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GENERALIZED W-ALGEBRAS AND INTEGRABLE HIERARCHIES

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ABSTRACT

We report on generalizations of the KdV-type integrable hierarchies of Drinfel'd and Sokolov. These hierarchies lead to the existence of new classical W -algebras, which arise as the second Hamiltonian structure of the hierarchies. In particular, we present a construction of the $W_n^{(t)}$ algebras.

Several years ago, Drinfel'd and Sokolov obtained a very significant generalization of the usual Korteweg-de Vries (KdV) hierarchy [1]. This hierarchy, the first non-trivial flow of which is

$$\frac{\partial u}{\partial t_1} = -\frac{1}{4}u''' + \frac{3}{2}uu' \text{ where } u' \equiv \frac{\partial u}{\partial x}, \quad (1)$$

can be straightforwardly obtained in terms of pseudo-differential operators [2]. To this end one considers the scalar Lax equation

$$\frac{dL}{dt} = [A, L], \text{ where } L = D^k + \sum_{i=1}^k u_i D^{i-1}, \quad A = \sum_{i=1}^{m+1} v_i D^{i-1}. \quad (2)$$

It can be shown that A is determined by L , up to m arbitrary constants and one arbitrary function; also we note that since the right-hand side of the equation can be shown to have order less than or equal to $k-2$ we can consistently set $u_{k-1} = 0$. The case $k = 2$ then corresponds to the usual KdV equation.

Now, these scalar Lax equations above can be given a matrix representation without difficulty [1]. Choosing

$$\tilde{L} = \partial_x + q + \Lambda = \partial_x + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ u_1 & \cdots & u_n & 0 \end{pmatrix} + \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ z & & & \end{pmatrix} \quad (3)$$

with z a spectral parameter, the scalar Lax equation becomes the matrix Lax equation

$$\frac{d\tilde{L}}{dt} = [\tilde{A}, \tilde{L}], \quad (4)$$

where the differential operator A , of order m , becomes a matrix $\tilde{A}$ of the form $\tilde{A} = \sum_{i=0}^m z^i \tilde{v}_i$. (This corresponds to $k = n+1$ in (2), where we have also set $u_{n+1} = 0$ as allowed.)

A natural question is whether the above construction can be generalized. One could consider a more general q , for example allow it to be a lower triangular matrix. In this case, though, one discovers that the matrix $\tilde{A}$ whose commutator with $\tilde{L}$ would give the

evolution equation of the hierarchy at a certain order, is not uniquely defined. However, introducing the notion of gauge-equivalence $\tilde{L} \rightarrow \tilde{L}' = N\tilde{L}N^{-1}$ with N a lower-triangular matrix with 1's on the diagonal, allows one to circumvent this indeterminacy and by choosing an appropriate gauge-slice one can actually re-obtain the scalar Lax equation. The advantage of this somewhat complicated rewriting of the scalar Lax equation is that whereas the q in equation (3) has no group-theoretic interpretation, the lower-triangular q can be considered an element of the Borel subalgebra, while the N can be considered an exponentiated subalgebra, of $sl(n+1)$. In fact our earlier KdV equations are now seen to correspond to $sl(n+1, \mathbb{C}[z, z^{-1}])$ Kac-Moody algebras; this interpretation allows a natural generalization, and so KdV type equations can be defined for arbitrary Kac-Moody algebras [1].

Similar results were obtained in [3], though the association here was made between *modified* KdV equations and untwisted affine Lie algebras. In this same work a suggestion was made concerning the proper setting in which these developments were to be viewed and it was this suggestion that provided an impetus for our work. Apart from their being examples of new integrable systems, the new hierarchies are also of interest because of they lead to generalized W -algebras. Essentially, systems of the KdV-type admit two distinct, but *coordinated*, Hamiltonian structures. The 'second' Poisson bracket algebra is then the W -algebra associated to the hierarchy: for example, the $sl(n+1)$ hierarchies defined above, via (3), lead to the W_{n+1} -algebras, where the u_i 's are the generators. Our construction includes the $W_3^{(2)}$ -algebra [4], which is related to an example of a 'fractional' KdV hierarchy [5]. The generalized hierarchies and W -algebras that we obtain appear to be related to other approaches [6].

