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**DOUBLING VERSUS NON-DOUBLING OF EQUATIONS
AND PHASE SPACE STRUCTURE
IN ONE-HERMITEAN-MATRIX MODELS**

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Abstract

In the framework of the saddle-point approximation, we analyse the phase structure of one-hermitean-matrix models with non-even interaction potential. This enables us to reconsider the issue of the doubling phenomenon: the boundaries of the phase space are shown to correspond to critical behaviours either with or without doubling of equations depending on the location of the roots of the eigenvalue distribution. Critical domains involving the Painlevé *II* equation arise also in this non-even regime. As a corollary it appears that pure gravity could be realized within a quartic potential bounded below and leading to a one-arc distribution. Furthermore, this could enable flows between $m = 3$ and $m = 2$ -theories.

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1. Introduction

The orthogonal-polynomial method [1] provides a powerful tool for solving $N \times N$ matrix models. It is by now well-known that at $N \rightarrow \infty$ such models exhibit several critical behaviours, and can in particular describe a continuum random surface [2] with all genera contributing.

The analysis of multicriticality and consequently the issue of a possible double scaling limit [3] is based on a scaling ansatz for the recursion coefficients. In the case of one-hermitean-matrix models, assuming that at the limit $N \rightarrow \infty$ the orthogonal-polynomial constants R_n and S_n become smooth functions of $x \equiv Tn/N$, $R(x)$ and $S(x)$, and working in the vicinity of $x = 1 - \delta x$, $R(x) = R_c - \delta R$ and $S(x) = S_c - \delta S$, a natural generalization of Kazakov's ansatz is:

$$\delta x \sim \delta R^m \sim \delta S^m. \quad (1)$$

It was shown in [4, 5] that such an ansatz always leads to a pair of equations for the two scaling functions $f \pm g/\sqrt{R_c}$, both of type m in the KdV hierarchy.

However, it is possible to relax somewhat the multicriticality conditions we obtained in [5], by taking into account subleading corrections in the scaling ansatz (1). At the critical point, the system is thus described by a unique m th KdV equation for only one scaling function. This was pointed out by F. David and studied by C. Marzban and R. Raju Viswanathan [6].

In view of its close connection with string theory, the phase structure of the one-hermitean (and also unitary)-matrix model has attracted much attention recently [7 - 10]. This structure is extraordinarily rich: it exhibits m th multicritical points conjectured to correspond to $(2m - 1, 2)$ -minimal models coupled to gravity as well as critical behaviours leading to the Painlevé II equation [11]. However a complete understanding of this phase structure, even in few-parameter spaces, is still lacking. This is particularly important, both for understanding the allowed flows and the related important issue of a non-perturbative definition of string theory [13, 14], and because (for high degree potentials) there might be indications of new, as yet uncharted, possible critical behaviours [15].

In this letter we would like to present a *systematic* classification of critical behaviours for general matrix potentials of degree up to four. We will mostly rely on a saddle-point analysis, but will also compare the results with those obtained from the orthogonal-polynomial technique. We will in particular clarify the issue of *doubling of degrees of freedom*, which occurs at intersections of critical surfaces of *weaker conditions*. We will point out the existence of well-defined critical potentials, with *one-arc eigenvalue densities* which give rise to the Painlevé I equation under the period-one scaling ansatz, but it remains an open problem whether and how this equation must actually be modified.

In the following we will start by solving for a quartic potential the various multicriticality conditions that come out of the orthogonal-polynomial method.

Then we will turn to the large N phase structure analysis by means of computation of the eigenvalue distribution in the saddle-point approximation. We will conclude in the last section.

2. The orthogonal-polynomial approach

Let us start with a comparatively brief review of the multicriticality conditions that are necessary to get or avoid the doubling of string equations. We will not go through all details of the well-known orthogonal-polynomial method. We consider the $N \times N$ one-hermitean-matrix model with potential

$$V(\lambda) = \sum_{k=1}^{deg(V)} \frac{g_k}{k} \lambda^k. \quad (2)$$

The basic equations for the recursion coefficients R_n and S_n are:

$$T \frac{n}{N} = \mathcal{W}(R_n, S_n) \quad (3a)$$

$$0 = \mathcal{Y}(R_n, S_n) \quad (3b)$$

where T is the temperature and

$$\begin{aligned} \mathcal{W} &\equiv g_2 R_n + g_3 (R_n S_n + R_n S_{n-1}) + \dots, \\ \mathcal{Y} &\equiv g_1 + g_2 S_n + g_3 (S_n^2 + R_n + R_{n+1}) + \dots. \end{aligned}$$

To analyse these equations, we set $x = Tn/N$ and assume that for $N \rightarrow \infty$, R_n and S_n have a smooth limit $R(x)$ and $S(x)$. We know that this assumption could fail [7 - 12] and we will come back to this point later. In the naive $N \rightarrow \infty$ limit eqs. (3) read:

$$x = W \equiv \frac{\partial \Omega}{\partial S} \quad (4a)$$

$$0 = Y \equiv \frac{\partial \Omega}{\partial R} \quad (4b)$$

where

$$\Omega(R, S) = \frac{1}{2\pi i} \oint dz V(z + \frac{R}{z} + S). \quad (5)$$

We define:

$$\begin{aligned} \alpha_k &\equiv \frac{(-)^{k+1}}{k!} \frac{\partial^k W}{\partial S^k} \\ \beta_k &\equiv \frac{(-)^{k+1}}{k!} \frac{\partial^k Y}{\partial S^k}. \end{aligned} \quad (6)$$

