THE VARIATION OF THE PARAMETERS

OF A 25 GEV

ALTERNATING GRADIENT SYNCHROTRON

WITH $\mu$ AND $n$. 

CERN-PS/JBA-MGNH/22
July 1st, 1954.
LIST OF FIGURES

Figure 1. The magnet structures
2. The magnet unit lengths
3. Diamond lengths
4. Diamond widths
5. Q
6. Transition Energy
7. Closed Orbit deviations
8. Injected Beam radius
9. Stop-band widths
10. Beating Factor
11. Peak Synchrotron oscillation amplitudes
12. Effective misalignments due to remanent fields
13. Peak momentum compaction factor
14. Effective Diamond lengths
15. Effective Diamond widths
16. Closed orbit, coupling and beating phenomena
17. Additions
18. Horizontal excursion of particles
19. Vertical excursion of particles
20. Horizontal excursion of particles
21. Vertical excursion of particles
22. Conclusions
Before computing the parameters for the "Conference Machine" an attempt was made to find an optimum value for \( n \), the magnetic field index. Many factors, such as the closed orbit deviation for given random misalignments of the magnet units, and the stop-band width due to random \( n \) variations between the same units, were taken into consideration. Finally the vertical and horizontal beam dimensions inside the vacuum chamber, just after injection, were calculated assuming that the injected beam came from a linac with a given "Acceptance" (the product of beam radius and half-angle of divergence). The results of this study showed that the vertical height of the beam was roughly constant for a range of \( n \) from 200 to 1000 but that the horizontal width of the beam increased rapidly as \( n \) was reduced below 200 due to the ever increasing amplitude of synchrotron oscillation. To obtain these results it was necessary to add together in some way the amplitudes of oscillation of the particles and the displacement of the centre of this oscillation from the geometric centre of the vacuum chamber. Some allowance was made for coupling of the energy of oscillation from the vertical to the horizontal directions and back again. Exactly how these additions should be made was not fully understood at that time and some guesses were made. The \( n \) value for the "Conference Machine" was chosen from these results and a full set of parameters calculated. It was felt after the excitement of designing the first machine had died down, that a detailed examination of the assumptions used in arriving at the optimum \( n \) value was overdue. Furthermore no serious attempt had been made to determine the optimum value for \( \mu \), the phase shift of the betatron oscillation per magnet period, in the first design. Therefore a programme of investigation and computation was planned to present in an easily digestible form how each of the many parameters of an alternating gradient machine varies with \( \mu \) and with \( n \). Using a method of addition that could be justified statistically it was planned to present
the manner in which the final beam cross-section varied with \( \mu \) and \( n \), assuming practical values for the errors of construction.

This report gives the results of this investigation and summarizes the conclusions we have reached so far.

The results are presented in a series of graphs using for axes the variables \( \mu \) and \( n \). The curves on the graphs therefore represent lines of constant value of a given parameter; e.g. constant displacement of the closed orbit or constant number of betatron oscillation cycles per revolution. By using the whole series of graphs therefore the full implication of varying \( \mu \) or \( n \) can be seen fairly quickly. The range of \( \mu \) and \( n \) over which the parameters were calculated was chosen as \( \mu = 0.5 \) to 1.5 radians and \( n = 100 \) to 800. Care was taken to see that practical machines could be built with \( \mu \) and \( n \) values covering the chosen range. Twelve sample machines were calculated and checks applied on their practicability at all stages. These twelve machines had \( \mu \) values of 0.5, 1.0 and 1.5 and \( n \) values of 100, 200, 400 and 800. All the machines had the following constant or nearly constant parameters. The final total energy \( E_{\text{max}} \) was \( 25 \times 10^9 \) eV and the maximum magnetic field on the equilibrium orbit \( B_{\text{max}} \) was 12,000 gauss. The radius of curvature of the particles \( r_0 \) was then 7178 cms. It was arranged that the circumferential length of all the machines was roughly the same i.e. the mean radius \( r_m \) was approximately constant; \( r_m/r_0 = 1.33 \pm 5 \% \). The magnet unit used throughout was the \( 1/2F, 1/2D \) unit (i.e. Half-focusing sector joined rigidly to half-defocusing sector) with field free sectors added between the units of suitable length and periodicity to maintain \( r_m \) roughly constant. In each machine it was arranged that \( N \), the number of magnet periods per revolution, was an integer divisible by 2 or 3 so that correcting lenses, Radio Frequency accelerating units and beam pick-up electrodes could be added periodically round the circumference. The kinetic energy of the particles at injection was fixed at \( 50 \times 10^6 \) eV and the magnetic field rise time to \( 12 \times 10^3 \) gauss was fixed at 1 sec.

The following parameters were computed for all twelve machines:

a) The magnet structure, the length and periodicity of the field free sectors and
the length of the magnet units.

b) The length and width of the working diamond as a ratio to the \( n \) value at the centre of the diamond.

c) \( Q \), the number of betatron oscillation cycles per revolution.

d) The transition energy, \( E_{TR} \).

e) The closed orbit deviation from the ideal orbit for given errors in alignment of the magnet units.

f) The beam radius inside the vacuum chamber for a given injected beam Acceptance.

g) The stop-band widths for given random errors in \( n \) in the magnet units as a ratio to the sub-resonance separation.

h) The Beating Factor, i.e. the ratio of maximum to minimum amplitude of the betatron oscillation at injection after a few revolutions.

i) The maximum synchrotron oscillation amplitude at injection.

j) The effective misalignments due to random remanent fields in the magnet units at injection.

The above parameters were calculated where possible as ratios to the constructional errors so that they are as universal as possible. Putting in possible values of these errors the following additions were then made,

a) Effective lengths and widths of the working diamonds, i.e. the actual widths minus the stop-band widths.

b) The beam cross-section inside the vacuum chamber soon after injection for various errors in construction.

In presenting these results the methods and equations used in the computations are detailed so that independent checks can be made on the answers obtained and criticism made on the methods employed.
The report ends with a discussion of the results of the computations taking into account the magnet design, the overall mechanical design, the design of the R.F. accelerating units and the correcting lenses and the layout and use of the machine. Much more work has yet to be done on the assumptions used in compiling the beam cross-sections, the effective diamond widths, the random remanent fields obtained in practice, etc..., before any valid conclusions can be reached. These will be the subject of further studies.
SECTION 2

The Magnet Structure

As has been stated in the introduction the approximate values chosen for \( n \) and \( \mu \) for the twelve sample machines were \( n = 100, 200, 400 \) and 800 and \( \mu = 0.5, 1.0 \) and 1.5 radians.

Using the magnet period adopted for the Conference Machine namely, \( S, \sqrt{2}P, \sqrt{2}D, S, \sqrt{2}D, \sqrt{2}P \) (where \( F \) denotes a focusing sector, for say the horizontal motion, \( D \) denotes a defocusing sector and \( S \) a field free sector) curves of \( \mu \) against \( \sigma \) (where \( \sigma = \frac{\pi}{2M} \cdot n \sqrt{2} \) and \( M \) is the number of magnet periods round the orbit) were calculated from,

\[
\cos \mu = \cos 2\sigma \cosh 2\sigma + \varepsilon M \sqrt{2} \left[ \cos \sigma \sinh \sigma - \sin \sigma \cosh \sigma \right] x \left[ 2\cos \sigma \cosh \sigma + \frac{\varepsilon}{2^2} M \sqrt{2} (\cos \sigma \sinh \sigma - \sin \sigma \cosh \sigma) \right]
\]

(1)

with \( \varepsilon = \frac{l_s}{r_0} = 0.01679 \)

where \( l_s \) is the length of a field free sector

\( r_0 \) is the radius of curvature of the particles in the magnet.

From these curves the following table giving the number of magnet periods round the orbit for each sample machine was calculated.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( \sigma )</th>
<th>( n=100 )</th>
<th>( n=200 )</th>
<th>( n=400 )</th>
<th>( n=800 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.433</td>
<td>72.55</td>
<td>102.6</td>
<td>143.1</td>
<td>205.2</td>
</tr>
<tr>
<td>1.0</td>
<td>0.607</td>
<td>51.76</td>
<td>73.19</td>
<td>103.5</td>
<td>146.4</td>
</tr>
<tr>
<td>1.5</td>
<td>0.723</td>
<td>43.45</td>
<td>61.44</td>
<td>86.90</td>
<td>122.9</td>
</tr>
</tbody>
</table>

Remembering that the total field free sector length is equal to \( M \cdot l_s \), it can be
seen that with $\frac{1}{S}$ constant there is a variation of 5 to 1 in the total field free sector length over the range of machine considered. It was argued that since the same devices, namely RF units, lenses etc... had to be fitted round the orbit the total field free sector length should be roughly the same for all machines. The easiest way to keep this parameter constant is to alter the length of the individual field free sectors. But from the practical point of view that RF units and lenses have definite fixed dimensions a continuous variation of the length of individual S sectors is not permitted. For this reason the minimum length chosen was 142.4 cms which with $r_0 = 7173$ cms, gives $\epsilon = 0.01984$. The S sector length was allowed to increase in steps of $\sqrt{2}$ for those machines where the number of magnet periods, $M$, was small so that the total S sector length was kept roughly the same. For those machines, only three in number, where $M$ is large it was impossible to keep the total S sector length constant without going below the minimum individual length and for these machines another magnet period was used omitting half the S sectors, namely, $S, \sqrt{2}S, \sqrt{2}D \sqrt{2}S, \sqrt{2}D \sqrt{2}F, \sqrt{2}F, \sqrt{2}D, \sqrt{2}D, \sqrt{2}F$. To enable the new values of $\phi$ and $M$ for the sample machines to be determined it was necessary to compute the $\mu$ against $\phi$ curves for various values of $\epsilon$ for the first type of magnet period and then to repeat these calculations for the second type magnet period.