We now explain the general construction, a more complete treatment appears in refs. [7,8]. The central object in the construction will be the loop algebra associated to a finite Lie algebra, $\hat{g} = g \otimes \mathbb{C}[z, z^{-1}]$, equipped with a derivation $d = z(d/dz)$, which induces the *homogeneous gradation*. Other gradations may be introduced by choosing $\text{rank}(g) + 1$

non-negative integers $\mathbf{s} = (s_0, \dots, s_{\text{rank}(g)})$, and defining the derivation [9]

$$d_{\mathbf{s}} = N_{\mathbf{s}} d + \sum_{k=1}^{\text{rank}(g)} \left(\frac{2}{\alpha_k^2} \right) s_k \omega_k \cdot H, \quad (5)$$

where $N_{\mathbf{s}} = \sum_{i=0}^{\text{rank}(g)} k_i s_i$ (the k_i being the Kac-labels of g), the ω_i are the fundamental weights of g , and H is the Cartan subalgebra of g . The derivation $d_{\mathbf{s}}$ then defines the $\mathbf{s}$ -gradation: $\hat{g} = \bigoplus_{i \in \mathbb{Z}} \hat{g}_i(\mathbf{s})$.

For later use define $\hat{g}_{>i}(\mathbf{s}) = \bigoplus_{k>i} \hat{g}_k(\mathbf{s})$ and similarly for $\hat{g}_{<i}(\mathbf{s})$, $\hat{g}_{\leq i}(\mathbf{s})$ and $\hat{g}_{\geq i}(\mathbf{s})$; also we say that $\mathbf{s} \succeq \mathbf{s}'$ provided that $s_i \neq 0$ whenever $s'_i \neq 0$, and we thereby obtain a partial ordering on the set of gradations $\{\mathbf{s}\}$ of $\hat{g}$. For convenience we shall use superscripts to denote $\mathbf{s}[w]$ -grades and subscripts to denote $\mathbf{s}$ grades. To simplify notation we assume that we are working in some faithful matrix representation of g .

There is a class of subalgebras of $\hat{g}$ which will play an important rôle in what follows, these being the *Heisenberg subalgebras*. They are special types of maximally commuting subalgebras whose precise definition may be found in [10], in which it is further shown that the inequivalent Heisenberg subalgebras of $\hat{g}$ are classified by the conjugacy classes of the Weyl group of g . We denote the Heisenberg subalgebra corresponding to the conjugacy class containing the Weyl reflection w , as $\mathcal{H}[w]$. An important property of these algebras is that they are associated to a particular gradation of $\hat{g}$, which we denote $\mathbf{s}[w]$, such that $\mathcal{H}[w]$ is an invariant subspace of $\text{ad } d_{\mathbf{s}[w]}$. At this point we make the parenthetic remark, that for the following construction we need a particular gradation, and we then need to be able to choose an element which has a well-defined grade and is semi-simple, i.e. one such that the Kac-Moody algebra $\hat{g}$ equals the direct sum of the kernel plus the image of the action of this element on $\hat{g}$. The procedure outlined above, where we lift conjugacy classes of the Weyl group, and look at the associated gradations, is a means of accomplishing this in a systematic fashion.