We assume now the existence of a critical point at $x = x_c = 1$, $R = R_c$, $S = S_c$ and $Y = Y_c$. Eq. (5) suggests that a translation of the potential - which actually does

not modify any of its critical properties - makes it possible to choose $S_c = 0$ which we will do in the sequel⁽¹⁾. With this choice, the potential dependent quantities at criticality, α_{even}^c and β_{odd}^c depend on g_{even} while α_{odd}^c and β_{even}^c depend on g_{odd} only. Eq. (4a) now determines R_c :

$$1 = g_2 R_c + 3g_4 R_c^2 + \dots \quad (7)$$

On the other hand eq. (4b) shows that one can always make Y_c vanish by choosing appropriately g_1 ⁽²⁾:

$$0 = g_1 + 2g_3 R_c + \dots \quad (8)$$

In reference [5] we made the scaling ansatz (1) and defined:

$$t \equiv \delta x N^{\frac{2m}{2m+1}} \quad (9)$$

as well as the scaling functions

$$\begin{aligned} f(t) &\equiv \delta R N^{\frac{2}{2m+1}} \\ g(t) &\equiv \delta S N^{\frac{2}{2m+1}}. \end{aligned} \quad (10)$$

Within these assumptions, the only consistent multicriticality conditions we found were:

$$\begin{aligned} \alpha_1^c &= \dots = \alpha_{m-1}^c = 0 \\ \beta_1^c &= \dots = \beta_{m-1}^c = 0; \end{aligned} \quad (11)$$

these will be called in the following *strong conditions*. They lead precisely to the doubling of string equations:

$$t = \epsilon_{\pm}^{(m)} \mathcal{D}_m \left[f \pm g \sqrt{R_c}; R_c \right] \quad (12)$$

with

$$\epsilon_+^{(m)} = \frac{\alpha_m^c + \beta_m^c \sqrt{R_c}}{R_c^{m/2}}, \quad \epsilon_-^{(m)} = (-)^m \frac{\alpha_m^c - \beta_m^c \sqrt{R_c}}{R_c^{m/2}} \quad (13)$$

and $\mathcal{D}_m$ the Gelfand-Dikii operators⁽³⁾. For instance $\mathcal{D}_2[h; R_c] = h^2 - \frac{R_c}{3} h''$. As a consequence, this doubling of equations leads to a doubling of the relevant operators. These belong to two disconnected sectors, making the Hilbert space factorize into two non-interacting pieces⁽⁴⁾. In ref. [16] C. Bachas discusses in a

⁽¹⁾ For pure odd potentials such a choice is however incompatible with eq. (7). One has then to keep $S_c \neq 0$.

⁽²⁾ In any case, Y_c scales as δx and must vanish.

⁽³⁾ For even potentials and with the choice $S_c = 0$, we have: $\epsilon_{\pm}^{(m)} = \epsilon_{\mp}^{(m)}$.

⁽⁴⁾ The number of relevant operators is roughly the number of constraints we must put on the potential in order to make it critical. In that respect $\alpha_1, \dots, \alpha_{m-1}, \beta_1, \dots, \beta_{m-1}$ play the role of sources for such operators. It is easy to show that the operators generated respectively by the sources $\epsilon_{\pm}^{(i)}$ $i=1, \dots, m-1$ are precisely the decoupled ones.

pure statistical mechanical framework this tensor product structure for the $m = 2$ - case (pure gravity) reached with quadrangulations of the $2D$ surfaces versus the one obtained by triangulations which does not lead to the doubling phenomenon as we will see later. We would like to stress here that there exist potentials that satisfy conditions (11) and have *no definite parity*. In other words, the *doubling phenomenon is not a particular feature of even potentials*⁽⁵⁾. To be more concrete we give in fig. 1 an example of fifth degree $m = 2$ -critical potential that leads to this doubling of equations.

We now turn to another situation, where doubling does not occur. Instead of (10) we consider:

$$\begin{aligned} \delta R &= f N^{-\frac{2}{2m+1}} + \phi_1 N^{-\frac{2}{2m+1}} + \dots + \phi_{2m-2} N^{-\frac{2m}{2m+1}} \\ \delta S &= g N^{-\frac{2}{2m+1}} + \gamma_1 N^{-\frac{2}{2m+1}} + \dots + \gamma_{2m-2} N^{-\frac{2m}{2m+1}}. \end{aligned} \quad (14)$$

As usually, we get multicriticality conditions after implementing (14) in eqs. (3) and demanding their consistency. Again we will not work out all the details: these can be found in [6]. In that reference the authors consider the critical behaviours $m = 2, 3$ and 4 and analyse them respectively in cubic, quartic and fifth degree potentials. We will present here the results for the $m = 2$ and 3 -cases for a potential of any degree. A particular feature of the ansatz (14) is that it always leads to two branches of multicriticality conditions (we will refer to them as *weak conditions*). This is not surprising since it corresponds to the choice we have to make one of the two equations (3a $\pm$ 3b) become the m th member of the KdV hierarchy for the scaling function f ; the other is some (complicated) equation relating different ϕ_i 's and γ_j 's. Of course, the strong conditions (11) are restored at the intersection of these branches.

