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THE $\pi^+ - \pi^0$ MASS DIFFERENCE IN THE QCD EFFECTIVE ACTION APPROACH

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Abstract

We present a calculation of the electromagnetic $\pi^+ - \pi^0$ mass difference within the framework of the QCD effective action approach. The dependence on the constituent chiral quark mass is discussed.

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There have been some recent attempts to derive effective chiral Lagrangians from QCD at some approximate level (See Ref. [1] and references therein.) These effective Lagrangians are expected to describe the strong interactions of the ground state $SU(3)$ -octet of pseudoscalar particles (π , K and η) at low energies and in the presence of external gauge fields, electromagnetism in particular. The rather successful results of these attempts have prompted further work in this direction. The same method has been used to calculate the imaginary part ($\mathcal{O}(p^6)$ terms) of the effective action [2] and in Ref. [3] this approach has been extended to the evaluation of the effective action of four-quark operators in the sector of non-leptonic weak interactions.

The purpose of this letter is to discuss the electromagnetic $\pi^+-\pi^0$ mass difference within the QCD effective action approach. The feature which makes this example particularly interesting is the fact that the integral over the virtual photon brings contributions both from the short-distance and the long-distance regimes. Therefore it forces one to discuss how the matching of the two asymptotic regimes is made. In the case of the large N_c approach of Bardeen, Buras and Gérard [4], the discussion of the $\pi^+-\pi^0$ mass difference in [5] was particularly useful to illustrate the philosophy of their method. We wish to illustrate with the same example some features of the effective action approach.

The appropriate framework to discuss the QCD effective action approach is the large N_c limit [6]. Within this framework, the non-perturbative gluonic contributions to the various coupling constants of the effective chiral Lagrangian can be seen as successive powers of $\alpha_s N_c$ corrections which appear as vacuum expectation values of gluonic operators normalized to a scale associated with the spontaneous breaking of the chiral $SU(3)_L \times SU(3)_R$ symmetry. The mechanism of spontaneous chiral symmetry breaking is introduced, however, at a phenomenological level in Refs. [1, 2] and [3]. To the usual QCD Lagrangian a new term is added ($q_{R,L} = \frac{1}{2}(1 \pm \gamma_5)q$; $\bar{q} = (\bar{u} \ \bar{d} \ \bar{s})$ and summation over colour is understood):

$$-M_Q (\bar{q}_R(x)U(x)q_L(x) + \bar{q}_L(x)U^\dagger(x)q_R(x)), \quad (1)$$

where $U(x)$ denotes a unitary 3×3 matrix in flavour space with $\det U = 1$, which collects the octet of pseudoscalar fields which parametrize the deviation from the ground state $U >= 1$. Under chiral $SU(3)_L \times SU(3)_R$ transformations (V_L, V_R)

$$U \rightarrow V_R U V_L^\dagger, \quad (2)$$

and the new term in eq. (1) is invariant. The mass parameter M_Q is a free parameter here which acts as an infra-red regulator in the derivation of the low-energy effective action. As discussed in [1] M_Q can be interpreted as a "constituent chiral quark mass", much the same as in the chiral quark model of Georgi and Manohar [7]. The new term in (1) is expected to parametrize the bulk of the dynamics which is at the origin of spontaneous chiral symmetry breaking in QCD. How this term arises in QCD remains an open question. It can be shown, however, that the term in (1) is in fact equivalent to a Nambu-Jona-Lasinio [8] mechanism triggered by a four-quark operator of the type $(\bar{q}_R q_L)(\bar{q}_L q_R)$ (i, j are flavour indices, summation over colour within each bracket is understood), at the mean field approximation where the auxiliary scalar field is set to its vacuum expectation value equal to M_Q .

The QCD Lagrangian with three light flavours, in the presence of external $SU(3)_L \times SU(3)_R$ gauge fields l_μ and r_μ and with inclusion of the constituent chiral quark mass term in (1) is defined by

$$\mathcal{L}_{QCD} = \mathcal{L}_{QCD}^0 + \bar{q}_L \gamma^\mu l_\mu q_L + \bar{q}_R \gamma^\mu r_\mu q_R - \bar{q}_L \mathcal{M} q_R - M_Q (\bar{q}_R U q_L + \bar{q}_L U^\dagger q_R), \quad (3)$$

where $G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g s f_{abc} G_\mu^b G_\nu^c$

$$\mathcal{L}_{QCD}^0 = \sum_{a=1}^8 \left\{ -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu} + i \bar{q} \gamma^\mu (\partial_\mu + i \frac{g s}{2} \lambda^a G_\mu^a) q \right\}. \quad (4)$$

Here $\mathcal{M}$ denotes the current quark mass matrix $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ and l_μ, r_μ are Hermitian 3×3 traceless matrices in flavour space. The electromagnetic interactions correspond to the choice $l_\mu = r_\mu = e Q A_\mu(x)$, with $Q = \text{diag}(2/3, -1/3, -1/3)$ and A_μ the photon gauge field.