But of course after completing this stage in the computing the value of $N$ arrived at was not necessarily a whole number, as it must be in any practical machine and also it could not be divided by 2 or 3 which is vital if the RF units and lenses are to be inserted regularly around the orbit in the S sectors. However since $M = \frac{N}{2} \phi^{1/2}$ for the type 1 magnet period and $M = \frac{N}{4} \phi^{1/2}$ for the type 2 period it is possible to adjust $M$ to be an integer divisible by 2 or 3 (or preferably both) by slightly altering $n$, leaving $\phi$ unaltered.

As a final check on the practicability of the sample machines the magnet structure for each machine was drawn up completely with the RF accelerating units, correcting lenses and pick-up electrodes. The basis for arranging the various devices...
in the 3 sectors is shown in Figure 1. The constants for the twelve machines are given in Table 1. Some further adjustments have been made to \( n \) and \( \mu \) as a result of calculating the working diamonds for the machines as is described in the next section.
### Table 1.

Data on the twelve Machines

<table>
<thead>
<tr>
<th>Machine</th>
<th>n</th>
<th>$\mu$</th>
<th>M</th>
<th>Q</th>
<th>$\phi$</th>
<th>$l_s/r_o$</th>
<th>$r_m/r_o$</th>
<th>$L_s/r_o$</th>
<th>X</th>
<th>m</th>
<th>Structure</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>112</td>
<td>0.57</td>
<td>35</td>
<td>3.28</td>
<td>4618</td>
<td>0.0281</td>
<td>1.3214</td>
<td>102.1</td>
<td>1</td>
<td>36</td>
<td>A(3)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>105.5</td>
<td>0.99</td>
<td>27</td>
<td>4.25</td>
<td>5976</td>
<td>0.0397</td>
<td>1.3410</td>
<td>104.4</td>
<td>18</td>
<td>36</td>
<td>A(2)</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>98.5</td>
<td>1.57</td>
<td>21</td>
<td>5.25</td>
<td>7424</td>
<td>0.0397</td>
<td>1.2652</td>
<td>86.7</td>
<td>14</td>
<td>42</td>
<td>A(2)</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>206</td>
<td>0.52</td>
<td>51</td>
<td>4.25</td>
<td>4421</td>
<td>0.0198</td>
<td>1.3221</td>
<td>102.1</td>
<td>34</td>
<td>34</td>
<td>A(1)</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>207.5</td>
<td>1.09</td>
<td>35</td>
<td>6.27</td>
<td>6285</td>
<td>0.0281</td>
<td>1.3214</td>
<td>104.4</td>
<td>24</td>
<td>36</td>
<td>A(3)</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>206</td>
<td>1.62</td>
<td>30</td>
<td>7.74</td>
<td>7515</td>
<td>0.0281</td>
<td>1.2679</td>
<td>86.7</td>
<td>20</td>
<td>40</td>
<td>A(2)</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>394</td>
<td>0.50</td>
<td>35</td>
<td>5.75</td>
<td>4330</td>
<td>0.0281</td>
<td>1.3214</td>
<td>103.3</td>
<td>24</td>
<td>36</td>
<td>A(3)</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>411</td>
<td>1.06</td>
<td>51</td>
<td>8.76</td>
<td>6244</td>
<td>0.0198</td>
<td>1.3221</td>
<td>104.4</td>
<td>34</td>
<td>34</td>
<td>A(1)</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>417</td>
<td>1.61</td>
<td>42</td>
<td>10.76</td>
<td>7499</td>
<td>0.0198</td>
<td>1.2652</td>
<td>86.6</td>
<td>28</td>
<td>42</td>
<td>A(3)</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>790</td>
<td>0.51</td>
<td>51</td>
<td>8.25</td>
<td>4350</td>
<td>0.0198</td>
<td>1.3221</td>
<td>103.3</td>
<td>34</td>
<td>34</td>
<td>A(1)</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>806</td>
<td>1.03</td>
<td>36</td>
<td>11.75</td>
<td>6194</td>
<td>0.0198</td>
<td>1.2273</td>
<td>72.2</td>
<td>24</td>
<td>36</td>
<td>A(3)</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>801</td>
<td>1.47</td>
<td>63</td>
<td>14.73</td>
<td>7057</td>
<td>0.0198</td>
<td>1.3979</td>
<td>125.1</td>
<td>42</td>
<td>42</td>
<td>A(1)</td>
<td>1</td>
</tr>
</tbody>
</table>

$M$ = number of magnet periods round the circumference

$Q$ = number of betatron oscillation cycles per revolution

$\phi = \frac{n}{2 \pi}$

$l_s$ = length of the $S$ sectors

$r_m$ = mean radius

$r_o$ = radius of curvature

$L_s$ = total length of $S$ sectors round the circumference = $2.1M$

$X$ = number of RF units round the circumference

$\gamma$ = number of correcting lenses round the circumference

$\delta$ = number of auxiliary gaps round the circumference

$m$ = RF harmonic number

Structure, Letter refers to Type of Structure and figure to method of exciting the RF units (see Figure.1).

Period, refers to type of period used (see "Magnet Structure" section).
### Possible Machine Structures

#### Structure Type A

<table>
<thead>
<tr>
<th>Period</th>
<th>Lens</th>
<th>AUX: R.F.</th>
<th>Phasing of R.F. Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>$\frac{3}{3} M$</td>
<td>$\frac{3}{3} M$</td>
<td>$\frac{3}{3} M$</td>
</tr>
</tbody>
</table>

**Method:**
- 1
- 2
- 3

#### Structure Type B

<table>
<thead>
<tr>
<th>Period</th>
<th>Lens</th>
<th>AUX: R.F.</th>
<th>Phasing of R.F. Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>$M$</td>
<td>$\frac{1}{3} M$</td>
<td>$\frac{1}{3} M$</td>
</tr>
</tbody>
</table>

**Method:**
- 1
- 2
- 3

**Phasing:**
- In Phase
- $\frac{3}{3} M$
- $\frac{4}{3} M$
- $180^\circ$ Phase Shift
- $M$
PHYSICAL LENGTH OF THE MAGNET UNITS

Fig 2

![Diagram showing physical length units with numerical labels and a legend for 1/2 F and 1/2 D.]
For each of the twelve machines given in Table 1 the value of 
\( \phi = \frac{\pi}{2 M} n^{1/2} \) was found for three values of \( n \) around the chosen value. Using 
these values the trace of the matrix obtained by multiplying the individual 
matrices of the \( S \), \( B \) and \( D \) sectors was calculated and so \( \mu = \cos^{-1} \frac{\text{trace}}{2} \) 
and finally \( Q = \frac{\mu \pi}{2} \) found for each case. For each machine a graph of \( Q \) versus 
\( n \) was plotted and the length of the working diamond and mid \( n \) point established. 
In the matrix elements for focusing and defocusing for this calculation \( n_1 = n_2 \), 
where \( n_1 \) is the \( n \) value in the focusing sector and \( n_2 \) in the defocusing sector.

To find the width of the diamonds calculations were made for the case 
of \( n_1 = k n_2 \), where \( k \) is about 0.95. This gave another curve of \( Q \) versus \( n_1 \).
The calculations were repeated for \( n_2 = k n_1 \) and finally on a plot of \( n_1, n_2 \) the 
three straight lines \( n_1 = n_2, n_1 = k n_2, n_2 = k n_1 \) were drawn along which the 
integer and half-integer \( Q \) points could be marked. By joining up similar \( Q \) points 
the working diamond was formed and its width measured. From the same diamond shapes 
the value of \( Q \) and \( n \) at the mid point of the diamonds for each of the twelve machines 
was noted. Finally the diamond lengths, diamond widths and \( Q \) values were plotted 
on the \( n, \mu \) plane and the results are presented in Figure 3 and 4 respectively.

As an independent check on the final result and the matrix multiplications 
the mid point \( n \) values were taken for each machine and \( Q \) calculated from 
\( Q = \frac{\mu \pi}{2} \) 
where \( \mu \) was obtained by substitution in Equation 1.
Note on the matrices used.

