Jet Topology and New Jet Counting Algorithms *

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Abstract

A QCD theoretical analysis of jet cross sections is presented. We discuss in detail the JADE (invariant mass-type) and $k_{\perp}$ -algorithms for multijet cross sections in e^+e^- -annihilation, with emphasis on hadronization effects and large perturbative corrections at small values of the jet resolution parameter y_{cut} . For the $k_{\perp}$ -algorithm we present results of calculations which include resummation of leading and next-to-leading logarithms of y_{cut} to all orders in QCD perturbation theory. We also discuss the recently-proposed Geneva algorithm and show that it suffers from some of the same difficulties as the JADE algorithm.

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1. Introduction

A jet can be defined as a large amount of hadronic energy in a small angular region. According to this *qualitative* definition, the first *Evidence for Jet Structure in Hadron Production by e^+e^- Annihilation* was reported in 1975 [1]. Since then, hadronic jet production in lepton [2,3] and hadron [4,5] colliders has become a main tool in investigating strong interaction physics and testing Quantum Chromodynamics (QCD).

We are nowadays in a position to use jet cross section data both for

- precise quantitative tests of QCD (measurement of the QCD coupling α_s and scale $\Lambda_{\overline{MS}}$ [3], study of QCD coherence [6]);
- looking for breakdown of the standard model and new physics [5].

In order to do that, the qualitative definition of a jet given above is no longer sufficient and must be replaced by a precise *quantitative* definition. We need a jet algorithm able to specify unambiguously a jet configuration starting from hadrons detected in the final state. A *jet algorithm* is defined giving

- j_1) a test variable (energy-angle resolution) y_M ;
- j_2) a recombination procedure.

The test variable y_M is needed in order to specify whether or not two hadrons h_k, h_l belong to the same jet, whilst the recombination procedure tells us how jet properties are related to the ones of hadrons belonging to it.

The jet defining conditions j_1) and j_2) have to fulfill the requirements of being

- i) infrared (IR) and collinear safe
- ii) simple to implement in the experimental analysis
- iii) simple to implement in the theoretical calculation
- iv) subject to small hadronization corrections.

Requirements ii) and iii) are self-evident.

The other requirements follow from the motivation of comparing data with theory, namely QCD. So far, we are able to perform QCD calculations, using perturbation theory, only for the region of small distances in which high energy collisions produce partons (quarks, gluons). At larger distances the produced quarks and gluons are confined by the colour force field and are forced to dress themselves up into colourless hadrons. Although it is not possible to describe the hadronization process by perturbation theory, the "preconfinement" property [7] (or "local parton hadron duality" [8]) of QCD implies that hadron jets should maintain the kinematic features of the underlying quarks and gluons. That is because, according to preconfinement, the hadronic flow in the final state follows the partonic flow quite closely, with transfers of momentum and other quantum numbers that are local in phase space.

The way to enhance preconfinement effects (or, equivalently, to reduce hadronization corrections) is to define jet cross sections which at parton level are not very sensitive to large distance physics, namely any small value of parton masses. Therefore jet algorithms must satisfy the property i) above, i.e. jet cross sections at parton level must be

finite order by order in perturbation theory in the limiting case of final state massless partons [9,10]. Obviously, hadronization corrections still affect jet algorithm satisfying the property i). Therefore one should try to minimize hadronization effects for instance using Monte Carlo event generators to compare parton level and hadron level results for different jet algorithms.

Still within the class of jet algorithms fulfilling the requirements i) - iv), one has ample freedom of choosing *test variables* and *recombination procedures*. The aim of this paper is not to provide a comprehensive review of existing jet algorithms [11] but to present a QCD inspired critical analysis of them. Our goal is to advocate a QCD-based jet algorithm, i.e. a definition of a jet whose dynamics follows QCD theory as closely as possible.

We start our analysis in Sec. 2, discussing the jet algorithm mostly used at exclusive level, namely the "JADE algorithm" for e^+e^- annihilation [12]. The description of good and bad features of the JADE algorithm performed in Secs. 2 and 3 will provide us with the physical motivations to introduce the QCD inspired jet algorithm which we call k_T -algorithm [13]. The k_T -algorithm is defined in Sec. 4 where we present our main theoretical results. In Sec. 5 we draw some conclusion. Some results on another jet algorithm, the recently proposed "Geneva algorithm" [14,3], are presented in Appendix A.

2. The JADE algorithm

2.1 Definition

The JADE algorithm is a full exclusive algorithm to define jet cross sections in e^+e^- annihilation. Let us consider a n-hadron final state

$$e^+(p) + e^-(\bar{p}) \rightarrow h_1(p_1) + \dots + h_n(p_n), \quad p + \bar{p} \equiv Q. \quad (1)$$

The JADE algorithm is defined according to the following iterative procedure.

- 1) Define a resolution parameter y_{cut} .

- 2) For every pair of hadrons h_k, h_l compute the corresponding "invariant mass" M_{kl}^2 and define

$$y_{kl} = M_{kl}^2/Q^2. \quad (2)$$

- 3) If y_{ij} is the smallest value of y_{kl} computed in 2) and $y_{kl} < y_{cut}$, combine (p_i, p_j) in a single jet ('pseudoparticle') p_{ij} according to a recombination prescription.

- 4) Repeat this procedure from step 2) until all pairs of objects (particles and/or pseudoparticles) have $y_{kl} > y_{cut}$. Whatever objects remain at this stage are called jets.

According to the classification given by j_1), j_2) in Sec. 1, the JADE algorithm uses the scaled invariant mass (2) as test variable. In the experimental as well as theoretical analyses several different recombination procedures may be used [12,15]. They differ among themselves in the way in which the "invariant mass" and the 4-momentum p_{ij} of a pseudoparticle are defined in terms of the momenta (p_i, p_j) of the recombined particles and/or pseudoparticles. There are essentially two classes of recombination schemes (Table 1) whose main difference is in the definition of the "invariant mass". In the first class (E-scheme) the invariant mass M_{ij}^2 is the true invariant mass, whilst in the massless recombination schemes (JADE, E0, P, P0) M_{ij}^2 is computed defining the pseudoparticle momenta in such a way to have massless pseudoparticles. The various

schemes in this latter class correspond to various possibilities of rescaling energy and 3-momentum in order to define a massless pseudoparticle.