The effective action at low energies is defined by integrating out the quarks and gluons from the functional integral

$$\exp W_E(U, l, r, \mathcal{M}, M_Q) = \int [dG_\mu][d\bar{q}][dq] \exp \left(\int d^4x \mathcal{L}_{QCD}^0 \right). \quad (5)$$

The integral over quarks can formally be done resulting in

$$\exp W_E(U, l, r, \mathcal{M}, M_Q) = \int [dG_\mu] \exp \left(- \int d^4x \frac{1}{4} \sum_a G_{\mu\nu}^a G_{\mu\nu}^a \right) \det \mathcal{D}_E \quad (6)$$

with $\mathcal{D}_E$ the Euclidean Dirac operator

$$\mathcal{D}_E = \gamma_\mu \nabla_\mu + \mathcal{M} = \gamma_\mu (\partial_\mu + A_\mu) + \mathcal{M}, \quad (7)$$

where

$$A_\mu = i G_\mu - i \frac{1 + \gamma_5}{2} r_\mu - i \frac{1 - \gamma_5}{2} l_\mu, \quad (8)$$

and

$$\mathcal{M} = -M_Q \left(U \frac{1 - \gamma_5}{2} + \frac{1 + \gamma_5}{2} U^\dagger \right) - \mathcal{M}. \quad (9)$$

The quantity of interest for our purposes is the real part of the effective action in an external $A_\mu, \mathcal{M}$ field background, i.e.,

$$\Gamma_E(\mathcal{A}, \mathcal{M}) = \frac{1}{2} \log \det \mathcal{D}_E^\dagger \mathcal{D}_E. \quad (10)$$

This determinant requires a regularization to be defined in the ultra-violet regime. The ζ -function regularization method was used in Ref. [1]. Unfortunately this method does not keep track directly of the ultra-violet quadratic divergences and therefore becomes rather cumbersome to apply to the $\pi^+-\pi^0$ electromagnetic mass difference calculation. Here we shall adopt the proper time cut-off regularization which is better adapted to the underlying physics. The real part of the effective action at low energies is then defined as

$$\Gamma_E(\mathcal{A}, \mathcal{M}) = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \text{Tr} \exp(-\tau \mathcal{D}_E^\dagger \mathcal{D}_E), \quad (11)$$

where Tr stands for trace over Dirac- γ matrices, colour $SU(3)$ matrices and Euclidean space. The presence of the cut-off Λ_x regulates the small τ behaviour of the integrand, i.e., the ultra-violet behaviour. The cut-off $1/\Lambda_x$ should be viewed as a physical length. In a renormalization of QCD à la Wilson, the rest of the integration over quarks and gluons coming from the interval $0 \leq \tau < 1/\Lambda_x^2$ is expected to be at the origin of the phenomenological term in (1).

In the soft-pion limit with PCAC and using the equal-time commutation relations of hadronic currents, Das et al. [9] derived an integral representation for the electromagnetic $\pi^+\pi^0$ mass difference

$$\Delta m_\pi^2 = (m_\pi^2 - m_\pi^2)_{EM} = -i \frac{e^2}{F_\pi^2} \int \frac{d^4 q}{(2\pi)^4} \frac{g_{\mu\nu} - 4 \frac{q_\mu q_\nu}{q^2}}{q^2 - i\epsilon} q^\mu q^\nu [\Pi_{LR}^{(1)}(q^2) + \Pi_{LR}^{(0)}(q^2)] \quad (12)$$

with $\Pi_{LR}^{(J)}(q^2)$, $J = 0, 1$ the two-point functions defined by the relations $(\psi_{R,L} = \frac{1 \pm \gamma_5}{2} \psi$ with ψ a quark Dirac spinor of specific flavour; colour summation within each current is understood)

$$\begin{aligned} \Pi_{LR}^{\mu\nu}(q^2) &= i \int d^4 x e^{iq \cdot x} \langle 0 | T(\bar{\psi}_L(x) \gamma^\mu \psi_L(x) \bar{\psi}_R(0) \gamma^\nu \psi_R(0)) | 0 \rangle > \\ &= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{LR}^{(1)}(q^2) + q^\mu q^\nu \Pi_{LR}^{(0)}(q^2). \end{aligned} \quad (13)$$

