which are sections of an arbitrary holomorphic vector bundle over a compact Riemann surface without boundary.

Submitted to Phys. Lett. B

IPNO/TH 91-70
SISSA-140/91/FM
LAPP-TH-351/91
CERN-TH.6262/91
September 1991

*Unité de Recherche des Universités Paris XI et Paris VI associée au CNRS.

[†]URA 14-36 du CNRS, associée à l'Ecole Normale Supérieure de Lyon, et au Laboratoire d'Annecy-le-Vieux de Physique des Particules.

1 Introduction

In a previous letter [1], we have presented a local extension of the Belavin-Knizhnik [2]-[8] holomorphic factorization theorem for free spin j conformal fields. In the present letter, the local holomorphic factorization property is shown to extend to fields which are sections of an arbitrary holomorphic vector bundle over a compact Riemann surface without boundary [9] and the corresponding anomalies are displayed.

2 Notations

We consider a holomorphic vector bundle $\mathcal{U}$ over a compact Riemann surface Σ without boundary [10]. Let $\Psi \in C^\infty(\Sigma, \mathcal{U})$, $\Psi^* \in C^\infty(\Sigma, \mathcal{U}^* \otimes K)$ where K is the canonical bundle of Σ for the complex structure corresponding to the local complex analytic coordinates Z_α defined in open sets U_α of an open covering $\bigcup_\alpha U_\alpha$ of Σ , and where $*$ denotes the dual.

Consider the action

$$S = \int_{\Sigma} \frac{d\bar{Z} \wedge dZ}{2i} \Psi_Z^* \partial_{\bar{Z}} \Psi , \quad (1)$$

where we have used the notation

$$\Psi^* = \Psi_Z^* dZ . \quad (2)$$

We shall parametrize the complex structure of $\mathcal{U}$ with respect to a copy v defined by local complex analytic coordinates z_α on Σ and holomorphic transition functions $\varphi_{\alpha\beta}(z_\beta)$. Similarly κ will denote the canonical bundle of Σ for $\mu = 0$. From now on, unless otherwise specified, we omit the indices α , all formulae glueing through holomorphic changes of coordinates. Let

$$\begin{aligned} dZ &= \lambda(dz + \mu d\bar{z}) \\ \lambda &= \partial_z Z \equiv \partial Z \\ (\bar{\partial} - \mu \partial) \ln \lambda &= \partial \mu . \end{aligned} \quad (3)$$

μ , the coefficient of a Beltrami differential, fulfills $|\mu| < 1$. Define

$$\Psi = V^{-1} \psi , \quad \Psi_Z^* = \frac{\psi_z^*}{\lambda} V , \quad (4)$$

where

$$(\partial_{\bar{Z}} + \mathcal{A}_{\bar{Z}})V = 0 , \quad (5)$$

i.e.

$$(\bar{\partial} - \mu \partial + \bar{\mathcal{A}})V = 0 , \quad (6)$$

with

$$\mathcal{A}_{\bar{Z}} = \frac{\bar{\mathcal{A}}}{\bar{\lambda}(1 - \mu \bar{\mu})} . \quad (7)$$

The variables μ and $\bar{\mathcal{A}}$ parametrize the complex structure of $\mathcal{U}$. Note the glueing property of $\bar{\mathcal{A}}$

$$\bar{\mathcal{A}}_\alpha = \frac{1}{\bar{\partial}_\beta z_\alpha} \left[\varphi_{\alpha\beta} \bar{\mathcal{A}}_\beta \varphi_{\alpha\beta}^{-1} - \mu_\beta \varphi_{\alpha\beta} \partial_\beta \varphi_{\alpha\beta}^{-1} \right]. \quad (8)$$

Choosing a reference type $(1, 0)$ connection A and defining

$$\bar{A} = \bar{\mathcal{A}} + \mu A, \quad (9)$$

one has

$$\bar{A}_\alpha = \frac{1}{\bar{\partial}_\beta z_\alpha} \varphi_{\alpha\beta} \bar{\mathcal{A}}_\beta \varphi_{\alpha\beta}^{-1}. \quad (10)$$

The action (1) now reads

$$S(\psi, \psi_z^*; \mu, \bar{\mathcal{A}}) = \int_\Sigma \frac{d\bar{z} \wedge dz}{2i} \psi_z^* (\bar{\partial} - \mu \partial + \bar{\mathcal{A}}) \psi. \quad (11)$$