The JADE algorithm obviously fulfils the requirements (i) and (ii) in Sec. 1. In the following we discuss in detail the points (iii) and (iv).

2.2 Hadronization corrections

Hadronization effects for the JADE algorithm can be investigated using Monte Carlo event generators [16-18]. These Monte Carlo generators include a parton shower, describing parton production according to perturbative QCD, plus hadronization models. Comparing jet cross sections obtained by Monte Carlo simulations both from particles at the end of the QCD shower and from particles after hadronization, one is able to estimate the size of hadronization corrections.

Fig. 1 shows the results of the analysis carried out by the OPAL collaboration [19] at LEP using the JETSET QCD shower model. Similar results have been obtained from other Collaborations at LEP and lower energies as well as using different Monte Carlo generators.

It can be seen from Fig. 1 that hadronization corrections are large for the E-recombination scheme, moderate for the P- and P0-scheme and small for the E0-scheme (the results for the JADE-scheme are similar to the ones for the E0-scheme). The fact that hadronization effects are relevant for most of the recombination schemes may signal that something is going wrong with the JADE algorithm and call for a better theoretical understanding of jet definition.

2.3 Theoretical calculations and renormalization scale problem

Let R_n represent the n -jet fraction

$$R_n = \frac{\sigma_{n\text{-jet}}}{\sigma_{\text{TOT}}} \quad (3)$$

Since the JADE algorithm is IR and collinear safe, R_n can be computed in QCD perturbation theory. So far the n -jet fractions ($n = 2, 3, 4$) have been evaluated up to the second order in α_S and the result can be written as follows

$$R_n(\alpha_S(\mu), Q^2/\mu^2; y_{\text{cut}}) = \delta_{n2} + \alpha_S(\mu) C_n^{(1)}(y_{\text{cut}}) + \alpha_S^2(\mu) [C_n^{(2)}(y_{\text{cut}}) - C_n^{(1)}(y_{\text{cut}}) \beta_0 \ln Q^2/\mu^2] + O(\alpha_S^3(\mu), \alpha_S^2(\mu) \ln^2 Q^2/\mu^2), \quad (4)$$

where $12\pi\beta_0 = 11C_A - 2N_f$, N_f is the number of flavours, $C_A = N_c = 3$ is the number of colours and the coupling $\alpha_S(\mu)$ can be expressed as a function of the QCD scale $\Lambda_{\overline{\text{MS}}}$ according to the two loop expression

$$\alpha_S(\mu) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_{\overline{\text{MS}}}^2)} \left[1 - \frac{6(153 - 19N_f) \ln[\ln(\mu^2/\Lambda_{\overline{\text{MS}}}^2)]}{(33 - 2N_f)^2 \ln(\mu^2/\Lambda_{\overline{\text{MS}}}^2)} \right] \quad (5)$$

In one loop order $C_4^{(1)} = 0$, and $C_2^{(1)}, C_3^{(1)}$ are given by [†]

$$C_2^{(1)}(y_{\text{cut}}) = -C_3^{(1)}(y_{\text{cut}}) = \frac{C_F}{2\pi} \left[-2 \ln^2 y_{\text{cut}} - 3 \ln y_{\text{cut}} + \frac{\pi^2}{3} - \frac{5}{2} + 6y_{\text{cut}}(1 + \ln y_{\text{cut}}) + 4 \ln y_{\text{cut}} \ln(1 - y_{\text{cut}}) - 2 \ln^2(1 - y_{\text{cut}}) + 3(1 - 2y_{\text{cut}}) \ln(1 - 2y_{\text{cut}}) + \frac{9}{2} y_{\text{cut}}^2 - 4 \text{Li}_2\left(\frac{y_{\text{cut}}}{1 - y_{\text{cut}}}\right) \right], \quad (6)$$

[†]Our result (6) differs from that in Ref. [20] in the coefficient of the dilogarithm Li_2 .

where

$$C_F = \frac{N_c^2 - 1}{2N_c}, \quad \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}. \quad (7)$$

The two loop functions $C_n^{(2)}(y_{\text{cut}})$ (note that $C_2^{(2)}$ and $C_3^{(2)}$ are recombination scheme dependent) have been computed numerically [15,20] using the four-parton matrix elements in Ref. [21].

The two loop expression in eq. (4) depends on the renormalization scale μ^2 at which the running coupling $\alpha_S(\mu)$ is evaluated. This dependence is an artifact of the fixed order perturbative expansion in α_S because $R_n(\alpha_S(\mu), Q^2/\mu^2; y_{\text{cut}})$ is a renormalization group invariant quantity if computed to all orders in α_S . We have taken into account this property writing explicitly a correction term of order $\alpha_S^3(\mu) \ln^2 Q^2/\mu^2$ on the r.h.s. of eq. (4).

In comparing eq. (4) with data, one has to fix the value of μ^2 . Since the only physical scale in R_n is the centre of mass energy Q^2 , the "natural" value to be used for μ^2 is $\mu^2 \simeq Q^2$.

The two loop QCD prediction for R_n has been compared with data (after correction for detector acceptance and hadronization effects) at Z^0 energies [19,22] and below [12,23,24]. As an illustrative example we show in Fig. 2 the result for $D_2(y_{\text{cut}}) = dR_2(y_{\text{cut}})/dy_{\text{cut}}$ of the OPAL Collaboration [19].

The main conclusion is that eq. (4) with $\mu^2 = Q^2$ agrees with data in the large y_{cut} region ($y_{\text{cut}} > 0.05$), whilst a good description of the data in the small y_{cut} region ($y_{\text{cut}} < 0.05$) requires the use of very small renormalization scales $\mu^2 \ll Q^2$ [19]. The need of such "unnatural" renormalization scale to fit experimental data, already noticed for other quantities [25,19], requires some physical explanation.