Structure type A can be drawn symbolically as,

\[
\begin{array}{ccc}
S & - & M_1 & - & S & - & M_2 \\
\frac{1}{2}D & - & \frac{1}{2}F
\end{array}
\]

and the matrix for the whole period is,

\[
\begin{bmatrix}
M_2 & S & M_1 & S
\end{bmatrix}
\]

Structure type B can be drawn symbolically as,

\[
\begin{array}{ccc}
S & - & M_1 & - & S & - & M_2 & - & M_1 & - & M_2 \end{array}
\]

and the matrix for the whole period is,

\[
\begin{bmatrix}
M_2 & M_1 & M_2 & S & M_1 & S
\end{bmatrix}
\]

The individual matrices are given by,

\[
|M_1| = \begin{bmatrix}
\left( c \cdot \text{ch} + \frac{n_2}{n_1} \cdot s \cdot \text{sh} \right) & \left( \frac{1}{n_2} c \cdot \text{sh} + \frac{1}{n_1} s \cdot \text{ch} \right) \\
\left( \frac{1}{n_2} c \cdot \text{sh} - \frac{1}{n_1} s \cdot \text{ch} \right) & \left( c \cdot \text{ch} - \frac{n_2}{n_1} s \cdot \text{sh} \right)
\end{bmatrix}
\]

\[
|M_2| = \text{same as } M_1 \text{ with elements } M_{11} \text{ and } M_{22} \text{ interchanged}
\]

\[
|S| = \begin{bmatrix}
1 & \varepsilon \\
0 & 1
\end{bmatrix}
\]

where,

\[
c = \cos \varphi_1 , \quad s = \sin \varphi_1
\]

\[
\text{ch} = \cosh \varphi_2 , \quad \text{sh} = \sinh \varphi_2
\]

\[
\varepsilon = \frac{l_s}{r_0}
\]

and \( l_s \) is the \( S \) sector length for the particular machine being calculated.

and,

\[
\varphi_1 = \frac{\pi}{2M} \cdot \frac{n_1}{n_1} , \quad \varphi_2 = \frac{\pi}{2M} \cdot \frac{n_2}{n_2}
\]
LENGTH OF THE WORKING DIAMOND

\[
\Delta n = \frac{\Delta n}{n_0}
\]

Length of diamond

\[
\Delta n = \frac{\Delta n}{n_0}
\]
NUMBER OF BETATRON CYCLES PER REVOLUTION, Q. FIGS

Q = 3
The Transition Energy

The total Transition Energy is given by,

\[
\left( \frac{E_{\text{tr}}}{E_{\text{o}}} \right)^2 = \frac{n}{2} \cdot \frac{r_m}{r_o} \cdot \left[ \frac{\pi}{2M} \cdot n^2 \right] \left[ \coth \left( \frac{\pi}{2M} \cdot \frac{n}{2} \right) - \cot \left( \frac{\pi}{2M} \cdot \frac{n}{2} \right) \right]
\]

where,

- \( E_{\text{o}} \) = rest of energy of proton = 938.23 \( \times \) 10^6 eV
- \( \frac{r_m}{r_o} = \left( 1 + \frac{1}{n} \cdot \frac{M}{n} \right) \)
- \( M \) = number of magnet periods in the circumference of the machine
- \( r_o = 7178 \) cm

The transition energy was calculated for each machine and the results plotted on the \( n, \mu \) plane. They are given in Figure 6.
SECTION 5

Closed Orbit Deviations

Figure 7 gives curves for the deviation of the closed orbit due to mechanical misalignment of otherwise perfect magnet units. It is assumed that the ends of each magnet, which is a rigidly coupled half-focusing-half-defocusing unit, are independently aligned to a perfect circle to within a certain mechanical tolerance specified by its rms value $a$. The effect of twist misalignments can be included, as the resulting small horizontal components of the guiding field are equivalent to vertical misalignments of untwisted magnet units.

The method of calculating the closed orbit deviations is the same as that used in our earlier report (JBA-MGNH/17): the displacement and the slope, $y, y'$, of the trajectory of a particle leaving a misaligned unit of the magnet are used to calculate the quadratic form

$$
U = a_{12}y'^2 + (a_{11} - a_{12})y^2 - a_{21}y^2
$$

where $a_{12}$ etc... are the matrix elements for the period starting at the end of the displaced unit, and $U$ corresponds to the energy of oscillation of a particle in a conventional focusing potential well. The sum of the $U$ for all the independent units in one magnet period, (calculated with the appropriate matrix elements in each case), multiplied by the number of magnet periods, $M$, in one revolution, gives the average total "energy" of oscillation $U_M$ given to a particle during one revolution. Near a main resonance the closed orbit closely resembles a free betatron oscillation, and the corresponding quadratic form $U_c = a_{12}y'^2 + (a_{11} - a_{12})y^2 - a_{21}y^2$ is very nearly constant at all points round the machine, and is related to $U_M$ by

$$
U_c = \frac{U_M}{2(1 - \cos \phi)}
$$

where $\phi$ is the phase shift round the machine, $= 0$ on resonance.
The peak deviation of the closed orbit is given in this approximation by

\[ \langle y^2 \rangle = \frac{N_k}{a_{21, \text{min}}} \]

where \( a_{21} \) is calculated at the point where it is a minimum, i.e., in the middle of a focusing sector.

Away from a resonance, this approximation is not very good, as the closed orbit becomes less regular in its oscillation and the peak deviation fluctuates round the machine. The calculated figure for \( \langle y^2 \rangle \) seems to need increasing by about 15 o/o at the usual working point half way between main and sub-resonances, to judge from computation of closed orbits in actual cases. This correction factor \( K \) is not applied in these curves, but is considered later in the section on determining apertures.

A more serious weakness of these results is that they are based on the assumption of independent alignment errors: in fact, any practical method of setting up the magnet units will introduce correlation of alignment errors of neighbouring units, and this may alter the closed orbit deviations considerably. Until statistical results from the computation of closed orbits with correlated errors are available, these curves must be used with caution.

In using these curves, if the machine has different radial and vertical frequencies, appropriate values of \( \mu_x, \mu_y \) should be used, and the average \( n_x \mu_x, n_y \mu_y \) should be chosen to give the right values of \( Q_x, Q_y \).
The numbers on the curves are the closed orbit deviations \( \frac{\hat{y}_c}{a} \) in terms of the r.m.s. error, \( a \), of the ends of the magnet units.
SECTION 6

The Injected Beam Radius

Figure 8 gives the radius of beam in the synchrotron when the latter is properly "matched" to the source, which is assumed to give a beam filling an ellipse in phase space whose Acceptance is $\frac{1}{2} \times 10^{-3}$ cm radians. (For example a linac whose final spot size is $\frac{1}{2}$ cm radius with an incoherent half-angle of divergence of $1.10^{-3}$ radians)

When properly matched into the synchrotron, by suitable lenses, the beam is of nearly constant diameter, with only a small amount of sausaging whose wavelength is equal to the length of the magnet period. The figures given are for the maximum radius of such a beam.

The formula used for the radius of a properly matched beam, in the middle of a focusing or defocusing sector is:

$$ r_b = \sqrt{\frac{r_A}{1 - \frac{a_{21}}{a_{12}}}} $$

where $r_o$ = radius of curvature = 7178 cms

$A$ = Acceptance cm x radians

$a_{21}$, $a_{12}$ are matrix elements for the magnet period in question.
The numbers on the curves are the beam radii in cms for an injected beam of acceptance \( \frac{1}{200} \) cms radians.
SECTION 7

The Stop-band widths

Figure 9 shows curves of equal stop-band width, on the assumption that the various half sectors have uncorrelated random errors of 1 o/o rms in \( n \). The numbers of the curves show the percentage fraction of the space between main and sub-resonance points which the stop-bands occupy, with a safety factor to cover 95 o/o of randomly chosen machines, and the effect of "beating" near stop-band edges.

The expression for the rms fraction of main to sub-resonance spacing occupied by stop-bands, without allowing any safety factor is

\[
\bar{f}_{\text{stop}} = \frac{\sqrt{\Delta \mu^2}}{\pi} \quad \text{where}
\]

\[
\frac{\Delta \mu^2}{\mu} = M \sum_{i} \sin^2 \mu \left[ \frac{1}{2} (\text{Trace} \frac{\delta T}{\delta n_i})^2 - 4(\text{Det} \frac{\delta T}{\delta n_i}) \right]
\]

Here \( M \) = No of periods in circumference of machine

\( T \) = Matrix of period

\( i \) = subscript specifying independent elements in a period, each of which has an error.

The derivation of this formula is given in an appendix to this report.

The usable space between two stop-bands is the space between the nominal resonant and sub-resonant points, less half the sum of the two stop-band widths, which have statistically independent Rayleigh distributions. (The displacement of the stop-band centres from their nominal positions does not enter here, as neighbouring stop-bands are displaced by almost exactly the same amount). The space which is unusable if 95 o/o of randomly chosen machines are to work is 1.47 times the rms stop-band width. Further, if the Beating Factor, the ratio of maximum to minimum amplitudes of betatron oscillations close to a stop-band edge, is to remain less than 2, the stop-bands are effectively 1.67 times as wide as the...
actual unstable region. Thus the unusable region is $1.47 \times 1.67 = 2.45$ times the rms stop-band width, and this factor is included in the widths shown for the various curves.
STOP BAND WIDTHS

![Diagram showing stop band widths with labeled bands at 17.5%, 20%, 22.5%, 25%, 27.5%, and 30%.](image)

Stop band width

\[ \frac{A}{B} \% \]

Ratio \[ \frac{\text{max}}{\text{min}} = 3 \]

Max amplitude of waveform \( \text{rms} \)

Min amplitude \( \text{rms} \)
Due to the presence of stop-bands, the amplitude of free betatron oscillations, even in the stable region, is no longer constant in successive revolutions, but beats up and down rhythmically. The ratio of maximum to minimum amplitude is called the beating factor, \( F \). At the working point midway between main and sub-resonances \( F = 1 \) is proportional to the width of neighbouring stop-bands. (See Appendix to this report). The curves of constant Beating Factor, Figure 10, are thus the same as those of constant stop-band width with different numbers attached.