3 Anomalies : local theory

The classical action (11) is invariant under the action on the fields of the automorphisms of v , the infinitesimal form of which reads

$$\begin{aligned} s\psi &= (c \cdot \partial)\psi + M\psi + \Omega\psi \\ s\psi_z^* &= (c \cdot \partial)\psi_z^* - \psi_z^* M + (\partial c + \mu \partial \bar{c})\psi_z^* - \psi_z^* \Omega \\ sM &= (c \cdot \partial)M + M^2 + [M, \Omega] \\ s\Omega &= (c \cdot \partial)\Omega + \Omega^2 \\ sc &= (c \cdot \partial)c \\ s\bar{\mathcal{A}} &= (c \cdot \partial)\bar{\mathcal{A}} + (\bar{\partial} \bar{c} - \mu \partial \bar{c})\bar{\mathcal{A}} - (\bar{\partial} - \mu \partial)(M + \Omega) - [\bar{\mathcal{A}}, M + \Omega] \\ sA &= 0 \\ s\mu &= \bar{\partial} C + C \partial \mu - \mu \partial C \\ sC &= C \partial C, \end{aligned} \quad (12)$$

where $C = c + \mu \bar{c}$ [11]. Ω generates gauge transformations and M generates parallel transport along the vector field $c \cdot \partial \equiv c \partial + \bar{c} \bar{\partial}$. $\Omega, M, c \cdot \partial$ are Faddeev-Popov ghosts and $s^2 = 0$.

In terms of

$$\hat{\Omega} = \Omega + M - \bar{c} \bar{\mathcal{A}} \quad (13)$$

one has

$$\begin{aligned} s\bar{\mathcal{A}} &= C \partial \bar{\mathcal{A}} - [\bar{\mathcal{A}}, \hat{\Omega}] - (\bar{\partial} - \mu \partial) \hat{\Omega} \\ s\hat{\Omega} &= C \partial \hat{\Omega} + \hat{\Omega}^2. \end{aligned} \quad (14)$$

Note however the glueing properties

$$\hat{\Omega}_\alpha = \varphi_{\alpha\beta} \hat{\Omega}_\beta \varphi_{\alpha\beta}^{-1} + C_\beta \varphi_{\alpha\beta} \partial_\beta \varphi_{\alpha\beta}^{-1} . \quad (15)$$

It is thus convenient to use

$$H = \hat{\Omega} - AC = M + \Omega - c \cdot A \equiv M + \Omega - (cA + \bar{c}\bar{A}) , \quad (16)$$

which transforms homogeneously according to

$$H_\alpha = \varphi_{\alpha\beta} H_\beta \varphi_{\alpha\beta}^{-1} . \quad (17)$$

The s operation then reads

$$\begin{aligned} s\psi &= c \cdot (\partial + A)\psi + H\psi \\ s\psi_z^* &= (c \cdot \partial)\psi_z^* - \psi_z^*(c \cdot A) + (\partial c + \mu\partial\bar{c})\psi_z^* - \psi_z^* H \\ sH &= C\partial H + C[A, H] + H^2 \\ s\bar{A} &= C(\partial\bar{A} - \bar{\partial}A + [A, \bar{A}]) - (\bar{\partial} - \mu\partial)H - [\bar{A} - \mu A, H] \\ &\equiv CF - (\bar{\partial} - \mu\partial)H - [\bar{A} - \mu A, H] \\ sA &= 0 \\ s\mu &= \bar{\partial}C + C\partial\mu - \mu\partial C \\ sC &= C\partial C . \end{aligned} \quad (18)$$

A quantization of this classical situation is expected to exhibit two types of anomalies : the diffeomorphism anomaly

$$\int_\Sigma \frac{d\bar{z} \wedge dz}{2i} C (\partial^3 + 2\mathcal{R}\partial + \mathcal{R}')\mu , \quad (19)$$

where $\mathcal{R}$ is the coefficient of a (holomorphic) projective connection [12], and the gauge anomaly which can be checked to be a sum of terms of the form

$$\int_\Sigma \frac{d\bar{z} \wedge dz}{2i} \text{Tr} \tilde{H} \tilde{F} . \quad (20)$$

Here the objects $\tilde{H}$ and $\tilde{F}$ refer to the tensor product of v with some power of κ . Since the initial problem depends symmetrically on v and $v^* \otimes \kappa$, that symmetry should be preserved and we shall see that, in fact, there is only one such term pertaining to the bundle $v \otimes \kappa^{-\frac{1}{2}}$. One may check that under a variation of the background connection A the gauge anomaly is altered by a s -exact term, at fixed $\hat{\Omega}, \bar{A}$.