The choice of gauge in eq. (12) is the one made in Ref. [9]. It eliminates the contribution from seagull-like terms proportional to $g^{\mu\nu}$. The question of gauge-independence of Δm_π^2 is one point we shall discuss below. The convergence of the integral representation in (12) is guaranteed by the first and second Weinberg sum rules [10]. In QCD, the first Weinberg sum rule holds even in the presence of explicit chiral symmetry breaking due to quark masses; the second Weinberg sum rule only holds in the chiral limit where $M \rightarrow 0$ [11]. If, furthermore, one assumes that the spectral representation of $\Pi_{LR}^{(1)}$ is dominated by the low-lying resonances (the ρ and a_1 particularly) with the mass relation $M_{a_1} = \sqrt{2} M_\rho$, one finds

$$\Delta m_\pi^2 = \frac{\alpha^3}{\pi^2} M_\rho^2 \log 2, \quad (14)$$

which is the result of Das et al. [9]. A more recent phenomenological evaluation of the Δm_π^2 sum rule has been made in Ref. [12].

We shall next briefly recall how the $\pi^+\pi^0$ electromagnetic mass difference emerges within the framework of effective chiral Lagrangians [13]. From the onset, we restrict the discussion to the limit where $M \rightarrow 0$ (the chiral limit). Then, to lowest order in the number of derivatives and external l_μ and r_μ fields, the effective Lagrangian invariant under local $SU(3)_L \times SU(3)_R$ is given by the term

$$\mathcal{L} = \frac{F_\pi^2}{4} \text{tr} D_\mu U^\dagger D^\mu U, \quad (15)$$

with U the same 3×3 matrix as in (1) and the covariant derivative

$$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu. \quad (16)$$

A useful parametrization for U is

$$U = \exp \left(-i \frac{\sqrt{2}}{F_\pi} \Phi \right), \text{ where } \Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}} \eta \end{pmatrix}. \quad (17)$$

The constant F_π is not fixed by symmetry requirements. In our normalization $F_\pi = 93.3 \text{ MeV}$.

In order to take into account virtual electromagnetic effects we add to the term in (15) the relevant photon kinetic term $-1/4 F_{\mu\nu} F^{\mu\nu}$ ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$). The presence of external fields, which themselves are propagating, allows then for two more invariants in the effective Lagrangian without derivatives. In terms of the $SU(3)_L$ and $SU(3)_R$ generators Q_L and Q_R they are:

$$C_1 \text{tr} Q_R U Q_L U^\dagger + C_2 \text{tr} (Q_L^2 + Q_R^2). \quad (18)$$

In the case of electromagnetism $Q_L = Q_R = eQ$. The term proportional to C_2 does not have any effect for physical processes but is relevant when we extract C_1 from vacuum to vacuum amplitudes. In the parametrization (17) the term proportional to C_1 has the form

$$e^2 C_1 \text{tr} Q U Q U^\dagger = -\frac{2e^2 C_1}{F_\pi^2} (\pi^+ \pi^- + K^+ K^-) + \mathcal{O}(\Phi^4), \quad (19)$$

which corresponds to a $\pi^+\pi^0$ mass splitting

$$\Delta m_\pi^2 = \frac{2e^2 C_1}{F_\pi^2}. \quad (20)$$

The constant C_1 , like F_π^2 , is not fixed by symmetry requirements alone.

At the same order in the low energy expansion there is also the contribution from the one-photon loop in Fig. 1a and 1b to the pion mass. The black dots in these figures indicate the relevant vertices from the Lagrangian in eq. (15). In terms of the two-point functions $\Pi_{LR}^{(J)}$ which appear in eq. (12) this corresponds to the low energy representation (in the chiral limit and with $Q^2 = -q^2 \geq 0$)

$$(\Pi_{LR}^{(1)} + \Pi_{LR}^{(0)})_{\text{CHPT}} = -\frac{1}{2} (\Pi_{AA}^{(1)} + \Pi_{AA}^{(0)})_{\text{CHPT}} = -\frac{F_\pi^2}{Q^2} + \mathcal{O}(1). \quad (21)$$

The contribution from this term to Δm_π^2 vanishes in a dimensional regularization of the loop integral [5, 13]. It is, however, proportional to Λ_x^2 in the proper time regularization, where the photon propagator becomes

$$\frac{1}{Q^2} \rightarrow \int_{1/\Lambda_x^2}^\infty d\tau e^{-\tau Q^2}, \quad (22)$$

with the result

$$\Delta m_\pi^2(0^{++}) = \frac{3\alpha}{4\pi} \int_0^\infty dQ^2 e^{-Q^2/\Lambda_x^2} = \frac{3\alpha}{4\pi} \Lambda_x^2. \quad (23)$$