The relation used is

\[
F = 1 + 1.7G
\]

where \( G = \frac{\text{r.m.s. stop-band width}}{\text{main - sub resonance spacing}} \)
BEATING FACTORS

\[ F = \text{Beating Factor} = \frac{A}{B} \]

(for average or rms machine)
The Synchrotron Oscillation Amplitudes at Injection

The Synchrotron Oscillation amplitude is given by,

\[ \Delta r \approx 2 \left[ \frac{E}{E_{TR}} \right]^{3/2} \left[ \frac{E_0}{E_{TR}} \right]^{3/2} \left[ 1 - \left( \frac{E_0}{E_{TR}} \right)^2 \right]^{1/2} \left[ \left( \frac{E}{E_{TR}} \right)^2 - \left( \frac{E_0}{E_{TR}} \right)^2 \right]^{1/2} \cdot \left[ \frac{r_0 r_m B_0}{E_0 m} \right] \cdot 0.683^{1/2} \]

where,

- \( E \) = total injection energy = 988.23 \cdot 10^6 eV
- \( E_0 \) = rest energy of proton = 938.23 \cdot 10^6 eV
- \( E_{TR} \) = total transition energy (see note on transition energy)
- \( B_0 \) = dB/dt at injection = 12.10^3 gauss/sec.
- \( r_0 \) = 7178 cms.
- \( \frac{r_m}{r_0} = (1 + \frac{E}{E_0} \cdot \frac{m}{n}) \) and \( r_m \) is the mean radius.
- \( m \) = harmonic number of the RF.
- \( 0.683 \) = obtained for a stable phase angle of 150° at injection.

The synchrotron amplitudes were calculated for each machine but these amplitudes are mean amplitudes (see Figure 11 below). The maximum radial swing of particles during the synchrotron oscillation is given by,
\[
\frac{\Delta r_{\text{max}}}{r_0} = \frac{1}{n} \cdot \frac{\Delta \rho}{\rho} \cdot \left[ 1 + \frac{2}{\sin \phi (\coth \phi - \cot \phi)} \right]
\]

and the mean radial swing by,

\[
\frac{\Delta r_{\text{mean}}}{r_0} = \frac{1}{n} \cdot \frac{\Delta \rho}{\rho} \cdot \frac{2}{\phi} \cdot \left[ \frac{1}{(\coth \phi - \cot \phi)} \right]
\]

where,

\[
\phi = \frac{\pi}{2M} \cdot n
\]

By using the mean values obtained from the first equation and multiplying them by a function \( f(\phi) = \frac{r_{\text{max}}}{r_{\text{mean}}} \) obtained from the last two equations the maximum swing of the particles was found.

The results are plotted on the \( \mu, n \) plane in Figure 11.
MAXIMUM AMPLITUDE OF SYNCHROTRON OSCILLATION.

FIG. 11

Maximum Amplitude of Synchrotron Oscillation

Mean amplitude of Synchrotron Oscillation

Equilibrium Orbit
SECTION 10

The Effect of Random Variations in the Remanent Magnetic Field

Although the coils of all the magnet units are in series, the magnetic field on the equilibrium orbit will not be exactly the same in all the magnet units. At injection about one third of the field is due to remanence, which is known to be a rather variable quantity; at high fields variations in the permeability of the iron from block to block also will cause slight changes in the guiding field.

These variations of guiding field round the machine are equivalent to extra radial misalignments of individual short lengths of the magnet, and will cause extra deviations of the closed orbit. The effective radial displacement $\delta r$ of a short length whose guiding field is in error by $\delta B$ is

$$\delta r = \frac{r_0 \delta B}{n B_0}$$

If a magnet unit contains $R$ parts having random field errors $\delta B$, the equivalent tolerance on the ends of the unit appears, from an approximate statistical treatment, to be

$$a = 2 \frac{r_0}{n} \frac{\delta B}{B_0} \frac{1}{\sqrt{R}}$$

If the parts of the magnet unit have length $L$, then

$$R = \frac{2\pi r_0}{N L}$$

$$N = \text{No of magnet units}$$

or

$$a = \sqrt{\frac{2r_0}{\pi n}} \frac{\sqrt{N}}{L} \frac{L \delta B}{B_0}$$

The curves of Figure 12 show the value of $a$, in mm., for $\frac{L dB}{B_0} = 0.01$, $L$ measured in cms. The value of $a$ in any particular case, for which $\frac{L dB}{B_0}$ is known,
can be added (in square) to the mechanical tolerance used in Figure 7 to find the closed orbit deviations.

It should be noted that, even if the machine were aligned with the injected beam at the lowest field used, the extra misalignments due to remanent field will have disappeared before the betatron damping has greatly reduced the injection oscillations, and the resulting change in the closed orbit might be dangerous.

These curves can, of course, also be used to estimate changes of the closed orbit deviation due to other differences in the magnetization curves of the various parts of the magnet, for example near saturation.
EFFECTIVE RADIAL MISALIGNMENTS DUE TO REMANENT FIELD ERRORS.
SECTION 11

The peak Momentum Compaction factor

The momentum compaction factor, that is the change in radius for a given charge in particle momentum, is usually calculated for the mean value of the radial displacement and ignores the superposed roughly sinusoidal motion of periodicity equal to the magnet period. The compaction factor using the maximum radial displacement at the peak of this sine wave (at the centre of a focusing sector) we have called the peak momentum compaction factor and this is given by,

$$\frac{\Delta r_{\text{max}}}{r_0} = \frac{1}{\mu} \left[ 1 + \frac{2}{\sin \phi (\coth \phi - \cot \phi)} \right]$$

Figure 13 gives the equi-values of this factor in the $\mu$, $\phi$ plane.
Peak momentum compaction factor

\[
\frac{\Delta \tau}{\tau_0} = \frac{1}{\eta} \left[ 1 + \frac{2}{\sin \phi (\cot \phi - \cot \phi)} \right]
\]
SECTION 12

The effective size of the working diamond

The effective length and width of the working diamond is found by subtracting the stop-band widths given in Figure 9 from the size of the working diamond given in Figures 3 and 4. The results in the μ, n plane are given in Figures 14 and 15.
Effective Diamond Lengths

Effective length of Diamond = \( \frac{\Delta n}{n_o} \) %
Effective Diamond Widths

Effective Width of Diamond = \( \frac{\Delta n'}{n_o} \) %
SECTION 13

The determination of the minimum vacuum chamber aperture

To choose the size of the vacuum chamber for an A-G synchrotron we must be able to find the maximum deviation of an injected proton from the ideal circular orbit in any given machine, and then use this data statistically to fix an aperture which will hold the particles safely in, say, 95 o/o of practical cases.

The main phenomena which determine the displacement of particles at injection are:

a) The deviations of the closed orbit from the ideal circle.

b) The amplitude of oscillation about the closed orbit of the centre particle of the injected beam.

c) The radius of the injected beam.

d) The beating factor for betatron oscillations.

e) Coupling of oscillations between radial and vertical directions.

f) Frequency and energy errors, and synchrotron oscillations.

g) Gas scattering.

h) Space charge effects.

To deal with the last two effects first: the increase in beam size caused by gas scattering does not occur at once, and in the mean time the other betatron oscillations seem to be damped sufficiently to make room for the gas scattering. Again, with practical values of beam current, the effect of space-charge seems only to cause a small increase in the diameter of a properly matched injected beam.
Both these effects, if found in future cases to be more important, can be included by modifying the radius of the injected beam.

We will first consider quantitatively what happens to the beam injected into a particular machine.

We will assume that the working point in the stability diagram is chosen to be in the middle of one of the diamonds not crossed by the unstable type of coupling resonance. This choice of working point keeps the build-up of betatron oscillations as small as possible, but does involve strong coupling back and forth between vertical and radial oscillations, as the working point is right on the other, stable, coupling resonance (Figure 16a). We will further assume that the beam is injected tangential to the ideal circular orbit in the middle of the vacuum chamber.

At injection, the centre particle of the injected beam will start to oscillate about the closed orbit with vertical and radial amplitudes which we will call \( z', \rho' \), and which depend on the position and slope of the closed orbit at the point of injection. The closed orbit itself will have maximum vertical and radial deviations \( \hat{z}_c, \hat{\rho}_c \), at some two points on the circumference of the machine, and in the absence of any coupling or n type errors, the maximum displacements of the particle after a sufficient number of revolutions for the phases to come right, would be \( z' + \hat{z}_c, \rho' + \hat{\rho}_c \) (Figures 16b, c).

The behaviour in the presence of linear coupling has been discussed in detail by Lüders (CERN-PS/GL 9). He shows that the oscillations, which in the absence of coupling would produce a Lissajou figure covering a fixed rectangle in the \((z, r)\) plane, (they are assumed to have slightly different frequencies), now fill a rectangle whose proportions change with time, and which always remains inscribed in the same circle as the fixed Lissajou rectangle in the uncoupled case (Figure 16d, e). The limits of movement of the outer corner of the Lissajou envelope are symmetrically placed about the \(45^\circ\) line, and depend on the precise coupling errors, and on the starting conditions for the trajectory chosen. The maximum value of the vertical or radial component of the oscillation can then be written as \( \hat{\delta} \sqrt{z'^2 + \rho'^2} \), where
is a parameter depending on the particular machine and on the starting conditions, and which will be between $\frac{1}{12}$ and 1.

In our particular machine, then, the maximum displacements of the beam centre due to the effects so far considered are

$$ \delta_c + \tilde{\delta} \sqrt{z'^2 + \rho'^2} \quad \text{vertically} $$

and

$$ \tilde{\delta}_c + \tilde{\delta} \sqrt{z'^2 + \rho'^2} \quad \text{radially}. $$

In a machine with $n$ type errors, (variations in sector length or $n$ value), all betatron oscillations are modulated in amplitude, the ratio of maximum to minimum amplitude being $F$, the Beating Factor. (See Sect. 8 and Appendix II). In a particular case, the increase in amplitude after injection will be some factor $F$, less than the full amount $F$, as injection will not in general occur at a point of minimum oscillation (Figure 16 f). This amplitude increase will also affect the radius of the injected beam $r_b$, which has to be added on to the displacement of the beam centre. The final values for the displacements of the particles on the outside of the beam are then

$$ \delta_c + F \tilde{\delta} \sqrt{z'^2 + \rho'^2} + Fr_b \quad \text{vertically} $$

$$ \tilde{\delta}_c + F \tilde{\delta} \sqrt{z'^2 + \rho'^2} + Fr_b \quad \text{radially}. $$

Before considering the statistical distributions of these displacements to estimate the apertures needed for machines whose precise errors are not known, we must split up some of the quantities into parts due to different physical causes.