A first tentative explanation is that for $y_{\text{cut}} \ll 1$, jet cross section definition involves two very different scales Q^2 and $y_{\text{cut}}Q^2$. Therefore one might argue that $\mu^2 \sim y_{\text{cut}}Q^2$. However that implies that R_n are functions of $\alpha_S(\sqrt{y_{\text{cut}}}Q)$ and hence for $y_{\text{cut}} \ll 1$ they depend on large distance physics and, eventually, become IR sensitive. It follows that $\mu^2 \sim y_{\text{cut}}Q^2$ has to be excluded on theoretical grounds.

A second tentative explanation might use the recourse to some optimization procedure. Several prescriptions to fix the value of the renormalization scale μ^2 have been proposed [26-28] and are nowadays widely used. Their application to jet cross sections in the JADE algorithm has been investigated in [29]. It turns out that the optimized renormalization scale μ_{op}^2 is much smaller than Q^2 for $y_{\text{cut}} > 0.05$ but one can get $\mu_{\text{op}}^2 \gg Q^2$ reducing y_{cut} to very small values (Fig. 3).

Although neither $\mu^2 \sim y_{\text{cut}}Q^2$ nor $\mu^2 \sim \mu_{\text{op}}^2$ have theoretical justifications, one can try to compare the corresponding predictions with data. From the analysis of Ref. [29], we conclude (Fig. 4) that both prescriptions are not able to improve the agreement with data in the relevant region of small $y_{\text{cut}} < 0.05$.

We regard the results described above as the failure of phenomenological attempts to justify the choice of values of μ^2 very different from Q^2 and we think we should offer our theoretical explanation of that.

Let us state more precisely the argument that the physical value for the renormalization scale is $\mu^2 \simeq Q^2$. In order to compare the $O(\alpha_S^2)$ -expression (4) with data one has to assume that the perturbative expansion for R_n is convergent. If μ^2 is very

different from Q^2 , we know that the two loop result for R_n neglects large higher order logarithmic corrections of the type $(\ln Q^2/\mu^2)^k$ (for instance the $O(\alpha_s^3(\mu) \ln^2 Q^2/\mu^2)$ term in eq. (4)). Therefore, modulo fortuitous cancellations, when $\mu^2 \neq Q^2$ one expects the perturbative expansion for R_n is badly convergent and one cannot safely compare the fixed order truncation with data. And vice versa, by studying the μ^2 dependence of the perturbative series in $\alpha_s(\mu)$ one can test its convergence: if the series is convergent, R_n should be a stable function of $\mu^2 \simeq Q^2$ and a rapidly varying function for μ^2 very different from Q^2 .

It turns out that in second order $dR_n/d\ln \mu^2$ for $\mu^2 \simeq Q^2$ strongly depends on y_{cut} for small y_{cut} values. This means that the perturbative series in α_s is not convergent for $y_{cut} \ll 1$. The choices $\mu^2 \simeq y_{cut}Q^2$ or $\mu^2 \simeq \mu_p^2$ are just guesses to improve the convergence of the perturbative expansion. We think the only reliable procedure is to identify the origin of the terms spoiling the convergence and to compute them in higher orders.

Large coefficients in the α_s -expansion for R_n already appear in one loop order. From eqs. (4), (6) we have ($L \equiv \ln 1/y_{cut}$)

$$R_2 \simeq 1 - \frac{C_F \alpha_s}{\pi} L^2, \quad R_3 \simeq \frac{C_F \alpha_s}{\pi} L^2, \quad (y_{cut} \ll 1). \quad (8)$$

The double logarithmic term in eq. (8) comes from the bremsstrahlung spectrum for emission of a soft (with energy fraction $\epsilon = \omega/E \ll 1$) and collinear (with angle $\theta \ll 1$) gluon from the $q\bar{q}$ pair (Fig. 5). In the soft and collinear limit the single gluon emission probability is

$$dw(1) = \frac{C_F \alpha_s}{\pi} \frac{d\epsilon}{\epsilon} \frac{d\theta^2}{\theta^2}, \quad (9)$$

which, integrated over the phase space for the JADE algorithm ($y_{gg} \simeq \epsilon\theta^2$), gives

$$\int dw(1) \Theta(\epsilon\theta^2 - y_{cut}) = \frac{C_F \alpha_s}{\pi} \int_0^1 \frac{d\epsilon}{\epsilon} \int_0^1 \frac{d\theta^2}{\theta^2} \Theta(\epsilon\theta^2 - y_{cut}) = \frac{C_F \alpha_s}{2\pi} L^2. \quad (10)$$

The large contributions $\alpha_s L^2$ are responsible for (i) the unphysical shape of the n -jet rates computed in fixed order perturbation theory (for instance in one loop order $R_2 \rightarrow -\infty$ for $y_{cut} \rightarrow 0$) and (ii) the instability of the perturbative expansion with respect to renormalization scale variations [30].

At small values of y_{cut} , higher order terms in α_s are enhanced by powers of $\ln^2 y_{cut}$. In this kinematical region the real expansion parameter is the large effective coupling $\alpha_s L^2$ and therefore any finite-order perturbative calculation cannot give an accurate evaluation of the cross section. The logarithmic terms must be resummed to all orders in α_s before a reliable prediction can be made. No choice of μ^2 in a fixed order result can account for such a resummation because changing μ^2 one can only introduce single logarithmic corrections $(\alpha_s L)^k$ in higher orders. A particular choice of μ^2 can at most simulate the double logarithmic resummation over a limited range of y_{cut} [30,31].

3. The small y_{cut} region

3.1 Exponentiation of Sudakov logarithms

The appearance of large double logarithmic terms is a common feature of any hard process in the semi-inclusive or Sudakov region, where emission of radiation is inhibited by the kinematics [32,33]. In the case of jet cross sections at $y_{cut} \ll 1$, the jet invariant mass is constrained to be so small as to allow only emission of gluons that are soft and

collinear with respect to the parton generating the jet. The double logarithmic terms $\alpha_s \ln^2 y_{cut}$ are due to such soft and collinear gluons.

