ON THE MEAN FIELD ANSATZ IN ANYON THEORIES

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We study the physics of the mean field ansatz in a simplified anyon model. The model is solved exactly in a finite geometry with particular attention to boundary conditions compatible with a mean statistical magnetic field. The spectrum consists of bosonic plasma-like density fluctuations with a gap. We find that the repulsive Coulomb interactions introduced by the fluctuations around the mean field force the statistical charge and magnetic field to become localized near the boundaries. Translational invariance, broken by the mean field, is restored in the infinite volume limit as the statistical charge and magnetic field escape to infinity. The polarization effects of an external magnetic field and the role of "large" gauge transformations are also studied.

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I. INTRODUCTION AND MOTIVATION

In the last few years, the concept of particles obeying fractional statistics[1], i.e. anyons[2–4], has attracted much attention in condensed matter physics and field theory. In condensed matter physics, anyons may describe the quasiparticle excitations in the Fractional Quantum Hall Effect (FQHE). Halperin[6] observed that the quasiparticles described by Laughlin’s[7] wave function for the (FQHE) obey fractional statistics. Wilczek[2] and Wilczek and Zee[3] proposed that fractional statistics may be naturally incorporated in a theory by attaching “statistical” solenoids permanently to the particles. The phase under exchange in the many-particle wave-function is thus a consequence of the Aharonov-Bohm effect of the particles in the presence of the gauge potential produced by the statistical solenoids of the other particles. At the level of a field theory, attaching permanently flux tubes to charged particles is achieved by introducing a Chern-Simons (CS)[5] term in the Lagrangian for the statistical gauge fields. This (CS) term gives rise to a modified Gauss’ law that attaches magnetic fields (rather than electric field) to the charge density. Effective (CS) field theories were proposed to describe the (FQHE) by Girvin[8], Girvin and MacDonald[9], Zhang, Hansson and Kivelson[10] and Read[11].

In a different development, Laughlin[13] and Fetter et. al.[12] developed a mean field theory for the anyon gas and showed that fluctuations treated in the RPA approximation produce a massless pole in the current-current correlation function, and consequently a Meissner effect. Chen et. al.[14] argued that this pole is exact. The thermal and magnetic properties of anyon superconductors have been recently examined within the context of this mean field approach[15]. The approach followed by the above authors, referred to in this note as the mean field ansatz, consists in the following: the statistical charge density is split into a homogeneous component (zero momentum mode in a Fourier expansion) and a component whose Fourier expansion

contains all the non-zero momentum components and whose integral over the spatial volume vanishes.

Gauss' law associates a constant magnetic field with the homogeneous component of the charge density, this constant magnetic field is treated exactly and the components with non-zero momentum components are treated in the Random Phase Approximation (RPA). Thus the distinctive feature in this approach is that the mean field ansatz corresponds to assuming that the charge density is homogeneously distributed in the system thus smearing the statistical fluxes of individual particles homogeneously. The net effect is that the particles now move in an homogeneous mean statistical magnetic field and the mean field states are Landau levels.

The constant magnetic field breaks the original translational invariance of the system down to the magnetic translation group whose representations are projective. This results from the fact that a constant magnetic field introduces a term linear in the coordinates in the gauge fields. It has been argued by Chen et. al.[14] that the massless pole obtained in the RPA approximation is the degree of freedom that is necessary to restore the original translational invariance of the theory.

Writing the charge density as a homogeneous component plus fluctuations, it turns out that the statistical interactions give rise to a two-body *repulsive Coulomb interaction*. In the fermionic case of references([12, 14]) this two-body Coulomb interaction is a result of replacing one of the charge densities in the three body interactions by its mean field (homogeneous) value.

The purpose of this note is to study the stability of the mean field ansatz when the Coulomb interaction is introduced. Furthermore, we would like to investigate the mechanism by which translational invariance may be restored by fluctuations.