We begin by defining a differential operator $L \equiv \partial_x + q + \Lambda$, associated to the data $(\Lambda, w, \mathbf{s})$, in the space $C^\infty(\mathbf{R}, \hat{g})$, where $\hat{g}$ is the loop algebra of a finite Lie algebra g . The

element Λ is a regular element of $\mathcal{H}[w]$ which is constant ($\partial_x \Lambda = 0$) with positive $\mathbf{s}[w]$ -grade i , where w is some element of the Weyl group of g — conjugate elements of the Weyl group leading to isomorphic hierarchies. Here by regular we mean that $\text{Ker}(\text{ad } \Lambda) = \mathcal{H}[w]$. (If this latter condition is not satisfied, one can still define a hierarchy but a bit more care must be taken [3,7].) Lastly the *potential* $q \in C^\infty(\mathbf{R}, Q)$, where Q is defined to be the following subspace of $\hat{g}$, $Q = \hat{g}_{\geq 0}(\mathbf{s}) \cap \hat{g}^{< i}(\mathbf{s}[w])$, $\mathbf{s}$ some other gradation of $\hat{g}$ such that $\mathbf{s} \preceq \mathbf{s}[w]$.

Having imposed the requisite conditions on the terms in L we now assert that there exist an infinite number of commuting flows of the form

$$\frac{\partial L}{\partial t_b} = [A(b), L]. \quad (6)$$

Here b is a constant element of the subspace of $\mathcal{H}[w]$ with positive $\mathbf{s}[w]$ -grade, and $A(b) \in C^\infty(\mathbf{R}, \hat{g})$.

To prove this, define the following exponentiated action of the loop algebra on L , $\mathcal{L} = \Phi L \Phi^{-1}$. It can be shown that there exists a (non-unique) transformation of this type, for which $\mathcal{L}$ has the form

$$\mathcal{L} = \partial_x + \Lambda + \sum_{j < i} h^j, \quad (7)$$

where $h^j \in C^\infty(\mathbf{R}, \mathcal{H}^j[w])$. If we then have an M such that $[M, L] = 0$, setting $A = M_{\geq 0}$ or $A = M^{\geq 0}$, we find that $[A, L] \in C^\infty(\mathbf{R}, Q)$, see [7]. Since a constant element of $\mathcal{H}[w]$, say b , commutes with $\mathcal{L}$, the quantity $\Phi^{-1} b \Phi$ commutes with L . Using the above result we deduce that two sets of flows of the form (6) exist with $A(b) = (\Phi^{-1} b \Phi)_{\geq 0}$ or $A(b) = (\Phi^{-1} b \Phi)^{\geq 0}$, respectively.

We now introduce the key concept of a gauge symmetry. The form of L is preserved under the transformation $L' = S L S^{-1}$, S being generated by the subalgebra of functions on $P = \hat{g}_0 \cap \hat{g}^{< 0}$. Under this action:

$$q \longmapsto S(\partial_x + q + \Lambda)S^{-1} - \partial_x - \Lambda. \quad (8)$$

The phase space of the hierarchy is actually equal to $C^\infty(\mathbf{R}, Q)$ modulo the action of these gauge transformations.

We can write consistent flows for a gauge fixed operator $\tilde{L}$, by choosing $\tilde{q} \in C^\infty(\mathbf{R}, \tilde{Q})$, where $\tilde{Q}$ is some consistent gauge slice. However, if we do this then there is no guarantee that the flows (6) will preserve the gauge slice chosen. To compensate for this it will be necessary to modify the flows by gauge transformations:

$$\frac{\partial \tilde{L}}{\partial t} = \left[(\Phi^{-1} b \Phi)_{\geq 0} + \theta, \tilde{L} \right] \text{ and } \frac{\partial \tilde{L}}{\partial t'} = \left[(\Phi^{-1} b \Phi)^{\geq 0} + \theta', \tilde{L} \right], \quad (9)$$

for some $\theta, \theta' \in C^\infty(\mathbf{R}, P)$. It can be demonstrated that the equations in (9) lead to the same evolution equation for $\tilde{q}$, and also that the quantities h^j of (7) are the conserved densities for the hierarchy. In addition, the finite set of h^j 's with $i > j \geq 0$ are constant along all the flows (recall that i is defined by the fact that $\Lambda \in \mathcal{H}^i[w]$), *i.e.* for any flow

$$\partial_t h^j = 0 \quad i > j \geq 0. \quad (10)$$

This result means that some components of q can be consistently set to zero, since they are constant under the flows.