In recent years, following the pioneering calculations [34] of double logarithms, the programme of resummation of Sudakov logarithms has been successfully carried out to next-to-leading order for transverse [35] and longitudinal momentum distributions [36,37] in hadron-initiated processes, as well as for quantities like the energy-energy correlation [38] and the thrust [30] and heavy jet mass distribution [31] in e^+e^- annihilation. All these successful resumptions depend on the fact that the large logarithmic corrections to the relevant quantity exponentiate. By this we mean that terms of the form $\alpha_s^m \ln^m y_{cut}$ with $m \leq 2n$ can be combined into an exponential function of less singular terms, in fact with $m \leq n+1$. Once exponentiation has been established, the resummation of leading and next-to-leading logarithms to all orders is reduced to a much simpler, finite-order calculation of the relevant exponent.

Let us recall the physical basis for exponentiation starting from the simpler case of Quantum Electrodynamics (QED). In QED photons are not charged and this implies that multiple soft photon emission is an independent process. Thus the probability $dw(1, \dots, n)$ for n -soft photon emission factorizes in the product of single photon factors $dw(i)$ in eq. (9)

$$dw(1, \dots, n) = \frac{1}{n!} \prod_{i=1}^n dw(i) = \frac{1}{n!} \prod_{i=1}^n \frac{\alpha}{\pi} \frac{d\epsilon_i}{\epsilon_i} \frac{d\theta_i^2}{\theta_i^2}. \quad (11)$$

Starting from the factorized result (11) one can get the corresponding cross sections contribution by integration over the relevant phase-space $\Theta(1, \dots, n; y_{cut})$. Therefore the factorization of multiphoton amplitudes leads to the exponentiation of the cross section if and only if the phase space constraint factorizes in the soft limit, i.e.

$$\Theta(1, \dots, n; y_{cut}) \simeq \prod_{i=1}^n \Theta(i; y_{cut}), \quad (12)$$

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \int dw(1, \dots, n) \Theta(1, \dots, n; y_{cut}) \\ \simeq 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{i=1}^n \int dw(i) \Theta(i; y_{cut}) = \exp \left[\int dw(i) \Theta(i; y_{cut}) \right]. \end{aligned} \quad (13)$$

The property (11) comes from QED dynamics whilst (12) depends on the kinematic definition of the cross section. The cross section exponentiates in the Sudakov region iff its definition does not induce kinematic correlations among soft photons.

Let us come to the QCD case. The main difference with respect to QED is due to the fact that gluons have colour charge. Therefore they can radiate in cascade and soft gluon emission is no longer an independent process. Strong gluon correlations are enforced by the dynamics, multiple gluon emission is not factorized into single emission contributions and we can only expect some kind of generalized exponentiation.

Nevertheless a simple exponentiation structure is still valid for highly inclusive cross sections like a large class of two jet dominated quantities [34-38,30,31]. Let us consider for example the two loop non-factorized contributions in Fig. 6. If the two-jet cross section is defined in such a way that the integration over the soft gluon degrees of freedom is unconstrained, the real (Fig. 6a) and virtual (Fig. 6b) contribution cancel each other in the IR and collinear regions. The only logarithmic term which survives

comes from the ultraviolet region for the virtual diagram. It leads to the running of the QCD coupling $\alpha_S \rightarrow \alpha_S(k_\perp)$ with an argument as given by the transverse momentum $k_\perp$ of the emitted gluon [39].

The cancellation described so far is preserved to higher orders due to *unitarity* and *gauge invariance* properties [40] and leads to the factorization of the one gluon emission probability (9) with $\alpha_S \rightarrow \alpha_S(k_\perp)$ ($k_\perp \sim \epsilon\theta Q$). Exponentiation then follows in the absence of kinematic correlations among soft partons, related to the actual definition of the cross section.

3.2 Leading and next-to-leading logarithms for the JADE algorithm

In order to check whether exponentiation works for jet cross sections defined according to the JADE algorithm, one can start computing n -jet rates to $O(\alpha_S^2)$. In the small y_{cut} region one can write ($L \equiv \ln 1/y_{\text{cut}}$)

$$R_n = \delta_n + \frac{\alpha_S(Q)}{\pi} [A_2^{(n)} L^2 + A_1^{(n)} L + O(1)] + \left(\frac{\alpha_S(Q)}{\pi} \right)^2 [B_4^{(n)} L^4 + B_3^{(n)} L^3 + O(L^2)] \quad (14)$$

We have computed the leading ($A_2^{(n)}, B_4^{(n)}$) and next-to-leading ($A_1^{(n)}, B_3^{(n)}$) logarithmic coefficients for the JADE algorithm in the E-recombination scheme and the results [41] are reported in Table 2 (the C_F^2 contributions for $B_4^{(n)}$ were first computed by Brown and Stirling [42]).

One has the surprising result that the leading double logarithmic contributions do not exponentiate even in the abelian limit $C_A \rightarrow 0$ [42]. In Table 2 we have split the C_F^2 contributions for the two loop coefficient $B_4^{(n)}$ into two terms. The first one in parentheses is the term corresponding to the exponentiation and the second one leads to the violation of exponentiation.

According to our discussion in Sec. 3.1, these results imply that the JADE algorithm leads to non-factorized phase space which, in the E-recombination scheme, produces *attractive* kinematic correlations among soft gluons. For the two jet cross section these correlations appear in the kinematic configuration in which the $q\bar{q}$ pair emits two back-to-back ($\theta_1^2, \theta_2^2 \ll 1$) and soft ($\epsilon_1, \epsilon_2 \ll 1$) gluons (Fig. 7). The corresponding factorized phase space is

$$\epsilon_1 \theta_1^2, \epsilon_2 \theta_2^2 < y_{\text{cut}} \quad (15)$$

but, following the E-recombination procedure, the sub-region

$$\epsilon_1 \theta_1^2, \epsilon_2 \theta_2^2 > \epsilon_1 \epsilon_2; \quad \epsilon_1, \epsilon_2 > y_{\text{cut}} \quad (16)$$

has to be subtracted from the two jet region and included in the three jet rate. We see that the E-scheme gives rise in the region of eqs. (15), (16) to an "anomalous" three jet configuration in which a jet is given by two very soft partons with a large relative angle. One may say that the JADE algorithm counts the number of lumps instead of jets.[†]

The attractive kinematic correlations induced by the E-recombination scheme imply the following consequences.