Our motivation to study these issues stem from the observation that the situation of translational invariance broken by a homogeneous charge density would also

be relevant if one were to quantize Quantum Electrodynamics in a sector with a *finite charge density*. The reason for this is as follows, solving Gauss' law in QED for a constant charge density would yield an electric field that is linear in the coordinates thus breaking explicitly the translational invariance of the system. However, there is an instantaneous repulsive Coulomb interaction that unstabilizes this state with homogeneous charge density. At finite density, the Coulomb interaction will be screened and in a finite volume we would find that the charge is localized within a screening length near the boundaries. In the infinite volume limit the charge escapes to infinity thus restoring the translational invariance of the system. This is the situation in the massless Schwinger model, QED in $1 + 1$ dimensions[16]. Our experience with this model, motivates us to try to study the mean field ansatz for anyone looking for a similar phenomenon.

In the usual studies of the electron gas at high density in the RPA approximation, this situation is avoided by incorporating a neutralizing background. This effectively subtracts the homogeneous component of the charge density thus avoiding the infrared divergences associated with the long range Coulomb interaction.

The model that we study in this note is that of a scalar field coupled to a gauge field whose dynamics is determined by a Chern-Simons term. This model is simple enough that may be solved exactly, however, it has the important features of more realistic models, in particular it allows to study the mean field ansatz and the role of the Coulomb interaction.

We quantize this model in a finite volume (a square of length L) with particular attention to the subtle issue of boundary conditions. The reason for studying this theory in finite volume is that we will not introduce any neutralizing background but the finite volume allows us to handle the infrared behavior. By isolating the homogeneous mode of the charge density, we analyze the role played by the mean

statistical magnetic field. We find that only a certain class of boundary conditions are compatible with a constant magnetic field but some (or all) of the translational symmetries available for the particular geometry and boundary conditions are broken.

Fluctuations around the mean field are responsible for redistributing the charge, which becomes localized near the boundaries, and in the infinite volume limit the charge density becomes vanishingly small in the bulk. This is the mechanism by which the translational symmetries become manifest in this limit.

In the next section we introduce the model and proceed to solve it exactly for different boundary conditions, we show explicitly the phenomenon of charge redistribution. We also discuss in detail the phenomenon of charge polarization by an external magnetic field and study the role of “large” gauge transformations.

II. THE MODEL

As mentioned in the introduction, one of the motivations of this note is to understand the stability of the mean field approximation. In particular, we want to understand how the state with a constant charge density survives the Coulomb repulsion arising from the interaction. Certainly trying to understand this question within the context of the usual fermionic theories will be very complicated. We propose to study a simpler model which bears all the intricacies of the more familiar models but exposes the main physical mechanisms in a clear manner. The model is described by a scalar field interacting with a Chern-Simons statistical gauge field. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi + eA_\mu)^2 + \frac{\theta}{2}\epsilon^{\mu\nu\lambda}\partial_\mu A_\nu A_\lambda \quad (2.1)$$

The Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\pi_\phi^2 + \frac{1}{2}(\vec{\nabla}\phi - e\vec{A})^2 + A_0\mathcal{G}(\vec{x}) \quad (2.2)$$

$$\pi_\phi = (\dot{\phi} + eA_0) \quad (2.3)$$

$$\mathcal{G}(\vec{x}) = \theta B(\vec{x}) - e\pi_\phi(\vec{x}) \quad (2.4)$$

$$B(\vec{x}) = (\partial_x A_y - \partial_y A_x) \quad (2.5)$$

The canonical momentum π_ϕ is identified with the charge density because it generates the gauge transformations on ϕ . The virtue of this model is that it is simple enough to permit an exact solution, but at the same time maintains some of the features (long range interactions etc.) of the more familiar models. The solution to the *classical* equations of motion corresponds to a free scalar field with mass $M = e^2/\theta$. However, quantization of the system, requires a deeper understanding of gauge invariance and boundary conditions. Furthermore, the solution to the classical equations of motion do not yield to an immediate understanding of the role played by the Coulomb interaction. At the quantum level we are only interested in physical states, these are obtained by carefully analyzing the constraints of the theory.