We can consider the spectrum of hierarchies for fixed w and Λ but allowing the gradation s to vary — it has only to satisfy $s \preceq s[w]$. For $s = s[w]$ it is apparent from the definition of P that there is no gauge invariance and so we have a generalization of the Drinfel'd-Sokolov *modified*-KdV hierarchies. On the contrary, when s is as 'small' as it can be, for instance the homogeneous gradation, the gauge invariance is maximal, and then we have a generalization of the Drinfel'd-Sokolov KdV-type hierarchies. Choosing a gradation s between these two extremes leads to what we call a *partially modified* KdV hierarchy (pmKdV). It turns out that all these hierarchies are related by a series of Miura maps [8].

The hierarchies constructed by Drinfel'd and Sokolov are recovered from our formalism by choosing $[w]$ to be the Coxeter class $[w_c]$, and taking Λ to have $s[w]$ -grade 1. There are two ways to generalize these known cases. Firstly, we can take Λ to be a different

element of the Heisenberg subalgebra corresponding to the Coxeter class, and secondly we can choose Λ to be an element of a new conjugacy class of the Weyl group. The former generalization leads to the fractional KdV hierarchies [5].

As an example of the above analysis, we consider the fractional KdV hierarchies for the $sl(n+1)$ -algebras. For $g = sl(n+1)$, the Heisenberg subalgebra $\mathcal{H}[w_c]$ is spanned by the elements

$$\Lambda_{j,m} = z^m \begin{pmatrix} \mathbf{0} & \mathbf{I}_{n+1-j} \\ z\mathbf{I}_j & \mathbf{0} \end{pmatrix} \quad j = 1, 2, \dots, n, \quad m \in \mathbf{Z}, \quad (11)$$

$\mathbf{I}_i$ being the $i \times i$ unit matrix. So for a given n we have a whole tower of hierarchies, for which $\Lambda = \Lambda_{j,m}$ $m \geq 0$; the original hierarchy of Drinfel'd and Sokolov being at the bottom. If we consider $sl(2)$ and $i = 3$, we find that the gauge-fixed form differs from that of the usual KdV form, explicitly

$$\tilde{q} = \begin{pmatrix} \tilde{a} & -\tilde{c} \\ \tilde{b} + \tilde{c}z & -\tilde{a} \end{pmatrix} \text{ as compared to } \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}. \quad (12)$$

Proceeding as outlined above, we find that the equations of motion for the first and second flow of the fractional hierarchy are

$$\partial_{t_1} \begin{pmatrix} \tilde{a} & -\tilde{c} \\ \tilde{b} + \tilde{c}z & -\tilde{a} \end{pmatrix} = \begin{pmatrix} \tilde{b} + 2\tilde{c}^2 & -2\tilde{a} \\ 2\tilde{a}z - 2\partial_{\tilde{x}}\tilde{c} + 4\tilde{c}\tilde{a} & -\tilde{b} - 2\tilde{c}^2 \end{pmatrix} \quad (13)$$

for the first, while the second is trivial (it identifies t_3 with $-x$).

We now turn to a discussion of the canonical formalism for the hierarchies. A feature often encountered in the Hamiltonian analysis of integrable hierarchies, is the presence of two *coordinated* Poisson structures which we designate $\{\phi, \psi\}_1$ and $\{\phi, \psi\}_2$. The property of *coordination* implies that the one-parameter family of brackets

$$\{\phi, \psi\} = \{\phi, \psi\}_1 + \mu\{\phi, \psi\}_2, \quad (14)$$

μ arbitrary, is also a Poisson structure, which is a non-trivial statement as regards the Jacobi identity. We say a system has a *bi-Hamiltonian structure* if the brackets are coordinated and if the Hamiltonian flow can be written in two equivalent ways

$$\dot{\phi} = \{H_2, \phi\}_1 = \{H_1, \phi\}_2. \quad (15)$$

Under various general assumptions, the existence of a bi-Hamiltonian structure implies the existence of an infinite hierarchy of flows, that is, an infinite set of Hamiltonians $\{H_i\}$, such that

$$\partial_{t_i}\phi = \{H_{i+1}, \phi\}_1 = \{H_i, \phi\}_2,$$

where the Hamiltonians are in involution with respect to both Poisson brackets, whence the flows ∂_{t_i} commute.