4 The local index theorem for families [13,14]

The only way which is known at present to quantize our free field system is to introduce hermitian metrics ρ on Σ and e on v . We consider the family $\mathcal{U}$ over Σ , endowed with metrics

$$E = V^\dagger e V, \quad \rho_{Z\bar{Z}} = \frac{\rho}{\lambda\bar{\lambda}} \quad (21)$$

over the complex analytic parameter space parametrized by $(\mu, \bar{A})$.

Now define [13,14]

$$\Gamma(\mu, \bar{A}; \text{c.c.}; \rho, e) = -\frac{1}{2} \ln \left(\frac{\det'_\zeta(-\Delta_{\rho, e; \mu, \bar{A}; \text{c.c.}})}{\det \| \langle \Phi_m | \Phi_n \rangle_v \| \det \| \langle \Phi_p^* | \Phi_q^* \rangle_{v^* \otimes K} \|} \right), \quad (22)$$

where $\Delta_{\rho, e; \mu, \bar{A}; \text{c.c.}}$ is the Laplacian on $C^\infty(\Sigma, \mathcal{U})$ and $\det'_\zeta$ its ζ -renormalized determinant [15], Φ_n (resp. Φ_p^*) span the kernel of $\bar{\partial}v$ (resp. $\bar{\partial}v^* \otimes K$), assumed to depend holomorphically on $(\mu, \bar{A})$, and $\langle | \rangle_v$ (resp. $\langle | \rangle_{v^* \otimes K}$) denotes the scalar product computed from the metrics $E, \rho_{Z\bar{Z}}$.

The local index theorem for families [14] states

$$\bar{\delta} \delta \left(\Gamma(\mu, \bar{A}; \text{c.c.}; \rho, e) - \Gamma(0, 0; 0, 0; \rho, e) \right) = -\frac{1}{8i\pi} \int_\Sigma \text{Tr} \left[(\mathcal{E} + \frac{1}{2} \mathcal{K} \mathbf{1})^2 - \frac{1}{12} (\mathcal{K} \mathbf{1})^2 \right]^{(2,2)}, \quad (23)$$

where δ (resp. $\bar{\delta}$) denotes the differential with respect to $\mu, \bar{A}$ (resp. $\bar{\mu}, \bar{A}^\dagger$), $\mathcal{E} + \frac{1}{2} \mathcal{K} \mathbf{1}$, resp. $\mathcal{K}$, is the curvature 2-form of the family $\mathcal{U} \otimes K^{-\frac{1}{2}}$, resp. K^{-1} , associated with the metric $E_{\sqrt{\rho_{Z\bar{Z}}}}$, resp. $\rho_{Z\bar{Z}}$, and the $(2, 2)$ superscript selects out the term of degree 2 on Σ and of degree 2 on the parameter space.

$\mathcal{K}$ was evaluated in [1]. A similar calculation yields

$$\begin{aligned} \frac{1}{4i} \int_\Sigma \text{Tr} \left[(\mathcal{E} + \frac{1}{2} \mathcal{K} \mathbf{1})^2 \right]^{(2,2)} &= \int_\Sigma \frac{d\bar{z} \wedge dz}{2i} (1 - \mu \bar{\mu}) \text{Tr} \delta \left(\frac{\bar{\tilde{a}} - \mu \tilde{e}^{-1} \bar{\tilde{a}}^\dagger \tilde{e}}{1 - \mu \bar{\mu}} \right) \\ &\times \bar{\delta} \left(\frac{\tilde{e}^{-1} \bar{\tilde{a}}^\dagger \tilde{e} - \bar{\mu} \bar{\tilde{a}}}{1 - \mu \bar{\mu}} \right), \end{aligned} \quad (24)$$

where

$$\bar{\tilde{a}} = \bar{A} + \frac{1}{2} \partial \mu + \mu \tilde{e}^{-1} \partial \tilde{e}, \quad \tilde{e} = e \sqrt{\rho}. \quad (25)$$