If, furthermore, one includes virtual effects due to vector and axial-vector particles as discussed, e.g., in refs. [5] and [13], the overall result is then

$$\Delta m_\pi^2(0^{++}, 1^{--}, 1^{++}) = \frac{3\alpha}{4\pi} \int_0^\infty dQ^2 e^{-Q^2/\Lambda^2} \frac{M_\pi^2 M_V^2}{(Q^2 + M_\pi^2)(Q^2 + M_V^2)}. \quad (24)$$

For $\Lambda_\chi \rightarrow \infty$ and $M_A = \sqrt{2}M_V$ this coincides with the result of Das et al [9]. For $M_A, M_V \rightarrow \infty$ and Λ_χ fixed we recover the pion result above, eq. (23).

At this stage it is instructive to perform an explicit calculation of Δm_π^2 using the Feynman diagrams which follow from the Lagrangian in (3). We set $\mathcal{M} = 0$, $t_\mu = \tau_\mu = eQ A_\mu$ and ignore gluonic corrections $\mathcal{O}(\alpha_s N_c)$ in the large N_c expansion. The first problem we have to deal with is that the photon propagator appears in loops. Therefore, an expansion in powers of q/M_Q with q the photon momentum is not adequate. The second problem is that we cannot consider the pseudoscalars as purely external fields any longer. In order to have a gauge invariant result, the pseudoscalars have to be treated as propagating fields as well. We do still assume, of course, that they only propagate because of the kinetic term produced by the quark loop. Again, because internal momenta can be large, we need the quark propagator to all orders in p^2/M_Q^2 as well. The final result will be most easily obtained by keeping the Feynman momentum integrals explicitly in all intermediate expressions. We calculate first the pseudoscalar propagator in the absence of electromagnetism. We shall afterwards take into account the corrections due to photon exchange. This will give us the electromagnetic mass difference.

The inverse pion propagator is given by the diagrams in fig. 2. With external momentum q running through, this gives

$$\begin{aligned} i\Pi^{-1}(q^2) &= q^2 \frac{N_c}{F_\pi^2} 4M_Q^2 \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - M_Q^2)((p+q)^2 - M_Q^2)} \\ &= q^2 \frac{N_c}{F_\pi^2} 4M_Q^2 iF(q^2) \end{aligned} \quad (25)$$

where the last equality sign implies the definition of $F(q^2)$. The properly normalized inverse pion propagator at $q^2 \approx 0$ is iq^2 . Requiring this leads to the same expression for F_π^2 as in the kinetic term of the effective Lagrangian. In the large N_c counting rules, the pion propagator is $\mathcal{O}(1)$ and F_π of order $\sqrt{N_c}$. That means that an n pion vertex through one quark loop is suppressed by $1/N_c^{n/2-1}$. There are then two types of diagrams that can contribute to the electromagnetic correction to the inverse pion propagator. They are those in fig. 3, where all possible one quark loops are designated by a hatched circle. The thick line is the full pion propagator of eq. (25). They reduce to those in fig. 1 if both the quark loop and the pion propagator are only kept to leading order in $1/M_Q$. To obtain the mass difference it is sufficient to calculate the corrections at $q^2 = 0$. Fig. 3a corresponds to three diagrams and for a photon propagator in an arbitrary gauge, $-i(g_{\mu\nu} - (1-a)q_\mu q_\nu/q^2)/q^2$, this contribution is given by

$$-\frac{e^2 N_c}{F_\pi^2} 4M_Q^2 \int \frac{d^4 q}{(2\pi)^4} \frac{F(q^2)}{q^2} (3+a). \quad (26)$$

Fig. 3b corresponds to 9 diagrams and the expression simplifies considerably for $q^2 = 0$.

The vertex $\pi^+ \pi^- \pi^- \gamma$ in the limit of one of the pion momenta vanishing is

$$\frac{ieN_c}{F_\pi^2} 4M_Q^2 q_\mu F(q^2). \quad (27)$$

Using this, the set of diagrams corresponding to fig. 3b contribute

$$\frac{e^2 N_c}{F_\pi^2} 4M_Q^2 \int \frac{d^4 q}{(2\pi)^4} \frac{F(q^2)}{q^2} a, \quad (28)$$

and the electromagnetic mass difference is given by the integral

$$\Delta m_\pi^2 = \frac{12e^2 N_c M_Q^2}{F_\pi^2} \int \frac{d^4 q}{(2\pi)^4} \frac{iF(q^2)}{q^2}. \quad (29)$$

The mass difference is positive since the two-loop integral is positive. Notice that eq. (29) is correct to all orders in q^2/M_Q^2 since we have kept the full momentum dependence of the quark loops and the pion propagator. The electromagnetic gauge dependence cancelled between the two classes of diagrams in fig. 3. The full dependence on q^2 of the pion propagator was obviously needed to ensure gauge invariance to all orders. We defer further discussion of this calculation till later.