An example of these features is provided by the original KdV hierarchy which has the two Poisson structures

$$\begin{aligned} \{u(x), u(y)\}_1 &= 2\delta'(x-y) \\ \{u(x), u(y)\}_2 &= \frac{1}{2}\delta'''(x-y) - 2u(x)\delta'(x-y) - u'(x)\delta(x-y). \end{aligned} \tag{16}$$

We can similarly obtain Poisson brackets for the hierarchies introduced above and shall find amongst other things that, in the same way as the second structure of the KdV equation is a Virasoro algebra, so the second structures of the new hierarchies constitute generalized W-algebras (for details see [8]). There are hierarchies that do not admit a bi-Hamiltonian structure, for example the *modified* KdV hierarchy (mKdV) and its generalizations. However, it is known that the single Hamiltonian structure is the pull-back of the second Hamiltonian structure of the associated KdV hierarchy by the Miura map connecting the hierarchies. This property extends to the generalized hierarchies; each KdV-type hierarchy admits a bi-Hamiltonian structure, whilst the associated pmKdV and mKdV hierarchies admit only a single Hamiltonian structure [8].

The function $q(x)$ plays the rôle of the phase space coordinate in this system. However, as we saw earlier, there exist symmetries in the system corresponding to the gauge transformations $L \rightarrow SLS^{-1}$. The *phase space* of the system $\mathcal{M}$ is the set of gauge equivalence classes of operators of the form $L = \partial_x + q + \Lambda$. The space of functions $\mathcal{F}$ on $\mathcal{M}$ is the set of gauge invariant functionals of q of the form

$$\varphi[q] = \int_{\mathbf{S}^1} dx f\left(x, q(x), q'(x), \dots, q^{(n)}(x), \dots\right), \tag{17}$$

where we now take periodic functions for convenience (following [1]). It is straightforward to find a basis for $\mathcal{F}$, the gauge invariant functionals. One simply performs a non-singular gauge transformation to take q to some canonical form $\tilde{q}$. The coordinates $\tilde{q}$ and their derivatives then provide the desired basis. For instance, for the generalized $sl(n+1)$ -KdV hierarchies of Drinfel'd and Sokolov, q consists of lower triangular $n+1$ by $n+1$ dimensional matrices, while the gauge group is generated by strictly lower triangular matrices. A good gauge slice consists of matrices of the form of q in (3), the u_i 's and their derivatives providing a basis for $\mathcal{F}$.

There is a natural inner product on the functions $C^\infty(\mathbf{S}^1, \hat{g})$, defined as follows

$$(A, B) = \int_{\mathbf{S}^1} dx \langle A(x), B(x) \rangle_{\hat{g}}, \quad (18)$$

where $\langle \cdot, \cdot \rangle_{\hat{g}}$ is the Killing form of $\hat{g}$. Explicitly $\langle a \otimes z^n, b \otimes z^m \rangle_{\hat{g}} = \langle a, b \rangle_g \delta_{n+m, 0}$, where $\langle \cdot, \cdot \rangle_g$ is the Killing form of g .