[†]We owe Nick Brown this nice picture.

- In two loop order, R_2 and R_4 decrease and R_3 increases with respect to the result for independent emission.
- The kinematic correlations strongly increase with the number of final state partons and therefore even the resummation of the leading $\ln y_{\text{cut}}$ terms to all orders appears hopeless.
- The recombination procedure produces a lot of "anomalous" jets: soft and large angle jets.
- Since, according to QCD preconfinement, hadronization is supposed to be a local process in energy and angle, we expect (and we do have) large hadronization corrections.

In order to check whether these features are just a pathology of the E-scheme, we have evaluated the jet rates (14) for the other recombination schemes. We found [41] that exponentiation is still violated and that leading and next-to-leading terms are the same for any massless recombination scheme (JADE, E0, P, P0). The results are reported in Table 3. Some comments are in order.

Firstly, the next-to-leading coefficients $B_3^{(n)}$ are equal to the ones for the E-scheme. This is due to the fact that the kinematic correlations related to the jet definition affect only the soft gluon phase space region.

Secondly, JADE, E0, P and P0-schemes have the same leading behaviour because the corresponding pseudoparticles are massless. That provides a first explanation of the different size of hadronization corrections for the E-scheme and the massless recombination schemes (cf. Sec. 2.2).

Finally, soft gluon kinematic correlations for the 2 and 3 jet rates in the massless schemes are weaker than for the E-scheme (compare the second contribution in parentheses for $B_4^{(n)}$ in Tables 2 and 3). That supports our explanation of the smaller hadronization effects observed for massless schemes. Note also that for massless schemes, the attractive gluon correlations only reduce R_4 , whilst both R_2 and R_3 are enhanced.

On balance, our conclusions about the JADE algorithm in the E-scheme are not changed very much by our results for the other schemes. The only point is that massless recombination procedures slightly reduce some bad features of the algorithm.

4. A new clustering algorithm: the $k_\perp$ -algorithm

4.1 Definition

We believe that the analysis of the JADE algorithm carried out in Secs. 2 and 3 sufficiently motivates the need of a different jet definition. The main goal is to define a clustering procedure which destroys soft gluon kinematic correlations in order to make the theoretical calculation possible for $y_{\text{cut}} \ll 1$ and to reduce hadronization effects.

We are looking for a jet algorithm whose dynamics features follow as close as possible QCD theory. In the naive parton model, hadronic jets have a *cylindrical shape*. By this we mean that they are produced with a total energy increasing with the centre of mass energy and with a limited fixed transverse momentum $< k_\perp$ (of order of some hundred MeV) with respect to the jet axis. The QCD jets have instead an *almost conical shape*: parton multiplication due to the fragmentation of the parent parton leads to hadron jets whose mean transverse momentum increases with the jet energy [32, 33, 43].

Thus it is natural to introduce a jet algorithm[†] in which the transverse momentum replaces the invariant mass of the original JADE algorithm as the jet resolution variable. To emphasise this change of variable, we call the new algorithm the $k_{\perp}$ -algorithm [13].

The $k_{\perp}$ -algorithm is defined in a similar way to the JADE algorithm, following the iterative procedure 1)-4) in Sec. 2.1 and just replacing the invariant mass by the transverse momentum as the jet resolution variable. Therefore, the test variable y_H at the step 2) becomes

$$y_H = \frac{2(1 - \cos \theta_H)}{Q^2} \min(E_k^2, E_l^2) = 2 \min(\epsilon_k^2, \epsilon_l^2) (1 - \cos \theta_H). \quad (17)$$

For the recombination procedure one can define several schemes (E, E0, P, P0) as in the JADE algorithm.

The main features of the $k_{\perp}$ -algorithm (in any recombination scheme) we expect are (i) small hadronisation effects (positive preliminary results have been already reported at this meeting [3]) and (ii) the possibility of resumming large $\ln y_{\text{cut}}$ terms to all orders in perturbation theory. In fact, since the test variable (17) is diagonal with respect to parton energies, it is a much more local energy-angle resolution variable than the invariant mass (which depends on both parton energies). For instance, it is trivial to check that the back-to-back configuration in Fig. 7 leads only to two jets in the factorized phase space region $\epsilon_1^2 \theta_1^2, \epsilon_2^2 \theta_2^2 < y_{\text{cut}}$.

4.2 Resummation in the small y_{cut} region

The n -jet rates

$$R_n(y_{\text{cut}}) = \sum_{m \geq n} \frac{1}{\sigma_{\text{TOT}}} \int d\sigma_m^{\text{ex}} \Theta_{n\text{-jet}}(y_{\text{cut}}) \quad (18)$$

are obtained by integration of the m -parton exclusive cross sections $d\sigma_m^{\text{ex}}$ ($m \geq n$) over the n -jet phase space $\Theta_{n\text{-jet}}(y_{\text{cut}})$ as given by the clustering procedure. We have computed the n -jet fractions using the coherent branching formalism [44,45,33] to evaluate the exclusive cross sections and working out the n -jet phase space to next-to-leading accuracy.

Using the $k_{\perp}$ -algorithm we are able to resum the leading $\alpha_S^m L^{2m}$ and next-to-leading $\alpha_S^m L^{2m-1}$ logarithmic contributions to all orders in α_S for any number of jets[‡]. The result [13] can be simply stated in terms of the generating function $\phi(Q, Q_0; u)$ for jet rates ($Q_0^2 \equiv y_{\text{cut}} Q^2$)

$$R_n(y_{\text{cut}} = Q_0^2/Q^2) = \frac{1}{n!} \left(\frac{\partial}{\partial u} \right)^n \phi(Q, Q_0; u) \Big|_{u=0}. \quad (19)$$

Defining the following emission probabilities

$$\begin{aligned} \Gamma_q(Q, q) &= \frac{2C_F \alpha_S(q)}{\pi} \frac{1}{q} \left(\log \frac{Q}{q} - \frac{3}{4} \right), \\ \Gamma_g(Q, q) &= \frac{2C_A \alpha_S(q)}{\pi} \frac{1}{q} \left(\log \frac{Q}{q} - \frac{11}{12} \right), \\ \Gamma_f(q) &= \frac{N_f \alpha_S(q)}{3\pi} \frac{1}{q}, \end{aligned} \quad (20)$$

[†]This algorithm was discussed at the Durham Workshop on JET Studies at LEP and HERA, December 1990, and is sometimes referred to as the Durham algorithm.