The closed orbit is displaced from the ideal circle near injection for several reasons:
a) Radial and vertical mechanical misalignment of magnet units due to inaccuracies in survey and levelling.

b) Errors of location of the median plane and guiding field with respect to the fiduciary marks on the magnet.

c) Effective extra vertical displacements of magnet units due to the radial component of the guiding field of a twisted sector or of one with a tilted median plane.

d) Effective extra radial misalignments due to random variations in the remanent field.

e) Radial displacements of the equilibrium orbit due to momentum errors and phase oscillations.

The effects of a), b) on the closed orbit can be estimated using the curves of Figure 7, assuming that the errors can be regarded as independent, uncorrelated errors of the ends of the magnet units. Figure 7 gives $\frac{\gamma c}{a}$, the expectation value of the peak deviation of the closed orbit for unit errors, i.e. the rms average over an ensemble of different machines and over all the peaks round the circumference of each machine, and this must be multiplied by the factor $K \approx 1.15$ mentioned in Section 5 to give $\frac{\gamma y}{a}$ the expectation value of the largest peak anywhere on the circumference (see Figure 16b).

Twists of the median plane, c), have the same effect on the closed orbit as parallel vertical displacements of the magnet units, and the $\sqrt{2F} \sqrt{2D}$ structure we consider is very insensitive to this kind of error. An angle of twist of $\sqrt{2} \times 10^{-3}$ radian corresponds to a displacement of $\frac{1}{6}$ mm. in a $n = 300$ machine, and the effect of this on the closed orbit can be neglected, being equivalent to about $\frac{1}{40}$ mm. tolerance on sector ends.
The equivalent "end alignment tolerance" for the effect of random injection field variations can be determined from Figure 12, if an estimate is available of these variations. If the results of the cosmotron magnet testing programme can be used as a guide, the random injection field variations between blocks 15 cms long might be of the order of 1 gauss (in 140) and reference to the curves of Figure 12 shows that the corresponding end tolerance, for \( \frac{\delta B}{B} = 3 \text{o/o} \), is about 1 mm, three times as large as the mechanical tolerances contemplated. If the blocks are stacked in the same way as in the cosmotron, with 2000 blocks to play with it should be possible to reduce the scatter between paired blocks 30 cms long by a factor of ten or more, and the whole effect may become insignificant. It has not been included in the later calculations.

At the instant of injection, the closed orbit of an injected particle is displaced outwards from the ideal circle by an amount \( \rho_{\text{cp}} = \alpha r_0 \delta p \) where \( \alpha \) is the momentum compaction factor given in Figure 13, and \( \delta p \) is the difference between the actual momentum of the particle and that determined by the magnetic field at that instant. The extreme displacement of the closed orbit of a particle which is on the edge of the phase stable region is the sum of the radial displacement due to the frequency error \( \rho_{\text{cf}} = \alpha r_0 \frac{\delta f}{f} \) and the radial synchrotron amplitude, \( \rho_{\text{cs}} \) (see Figure 17a).

In all, the important terms making up the closed orbit deviations are:

Vertically:

a), b) giving a total misalignment tolerance \( a_z \), with ensemble average peak displacement \( \left\langle \frac{\rho_{\text{cm}}}{a_z} \right\rangle = \bar{K} \frac{\rho_{\text{c}}}{{\bar{a}}} \ a_z \) (Figure 16b)

Radially:

a), b) as above \( \left\langle \frac{\rho_{\text{cm}}}{a} \right\rangle = \bar{K} \frac{\rho_{\text{c}}}{a} \ a \)

e) at instant of injection \( \rho_{\text{cp}} = \alpha r_c \frac{\delta p}{p} \), increasing at some later time to \( \rho_{\text{cf}} + \rho_{\text{cs}} \) (Figure 17b)
The betatron oscillations about the closed orbit are determined by the position of the closed orbit at the point of injection. If the beam is injected in the middle of a sector, the amplitude of oscillation is

$$ y' = \sqrt{y_C^2 - \frac{a_{12}}{a_{24}} y_C^2} $$

where $y_C$, $\dot{y}_C$ are the displacement and slope of the closed orbit at the injection point.

For the vertical component, where the closed orbit deviations are due only to random misalignments, $y_C^2 - \frac{a_{12}}{a_{24}} y_C^2$ is nearly constant round the machine, and is equal to the peak closed orbit deviation $\dot{y}_C$ (without the extra factor $k$), i.e. the vertical betatron oscillation amplitude is $z' = \frac{\dot{y}_C}{y_C}$.

Horizontally, where $y_C$ is the sum of the deviations due to misalignments and to the momentum error, the amplitude of oscillation will be

$$ \rho' = \sqrt{\left(\rho_{cm} + \rho_{cp}\right)^2 - \frac{a_{12}}{a_{24}} \left(\frac{\dot{y}_C}{y_C}\right)^2} $$

$$ = \frac{\dot{y}_C}{y_C} \sqrt{1 + 2 \frac{\rho_{cm}^2 \rho_{cp}^2}{\rho_{cm}^2}} $$

This amplitude will be unaffected by the later movements of the closed orbit due to phase oscillations, as these movements are adiabatic.

Using all these results, the maximum excursions of the particles at any time after injection in a particular machine are:

vertically

$$ F \frac{\dot{y}_C}{y_C} \sqrt{\frac{y_C^2}{y_C^2} + \frac{\rho_{cm}^2}{y_C^2} \left(1 + \frac{2\rho_{cm}^2 \rho_{cp}^2}{\rho_{cm}^2}\right)} + F r_b $$

radially

$$ \frac{\rho_{cm} + \rho_{cf} + \rho_{cs}}{\rho_{cm}} + \frac{\dot{y}_C}{y_C} \sqrt{\frac{a_{12}^2}{y_C^2} + \frac{\rho_{cm}^2}{y_C^2} \left(1 + \frac{2\rho_{cm}^2 \rho_{cp}^2}{\rho_{cm}^2}\right)} + F r_b $$

where all the quantities refer to the machine considered, (Figure 17c).
All one can hope to know in practice is the mean value and probability distribution of each of the various parameters; with these one can find the proportion of machines in which the particle excursion exceeds some stated amount, and then choose the vacuum chamber dimensions to hold the particles in, say 95% of the possible cases. To do this accurately would be difficult, and unnecessarily refined. Several of the parameters have only a small range of variation, or a small effect on the excursion of the particle, and it is justifiable to replace these with their mean values, and regard them as constants. \( \bar{\mu}, \bar{\sigma}, \bar{\beta}, \) and the frequency and momentum error terms \( \rho \) and \( \rho_c \) can be treated in this way. Since the mean values of the latter terms are zero, they can be omitted; further, \( \rho_c \), the synchrotron amplitude, and \( r_b \) the beam radius are known quantities, which can be added to the chosen aperture at the end of the calculations.

The remaining quantities, whose statistical variation must be taken into account, are \( \tilde{z}, \tilde{\beta} \). Läders (CERN-PS/GL 6) has shown that the distributions of these quantities are Rayleigh distributions, i.e. the probability of finding \( \tilde{z} \) in a certain region \( dz \) is

\[
P(d \tilde{z} | \tilde{z}) = \frac{2}{a^3} e^{-\frac{1}{2} \tilde{z}^2 / a^2} dz
\]

where \( a \) is the rms value of \( \tilde{z} = \langle \tilde{z} \rangle \).

Similarly

\[
P(dp | \beta) = \frac{2}{b^3} e^{-\frac{1}{2} \beta^2 / b^2} dp \quad b = \langle \beta \rangle
\]

The problem is then reduced to that of finding the probability distributions of

\[
\bar{\mu} \tilde{z} + \bar{\sigma} \tilde{c} \sqrt{\tilde{z}^2 + \tilde{\beta}^2}
\]

and

\[
\bar{\mu} \beta + \bar{\sigma} \tilde{c} \sqrt{\tilde{z}^2 + \tilde{\beta}^2}
\]

or, with appropriate substitutions, of
The cumulative distribution of this can be found analytically, and the values of \( x + \alpha \sqrt{x^2 + y^2} \) which include 95% of cases, for various choices of \( \alpha \) and of \( \langle \hat{x} \rangle, \langle \hat{y} \rangle \) have been computed numerically (Appendix III). When the parameters \( K, \hat{z}, \) etc. are known for a machine, the critical value of \( x + \alpha \sqrt{x^2 + y^2} \) can be transformed into the maximum displacement a particle is likely to make, with the 95% safety factor.

\( K, \hat{F}, \hat{C}, \) are now ensemble averages which must be related to the \( n \) value and other parameters of the machine. From the computation of the closed orbits in an ensemble of machines with random misalignments, we found \( \hat{K} \approx 1.15 \), for a rather low \( n \) machine. There seems no good theoretical estimate for it, but probably it does not change rapidly with \( n \). For simplicity it has been assumed constant, \( = 1.15 \) for all the trial machines.

