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ALIVE WITH LIOUVILLE IN NON-CRITICAL SPACE-TIME

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ABSTRACT

Four-point correlation functions for non-critical strings are calculated in $d = 1$ and $d = 25$ space-time dimensions by operator methods using a Liouville vertex operator.

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The Liouville theory remarkably often exited considerations in mathematics [1], [2] as well as in physics [3]-[5], and it did not loose its attractiveness although it was only poorly understood all the time. But trusting the message which this interacting theory has mediated since Liouville, we recognize that it can indeed be described by free fields [6],[7]. The quantum structure of the Liouville theory is then defined by the conformal and local properties which govern the classical theory.

One may hope, therefore, that the recent results in non-critical string theory [8]-[10] can be understood by exact calculations based on a Liouville theory which is canonically quantized in terms of free fields. In this letter, four-point correlation functions for non-critical strings will be discussed for the space-time dimensions $d = 1$ and $d = 25$. The more general treatment of this problem and the details of the derivations will be given elsewhere. The correlation functions can be calculated quite conventionally by operator methods using integrated vertex operators

$$V(\alpha_K, p_K) = \int d^2 z e^{\alpha_K \varphi(z)} e^{ip_K \cdot x(z)} \quad (1)$$

The string coordinates $x(z)$ and the Liouville field $\varphi(z)$ are coupled such that (1) has conformal weight zero. This connects the parameters α_K and p_K with the space-time dimension d [11]-[13]

$$\alpha_K = (\sqrt{25-d} - \sqrt{1-d+12\vec{p}_K^2})/(\sqrt{25-d} - \sqrt{1-d}) \quad (2)$$

The vertex operator for the string coordinates is well known and its Liouville counterpart will be defined by the quantum solution of the Liouville equation of motion. Taking as the free field $\psi(z)$ a genuine scalar, instead of the (so-called) pseudoscalar one of Refs. [6],[7], the Liouville vertex operator is given by [13]

$$e^{\alpha \varphi(z)} =: e^{\alpha \psi(z)} : (1 - : F(\alpha, P) \int_1^z dz' \int_1^{\bar{z}} d\bar{z}' e^{\psi^+(z')} e^{\psi^-(\bar{z}')} :)^{-2\alpha} \quad (3)$$

This expression looks simple, but it is rather involved due to quantum corrections, which are necessary in order to respect the conformal and the local properties of the Liouville theory quantum mechanically. The normal ordering prescription hides structures which are given by short-distance singularities [6] (for details, see Ref. [13]; n counts the expansion of (3) where these singularities appear after normal ordering the left- and right-moving fields $\psi^\pm(z)$)

$$\begin{aligned} f_{\pm}^{(\alpha, n)}(z, \{z'_K\}) &= \prod_{j=1}^n [|z'_j z|^{\alpha g} |z - z'_j|^{-2\alpha g}] \\ &\times \prod_{j < i}^n [|z'_j z'_i|^g |z'_j - z'_i|^{-2g}] \end{aligned} \quad (4)$$

Locality defines in addition momentum P -dependent quantum corrections (which again are more involved for the expanded and reordered form of (3) [13])

$$F(\alpha, P) = \frac{\sin(2\alpha\pi g)}{2\alpha\pi g} sh^2(\frac{1}{4}\gamma\eta P) sh^{-1}(\frac{1}{4}\gamma\eta P - i\alpha\pi g) sh^{-1}(\frac{1}{4}\gamma\eta P + i\alpha\pi g) \quad (5)$$

The different parameters are defined by the Liouville coupling $\gamma^2 = 48\pi/(25-d)$ and a suitable renormalization of it ($\gamma \rightarrow \gamma\eta$). The Liouville vertex operator $e^{\varphi(z)}$ will have conformal weight one if η fulfils the quadratic equation [7]

$$2\hbar\eta^2 - \eta + 1 = 0 \quad (6)$$

Here we use the abbreviation $\hbar = \gamma^2/(16\pi)$. The Liouville vertex operator (3), which includes integration over the singularities (4), is well defined only if

$$g \equiv 2\hbar\eta^2 = \frac{1}{12}(13 - d - \sqrt{(1-d)(25-d)}) \leq \frac{1}{2} \quad (7)$$

This in fact restricts the space-time dimension to $d \leq 1$ or $d \geq 25$. In general, the conformal weight of the Liouville vertex operator is

$$\Delta(e^{\alpha\varphi(z)}) = \alpha\eta(1 - 2\hbar\alpha\eta) \quad (8)$$

From this it follows that vertex operators of charge α and charge $\bar{\alpha}$

$$\bar{\alpha} = \frac{1}{2\hbar\eta} - \alpha \quad (9)$$

have the same dimension. For the short-distance behaviour, we get then the expected result

$$e^{\alpha\varphi(z)} e^{\bar{\alpha}\varphi(z')} = |z - z'|^{-2\Delta} \quad (10)$$

$2\beta_0 = (2\hbar\eta)^{-1}$ plays here the role of a “background charge” for which the Liouville vertex operator being the identity operator, has the dimension zero. For $d = 25$, this operator is the identity as $\beta_0 = 0$, and for $d = 1$ we have $\beta_0 = 1$. (I should mention that the origin of β_0 is different from the origin of the parameter Q of Refs. [11],[12] since we have fixed the improvement term of the energy momentum tensor as in the classical case [13].)