[‡]The resummation of leading logarithms for the two-jet rate has been also considered in Ref. [46]

the generating function ϕ in eq. (19) is given by

$$\phi(Q, Q_0; u) = u^2 \exp \left(2 \int_{Q_0}^Q dq \Gamma_q(Q, q) [\phi_q(q, Q_0; u) - 1] \right), \quad (21)$$

where the gluon functional ϕ_g satisfies the following implicit equation

$$\begin{aligned} \phi_g(Q, Q_0; u) &= u \exp \left(\int_{Q_0}^Q dq [\Gamma_g(Q, q) [\phi_g(q, Q_0; u) - 1] - \Gamma_f(q)] \right) \\ &\cdot \left\{ 1 + u \int_{Q_0}^Q dq \Gamma_f(q) \exp \left(\int_{Q_0}^q dq' [2\Gamma_g(q, q') - \Gamma_g(q, q')] \right) \right. \\ &\cdot \left. [\phi_g(q', Q_0; u) - 1] + \Gamma_f(q') \right\}. \end{aligned} \quad (22)$$

Solving eq. (22) as a power series in the jet label u , one can compute the jet rates R_n in eq. (19) for an arbitrary number of jets.

We refer to [13,41] for details of the calculations. Here, we want just to note that we are able to resum leading and next-to-leading $\ln y_{\text{cut}}$ terms because the $k_{\perp}$ -resolution variable in eq. (17), which is diagonal with respect to parton energies, does not induce soft gluon kinematic correlations in the clustering procedure. Moreover, although various recombination schemes (E, E0, P, P0) can differ significantly at finite y_{cut} , they all give the same leading and next-to-leading logarithms at small y_{cut} [41]. Therefore the formulae (21),(22) as well as the ones we shall give below are valid for any of the common recombination schemes.

The n -jet fractions one gets from eqs. (19)-(22) are (for $n \leq 4$)

$$R_2^{(e^+e^-)} = [\Delta_q(Q)]^2, \quad (23)$$

$$R_3^{(e^+e^-)} = 2 [\Delta_q(Q)]^2 \int_{Q_0}^Q dq \Gamma_q(Q, q) \Delta_g(q), \quad (24)$$

$$\begin{aligned} R_4^{(e^+e^-)} &= 2 [\Delta_q(Q)]^2 \left\{ \int_{Q_0}^Q dq \Gamma_q(Q, q) \Delta_g(q) \right. \\ &\quad + \int_{Q_0}^Q dq \Gamma_q(Q, q) \Delta_g(q) \int_{Q_0}^q dq' \Gamma_q(q, q') \Delta_g(q') \\ &\quad + \int_{Q_0}^Q dq \Gamma_q(Q, q) \Delta_g(q) \int_{Q_0}^q dq' \Gamma_f(q') \Delta_f(q') \left. \right\}, \end{aligned} \quad (25)$$

where we have introduced the quark and gluon Sudakov form factors

$$\Delta_q(Q) = \exp \left(- \int_{Q_0}^Q dq \Gamma_q(Q, q) \right), \quad (26)$$

$$\Delta_g(Q) = \exp \left(- \int_{Q_0}^Q dq [\Gamma_g(Q, q) + \Gamma_f(q)] \right), \quad (27)$$

which depend implicitly on y_{cut} via $Q_0 = Q \sqrt{y_{\text{cut}}}$, and we have defined

$$\Delta_f(q) = [\Delta_q(q)]^2 / \Delta_g(q). \quad (28)$$

The relevant two loop coefficients in eq. (14) for the $k_{\perp}$ -algorithm are given in Table 4.

The results in eqs. (23)-(28) are in agreement with the general discussion in Sec. 3.1. We see that a simple exponentiation of $\ln y_{\text{cut}}$ terms is valid for the two jet fraction R_2 .

The jet rates R_n with $n > 2$ satisfy simple exponentiation in the pseudo-abelian limit $\Gamma_s, \Gamma_f \rightarrow 0$

$$R_n = \frac{\Delta^2(Q)}{(n-2)!} \left[2 \int_{Q_0}^Q dq \Gamma_q(Q, q) \right]^{n-2} + (\Gamma_s \text{ and } \Gamma_f \text{ terms}), \quad (29)$$

whilst in the QCD case they follow a generalized exponentiation structure.

5. Discussion and outlook

In this paper we have presented a QCD theory based discussion of jet topology and jet algorithms. We have mainly focused on the small y_{cut} or Sudakov region. From our analysis we argued that soft gluon kinematic correlations induced by the JADE clustering procedure may be responsible for large hadronization corrections and for bad convergence properties of the QCD perturbative expansion (see also Appendix A for a different jet definition). In Sec. 4 we have discussed a new jet algorithm, the $k_{\perp}$ -algorithm, which does not suffer from the above problems.

Using the $k_{\perp}$ -algorithm, large higher order corrections at small values of y_{cut} can be easily evaluated. We were able to resum leading and next-to-leading logarithms of y_{cut} to all orders in perturbation theory for any number of jets. The results in eqs. (23)-(25) may be combined with the exact fixed order results, after subtraction of the terms that have been exponentiated, to obtain predictions for up to four jets over the full range of y_{cut} .