\( \hat{F} \) is the average increase in amplitude of betatron oscillation from the point of injection: this average will be about half the largest possible increase, from the minimum to the maximum of the "beat", (Figure 16f)

\[ \hat{F} \approx \sqrt{2} (1 + F) \]

\( F \) is given in Figure 10 if the random errors in \( n \) are known.

Referring to Figure 16e, showing the behavior of the oscillations when coupling is present, one can see that \( \hat{C} \), the ratio of the half diagonal of the Lissajou rectangle to its maximum component in \( z \) or \( \rho \) directions cannot be less than \( \frac{b}{\sqrt{a^2 + b^2}} \) where \( b \) is the larger of the two components at the start. Provided some phase angle happens to be right, any starting conditions can lead to the oscillation swinging completely from pure vertical to pure radial and back, so with the starting amplitudes \( a, b \) the average coupling factor \( \hat{C} \) will be about
\[
\frac{1}{2} \left( 1 + \frac{b}{\sqrt{a^2 + b^2}} \right) \quad (a < b)
\]

Instead of \( a, b \) we have used the vertical and radial tolerances, which determine the average starting amplitudes of oscillation directly.

Thus

\[
\bar{c} = \frac{1}{2} \left( 1 + \frac{a_p}{\sqrt{a_z^2 + a_p^2}} \right) \quad a_z \leq a_p
\]

and in the cases where \( a_p = a_z, \bar{c} = 0.85 \), and where \( a_p = 2a_z, \bar{c} = 0.95 \).

To illustrate this procedure, the minimum vacuum chamber dimensions have been calculated for the twelve trial machines, under two sets of assumptions about tolerances, and the results are presented as curves of equal chamber size in Figures 18 - 21.

The assumptions for Figures 18, 19 are:

- Magnet tolerances:
  - vertical \( 0.3 \text{ mm} \)
  - radial \( 0.3 \text{ mm} \) rms error of ends of \( \frac{1}{2}F \frac{1}{2}D \) units
  - \( n \) errors: \( 1 \% \) rms error in half sector \( n \) values

- Acceptance of injected beam: \( 0.5 \times 10^{-3} \text{ cm-rad} \) radius \( x \frac{1}{2} \) angle

- Energy and frequency errors: zero

and for Figures 20, 21:

- Magnet tolerances:
  - vertical \( 0.3 \text{ mm} \)
  - radial \( 0.6 \text{ mm} \)

- Other data as in the first case.

The synchrotron oscillation amplitudes from Figure 11 are included in these results.
CLOSED ORBIT, COUPLING AND BEATING PHENOMENA.

**FIG 16**

**Working point** Fig 16a

**Closed Orbit** Fig 16b

Additions in the simplest case. Fig 16c

\[(p, z)\] Lissajou Figure. No Coupling. Fig 16d

\[(p, z)\] Lissajou Figure. With coupling. Fig 16e

\[\text{Injection Ratio} = F\]

\[\text{Ratio} = F(>1)\]

Beating of the free oscillations. Fig 16f.
ADDITIONS

**Figure 17a.**

 energies specified by:

- a) Phase stability
- b) Frequency, \( E_f \)
- c) Linac, \( E_L \)
- d) Field, \( E_B \)

Energy phase diagram.

**Radius Diagram**

- a) Closed Orbit of just stable particles.
- b) Closed Orbit at injection.
- c) Injected beam.

**Radial Variation of closed orbit. Fig. 17b**

- \( P_{cp} \) injected beam
- \( R_b \) radius

\[ Q_0 = 3 \]
\[ Q = 3.25 \]

**Final additions in the radial direction. Fig 17c**

- \( \beta' = \beta_{cm} \left[ \frac{1 + 2 \beta_{cm} P_{cp}}{P_{cp}^2} \right] \)
The maximum horizontal displacement of particles includes the synchrotron oscillations but no frequency errors.

Horizontal errors = 0.3 mms r.m.s.
Vertical errors = 0.3 mms r.m.s.
Acceptance of the injected beam = \( \sqrt{0.00} \) cms radians.
The numbers on the curves are the maximum vertical displacement of the particles from the ideal orbit in ems.

The horizontal errors = 0.3 mm ± r.m.s.

The vertical errors = 0.3 mm ± r.m.s.

Acceptance of the injected beam = $\frac{1}{2000}$ ems ± radians.
The maximum horizontal displacement of particles includes the synchrotron oscillations but no frequency errors.

Horizontal errors = 0.6 mms rms
Vertical errors = 0.8 mms rms

Acceptance of the injected beam = $\frac{1}{2000}$ ems. radians.
The numbers on the curves are the maximum displacement of particles from the ideal orbit in cms.

The horizontal errors = 0.6 mm ± r.m.s.
The vertical errors = 0.3 mm ± r.m.s.

Linear acceptance of the injected beam (beam radius × ½ angle of divergence)

= 1/2000 cms. radians.
SECTION 14

Conclusions

The purpose of this study has been to present the evidence on which a decision can be made on the n and μ value for the CERN machine. The evidence is contained in the various figures.

One simple way to present the overall result is to consider each graph separately and to decide which of the constant value curves represents a boundary past which it would be inadvisable to go. For example we might say that we would not like the transition energy to be lower than, say 4 BeV. In that case we can consider the 4 BeV curve on the n, μ plane in Figure 6 as a boundary, not to be crossed. We can fix boundary curves on each of the graphs and by superimposing them on a common graph find the area in which we would like to see the n and μ value of the final machine. Each boundary that is fixed represents a judgment and where it is fixed is a matter of opinion. What is presented below is the authors' opinion.

Not all the boundaries determine the final target area and that is fortunate since it limits the boundary disputes to a few parameters. More will be said about this later. One fact should always be born in mind, namely that many graphs are based on assumptions, particularly about tolerances and correlations of errors of construction. It is not necessarily true as will be seen later that the final result depends only on the assumptions. However the assumptions should be remembered when exercising judgment.

We will now consider the various graphs and give our opinions on the boundaries.

1. Section 2. Figure 2. The length of the magnet units.

At present it is planned to erect the separate packages of magnet laminations on a welded up trolley the top of which forms a flat reference plane. This trolley is itself rigid and supports the weight of the magnet blocks. It is supported on three points from the floor of the trench and so can be adjusted for height, tilt, etc... In this way the 2P, 2D magnet unit can be considered as a
rigid unit whose ends can be adjusted by survey techniques to as near a perfect circle as possible. The flat top of the trolley enables the magnet blocks to be adjusted to get a correctly curved and aligned equilibrium orbit for the particles. If at any time the magnet units need readjustment then this trolley system can be readjusted without too much labour. However it is inconvenient to have the magnet units too long due to the weight of the units and difficulties with long span trolleys. So we say that we would not like the units to be longer than six meters.

2. Section 3. Figure 5. Q.

The survey system planned envisages using four radial trenches in which to lay an invar tape to set up four primary marker stations. From these a further four secondary stations are set up by angular measurements. Thus there might be four or eighth harmonic components of the structure exciting the betatron oscillations and Q values of 4 and 8 should be avoided. We therefore say that Q should be between 5 and 7. However if the present survey system were replaced by one using repetitive angular measurements to set up the marker stations without any central reference then the fourth harmonic excitation would be replaced by the harmonic corresponding to the number of marker stations used. Another criterion for limiting Q could be the question of ground subsidence. Unfortunately sags in the trench supporting the magnet ring are so rich in harmonics that machines with a wide range of Q values are more or less equally affected.

3. Section 4. Figure 6. Transition Energy.

There are three distinct stages of acceleration of the particles. First there is the injection stage, then the time while the particles are coming up to relativistic velocities, then the transition stage when the stable phase of the particles changes from that appropriate to a Linac to that for a synchrotron. It is best to separate the last two stages as much as possible. The particles are relativistic after about 2 GeV. It is advantageous to have some phase damping before the unsettling period of transition is reached. We therefore say that the Transition Energy should not be less than 4 GeV.
4. Section 10. Figure 12. Remanent Field Effects.

Cosmotron measurements show that in magnet blocks 15 cms long there might be variations in the field at injection of about 1 gauss in 140. Intelligent stacking of these blocks will reduce the scatter between paired blocks to considerably less than this amount. If the random scatter in the injection field is 0.3 gauss between blocks 25 cms long then the effective displacement of the ends of the magnet units should not exceed 0.3 mms. This boundary is shown as a broken line in Figure 22 since the random scatter should be better than 0.3 gauss but how much better is not known yet.


Clearly the bigger the final free space inside the working diamond the better. Non linearities may further reduce the free area either by increasing the widths of the stopbands or introducing subresonance bands. We feel that a diamond whose effective length $\Delta n/n = 5\%$ and whose effective width $\Delta n/n = 1\%$ is the smallest tolerable.


Reference to the Figures shows that the particle excursions do not vary very much over the $n$, $\mu$ plane. Which boundary one chooses is a matter of taste. We have chosen a maximum horizontal excursion of 3.5 cms and a maximum vertical excursion of 2.7 cms for the case of the horizontal and vertical tolerances both equal to 0.3 mms, and a maximum horizontal excursion of 5.0 cms and a maximum vertical excursion of 3.1 cms for the case of the horizontal tolerance = 0.6 mms and the vertical tolerance 0.3 mms. The important point is that these boundary lines can be put outside the interesting region at the price of very small changes in aperture. The vacuum chamber size depends on what tolerances can be kept when constructing the machine and the $n$ and $\mu$ values can be decided upon afterwards.

7. Magnet design.

From the magnet measurements completed so far it can be said that the lower the $n$ value the better is the uniformity in $n$ across the vacuum chamber.
aperture. This follows from general reasoning that there will be less saturation of the pole tips if the poles are more nearly parallel.