We are now in a position to discuss the effective action approach to calculate Δm_π^2 . We are interested in the effective action generated by integration over a virtual photon, i.e.,

$$\int d^4 x \int d^4 y \int \frac{d^4 q}{(2\pi)^4} e^{-i\phi(z-y)} \frac{-i(g_{\mu\nu} - (1-a)\frac{q_\mu q_\nu}{q^2})}{q^2 - i\epsilon} \frac{\delta^2}{\delta A_\mu(x) \delta A_\nu(y)} W_E(U, l, r, \mathcal{M} = 0, M_Q), \quad (30)$$

with W_E defined by the functional integrals in eqs. (5) to (11). The procedure to evaluate this expression is analogous to the one discussed in detail in ref. [3] for the effective action of four-quark operators. One has to calculate first the variation of the Euclidean effective action in eq. (11) with respect to the external photon fields $A_\mu(x)$ and $A_\nu(y)$, i.e.,

$$\frac{\delta^2}{\delta A_\mu(x) \delta A_\nu(y)} \exp \Gamma_E(A, M) = \int [dq] [d\bar{q}] (-ie) \bar{q}(x) \gamma^\mu Q q(x) (-ie) \bar{q}(y) \gamma^\nu q(y) \exp \left(\int d^4 z \bar{q} \mathcal{D}_E q \right), \quad (31)$$

and then perform the gluon average and the extra integral over the photon propagator as indicated in eqs. (6) and (30). The variation with respect to A_μ and A_ν has two terms :

$$\frac{\delta^2}{\delta A_\mu(x) \delta A_\nu(y)} \exp \Gamma_E = \left\{ \frac{\delta^2 \Gamma_E}{\delta A_\mu(x) \delta A_\nu(y)} + \frac{\delta \Gamma_E}{\delta A_\mu(x) \delta A_\nu(y)} \right\} \exp \Gamma_E. \quad (32)$$

The calculation of each term can now be made using the fermionic determinant representation in (10)

$$\begin{aligned} \frac{\delta \Gamma_E}{\delta A_\mu(x)} &= \frac{i}{2} \text{Tr} \left[\frac{\delta \mathcal{D}_E}{\delta A_\mu} \mathcal{D}_E^{-1} + \frac{\delta \mathcal{D}_E^\dagger}{\delta A_\mu} (\mathcal{D}_E^\dagger)^{-1} \right] \\ \frac{\delta^2 \Gamma_E}{\delta A_\mu(x) \delta A_\nu(y)} &= -\text{Tr} < x | \mathcal{D}_E^{-1} | y > \frac{\delta \mathcal{D}_E}{\delta A_\nu(y)} < y | \mathcal{D}_E^{-1} | x > \frac{\delta \mathcal{D}_E}{\delta A_\mu(x)}, \end{aligned} \quad (33)$$

and the proper-time representation of the Dirac operator

$$\langle x | \mathcal{D}_E^{-1} | y \rangle = \langle x | \left(\mathcal{D}_E^\dagger \mathcal{D}_E \right)^{-1} \mathcal{D}_E^\dagger | y \rangle = \int_0^\infty d\tau \langle x | e^{-\tau \mathcal{D}_E^\dagger \mathcal{D}_E} | y \rangle \mathcal{D}_E^\dagger | y \rangle. \quad (34)$$

For $z = y$, this representation needs a regularization. As for the determinant, we can use a proper time cut-off regularization. Explicit calculations can then be made using the heat kernel expansion

$$\langle x | e^{-\tau \mathcal{D}_E^\dagger \mathcal{D}_E} | y \rangle = \frac{1}{16\pi^2} e^{-(x-y)/(4\tau)} \sum_{n=0}^\infty H_n(x, y) \tau^n e^{-\tau M_Q^2}, \quad (35)$$

with $H_n(x, y)$ coefficient functions associated to the operator $\mathcal{D}_E^\dagger \mathcal{D}_E - M_Q^2 \mathbf{1}$.