A question we have discussed at length is the renormalization scale dependence. The coefficients of the leading and next-to-leading logarithmic contributions resummed in eqs. (23)-(25) are renormalization group invariant. Therefore, we cannot properly discuss the renormalization scale dependence in the $k_{\perp}$ -algorithm until the matching with fixed-order results and a smooth extrapolation between the region of small and large y_{cut} have been performed [41]. Anyway, according to our discussion in Sec. 2.3 and to our experience [30,31], the renormalization scale problem is not a problem at all! Once a given quantity has been correctly computed resumming classes of logarithmic higher order corrections, its expansion in this "improved perturbation theory" should be convergent for values of the renormalization scale μ^2 of the order of the physical scale of the process.

The $k_{\perp}$ -algorithm can also be used to compute inclusive and differential jet cross sections as well as being applicable to hadron collision processes. Calculations are in progress.

We hope experimental data for jets within the $k_{\perp}$ -algorithm will be soon available.

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Appendix A - The Geneva algorithm

A different jet algorithm called the Geneva or E_n -algorithm [14,3] has been recently

proposed to overcome the difficulty of resumming in y_{cut} terms in the JADE algorithm. In this clustering algorithm the jet resolution variable y_M is defined by

$$y_M = \frac{2E_k E_l}{(E_k + E_l)^2} (1 - \cos \theta_{kl}) = \frac{2e_k e_l}{(e_k + e_l)^2} (1 - \cos \theta_{kl}) \quad (30)$$

and the particle are recombined according to the E-scheme (although other recombination procedure can be used as well).

The test variable (30) is essentially the two parton invariant mass times a rescaling factor $1/(e_k + e_l)^2$. This rescaling factor acts as a repulsive potential counteracting the attractive soft gluon correlations arising in the JADE algorithm. In order to have a better understanding of this effect we have performed some calculations for the small y_{cut} region.

Let us start by considering a pseudo-abelian model in which "gluons" cannot radiate in cascade gg or $q\bar{q}$ real pairs. This model is similar to QED apart from the fact that the coupling constant runs as in QCD. For this model we were able to compute the generating function $\tilde{\phi}(Q, Q_0; u)$ of n -jet rates in the Geneva algorithm

$$\tilde{R}_n(y_{\text{cut}} = Q_0^2/Q^2) = \frac{1}{n!} \left(\frac{\partial}{\partial u} \right)^n \tilde{\phi}(Q, Q_0; u) \Big|_{u=0}. \quad (31)$$

For the generating function $\tilde{\phi}$ to leading and next-to-leading logarithmic accuracy we find [41]

$$\tilde{\phi}(Q, Q_0; u) = u^2 \exp \left\{ (u-1) \left[8 \int_{Q_0/Q}^1 \frac{dx}{x} \int_{xQ_0}^{x^2 Q} \frac{dq}{q} \alpha_S(q) - 3 \int_{Q_0}^Q \frac{dq}{q} \alpha_S(q) \right] \right\}. \quad (32)$$

The generating function (32) leads to a Poisson distribution and hence the corresponding n -jet rates exponentiate in this pseudo-abelian limit.

Let us come back to the QCD case. We have evaluated the two jet fraction R_2 resumming leading and next-to-leading in y_{cut} terms to all orders in α_S and we found the exponentiated expression

$$R_2 = \exp \left\{ - \left[8 \int_{Q_0/Q}^1 \frac{dx}{x} \int_{xQ_0}^{x^2 Q} \frac{dq}{q} \alpha_S(q) - 3 \int_{Q_0}^Q \frac{dq}{q} \alpha_S(q) \right] \right\}. \quad (33)$$

The results (32,33) confirm that the rescaling factor $1/(e_k + e_l)^2$ in eq. (30) is able to cancel the attractive kinematic correlations for the two jet cross section and for any jet rate in the abelian limit. However it looks surprising that the jet resolution variable (30), depending on both parton energies, may lead to a soft gluon phase space which is completely uncorrelated. A full two loop calculation confirms that kinematic correlation are still there. The result [41] for the two loop coefficients of eq. (14) in the Geneva algorithm is given in Table 5.

The abelian C_F^2 contributions to $B_1^{(n)}$ exponentiate, whilst the non-abelian terms $C_F C_A$ differ from those obtained for the JADE algorithm (these latter are reported in Table 5 as the first contribution to $B_1^{(n)}$ inside the parentheses). This difference is due to kinematic configurations like the one in Fig. 8 in which two soft gluons ($1 > e_1 > e_2$) are radiated in cascade ($1 > \theta_1^2 > \theta_2^2$) at large angle ($e_1 \theta_1^2 > y_{\text{cut}}$). In the JADE algorithm this configuration leads to a 4-jet event fraction $\{\bar{q}, q, g_1, g_2\}$ and to a 3-jet

event fraction $\{\bar{q}, q, (g_1, g_2)\}$. In the Geneva algorithm the rescaling factor $1/(e_k + e_l)^2$ in the test variable (30) induces a strong effective repulsion between the soft gluons g_1 and g_2 to produce a non-local 3-jet recombination $\{\bar{q}, (q, g_2), g_1\}$. This repulsive correlation enhances the 4-jet cross section and reduces the 3-jet cross section with respect to the JADE algorithm.

We conclude that the clustering procedure of the Geneva algorithm introduces repulsive kinematic correlations among soft gluons, thus leading to analogous problems to the ones arising in the JADE algorithm, viz.

- the resummation of the $\ln y_{cut}$ terms for R_n with $n \geq 3$ is (probably) hopeless;
- the jet recombination pattern is non-local in energy-angle and sizeable hadronization corrections are expected (preliminary results presented at this meeting [3] confirm this expectation).