The result of superimposing all these boundaries is shown in Figure 22. The final target area is shaded but it should be remembered that n and μ values for machines cannot be chosen at random inside this area. The Q values for machines have to have \( Q = \text{integer} + \frac{1}{2} \) or \( \frac{1}{4} \) to end up with the working point at the center of the diamond. Furthermore in the magnet structures used (see Figure 1) there is a superperiod equal to three magnet periods due to the correcting lens arrangement. This superperiod gives a wider stopband at \( \mu = \frac{\pi}{3} \) and as the lenses are energised to keep the working point in the middle of the working diamond during operation this stopband widens or narrows. Thus the top part of the target area is prohibited down to about \( \mu = 0.95 \). Both the permissible Q values and the \( \frac{\pi}{3} \) line are shown in Figure 22. The machine design given in CERN-PS/JBA-MGNH/19 is also marked on. (Point A

Acknowledgments

It is a pleasure to thank Miss Hanney who carried out all the computing for this study. We also thank Dr. Lüders for all the discussions on theoretical problems associated with the calculations and the free use we have made of his theoretical reports on the machine. Many other reports issued by CERN Staff have been used for which we make general acknowledgment.
CONCLUSION

Arrows point to the undesirable side of the curves.
Numbers refer to previous Figures, from which the curves are traced.
Machine design given in CERN/PS/JBA-MGNH 19, shown as point A.
SECTION 15

Appendices

These appendices contain more details of some of the calculations leading to the graphs in this report.

I. Stopband widths.

Three methods have been used for calculating the effect of n type errors, that of Courant and Snyder (EDC/HSS-1) and Twiss (Cambridge Design Study Report) using the \((\alpha, \beta)\) representation of the period matrix, the diagonal representation method of Bell (AEHE T/M 79), and the differential equation method of Lüders (CERN-PS/GL 4, 6). Results given by us earlier (CERN-PS/JBA-MGNH'10 and Report of CERN Geneva Conference) were found using Bell's method, which involves complex numbers, and is difficult to work with numerically. The \((\alpha, \beta)\) method has the disadvantage that for a magnet structure involving field free sections possibly placed asymmetrically, or unequal sector lengths, computing \(\frac{\partial \alpha}{\partial n}, \frac{\partial \beta}{\partial n}\) for all the possible errors \(n\), is a long job.

We have found that a more easily applied formula for the stopband width can be obtained using the straight-forward form of the transfer matrix, using the expression for the \(k\)th power of a 2 x 2 unimodular matrix,

\[
T^k = \frac{\sin k \mu}{\sin \mu} T - \frac{\sin (k-1) \mu}{\sin \mu} I
\]

which is by now widely known. Although this expression is not quite so simple as the corresponding ones in the \((\alpha, \beta)\) or diagonal representations, it contains the matrix \(T\) in its normal form as a product of factors each belonging to a separate part of the magnet period, and finding the effect of an error \(\delta n\) in one part is very simple numerically.

If one uses the subscripts \(\iota, j\) to refer to parameters, \(\alpha\), determining the properties of the magnet period, e.g. n values or lengths or sectors, and \(k, l\) to number the periods round the machine, then the matrix for an imperfect machine
can be written as

\[ M = M_0 + \sum_{k} \sum_{l} \frac{\partial M_0}{\partial x_{kl}} \delta x_{kl} + \frac{1}{2} \sum_{k} \sum_{l} \sum_{l'} \sum_{l''} \frac{\partial^2 M_0}{\partial x_{kl} \partial x_{l'l''}} \delta x_{kl} \delta x_{l'l''} \]

to the second order in a Taylor expansion in terms of the perfect machine matrix \( M_0 \).

If \( T \) is the matrix of a perfect magnet period, then

\[ M_0 = T^m \]

So

\[ \frac{\partial M_0}{\partial x_{kl}} = T^{-k-1} \frac{\partial T}{\partial x_{kl}} \]

and

\[ \frac{\partial^2 M_0}{\partial x_{kl} \partial x_{l'l''}} = T^{-k-2} \frac{\partial^2 T}{\partial x_{kl} \partial x_{l'l''}} \]

Using these expressions, the Trace of \( M \) becomes

\[ Tr M = Tr T^m + \sum_{k} \sum_{l} Tr \left( T^{-k-1} \frac{\partial T}{\partial x_{kl}} T^{-k} \right) \delta x_{kl} \]

\[ + \frac{1}{2} \sum_{k} \sum_{l} \sum_{l'} \sum_{l''} Tr \left( T^{-k-2} \frac{\partial^2 T}{\partial x_{kl} \partial x_{l'l''}} T^{-k} \right) \delta x_{kl} \delta x_{l'l''} \]

\[ + \frac{1}{2} \sum_{k} \sum_{l} \sum_{l'} \sum_{l''} \sum_{l'''} Tr \left( T^{-k-3} \frac{\partial^3 T}{\partial x_{kl} \partial x_{l'l''} \partial x_{l'''l''}} T^{-k} \right) \delta x_{kl} \delta x_{l'l''} \delta x_{l'''l''} \]

Near a resonance, we can write for the phase shift round the whole perfect machine, \( \tilde{\mu} \),

\[ \tilde{\mu} = m \mu - 2\pi Q_0 \quad (Q_0 \text{ integral}) \]

so the first term of \( Tr M \) is

\[ Tr T^m = 2 \cos \tilde{\mu} \approx 2 - \tilde{\mu}^2 \quad \text{since } \tilde{\mu} \ll 1 \]
Permuting the factors, the next term becomes

$$\sum_k \sum_l \mathrm{Tr} \left( \frac{\partial T}{\partial x_k} T^{-1} \right) \delta x_{lk}$$

and we can use the formula

$$T^m = \frac{\sin \mu}{\sin \mu} T - \frac{\sin (m-1) \mu}{\sin \mu} I = \frac{\sin \mu}{\sin \mu} T - \frac{\sin (m-1) \mu}{\sin \mu} I$$

and, from the third term, similarly

$$\mathrm{Tr} \left( \frac{\partial T}{\partial x_c} T^{-1} T^m \right) = \mathrm{Tr} \left( \frac{\sin \mu}{\sin \mu} \frac{\partial T}{\partial x_c} - \frac{\sin (m-1) \mu}{\sin \mu} \frac{\partial T}{\partial x_c} T^{-1} \right)$$

Now, since the determinant of any part of the machine, perfect or imperfect, is unity,

$$\det \left( T + \Delta T \right) T^{-1} = 1$$

or

$$\det \left( I + \sum_k \frac{\partial T}{\partial x_k} T^{-1} \delta x_k + \frac{1}{2} \sum_k \sum_l \frac{\partial^2 T}{\partial x_k \partial x_l} T^{-1} \delta x_k \delta x_l + \ldots \right) = 1$$

Expanding this determinant, and equating powers of $\delta x_k$ to zero, one obtains

$$\mathrm{Tr} \left( \frac{\partial T}{\partial x_c} T^{-1} \right) = 0$$

and

$$\frac{1}{2} \mathrm{Tr} \left( \frac{\partial^2 T}{\partial x_c^2} T^{-1} \right) + \det \left( \frac{\partial T}{\partial x_c} T^{-1} \right) = 0$$

Using the first of these results, and assuming $\tilde{\mu} \ll 1$

$$\mathrm{Tr} \left( \frac{\partial T}{\partial x_c} T^{-1} T^m \right) = \frac{\tilde{\mu}}{\sin \mu} \mathrm{Tr} \frac{\partial T}{\partial x_c}$$

$$\mathrm{Tr} \left( \frac{\partial T}{\partial x_c \partial x_j} T^{-1} T^m \right) = \mathrm{Tr} \left( \frac{\tilde{\mu}}{\sin \mu} \frac{\partial T}{\partial x_c \partial x_j} - \frac{\tilde{\mu} \cos \mu}{\sin \mu} \frac{\partial T}{\partial x_c \partial x_j} T^{-1} + \frac{\partial^2 T}{\partial x_c \partial x_j} T^{-1} \right)$$
Arranging $\text{Tr} M$ in powers of $\tilde{\mu}$, we have

$$
\text{Tr} M = -\tilde{\mu}^2 + \tilde{\mu} \left[ \sum_K \sum_l \frac{\delta x_{lk}}{\sin \mu} \text{Tr} \frac{\partial T}{\partial x_l} + \frac{1}{2} \sum_K \sum_l \sum_j \frac{\delta x_{lk} \delta x_{lj}}{\sin \mu} \left( \text{Tr} \frac{\partial T}{\partial x_l \partial x_j} - \cos \mu \text{Tr} \frac{\partial T}{\partial x_l \partial x_j} \right) \right] + 2 + \frac{1}{2} \sum_K \sum_l \sum_j \text{Tr} \frac{\partial T}{\partial x_l} T^{-1} \delta x_{lk} \delta x_{lj} + \frac{1}{2} \sum_K \sum_l \sum_j \delta x_{lk} \delta x_{lj} \left( \right)
$$

At the edge of a stopband $\text{Tr} M = 2$, which gives a quadratic equation in $\tilde{\mu}$ for the two edges.

$$
\tilde{\mu}^2 - \tilde{\mu} \left[ \frac{1}{2} \sum_K \sum_l \sum_j \sum_k \frac{\delta x_{lk}}{\sin \mu} \text{Tr} \frac{\partial T}{\partial x_l} \right] - \frac{1}{2} \sum_K \sum_l \sum_j \sum_k \frac{\delta x_{lk} \delta x_{lj}}{\sin \mu} \left( \text{Tr} \frac{\partial T}{\partial x_l \partial x_j} - \cos \mu \text{Tr} \frac{\partial T}{\partial x_l \partial x_j} \right) = 0
$$