We now proceed to define the two Poisson structures for the KdV hierarchies (for which s is the homogeneous gradation). For a constant element $b \in \mathcal{H}[w]^{>0}$, define the following functional of q , $H_b[q] = (b, h(q))$, where $h(q) = \sum_{j < i} h^j$ is the sum of the conserved densities of (7). For a functional φ of q define its functional derivative $d_q \varphi \in C^\infty(\mathbf{S}^1, \hat{g}_{\leq 0})$ via

$$\left. \frac{d}{d\varepsilon} \varphi[q + \varepsilon r] \right|_{\varepsilon=0} \equiv (d_q \varphi, r), \quad \forall r \in C^\infty(\mathbf{S}^1, Q). \quad (19)$$

Notice that there is an ambiguity in the definition of the functional derivative $d_q \varphi$ corresponding to the fact that terms in the annihilator of Q are not determined; despite this our subsequent definitions are well-defined [8]. The two coordinated Hamiltonian structures on $\mathcal{F}$ are

$$\begin{aligned} \{\varphi, \psi\}_1 &= - (d_q \varphi, z^{-1} [d_q \psi, L]), \\ \{\varphi, \psi\}_2 &= (q + \Lambda, [d_q \varphi, d_q \psi]_R) - (d_q \varphi, (d_q \psi)'), \end{aligned} \quad (20)$$

where $[x, y]_R \equiv [Rx, y] + [x, Ry]$, with $R(x) = (x_0 - x_{<0})/2$, for all $x \in \hat{g}$. It can be proved that both brackets satisfy the Jacobi identity and map gauge-invariant functionals to a

gauge-invariant functional, see [8]. As an aside, the appearance of an R-matrix here is, we believe, connected with a proper geometric understanding of all of the hierarchies above, which will be reported on elsewhere. Also it is found that under time evolution in the coordinate t_b , the following recursion relation holds

$$\frac{\partial \varphi}{\partial t_b} = \{\varphi, H_{zb}\}_1 = \{\varphi, H_b\}_2. \quad (21)$$

We can further show that the Poisson brackets defined above sometimes admit non-trivial centres. The existence of these centres is directly related to the Hamiltonian densities h^j with $i > j \geq 0$. As we have already remarked these densities are constant under all the flows of the hierarchy, and so not all the functionals on $\mathcal{M}$ are dynamical.

It is well-known that the KdV hierarchies exhibit a scale-invariance, i.e. under $x \rightarrow \lambda x$, for constant λ , each quantity can be assigned scaling dimensions such that the equations remains invariant. This actually generalizes to all the hierarchies considered above. In fact we can further generalize this result to arbitrary transformations $x \rightarrow y(x)$. This transformation turns out to be a Poisson mapping of the second symplectic structure. All this leads us to surmise that the second Poisson bracket contains (as a subalgebra) the algebra of conformal transformations, i.e. the Virasoro algebra, which would imply that the second Poisson bracket algebra is an extended chiral conformal algebra — thereby generalizing the occurrence of the W -algebras as the second Poisson bracket algebra of the hierarchies of Drinfel'd and Sokolov.

As an example of the above, consider the first fractional hierarchy associated to $sl(3)$. This corresponds to choosing $\Lambda = \Lambda_{2,0}$ in (11). Following our construction, q has the general form

$$q = \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} + z \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \star & 0 & 0 \end{pmatrix}, \quad (22)$$

The gauge invariant functionals are conveniently generated by the coordinates on the

following gauge slice

$$\tilde{q} = \begin{pmatrix} \frac{1}{2}U & 0 & 0 \\ G^+ & -U & 0 \\ T & G^- & \frac{1}{2}U \end{pmatrix} + \phi \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z & 0 & 0 \end{pmatrix}. \quad (23)$$

The second Poisson bracket algebra is easy to find using (20). One sees that ϕ is in the centre of the algebra, and so may be consistently set to zero, leaving the $W_3^{(2)}$ algebra discussed in [4,5]. More generally by considering $sl(k)$ with $\Lambda = \Lambda_{j,m}$, leads to the $W_k^{(j+mk)}$ -algebras.

As a final point, we mention that ‘Toda’ equations can be associated to these hierarchies, generalizing the relationship between the sine-Gordon equation and the usual mKdV equation ([3,1]) . This will be discussed elsewhere.

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