The two terms in eq. (32) correspond to the two configurations shown in fig. 3. The gauge dependence in the photon propagator disappears because of the cancellation between the non-transverse contributions from the first term with the second term which only produces non-transverse contributions. To lowest order in the chiral expansion, we find that the formal effective action defined in (30) has the structure

$$\int d^4x \left\{ e^2 C_1 \text{tr} U(x) Q U^\dagger(x) + e^2 C_2 \text{tr} Q^2 \right\}, \quad (36)$$

with the constant C_1 we are interested in given by the integral representation

$$2e^2 C_1 = -ie^2 \int \frac{d^4q}{(2\pi)^4} \frac{g_{\mu\nu} - \frac{2q_\mu q_\nu}{q^2}}{q^2 - i\epsilon} (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{LR}^{(1)}(q^2). \quad (37)$$

This equation is equivalent to the current algebra sum rule in eq. (12), but in the Landau gauge $a = 0$. (Recall that Δm_π^2 and C_1 are related by eq. (20).) For example, the one-photon loop contribution in figs. 1a and 1b corresponds here to the low-energy approximation

$$\Pi_{VV}^{(1)}|_{\text{ChPT}} = -\frac{1}{2} \Pi_{AA}^{(1)}|_{\text{ChPT}} = -\frac{F_\pi^2}{Q^2} + \mathcal{O}(1) \quad (38)$$

which, of course, leads to the same $\Delta m_\pi^2(0^+)$ result as in eq. (23). The new feature is that, within the approximated version of the underlying theory, we can now calculate F_π^2 as well as the two-point functions $\Pi_{LR}^{(j)}(q^2)$. To leading order in the large N_c limit and with neglect of $\mathcal{O}(\alpha_s N_c)$ corrections [1]

$$F_\pi^2 = \frac{N_c}{16\pi^2} 4M_Q^2 \Gamma(0, \frac{M_Q^2}{\Lambda_\chi^2}), \quad (39)$$

with $\Gamma(0, \epsilon)$ the incomplete Γ -function $\Gamma(0, \epsilon) = \int_\epsilon^\infty \frac{dz}{z} e^{-z}$. For $M_Q^2/\Lambda_\chi^2 \ll 1$, (γ_E is Euler's constant)

$$\Gamma(0, \frac{M_Q^2}{\Lambda_\chi^2}) = \log \frac{\Lambda_\chi^2}{M_Q^2} - \gamma_E + \mathcal{O}\left(\frac{M_Q^2}{\Lambda_\chi^2}\right). \quad (40)$$

The functions $\Pi_{LR}^{(j)}(q^2)$ are of course related to the function $F(q^2)$ introduced in the Feynman diagram calculation above in eq. (25). In the proper-time cut-off regularization, the integral

which defined $F(q^2)$ in eq. (25) becomes ($Q^2 = -q^2$):

$$\begin{aligned} F(q^2) &= -i \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - M_Q^2)((p+Q)^2 - M_Q^2)} \\ &\Rightarrow \frac{1}{16\pi^2} \int_0^1 dx \Gamma\left(0, \frac{Q^2 x(1-x) + M_Q^2}{\Lambda_\chi^2}\right), \end{aligned} \quad (41)$$

and the full combination of two-point functions $\Pi_{LR}^{(1)}(q^2) + \Pi_{LR}^{(0)}(q^2)$ which appears in the current-algebra integral representation in eq. (12), including the ChPT contribution in eq. (21), is now

$$\Pi_{LR}^{(1)}(q^2) + \Pi_{LR}^{(0)}(q^2) = -\frac{N_c}{16\pi^2} \frac{4M_Q^2}{Q^2} \int_0^1 dx \Gamma\left(0, \frac{Q^2 x(1-x) + M_Q^2}{\Lambda_\chi^2}\right). \quad (42)$$

When compared with the ChPT contribution alone in eq. (23), we see that the net effect of the underlying constituent quark structure has been the replacement

$$F_\pi^2 \rightarrow \frac{N_c}{16\pi^2} 4M_Q^2 \int_0^1 dx \Gamma\left(0, \frac{Q^2 x(1-x) + M_Q^2}{\Lambda_\chi^2}\right) \quad (43)$$

in the photon loop calculation of the lowest order chiral perturbation theory calculation.