The stopband width is the difference of the roots of this equation, which in the usual notation is $\sqrt{b^2 - 4ac}$.

$$
(D\tilde{\mu}) = \left[ \frac{1}{2} \sum_K \sum_l \sum_j \sum_k \frac{\delta x_{lk}}{\sin \mu} \text{Tr} \frac{\partial T}{\partial x_l} \right]^2 + 2 \sum_K \sum_l \sum_j \sum_k \frac{\delta x_{lk} \delta x_{lj}}{\sin \mu} \left( \text{Tr} \frac{\partial T}{\partial x_l \partial x_j} - \cos \mu \text{Tr} \frac{\partial T}{\partial x_l \partial x_j} \right)
$$

The mean of this will not contain terms with products of errors from different periods, or different parts of the same period, so

$$
\overline{D\tilde{\mu}^2} = \frac{1}{\sin^2 \mu} \sum_K \sum_l \left( \text{Tr} \frac{\partial T}{\partial x_l} \right)^2 \overline{\delta x_{lk}^2} + \frac{1}{\sin^2 \mu} \sum_K \sum_l \left( \text{Tr} \frac{\partial T}{\partial x_l} \right) \left( \overline{\delta x_{lk}^2} \right) + \frac{1}{4} \frac{1}{\sin^2 \mu} \sum_K \sum_l \sum_j \left( \overline{\delta x_{lk}^2} \right) + 2 \sum_K \sum_l \sum_j \sum_k \text{Tr} \frac{\partial T}{\partial x_l} \overline{\delta x_{lk}^2} \overline{\delta x_{lj}^2}
$$

The terms in $\overline{\delta x^2}$ and $\overline{\delta x^4}$ are beyond the order of accuracy of this analysis; dropping them, and replacing the last term by its equivalent found earlier,

$$
\overline{D\tilde{\mu}^2} = \sum_K \sum_l \left( \frac{1}{\sin^2 \mu} \left( \text{Tr} \frac{\partial T}{\partial x_l} \right)^2 \overline{\delta x_{lk}^2} - 4 \text{det} \frac{\partial T}{\partial x_l} \overline{\delta x_{lk}^2} \overline{\delta x_{lj}^2} \right)
$$
Since, finally, the range of errors will be the same in all periods, and also since \( \det T' = 1 \)

\[
\Delta \mu^2 = m \sum \left( \frac{1}{\sum \mu} \left( \frac{\partial T}{\partial x} \right)^2 - 4 \frac{\partial \mu}{\partial x} \right) \Delta x
\]

In practice, \( T = PQRSUV \ldots \)

\[
\frac{\partial T}{\partial x_k} = PQ \frac{\partial R}{\partial x_k} SUV + \det \frac{\partial T}{\partial x_k} \frac{\partial R}{\partial x_k}
\]

is quite simple to calculate, and forming the product for \( \frac{\partial T}{\partial x_k} \) is also quick numerically. If \( PQ, SUV \) are computed separately, these can be checked by the \( \det = 1 \) relation before forming the final product.

II. Beating Factors.

Liders (CERN-PS/GL 6) and Bell (AERE T/M 86) have shown that the exponential build up of the amplitude of betatron oscillations inside a stopband is foreshadowed by an up and down "beating" of the amplitude of oscillation, which can be considerable at a distance of less than one stopband width from the edge of the band. Liders and Bell give the formula for the Beating Factor, the ratio of maximum to minimum amplitude of oscillation, in the case where the working point is fairly close to one stopband edge, but do not treat the case where the working point is in its normal position, half way between two stopbands.

In the region where the influence of only the nearest stopband need be considered, the beating factor \( F \) is given by (Bell loc. cit.)

\[
F = \frac{1 + |D|}{1 - |D|}
\]

where

\[
\frac{1}{4}(|D| + |D|^{-1}) = \frac{\delta_n}{\delta_{n_{\text{stop}}}}
\]

\( 0 < |D| < 1 \)

and \( \delta_n \) is the distance of the working point from the center of the stopband. If this distance is several stopband widths

\[
\frac{\delta_n}{\delta_{n_{\text{stop}}}} = \frac{1}{4} |D|^{-1} \gg 1
\]
or \[ |D| \approx \frac{1}{4} \frac{\delta_{n \text{ stop}}}{\delta n} \ll 1 \]

\[ F \approx 1 + \frac{1}{2} \frac{\delta_{n \text{ stop}}}{\delta n} \]

At the normal working point, \( \delta n = \frac{1}{2} \delta n_{\text{ms}} \) where \( \delta n_{\text{ms}} \) is the spacing between the nearest main and subresonances, so this simplest treatment would give

\[ F = 1 + \frac{\delta_{n \text{ stop}}}{\delta n_{\text{ms}}} = 1 + \sigma_0 \]

To take into account the effect of the stopbands on either side of the working point, we have used some computing results for an ensemble of 30 machines with randomly chosen type errors in all their sectors, for which the stopband widths and the beating factors have been found numerically. These results showed that for that particular machine

\[ F = 1 + 1.7 \sigma_0 \]

and since if, as we have shown, the effect of one stopband only is considered, \( F \) is determined only by \( \sigma_0 \), it seems safe to consider

\[ F = 1 + 1.7 \sigma_0 \]

as a universal formula valid for all types of machine.

III. Probability Distributions.

The distributions of the peak closed orbit deviation are, according to Lüders, Rayleigh distributions, i.e.

\[ P(\Delta x | x) = \frac{2x}{a^2} e^{-x^2 / a^2} \, dx \]

for the differential distribution, and integrating,

\[ P(x < x_0) = 1 - e^{-x_0^2 / a^2} \]

for the cumulative distribution.

With two variates \( x, y \) with rms values \( a, b \) which are combined to give

\[ f = y + ax \sqrt{a^2 + y^2} \]

lines of constant \( f \) can be drawn in the \( x, y \) plane.
and the probability of finding $x, y$ in the strip $A$ is

$$P(x < x_0) \quad P(dy \mid y)$$

$$= (1 - e^{-x_0^2/a^2}) \frac{2y}{b} e^{-y^2/b^2} dy$$

with

$$f = y + \alpha \sqrt{x_0^2 + y^2}$$

or

$$\frac{x_0^2}{\alpha^2} = \frac{1}{\alpha^2} \left( \frac{f-y}{\alpha^2} - y \right)$$

Thus the probability of finding $x, y$ anywhere under the $f = \text{constant}$ line, i.e. the probability that $y + \alpha \sqrt{x+y}^2$ is less than $f$ is

$$\int \{ 1 - \exp \left[ -\frac{1}{\alpha^2} \left( \frac{f-y}{\alpha^2} - y \right) \right] \} \frac{2y}{b} e^{-y^2/b^2} dy$$

$$= \left\{ 1 - \exp \left[ -\frac{1}{\alpha^2} \right] \right\} - \frac{f}{\alpha^2} \int \frac{2y}{b} \exp \left[ -\frac{1}{\alpha^2} \left( \frac{f-y}{\alpha^2} - y \right) - \frac{y^2}{b^2} \right] dy$$

using $y/b$ as a variable, putting

$$\varphi = b^2 / \alpha^2$$

$$P = \varphi \left( 1 - \varphi^2 \right) + 1$$

and completing the square inside the $\exp\left[ \right]$ term, the integral can be evaluated as a sum of exponential functions and an error function integral.

$$\text{Prob} \left( y + \alpha \sqrt{x+y} < f \right) = 1 - \left( \frac{f}{b} \right) \exp \left[ -\frac{f^2}{b^2 \left( 1 + \alpha \right)} \right] - \frac{1}{\varphi} \exp \left[ -\frac{\alpha^2 f^2}{b \left( 1 + \alpha \right)} \right]$$

$$- \frac{2\varphi}{\sqrt{\varphi}} \frac{f}{b} \exp \left[ -\frac{\alpha^2 f^2}{b \left( 1 + \alpha \right)} \right] \int e^{-s^2} ds$$

where

$$S_1 = \sqrt{P} \frac{f}{b} \left( \frac{1}{1 + \alpha} - \frac{\alpha}{b} \right)$$

$$S_0 = -\sqrt{P} \frac{f}{b} \frac{\alpha}{b}$$
The particular cases which were considered were for $\alpha = 0.8, 0.9, 1.0$ which covered the range of values occurring in the twelve Machines, and for $\beta = 0.5, 1.0, 2.0$, i.e. where the expected closed orbit deviations in the two directions are either equal or where the deviation in one direction is twice that in the other. These correspond to the assumptions for Figures 18-21, that either the vertical radial alignment tolerances are equal, or that the radial errors are twice the vertical.

The values of $F_b$ which make the probability either 90% or 95% were found by tabulating the Probability function for varying $F_b$ with all the combinations of $\alpha$, $\beta$ specified above, and then interpolating inversely.

The results are:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$F_b = 0.90$</th>
<th>$F_b = 0.95$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4.39, 4.09, 3.79</td>
<td>4.86, 4.51, 4.17</td>
</tr>
<tr>
<td>1.0</td>
<td>3.38, 3.19, 3.00</td>
<td>3.78, 3.58, 3.37</td>
</tr>
<tr>
<td>2.0</td>
<td>3.12, 2.96, 2.80</td>
<td>3.54, 3.35, 3.10</td>
</tr>
</tbody>
</table>