$m = 2$: pure gravity

The criticality condition that the potential must satisfy is

$$\alpha_1^c = \pm \beta_1^c \sqrt{R_c}. \quad (15)$$

Furthermore, consistency requires two more constraints:

$$\begin{aligned} f \pm g \sqrt{R_c} &= 0 \\ \phi_1 \pm \gamma_1 \sqrt{R_c} &= \frac{f'}{2}. \end{aligned} \quad (16)$$

⁽⁵⁾ To avoid confusion note that what we called the "doubling phenomenon" and that we should have called the "degeneracy phenomenon" emphasizes that in some special domain of the phase space - the "strong critical domain" - one obtains two different continuum string theories described by two equations that are of the same kind and *decoupled*. Both of these theories contribute to the singular piece of the free energy.

Therefore the string equations (3) become:

$$t = \left(4\epsilon_{\mp}^{(2)} \mp \frac{\alpha_1^c}{2R_c^{3/2}} \right) \mathcal{D}_2 \left[f; \frac{R_c}{2} \right] \quad (17a)$$

$$t = \pm \frac{\alpha_1^c}{2R_c^{3/2}} \left(f^2 + \frac{R_c}{2} f'' + 4R_c \left(\phi_2 + \gamma_2 \sqrt{R_c - \frac{\phi_1^c}{2}} \right) \right). \quad (17b)$$

Among the six scaling functions $f, \phi_1, \phi_2, g, \gamma_1$ and γ_2 , satisfying constraints (16) and equations (17), two remain undetermined. In ref. [6] the authors chose $\phi_1 = \phi_2 = 0$; (16, 17b) become therefore algebraic equations for g, γ_1 and γ_2 .

$m = 3$: Lee-Yang singularity

The conditions we obtain in this case are:

$$\alpha_1^c = \pm \beta_1^c \sqrt{R_c} \quad (18a)$$

$$\alpha_1^c = \pm 8\sqrt{R_c} \left(\alpha_2^c \mp \beta_2^c \sqrt{R_c} \right). \quad (18b)$$

The constraints (16) also appear, together with two others relating the ϕ_i 's and γ_j 's. We will skip them together with one of the two eqs. (3a $\pm$ 3b) and present only the string equation for the scaling function f :

$$t = \left(8\epsilon_{\mp}^{(3)} + \frac{2\alpha_2^c}{3R_c^2} \mp \frac{\alpha_1^c}{8R_c^{5/2}} \right) \mathcal{D}_3 \left[f; \frac{R_c}{2} \right]. \quad (19)$$

As in the previous case, four scaling functions among a total number of ten remain undetermined. A consistent choice is: $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$.

The above method can be iterated: for the general m th multicritical point, the potential must satisfy - for any of the two branches - $m - 1$ weak conditions. As advertised before, this corresponds to half the strong conditions required in the doubling situation. Furthermore, "strong" string equations come with $\mathcal{D}_m[f; R_c]$ while "weak" ones contain $\mathcal{D}_3[f; \frac{R_c}{2}]$. An analogous situation arises in a slightly different approach [17].

It is important to stress here that since α_{odd}^c and β_{even}^c depend on g_{odd} only, they vanish identically if the potential is even. This means that for pure even potentials, the weak conditions (15) or (18) reduce systematically to the strong ones (11) (6). The doubling of equations then becomes unavoidable. On the other hand, when the potential has no definite parity we can either work with (10) and observe the doubling phenomenon, or start from (14) and avoid it.

(6) Formally this holds also for pure-odd potentials, where α_{even}^c and β_{odd}^c vanish.

We are now able to present the whole analysis of the *complete fourth degree potential*. As we will see, this case is quite interesting since it contains a large spectrum of situations while it remains mathematically tractable.

STRONG CONDITIONS

Non trivial solutions to eqs. (11) with a fourth degree potential arise only for $m = 2$. The realization of the "strong" $m = 3$ requires a sixth degree term, as was established a long time ago. Solving (7, 8, 11) leads to the standard inverted double-well potential (see fig. 2):

$$\begin{aligned} g_1 &= g_3 = 0, \quad g_2 > 0, \quad g_4 < 0, \\ g_2 &= \sqrt{-12g_4}, \quad R_c = \frac{2}{g_2}. \end{aligned} \quad (20)$$

Notice that the odd piece has to vanish; if we want it to remain non-zero we must keep a λ^5 term in the potential (see fig. 1).