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scheme	M_{ij}^2/Q^2	recombination
E	$(p_i + p_j)^2/Q^2$	$p_{ij} = p_i + p_j$
JADE		$\epsilon_i = E_i/Q$ $p_{ij} = p_i + p_j$
E0		$\epsilon_i = E_i/Q$ $E_{ij} = E_i + E_j$ $p_{ij} = \frac{E_{ij}}{ p_i + p_j } (p_i + p_j)$
P	$2\epsilon_i\epsilon_j(1 - \cos\theta_{ij})$	$\epsilon_i = E_i/Q$ $p_{ij} = p_i + p_j$ $E_{ij} = p_{ij} $
P0		$\epsilon_i = E_i/\sum_k E_k$ $p_{ij} = p_i + p_j$ $E_{ij} = p_{ij} $

Table 1 - Recombination procedures for the JADE algorithm: massive (E) and massless (JADE, E0, P, P0) recombination schemes.

n	$A_2^{(n)}$	$A_1^{(n)}$	$B_1^{(n)}$	$B_3^{(n)}$
2	$-C_F$	$+\frac{3}{2}C_F$	$(\frac{1}{2} - \frac{1}{12})C_F^2$	$-\frac{3}{2}C_F^2 - \pi\beta_0 C_F$
3	$+C_F$	$-\frac{3}{2}C_F$	$(-1 + \frac{5}{24})C_F^2 - \frac{1}{24}C_F C_A$	$+3C_F^2 + \frac{1}{4}C_F C_A + \frac{7}{6}\pi\beta_0 C_F$
4	0	0	$(\frac{1}{2} - \frac{1}{6})C_F^2 + \frac{1}{24}C_F C_A$	$-\frac{3}{2}C_F^2 - \frac{1}{4}C_F C_A - \frac{1}{6}\pi\beta_0 C_F$

Table 2 - Leading and next-to-leading two loop coefficients for the JADE algorithm in the E-recombination scheme.

n	$A_2^{(n)}$	$A_1^{(n)}$	$B_1^{(n)}$	$B_3^{(n)}$
2	$-C_F$	$+\frac{3}{2}C_F$	$(\frac{1}{2} + \frac{1}{24})C_F^2$	$-\frac{3}{2}C_F^2 - \pi\beta_0 C_F$
3	$+C_F$	$-\frac{3}{2}C_F$	$(-1 + \frac{1}{12})C_F^2 - \frac{1}{24}C_F C_A$	$+3C_F^2 + \frac{1}{4}C_F C_A + \frac{7}{6}\pi\beta_0 C_F$
4	0	0	$(\frac{1}{2} - \frac{1}{6})C_F^2 + \frac{1}{24}C_F C_A$	$-\frac{3}{2}C_F^2 - \frac{1}{4}C_F C_A - \frac{1}{6}\pi\beta_0 C_F$

Table 3 - Leading and next-to-leading two loop coefficients for the JADE algorithm in the massless recombination schemes.

n	$A_2^{(n)}$	$A_1^{(n)}$	$B_1^{(n)}$	$B_3^{(n)}$
2	$-\frac{1}{2}C_F$	$+\frac{3}{2}C_F$	$+\frac{1}{8}C_F^2$	$-\frac{3}{4}C_F^2 - \frac{1}{3}\pi\beta_0 C_F$
3	$+\frac{1}{2}C_F$	$-\frac{3}{2}C_F$	$-\frac{1}{4}C_F^2 - \frac{1}{48}C_F C_A$	$+\frac{3}{2}C_F^2 + \frac{1}{6}C_F C_A + \frac{1}{2}\pi\beta_0 C_F$
4	0	0	$+\frac{1}{8}C_F^2 + \frac{1}{48}C_F C_A$	$-\frac{3}{4}C_F^2 - \frac{1}{6}C_F C_A - \frac{1}{6}\pi\beta_0 C_F$

Table 4 - Leading and next-to-leading two loop coefficients for the k_T -algorithm in any (E, E0, P, P0) recombination scheme.

n	$A_2^{(n)}$	$A_1^{(n)}$	$B_1^{(n)}$	$B_3^{(n)}$
2	$-C_F$	$+\frac{3}{2}C_F$	$\frac{1}{2}C_F^2$	$-\frac{3}{2}C_F^2 - \pi\beta_0 C_F$
3	$+C_F$	$-\frac{3}{2}C_F$	$-C_F^2 - (\frac{1}{24} + \frac{1}{6})C_F C_A$	$+3C_F^2 + \frac{1}{4}C_F C_A + \frac{7}{6}\pi\beta_0 C_F$
4	0	0	$\frac{1}{2}C_F^2 + (\frac{1}{24} + \frac{1}{6})C_F C_A$	$-\frac{3}{2}C_F^2 - \frac{1}{4}C_F C_A - \frac{1}{6}\pi\beta_0 C_F$

Table 5 - Leading and next-to-leading two loop coefficients for the Geneva algorithm.

Figure Captions

- Figure 1: OPAL results on relative production rates of n -jet events, determined from model calculations before and after the hadronisation process. Jets are defined by the JADE algorithm and four different jet recombination schemes are used.
- Figure 2: Measured distributions of $D_2(y)$ for the JADE algorithm, corrected for detector acceptance and hadronisation effects, compared to the corresponding analytic $O(\alpha_s^2)$ QCD calculations. The QCD parameters are taken from the fit results of $\Lambda_{\overline{MS}}$ with $f = 1$ ($f \equiv \mu^2/Q^2$) and of $\Lambda_{\overline{MS}}$ and μ^2 in the region of y indicated by the arrows.
- Figure 3: The scale $\mu/\sqrt{s}$ ($s = Q^2$) computed in Ref. [29] according to the $\sqrt{y}$ procedure (dotted) and to the optimization procedures in [26] (dashed), [27] (full) and [28] (dashed-dotted). The results are obtained for $Q = 91$ GeV and $\Lambda_{\overline{MS}} = 0.1$ GeV.
- Figure 4: Comparison between OPAL data and $O(\alpha_s^2)$ QCD predictions for n -jet production rates. The renormalization scale μ^2 is computed [29] according to the optimization procedures in (a) [26], (b) [27], (c) [28] and (d) $\mu^2 = yQ^2$.
- Figure 5: Soft and collinear gluon emitted from the quark-antiquark pair.
- Figure 6: Soft gluon dynamics correlations in two loop order: (a) real and (b) virtual emission contributions.
- Figure 7: Two back-to-back gluons emitted from the quark-antiquark pair.
- Figure 8: Two soft gluons radiated in cascade.

Fig. 2

Fig. 1

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8