The function $F(q^2)$ has a well-defined absorptive part

$$\frac{1}{\pi} \text{Im} F(t) = \frac{1}{16\pi^2} \sqrt{1 - \frac{4M_Q^2}{t}} \theta(t - 4M_Q^2), \quad (44)$$

from which follows the integral representation (which is convergent for $Q^2 \rightarrow \infty$)

$$\Pi_{LR}^{(1)}|_{\text{d.s.p}} = \frac{N_c}{16\pi^2} \frac{4M_Q^2}{Q^2} \int_0^\infty \frac{dt}{t+Q^2} \sqrt{1 - \frac{4M_Q^2}{t}}. \quad (45)$$

By an appropriate change of variables we can rewrite this in the form

$$\begin{aligned} \Pi_{LR}^{(1)}|_{\text{d.s.p}} &= \frac{N_c}{16\pi^2} \frac{4M_Q^2}{Q^2} \int_0^1 dx \log \left[1 + \frac{Q^2}{M_Q^2} x(1-x) \right] \\ &= \frac{N_c}{16\pi^2} \frac{4M_Q^2}{Q^2} \int_0^1 dx \lim_{\Lambda_\chi \rightarrow \infty} \int_{1/\Lambda_\chi^2}^\infty \frac{d\tau}{\tau} e^{-\tau M_Q^2} (1 - e^{-Q^2 x(1-x)}), \end{aligned} \quad (46)$$

where in the last step we have established contact with the proper time regularization method. We recall that the limit $Q^2 \rightarrow 0$ of this function governs the value of the coupling constant L_{10} in the $\mathcal{O}(p^4)$ chiral effective Lagrangian, i.e., the term $L_{10} \text{tr} F_{\mu\nu}^\dagger U^\dagger F_{\mu\nu} U$ [14], with the result [1]

$$L_{10} = -\frac{1}{4} \Pi_{LR}^{(1)}(0) = -\frac{1}{6} \frac{N_c}{16\pi^2} e^{-m_\pi^2/\Lambda_\chi^2}. \quad (47)$$

The sum of $\Pi_{LR}^{(1)}|_{\text{d.s.p}}$ in eq. (46) and $\Pi_{LR}^{(1)}|_{\text{ChPT}}$ in eq. (21) gives the function to be inserted in the r.h.s. of eq. (37). With F_π^2 given by eq. (39), we find the result

$$2e^2 C_1 = \frac{3\alpha}{4\pi} \int_0^\infty dQ^2 e^{-Q^2/\Lambda_\chi^2} \frac{N_c}{16\pi^2} 4M_Q^2 \int_0^1 dx \Gamma\left(0, \frac{Q^2 x(1-x) + M_Q^2}{\Lambda_\chi^2}\right), \quad (48)$$

which corresponds to a $\pi^+\pi^0$ mass splitting

$$\Delta m_\pi^2|_{M_Q} = \frac{3\alpha}{4\pi} \int_0^\infty dQ^2 e^{-Q^2/\Lambda_\chi^2} \frac{1}{\Gamma(0, M_Q^2/\Lambda_\chi^2)} \int_0^1 dx \Gamma\left(0, \frac{Q^2 x(1-x) + M_Q^2}{\Lambda_\chi^2}\right) \quad (49)$$

In order to compare this result with the one in eq. (23), and also to calculate the integral numerically, it is convenient to use the representation of $\Gamma(0, \epsilon)$ above and to do first the Laplace transform over Q^2 , with the result

$$\Delta m_\pi^2|_{M_Q} = \frac{3\alpha}{4\pi} \Lambda_\chi^2 \frac{1}{x \Gamma(0, M_Q^2/\Lambda_\chi^2)} \int_0^1 dx \int_1^\infty \frac{dz}{z} e^{-zM_Q^2/\Lambda_\chi^2} \frac{1}{1+zx(1-x)} \quad (50)$$

In fig. 4 we show $\Delta m_\pi|_{M_Q} = \Delta m_\pi^2|_{M_Q}/(2m_\pi)$ as a function of Λ_χ for $M_Q = 250, 330$ and 400 MeV respectively. As a comparison we have also shown $\Delta m_\pi(0^+)$ following from eq. (23). As we can see, the result is somewhat less divergent than the pure pion result. In fig. 5 we show the resulting $\pi^+\pi^0$ mass difference including the short-distance contribution calculated in ref. [5]

$$\Delta m_\pi^2|_{QCD} = \frac{3\alpha}{F_\pi^2} \frac{\alpha_S(\Lambda_\chi^2)}{\Lambda_\chi^2} (<\bar{q}q>(\Lambda_\chi^2))^2 \quad (51)$$

for values of $<\bar{q}q>(\Lambda_{QCD}) = -(194 \text{ MeV})^3$ [15] and $-(226 \text{ MeV})^3$ and $\Lambda_{QCD} = 200 \text{ MeV}$. Using $<\bar{q}q>(\mu) \approx -F_\pi^2 m_\pi^2/m_\pi(\mu)$ these values correspond to $m_\pi(1 \text{ GeV}) \approx 234 \text{ MeV}$ and 150 MeV respectively. The values of the pion electromagnetic mass difference at the point of minimum sensitivity to Λ_χ are $\Delta m_\pi = 2.6 \text{ MeV}$ and 4.0 MeV . This should be compared to the experimental value $4.43 \pm 0.03 \text{ MeV}$. We note that the numerical value is not very sensitive to the value of M_Q .