WEAK CONDITIONS

Either $m = 2$ or $m = 3$ -multicritical behaviours are realized. In this case, the odd piece of the potential plays a crucial role: when it vanishes "weak" $m = 2$ becomes "strong" $m = 2$ and "weak" $m = 3$ disappears. Let us review them:

$m = 2$

Eqs. (7, 8, 15) give two two-dimensional critical surfaces in the space of couplings that intersect on the line (20). We have:

$$R_c = -\frac{g_2}{6g_4} \pm \sqrt{\left(\frac{g_2}{6g_4}\right)^2 + \frac{1}{3g_4}}. \quad (21)$$

Signs "+" and "-" correspond respectively to the regions $g_4 > 0$ and $g_4 < 0$. With this choice, R_c remains finite and continuous across the plane $g_4 = 0$ (this is required by the fact that at $g_4 = 0$, weak conditions still give a non-trivial solution $m = 2$). The following equations define the critical surfaces:

$$g_3^2 = \left(\frac{g_2^2}{4} + 3g_4\right) \left(\frac{g_2}{2} + \sqrt{\frac{g_2^2}{4} + 3g_4}\right) \quad (22)$$

$$\frac{g_1}{g_3} = \frac{g_2}{3g_4} \mp \sqrt{\left(\frac{g_2}{3g_4}\right)^2 + \frac{4}{3g_4}}. \quad (23)$$

Again the upper sign is suitable for positive g_4 , the lower one being for negative g_4 . Eq. (22) admits two solutions: the one with positive g_3 is the critical surface

corresponding to $\alpha_i = \beta_i \sqrt{R_c}$ while the g_3 -negative one satisfies $\alpha_i = -\beta_i \sqrt{R_c}$. At this stage there is no *a priori* obstruction to any point of the critical surface being a $m = 2$ -critical point. However, the coefficient of $\mathcal{D}_2[f; \frac{R_c}{2}]$ in eq. (17a) must be positive. This leads to the constraint: $g_2 > \sqrt{\frac{4}{5}} g_4$. In fig. 3 we plotted two examples of such critical potentials.

$m = 3$

We have now to solve eqs. (7, 8, 18). The only new equation with respect to the previous case is (18b). Consequently, relations (21-23) still hold and $m = 3$ -behaviour appears at a critical line which is the edge of the $m = 2$ -two dimensional critical surface. It satisfies:

$$g_2 = \sqrt{\frac{4}{5}} g_4, \quad g_2, g_4 > 0. \quad (24)$$

In this case, the non-universal coefficient that appears in eq. (19) is always positive. An example of such a potential is given in fig. 4. Fig. 5 gives a representation of conditions (22) and (24) in different subspaces of the space of couplings.

3. The saddle-point analysis

Let us turn now to a more systematic analysis of the phases of the one-hermitean-matrix model with quartic potential. Similar analyses, but confined to even potentials can be found in refs. [7, 8]. We will study the phase space in the large N limit by using the saddle-point approximation [18].

The quantity of interest is the eigenvalue density $\varrho(\lambda) \equiv \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i)$ where $\{\lambda_i\}$ are the saddle-point configuration of the eigenvalues of the random matrix. The (positive) function $\varrho(\lambda)$ satisfies the saddle-point equation:

$$V'(z) - 2P \int_{\mathcal{D}} d\zeta \frac{\varrho(\zeta)}{z - \zeta} = 0 \quad (25)$$

with the normalization condition

$$\int_{\mathcal{D}} d\lambda \varrho(\lambda) = 1. \quad (26)$$

$\mathcal{D}$ is the compact support of ϱ and may consist of many disconnected pieces (arcs), depending on the shape of the potential. For instance, when the potential has a unique minimum the eigenvalue density has only one arc in any case. However, in a two minimum case, one or two-arc distributions can appear depending on the height of the barrier between the minima. For more complicated potentials, even more complicated situations may arise.

For a potential of given degree, one has in the space of couplings different phases, each being characterized by a given number of arcs in the eigenvalue density. At the boundaries of these phases, the free energy, which is given by:

$$F = \int_{\mathcal{D}} d\lambda \varrho(\lambda) V(\lambda) - \iint_{\mathcal{D}} d\lambda d\mu \varrho(\lambda) \varrho(\mu) \ln |\lambda - \mu|, \quad (27)$$

is in general non-analytic. These are the phase boundaries that we will map out explicitly in this section; we will compare them to the critical surfaces obtained in the previous section.

In order to determine $\varrho(\lambda)$ we use the standard method described in [18]. This method consists of determining the generating function

$$G(z) = \int_{\mathcal{D}} d\zeta \frac{\varrho(\zeta)}{z - \zeta} \quad (28)$$

that satisfies a Schwinger-Dyson equation and is defined for any complex number outside $\mathcal{D}$. Eqs. (25, 28) imply:

$$\varrho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \Im G(\lambda + i\epsilon). \quad (29)$$

The analytic properties of $G(z)$ are sufficient to determine it uniquely:

$$G(z) = \frac{1}{2} \left(V'(z) + \sqrt{\Delta(z)} \right). \quad (30)$$

$\Delta(z)$ is a $2 \deg(V) - 2$ polynomial such that: (i) every odd degree zero of $\Delta(z)$ is a branch point of $G(z)$ and is real (ii) the branch cuts of $G(z)$ are the intervals between two adjacent odd degree zeros of $\Delta(z)$ with $\Delta(z)$ negative; hence these are the arcs of ϱ (iii) an odd number of even degree zeros of $\Delta(z)$ must be located between two consecutive arcs (iv) it ensures the correct asymptotics for $G(z)$, implied by of (26): $G(z) \sim \frac{1}{z}$ when $|z| \rightarrow \infty$. Any surface in the space of couplings across which any of these properties is violated, is a boundary for a given type of eigenvalue distribution.