One finds

$$\frac{1}{8i\pi} \int_\Sigma \text{Tr} \left[(\mathcal{E} + \frac{1}{2} \mathcal{K} \mathbf{1})^2 \right]^{(2,2)} = \bar{\delta} \delta \bar{\Delta} \bar{\Gamma}, \quad (26)$$

with

$$\bar{\Delta} \bar{\Gamma} = -\frac{1}{2\pi} \int_\Sigma \frac{d\bar{z} \wedge dz}{2i} \text{Tr} \frac{1}{1 - \mu \bar{\mu}} \left[\bar{\tilde{a}} \tilde{e}^{-1} \bar{\tilde{a}}^\dagger \tilde{e} - \frac{1}{2} \bar{\mu} \bar{\tilde{a}}^2 - \frac{1}{2} \mu \tilde{e}^{-1} \bar{\tilde{a}}^{\dagger 2} \tilde{e} \right]. \quad (27)$$

Thus

$$\bar{\delta}\delta\Gamma_{\text{fact}} = 0, \quad (28)$$

with

$$\Gamma_{\text{fact}} \equiv \Gamma(\mu, \bar{\mathcal{A}}; \text{c.c.}; \rho, e) - \Gamma(0, 0; 0, 0; \rho, e) + \widetilde{\Delta\Gamma} + \text{Tr} \mathbf{1} \Delta\Gamma, \quad (29)$$

where $\Delta\Gamma$ is given in [1](eq.22) with $C_j = -\frac{1}{2}$, corresponding to $j = \frac{1}{2}$. This term has been amply discussed in [1].

Besides being factorizable, Γ_{fact} has the following properties : modulo a slight modification of $\widetilde{\Delta\Gamma}$

$$\widetilde{\Delta\Gamma} \rightarrow \widetilde{\Delta\Gamma}_{\text{tot}} \equiv \widetilde{\Delta\Gamma} + \frac{1}{2\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \text{Tr} \left[(\tilde{e}^{-1} \partial \tilde{e} - \tilde{A})(\tilde{A} + \frac{\mu}{2}(\tilde{e}^{-1} \partial \tilde{e} - \tilde{A})) + \text{c.c.} \right], \quad (30)$$

where $\tilde{A}$ is a background connection on $v \otimes \kappa^{-\frac{1}{2}}$, it does not depend anymore on ρ, e : the generalized Weyl anomaly of $\Gamma(\mu, \bar{\mathcal{A}}; \text{c.c.}; \rho, e)$ is easily evaluated by standard heat kernel methods and does exhibit the structure

$$\int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \text{Tr} \left[(\tilde{E}^{-1} \delta_W \tilde{E}) \partial_{\bar{z}} (\tilde{E}^{-1} \partial_z \tilde{E}) - \frac{1}{12} (\delta_W \ln \rho_{z\bar{z}}) \partial_{\bar{z}} \partial_z \ln \rho_{z\bar{z}} \mathbf{1} \right]. \quad (31)$$

Upon including $\widetilde{\Delta\Gamma}_{\text{tot}}$, which does not affect the holomorphic factorization property since it is holomorphically split, one can check that

$$s\Gamma_{\text{fact}} = \frac{1}{2\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \text{Tr} \left[\tilde{H} \tilde{F} - \frac{1}{12} C (\partial^3 + 2\mathcal{R}\partial + \mathcal{R}') \mu \mathbf{1} + \text{c.c.} \right], \quad (32)$$

where (cf. eq.(20)) $\tilde{H}$ and $\tilde{F}$ refer to $v \otimes \kappa^{-\frac{1}{2}}$. The first term, the parallel transport anomaly, is produced by the first counterterm $\widetilde{\Delta\Gamma}_{\text{tot}}$, while the holomorphically split diffeomorphism anomaly is solely produced by the second counterterm $\Delta\Gamma$. Metric independence is crucial and can be used by omitting the variations