Let us now calculate the four-point function for charges $\alpha_i = \alpha$, $i = 1, 2, 3$ and $\alpha_4 = \bar{\alpha}$ which sum up to

$$3\alpha + \bar{\alpha} = 2\alpha + 2\beta_0 \quad \text{with} \quad 2\beta_0 = \frac{1}{2\hbar\eta} \quad (11)$$

For the Liouville theory the counting of charges is unusual. The integrals of the vertex operator (3) act like screening charges. As any concrete calculation with the vertex operator (3) takes into account its expanded form, we have arbitrarily many of them. The neutrality condition will take the form

$$2\alpha + 2\beta_0 + n_4 + n_2 + n_1 = 0 \quad (12)$$

(n_3 is missing because we have taken Möbius invariance to fix three of the four integrations of the Haar measure by choosing $z_4 = \infty$, $z_3 = 1$, $z_1 = 0$. For $z = 1$ the integrals in (3) do not contribute.) Taking both solutions $\eta_{\pm}$ of (5) and its properties

$$\eta_+ + \eta_- = \eta_+\eta_- = \frac{1}{2\hbar} \quad (13)$$

we obtain

$$2\alpha + 2\beta_0 = (1 - m)\eta_- + (1 - n)\eta_+ \quad (14)$$

From this it follows that only for $d = 1$, which gives $\eta_{\pm} = 2$, does the relation (12) have a non-trivial solution. For $d = 25$ $\eta_{\pm}$ and β_0 are zero. This implies that for $d = 25$ all n_i have to be zero otherwise the correlation function would be zero. The Liouville field $\varphi(z)$ then becomes the non-interacting free field $\psi(z)$ with central charge $c = 1$. In this case, α_K is non-linearly related to p_K by (2)

$$\alpha_K^2 = 1 - \frac{1}{2}\vec{p}_K^2 \quad (15)$$

and the renormalized Liouville coupling becomes purely imaginary

$$\gamma\eta = 2i\sqrt{2\pi} \quad (16)$$

The kinetic term of the Liouville theory then gets a negative sign which implies that the Liouville field $\varphi(z)$ can be interpreted for $d = 25$ as a time-like coordinate of a 26-dimensional Minkowski space theory [14]. The four-point function is in this case a Virasoro-Shapiro amplitude in a 26-dimensional Minkowski space, where the α and the p_K form a new vector p_K

$$V(p_1, p_2, p_3, p_4) = \int_0^1 dz |z|^{\frac{1}{2}p_1 p_2} |1-z|^{\frac{1}{2}p_2 p_3} \quad (17)$$

This amplitude describes the critical string.

Let us finally discuss the situation for $d = 1$. Straightforward calculations provide correlation functions with a huge complexity. The correlation function will again be mainly a Veneziano-Shapiro amplitude, but now the Liouville field will not play the rôle of a time coordinate. Instead the spectrum will be shifted in the momentum squared in accordance with the relation

$$\alpha_K = 1 - \frac{1}{\sqrt{2}} p_K \quad (18)$$

For $d = 1$ the integrand of (17) contains a factor $K(z)$ which is expressed by three sums (interrelated by condition (11)) of products of non-standard integrals which I shall exemplify. Before, I should mention that the theory possesses a unity operator of great complexity. This has no influence on the properties of the operators. But this complexity makes the interpretation of higher-order correlation functions difficult. It may show redundancies. For the four-point correlation function the following three types of integrals appear ($x = 1, z$)

$$\prod_{K=1}^n \left[\int_0^x du_K |u_K|^{a\alpha} |1-u_K|^{-2g\alpha} |z-u_K|^{-2g\alpha} \right] \prod_{j<i}^n \left[|u_j u_i|^g |u_j - u_i|^{-2g} \right]$$

and

$$\prod_{K=1}^m \left[\int_0^1 dv_K |v_K|^{b\alpha} |1-v_K|^{-2g\alpha} |1-zv_K|^{-2g\alpha} \right] \prod_{r<s}^m \left[|v_r v_s|^g |v_r - v_s|^{-2g} \right] \quad (19)$$

All these integrals are, in addition, coupled among each other by additional factors like

$$\prod_{K=1}^n \prod_{\ell=1}^m |1-u_K v_\ell|^{-2g} \quad \text{and} \quad \prod_{K=1}^n \prod_{\ell=1}^m |u_K - v_\ell|^{-2g} \quad (20)$$

These integrals have no poles in p^2 or α^2 . Possible "leg-poles" are cancelled, as far as I see it, by the different integrals and by the momentum corrections (5) which have zeros there (a general proof is missing at the moment!). If this holds true, the results found for $d = 1$ support the suggestion Polyakov made in Ref. [8].

In a forthcoming paper [15] I shall give proofs. There I shall also discuss the quantum group structure and other items.

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