Let us now apply the above method for the quartic interaction. In that case $\Delta(z)$ is a sixth degree polynomial. The location of its zeros leads to two possible configurations respectively, with a one and two-arc distribution. These are depicted in fig. 6.

ONE-ARC PHASE

For a one-arc distribution arising from a quartic potential, $\Delta(z)$ has the general form:

$$\Delta(z) = g_4^2 (z - 2a)(z - 2b)(z - 2d)^2 (z - 2\bar{d})^2; \quad (31)$$

a, b must be real while d and $\bar{d}$ are either complex conjugate or both real. We conventionally choose $b < a$, $\bar{d} \leq d$ (when these are real) and introduce the reduced real variables: $x = a + b$, $y = a - b > 0$, $u = d + \bar{d}$ and $v = d\bar{d}$. The constraint of the asymptotic behaviour of $G(z)$ leads to the following equations:

$$\begin{aligned} \frac{g_3}{g_4} + 2u + x &= 0 \\ \frac{g_2}{g_4} + \frac{y^2}{2} - 2ux - 4v &= 0 \\ \frac{g_1}{g_4} + \frac{xy^2}{2} - uy^2 + 4vx &= 0 \\ \frac{2}{g_4} - \frac{y^2(4x^2 + y^2)}{8} + uxy^2 - 2vy^2 &= 0. \end{aligned} \quad (32)$$

As we mentioned before, a translation of the potential $V(\lambda) \rightarrow V(\lambda - \gamma)$ does not modify any of its critical properties. In the present formalism we can use the freedom to fix x as we did in the orthogonal-polynomial method with S_c . This will reduce the dimension of the phase space from four to three. A very convenient choice is of course $x = 0$. As we will see later this choice is (not surprisingly) the one corresponding to $S_c = 0$: it will give the same multicriticality conditions.

Let us now describe the structure of the one-arc phase. We start by assuming that d and $\bar{d}$ are real. Any of the following situations represent a generic point of the one-arc phase: $\bar{d} \leq b < a \leq d$, $\bar{d} \leq d \leq b < a$, $b < a \leq \bar{d} \leq d$. The boundaries of that domain are reached when respectively $\bar{d} \rightarrow b$ and/or $d \rightarrow a$, $d \rightarrow b$, $\bar{d} \rightarrow a$. The analysis of eqs. (32) leads to the following results:

In the region $g_4 < 0$ the boundaries $\bar{d} = b < a \leq d$ and $\bar{d} \leq b < a = d$ correspond respectively to the positive and negative branches of the surface defined by:

$$g_3^2 = \left(\frac{g_2^2}{4} + 3g_4 \right) \left(\frac{g_2}{2} + \sqrt{\frac{g_2^2}{4} + 3g_4} \right) \quad (33)$$

$$\frac{g_1}{g_3} = \frac{g_2}{3g_4} + \sqrt{\left(\frac{g_2}{3g_4} \right)^2 + \frac{4}{3g_4}}. \quad (34)$$

The branches intersect when $\bar{d} = b < a = d$, at $g_3 = 0$ and $g_2 = \sqrt{-12g_4}$. For any g_3 between the two branches, a one-arc phase is allowed; outside these surfaces the local minimum is not deep enough to confine the eigenvalues into a compact domain. This situation is depicted in figs. 9a and b.

The above result is in agreement with the orthogonal-polynomial one: equations (33, 34) are nothing but (22, 23) when g_4 is negative. Moreover it gives a natural interpretation of the strong and weak multicriticalities in terms of the eigenvalue density structure: weak $m = 2$ -criticality is reached when one double zero of Δ collapses on the one-arc support of ϱ , either from the left or right side; on

the other hand strong conditions require the simultaneous collapse of one double zero from each side. Of course, this phenomenon is expected to be general and to occur with any potential. We checked in particular that the strong multicritical fifth degree potential (see fig. 1) leads to an eigenvalue distribution whose root structure is depicted in fig. 7. Furthermore this explanation enables us to understand geometrically why strong $m = 2$ -conditions require at least a quartic potential while a cubic one is enough for the weak conditions. When $g_4 \rightarrow 0^-$, $d \rightarrow +\infty$ for $g_3 \geq 0$ and $\bar{d} \rightarrow -\infty$ for $g_3 \leq 0$: the potential becomes of degree three and consequently $\Delta(z)$ loses a double root. The other quantities are continuous across the plane $g_4 = 0$ ensuring therefore the analyticity of the free energy.