From the comparison between eqs. (23) and (50) it emerges that the naive pion result now appears modulated by a factor which depends only on the ratio M_Q^2/Λ_χ^2 . The leading behaviour of Δm_π^2 on Λ_χ^2 is now smoother than it was for the naive pion pole result. However, $\Delta m_\pi^2|_{M_Q}$ still diverges if we let $\Lambda_\chi \rightarrow \infty$. This is a signal that, at least for the problem we are considering, the description via the constituent chiral quark mass in (1) alone does not contain the full physics. The fact that one is missing important contributions clearly appears when one tries to implement the Weinberg sum rules with the spectral representation in eq. (46). In QCD we know the short-distance behaviour of the two-point function $\Pi_{LR}^{(1)}(Q^2)$. Using the operator product expansion, the large Q behaviour of $\Pi_{LR}^{(1)}(Q^2)$ in the chiral limit is found to be [16]

$$\lim_{Q^2 \rightarrow \infty} \Pi_{LR}^{(1)}(Q^2) = -\frac{1}{Q^6} \frac{9\pi^2}{4} \frac{\alpha_S}{\pi} <\bar{\psi}\psi>^2 + \mathcal{O}\left(\frac{1}{Q^8}\right) \quad (52)$$

where we have made use of the large N_c limit relations

$$<(\bar{\psi}\gamma_\mu\lambda^a\psi)^2> = - <(\bar{\psi}\gamma_\mu\gamma_5\lambda^a\psi)^2> = -\frac{16}{9} <\bar{\psi}\psi>^2 \quad (53)$$

Inserting this result (52) in eq. (37) with a lower limit on Q^2 reproduces the short distance contribution to Δm_π^2 calculated in Ref. [5], eq. (51).

The two-point function in eq. (46) definitely does not have the QCD short-distance behaviour of eq. (52). In order to satisfy it new physical effects have to be included. These are mainly the vector and axial-vector degrees of freedom. In the QCD effective action approach, they can be taken into account consistently by adding new four-quark operators in a full-blown gauged Nambu–Jona-Lasinio model. Work in this direction is in progress. We have so far shown that the constituent chiral quark term in eq. (1), in addition to its other successes [1, 2, 3], produces the correct $\pi^+\pi^0$ mass difference for reasonable values of the parameter Λ_χ which separates the two integration regions below and above the scale of spontaneous chiral symmetry breaking.

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List of Figures

1	Feynman diagrams for the electromagnetic pion mass difference in the case of Chiral Perturbation theory.	12
2	Feynman diagrams for the inverse pion propagator.	12
3	Diagrams for the electromagnetic mass difference in the quark case.	12
4	The long-distance part of the pion electromagnetic mass difference as a function of Λ_χ for various values of M_Q	13
5	The full pion electromagnetic mass difference.	14

Figure 1: Feynman diagrams for the electromagnetic pion mass difference in the case of Chiral Perturbation theory. The dots are vertices from the Lagrangian (15). The dashed line are pions and the wavy line is a photon.

Figure 2: Feynman diagrams for the inverse pion propagator. The vertices are those from the Lagrangian (3). The full lines with arrows are quark lines. Dashed lines are pions.

Figure 3: Diagrams for the electromagnetic mass difference in the theory of the Lagrangian (3). The hatched circles are all possible one-loop quark diagram insertions. The full line is the full pion propagator of eq. (25). The dashed lines are external pions.

Figure 4: The long-distance part of the pion electromagnetic mass difference as a function of Λ_χ for various values of M_Q . Lines 1,2,3 correspond to $M_Q = 400, 330, 250$ MeV. The dashed line is the lowest order Chiral Perturbation Theory result (23).

Figure 5: The full pion electromagnetic mass difference as a function of Λ_χ for $\Lambda_{QCD} = 200$ MeV and $M_Q = 330$ MeV for two values of the quark condensate. The full line is for the value of Ref. [15]. The dashed line corresponds to $\langle \bar{q}q \rangle = -(226 \text{ MeV})^3$.