$$\begin{aligned} \tilde{e}^{-1} s \tilde{e} &= c(\tilde{e}^{-1} \partial \tilde{e} - \tilde{A} - \tilde{e}^{-1} \tilde{A}^{\dagger} \tilde{e}) + \bar{c}(\tilde{e}^{-1} \bar{\partial} \tilde{e} - \tilde{e}^{-1} \tilde{A}^{\dagger} \tilde{e} - \tilde{A}) - \tilde{H} - \tilde{e}^{-1} \tilde{H}^{\dagger} \tilde{e} \\ s \ln \rho &= (c \cdot \partial) \ln \rho + (\partial c + \mu \partial \bar{c}) + (\bar{\partial} \bar{c} + \bar{\mu} \bar{\partial} c) \end{aligned} \quad (33)$$

in the counterterms, compensating for the Weyl anomaly computed at these variations of the metrics.

The particular case $\mathcal{V} = K^j$ of ref.[1] is recovered with

$$\widetilde{\mathcal{A}} = -(j - \frac{1}{2}) \partial \mu, \quad \tilde{A} = (j - \frac{1}{2}) A_{\kappa}, \quad \tilde{M} = (j - \frac{1}{2}) (\partial c + \mu \partial \bar{c}), \quad (34)$$

and compensating the dependence on the background connection A_{κ} on the canonical bundle κ by the introduction of a projective connection within an additional counterterm of the form

$$\int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \left[\mu (\mathcal{R} + \partial A_{\kappa} + \frac{1}{2} A_{\kappa}^2) + \text{c.c.} \right]. \quad (35)$$

5 Conclusion

We have shown that the local holomorphic factorization property extends to the general situation where fields are sections of an arbitrary holomorphic vector bundle over Σ . This analysis is, however, at the moment local over moduli space. Details will be given elsewhere [16].

Acknowledgements : We wish to thank Frank Thuillier for a crucial contribution at an early stage of this work and François Gieres for his active interest and for generously investing some of his time in checking some lengthy computations.

References

- [1] M. Knecht, S. Lazzarini, R. Stora ; Phys. Lett. **B262** (1991) 25.
- [2] A.A. Belavin, V.G. Knizhnik ; Phys. Lett. **B168** (1986) 201 ;
A.A. Belavin, V.G. Knizhnik ; Sov. Phys. JETP **64** (1986) 214.
- [3] L. Alvarez-Gaumé, G. Moore, C. Vafa ; Comm. Math. Phys. **106** (1986) 1.
- [4] E. D'Hoker, D.H. Phong ; Rev. of Mod. Phys. **60** n^o4 (1988) 917.
- [5] J.-B. Bost, T. Jolicoeur ; Phys. Lett. **B174** (1986) 273.
- [6] J.-B. Bost ; Bourbaki Seminar, Astérisque Vol. **152-153** (1986) ;
D.S. Freed ; “On Determinant Line Bundles” in “Mathematical Aspects of String Theory”,
S.T. Yau ed., World Scientific (1987).
- [7] P. Nelson ; Phys. Rep. **149** (1987) 337.
- [8] P.G. Zograf, L.A. Takhtadzhyan ; Russian Math. Surveys **42** (1987) 169.
- [9] P.G. Zograf, L.A. Takhtadzhyan ; Math. USSR Izvestiya **35** (1990) 83.
- [10] R.C. Gunning ; “Lectures on Vector Bundles over Riemann Surfaces”, Princeton Univ. Press
(1967).
- [11] C. Becchi ; Nucl. Phys. **B304** (1988) 513.
- [12] M. Knecht, S. Lazzarini, F. Thuillier ; Phys. Lett. **B251** (1990) 279.
- [13] D. Quillen ; Funct. Anal. Appl. **19** (1985) 31 ;
D. Quillen ; Topology **24** (1985) 89.
- [14] J.-M. Bismut, H. Gillet, C. Soulé ; Comm. Math. Phys. **115** (1988) 79, 115 & 301.

- [15] D.B. Ray, I.M. Singer ; Adv. in Math. **7** (1971) 145 ;
D.B. Ray, I.M. Singer ; Proc. Symp. in Pure Math. "Partial Differential Equations", Vol.
XXIII, p.167, American Math. Soc. Providence (1973) ;
D.B. Ray, I.M. Singer ; Ann. Math. **98** (1973) 154.
- [16] M. Knecht, S. Lazzarini, R. Stora ; in preparation.