From the other side, $g_4 \rightarrow 0^+$, we obtain: $\bar{d} \rightarrow -\infty$ when $g_3 \geq 0$ and $d \rightarrow +\infty$ when $g_3 \leq 0$. This explains why in the region $g_4 > 0$ we can observe $\bar{d} \leq d \leq b < a$ ($g_3 > 0$) or $b < a \leq d \leq \bar{d}$ ($g_3 < 0$): the boundaries $\bar{d} \leq d \leq b < a$ and $b < a = \bar{d} \leq d$ are reached respectively on the positive and negative branch of the surface defined by (33) and

$$\frac{g_1}{g_3} = \frac{g_2}{3g_4} - \sqrt{\left(\frac{g_2}{3g_4} \right)^2 + \frac{4}{3g_4}}, \quad (35)$$

$$g_2 \geq \sqrt{\frac{4}{5}g_4}. \quad (36)$$

Again a one-arc phase is allowed between two branches which do not intersect any more. Eqs. (33, 35) are precisely the eqs. (22, 23) for $g_4 > 0$. We conclude that when these boundaries are reached from the one-arc phase, they correspond to the collapse of one double root from one side and are indeed the weak $m = 2$ -critical surfaces (see fig. 9c). Ineq. (36) is precisely the positivity condition we met in the orthogonal-polynomial approach. The line satisfying $g_2 = \sqrt{\frac{4}{5}g_4}$ and embedded in the surface (33, 35) is a boundary on which either $\bar{d} = d = b < a$ or $b < a = d = \bar{d}$. Below that line no "weak" $m = 2$ -behaviour is possible. Furthermore, the line itself is nothing but the weak $m = 3$ -multicritical domain that was found by orthogonal-polynomial arguments (see eq. (24)). Geometrically this critical behaviour is reached when two double roots collapse on one of the edges of ϱ . To obtain "strong" $m = 3$ -behaviour two pairs of double roots must simultaneously reach both edges.

We will now concentrate on a particular surface: the one defined by the requirement that $d = \bar{d}$ (see fig. 8). Such a surface appears only for $g_4 > 0$; eqs. (32) show that it satisfies:

$$g_3^2 = \frac{8}{3}g_2g_4 + \sqrt{\left(\frac{4}{3}g_2g_4 \right)^2 + \frac{64}{3}g_4^3} \quad (37)$$

together with (35). We can analyse the root configuration on this surface. For $g_2 > \sqrt{\frac{4}{3}g_4}$, $\bar{d} = d < b < a$ if $g_3 > 0$ and $b < a < \bar{d} = d$ if $g_3 < 0$. The surface is

then embedded in the one-arc phase and defines a boundary between the domain where $d, \bar{d}$ are respectively real and complex conjugate (see fig. 9c). No phase transition occur across that surface. On the line $g_2 = \sqrt{\frac{4}{3}}g_4$, $\bar{d} = d = b < a$ or $b < a = \bar{d}$ and we recover the $m = 3$ -multicritical behaviour. Finally, if $g_2 < \sqrt{\frac{4}{3}}g_4$ then $b < d = \bar{d} < a$. This part of the surface is a boundary of the one-arc phase but no singular behaviour appears when it is reached from the upper side.

TWO-ARC PHASE

In this case the more general polynomial $\Delta(z)$ is

$$\Delta(z) = g_4^2(z - 2a)(z - 2b)(z - 2c)(z - 2d)(z - 2f)^2; \quad (38)$$

a, b, c, d and f must be real. We can choose $b \leq d \leq c \leq a$ and introduce the variables $x = a + b$, $y = a - b$, $s = c + d$ and $t = c - d$; they must satisfy the following equations:

$$\begin{aligned} \frac{g_3}{g_4} + 2f + x + s &= 0 \\ \frac{g_2}{g_4} + \frac{y^2}{2} + \frac{t^2}{2} - xs - 2f(x + s) &= 0 \\ \frac{g_1}{g_4} + \frac{(x-s)(y^2 - t^2)}{2} - f(y^2 + t^2 - 2xs) &= 0 \\ \frac{2}{g_4} - \frac{(y^2 - t^2)^2}{8} - \frac{(x-s)(xy^2 - st^2)}{2} + f(x-s)(y^2 - t^2) &= 0. \end{aligned} \quad (39)$$

Thanks to the usual argument of translation invariance, we restrict ourselves to the three dimensional subspace of couplings, $x = 0$ and analyse eqs. (39) along the same lines as for eqs. (32). The results are the following:

As was shown in the pure even case, a two-arc eigenvalue distribution appears only for positive values of g_4 . When $g_2 \geq \sqrt{\frac{4}{3}}g_4$ the two-arc phase exists outside the surface defined by (33, 35) (see fig. 9c). On this boundary the root configuration is the following: $b = d \leq f = c < a$ or $b < d = f \leq c = a$ depending on the sign of g_3 (see figs. 10a and b). No singular behaviour occurs when this surface is reached from the two-arc phase. If $g_2 < \sqrt{\frac{4}{3}}g_4$, the boundary is nothing but the surface (35, 37) on which we have $b < d = f = c < a$ (see fig. 10c). By using the orthogonal-polynomial method one can prove that when this surface is reached from the two-arc phase, it develops a critical behaviour of the type $\Pi m = 1$ that leads to the Painlevé II string equation. This behaviour was first pointed out in [11] for pure even potentials. More recently C. R. Nappi [19] went further by adding a linear piece. However, according to the condition (35), such a

linear interaction can not survive without a cubic one. This explains why in [19] the author had to introduce the renormalized coupling $\hat{g}_1 = g_1 N^{4/5}$ which scales along with the temperature. On this type $\Pi m = 1$ -critical surface we observe the collapse of the two arcs onto each other, trapping a double root between them. Moreover since there is only one way to realize this collapse, we do not expect any degeneracy (or decoupling) phenomenon to arise in type $\Pi m = 1$ -behaviour: no "strong" versus "weak" situations appear. This can be checked by means of orthogonal-polynomials with a period-two scaling ansatz. We must however stress that for type Π multicritical behaviour, realized when a double root is trapped between two collapsing arcs while $m - 1$ double roots collapse from each side of the double arc, it could happen that weaker conditions still lead to that behaviour. We can guess that these weaker conditions would demand the collapse of $m - 1$ double roots from one side only. Fig. 11 summarizes this situation.

4. Conclusion

We would like to end up with some concluding remarks. Our previous analysis shows that in the case of a non-even quartic interaction it is possible to map any of the situations arising through the orthogonal-polynomial method to the phase space obtained in the planar limit by using the saddle-point approximation. Special features such as doubling versus non-doubling and type II multicritical behaviours are well reproduced. The very nice intuitive idea that was suggested to explain the doubling phenomenon as simultaneous leaking of the eigenvalues out both to the left and to the right of the one-arc distribution seems however not to be always the true interpretation. It works qualitatively in the $m = 2$ -critical point when the potential is quartic with $g_4 < 0$ as one can see on figs. 2 and 3a. But certainly not in all cases: fig. 1 shows precisely a fifth degree potential leading to a doubling where the above picture fails. A related question that we might try to answer is the following: *is it possible to find a critical behaviour such that the string equations are respectively of the type m and m' ?* In terms of eigenvalue distribution this would happen on a surface where $m - 1$ double roots collapse on one side of a one-arc support while $m' - 1$ collapse on the other side. Unfortunately it is easy to be convinced that such a configuration corresponds to the weak Sup (m, m') -multicritical behaviour. An example is given in fig. 12. We tried also from an orthogonal-polynomial point of view, to construct critical potentials leading to two different m, m' -type string equations. At least in the framework of the standard scaling ansätze we present here, we did not succeed.

For potentials leading to more than two-arc distributions things will be much more involved. Actually, it is well understood that a one-arc distribution fits with a period-one scaling ansatz while a two-arc distribution demands a period-two ansatz [20]. For higher degree potentials, this correspondence is less clear especially when extra ambiguities arise, due to coexistence of n and ℓ -arc phases [7, 8]. In our case however, we can trust the period-one and two ansätze made respectively in one and two-arc phases, which allowed us to obtain the string equations on the critical

surfaces.

We come now to the following crucial point: weak $m = 2$ -critical conditions (15) within a quartic potential possess solutions for positive g_4 . Fig. 3b shows an example of such a potential.

This implies that the corresponding statistical mechanical model is well-defined and may provide a true non-perturbative definition of pure 2-D gravity. If this is the case the question that arises is what is the corresponding string equation. If one assumes smooth-function limits for both leading and subleading terms of R_n and S_n , then this equation turns out to be Painlevé I. Given the arguments of F. David [13], which show that a real solution to this equation is incompatible with the loop equations, it is *a priori* more natural to assume that the scaling ansatz (14) fails and the Painlevé I equation gets modified. It is very interesting to find out how, but this probably requires an analysis beyond the planar limit. So far we have only proved that the scaling ansatz (14), leading to the "weak" multicritical behaviours, is in agreement with the phase space structure in the planar limit. This precisely means that loop corrections to that phase space near critical surfaces could raise disagreements with the above scaling ansatz, showing that the true non-perturbative equations are not exactly the ones derived from (14). In particular, we could expect the Painlevé I equation to receive corrections.

A second point we want to make is that flows between $m = 2$ and $m = 3$ -multicritical points are in this $g_4 > 0$ case allowed. More generally, sitting on a "weak" $m = 3$ -point we can remain in the one-arc phase and flow towards a "weak" $m = 2$ -point or flow to a type II $m = 1$ -point through the two-arc phase. In ref. [11] it was pointed out that flows starting from $m = 3$ -critical points necessarily led to type II $m = 1$ ones. The reason was that the sixth degree even potential that was needed to define "strong" $m = 3$ -critical behaviour actually develops several minima along the flow trajectory that would drive the system to the "strong" $m = 2$ -critical point (also realized within a sixth degree potential bounded below). In the above construction however, this problem seems to be circumvented, essentially because "weak" $m = 3$ and $m = 2$ -critical theories appear instead of "strong" ones⁽⁷⁾. Notice that such a conclusion is not very surprising if we consider the $m = 3$ -critical behaviour as a hard dimer model on a dynamical surface at some critical value of the dimer activity [21]. Away from this critical activity we can expect this model to behave as a $m = 2$ -theory.

At this stage the above question remains open. Nevertheless the existence of a well-defined potential realizing the $m = 2$ -critical behaviour and possessing a one-arc distribution is encouraging, despite the fact that we do not know the exact equation it leads to. It deserves to be further analysed whether the above stabilization scheme is actually a simpler alternative than the ones already proposed in [14], in order to give a non-perturbative definition of pure gravity.

⁽⁷⁾ In that respect the doubling phenomenon and the degeneracy it leads to would precisely be responsible for the non-perturbative instability that forbids flows from odd m to even m -theories.

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Figure captions

FIG. 1 The more general fifth degree potential that satisfies eqs. (7, 8, 11) is ($g_4 > 0$):

$$V(\lambda) = -\frac{2g_5}{g_4}\lambda + (-3g_4)^{1/2}\lambda^2 - \frac{2g_5}{(-3g_4)^{1/2}}\lambda^3 + \frac{g_4}{4}\lambda^4 + \frac{g_5}{5}\lambda^5$$

Notice that one must demand that $\epsilon_+^{(2)}$ and $\epsilon_-^{(2)}$ (see (13)) be positive. This is realized in the above example provided $|g_5| < (\frac{3}{16})^{\frac{1}{4}}(-g_4)^{\frac{3}{4}}$. The plot corresponds to $g_4 = -\frac{1}{4}$, $g_5 = \frac{1}{10}$.

FIG. 2 Quartic even potential satisfying strong $m = 2$ -critical conditions (see (20)) with $g_2 = 2$.

FIG. 3 (a) Non-even quartic potential satisfying weak $m = 2$ -conditions (see (22, 23)); $g_2 = 2$, $g_4 = -\frac{1}{4}$.

(b) Same situation with $g_2 = \frac{3}{2}$, $g_4 = 1$.

FIG. 4 Non-even quartic potential satisfying weak $m = 3$ -conditions (see (22, 23, 24)) with $g_4 = 1$.

(a) g_2 as a function of g_3 (eq. (22)) at $g_4 = -\frac{1}{4}$ ("weak" $m = 2$ -critical surfaces with "strong" $m = 2$ -behaviour at their intersection),

(b) $g_4 = 0$,

(c) $g_4 = \frac{1}{4}$; "weak" $m = 2$ -behaviour arise above the marked points that correspond to $g_2 = \sqrt{\frac{4}{3}}g_4$ and realize the "weak" $m = 3$ -behaviour.

FIG. 6 (a) Generic configuration of the zeros of the polynomial $\Delta(z)$ corresponding to a quartic potential and leading to a one-arc eigenvalue density $\varrho(\lambda)$. Single dots stand for simple roots while others represent double roots. Notice that the double roots d and $\bar{d}$ can also be real.

(b) Representation of a two-arc distribution.

FIG. 7 Generic root configuration associated with the fifth degree potential plotted in fig. 1.

FIG. 8 Three different situations with a pair of double roots collapsing on each other. The last one can also be realized with a two-arc distribution (see fig. 10c).

FIG. 9 (a) g_2 as a function of g_3 (eq. (33)) at $g_4 = -\frac{1}{4}$ ("weak" $m = 2$ -critical surfaces with "strong" $m = 2$ -behaviour at their intersection). The one-arc distribution domain is I . In the complementary domain no compact solution for $\varrho(\lambda)$ is available and the system is not defined any more.

(b) Same picture for $g_4 = 0$.

(c) g_2 as a function of g_3 at $g_4 = \frac{1}{4}$. Surfaces α correspond to (33, 36); "weak" $m = 2$ -behaviour appears when these surfaces are reached from the one-arc phase (I') - one-arc phase consists of domains I' and I'' . Surfaces β and β' correspond to (37); β do not lead to any phase transition while β' develops a type II $m = 1$ -behaviour when reached from the two-arc phase (II). Surfaces α , β and β' become tangent on the marked points γ ("weak" $m = 3$ -critical points) where $g_2 = \sqrt{\frac{4}{3}}g_4$. Double roots d and $\bar{d}$ are respectively real and complex conjugate inside domains I' and I'' separated by β .

(d) g_2 as a function of g_4 at $g_3 = 0$ (eq. (33)).

FIG. 10 (a) Root configuration for the two-arc distribution ($g_4 > 0$) on the boundary (33) with $g_2 \geq \sqrt{\frac{4}{3}}g_4$ (branch α on fig. 9c); $g_3 > 0$.

(b) Same situation with $g_3 < 0$.

(c) Root configuration for the two-arc distribution on the boundary (37) with $g_2 < \sqrt{\frac{4}{3}}g_4$ (branch β' on fig. 9c). Type II $m = 1$ -behaviour appears when this surface is reached from the two-arc phase.

FIG. 11 (a) Root configuration in "strong" type II m th multicritical behaviour.

(b) Root configurations in "weak" type II m th multicritical behaviour.

FIG. 12 Root configurations for a particular fifth degree potential leading to "weak" $m = 3$ -critical points ($g_2 > 0$):

$$V(\lambda) = \mp \frac{\sqrt{2}}{2} g_2^{1/2} \lambda + \frac{1}{2} g_2 \lambda^2 \pm \frac{\sqrt{2}}{12} g_2^{3/2} \lambda^3 - \frac{1}{48} g_2^2 \lambda^4 \mp \frac{\sqrt{2}}{240} g_2^{5/2} \lambda^5.$$

Figures

fig. 1

fig. 2

fig. 3a

fig. 3b

fig. 4

fig. 5a

fig. 5b

fig. 5c

fig. 6b

fig. 7

fig. 6a

fig. 8

fig. 9a

fig. 9b

fig. 9c

fig. 9d

fig. 12

(a)

(a)

(b)

(b)

(c)

(b)

fig. 10

fig. 11