From the above Lagrangian and Hamiltonian (2.1,2.2), one finds two first class constraints

$$\pi_0 = 0 \quad (2.6)$$

$$\mathcal{G}(\vec{x}) = 0 \quad (2.7)$$

with $\mathcal{G}(x)$ given by (2.4). The usual way of handling these constraints is to fix a gauge by introducing two gauge conditions, converting the above first class system to a second class system and using the Dirac brackets. Quantization is then achieved by replacing these Dirac brackets by (i) times equal time commutators. This procedure, though correct and perfectly valid, is in a sense roundabout and does not permit an immediate identification of the gauge invariant operators. We prefer to quantize the theory canonically, imposing the above first class constraints directly onto the states,

thus selecting the gauge invariant states out of the full Hilbert space[17]. We use this quantization approach because it illustrates at once the gauge invariant degrees of freedom and prevents the appearance of spurious gauge artifacts that may otherwise pervade a particular calculation in a fixed gauge.

Furthermore, this approach will allow us to expose clearly the role of “large” and “small” gauge transformations and their interplay with the boundary conditions. Fixing a particular gauge, obscures the physics of boundary conditions and gauge transformations, and our quantization approach circumvents these difficulties. This point will be elaborated upon in the next sections.

The quantization is achieved by imposing the following equal time commutators

$$[\pi_\phi(\vec{x}), \phi(\vec{y})] = -i\delta^2(\vec{x} - \vec{y}) \quad (2.8)$$

$$[A_i(\vec{x}), A_j(\vec{y})] = \frac{i}{\theta} \epsilon_{ij} \delta^2(\vec{x} - \vec{y}) \quad (2.9)$$

With these commutation relations, the Euler-Lagrange equations are obtained as Heisenberg equations from the Hamiltonian (2.2). Gauss’ law is either obtained as a secondary constraint (stability under time evolution of the primary constraints) from the primary constraint $\pi_0 \approx 0$ and canonical commutation relations between π_0 and A_0 or directly as a constraint (not an equation of motion) from the Euler-Lagrange equations. The primary constraint $\pi_0 = 0$ is straightforward to handle: the states (and wave-functionals associated with them) do not depend on A_0 . The reader unfamiliar with this terminology should not become confused, just think of π_0 as the functional derivative with respect to A_0 (an identification resulting from the canonical commutator), then the constraint π_0 on to the quantum states results in the above assertion. Before turning our attention to the second constraint, i.e. Gauss’ law (2.7), we analyze the role of the boundary conditions. We point out that a thorough treatment of the boundary conditions is necessary for a consistent study

of the physics associated with the homogeneous mean field. We quantize the theory in a square “box” (Ω) such that $-L/2 \leq x, y \leq L/2$. Introducing different lengths along the two directions does not modify essentially the results.

We separate the homogeneous (zero mode) component of the charge density π_ϕ and magnetic field $B(\vec{x})$ as

$$\pi_\phi(\vec{x}) = \frac{Q}{L^2} + \tilde{\pi}(\vec{x}) ; \quad \int_{\Omega} d^2x \tilde{\pi}(\vec{x}) = 0 \quad (2.10)$$

$$B(\vec{x}) = B_0 + \tilde{B}(\vec{x}) ; \quad \int_{\Omega} d^2x \tilde{B}(\vec{x}) = 0 ; \quad B_0 = \frac{eQ}{\theta L^2} \quad (2.11)$$

where $\tilde{\pi}_\phi$ and $\tilde{B}$ are periodic without a zero momentum Fourier component, Q is the total charge and we used Gauss’ law equation (2.4) in the second line.

The natural boundary conditions appropriate for this geometry would be toroidal, that is, periodic along the two directions, however, in a gauge theory these toroidal boundary conditions must be fulfilled up to gauge transformations:

$$\phi(x + L, y) = \phi(x, y) + \Lambda_1(x, y) \quad (2.12)$$

$$\phi(x, y + L) = \phi(x, y) + \Lambda_2(x, y) \quad (2.13)$$

$$A_i(x + L, y) = A_i(x, y) + \partial_i \Lambda_1(x, y) \quad (2.14)$$

$$A_i(x, y + L) = A_i(x, y) + \partial_i \Lambda_2(x, y) \quad (2.15)$$

We now prove that a constant magnetic field is not compatible with these toroidal boundary conditions: a constant magnetic field implies a linear dependence on the coordinates for the gauge fields, and by the above boundary conditions these in turn require a quadratic dependence on the coordinates for the matter field. The most general expansion thus becomes

$$A_i(x, y) = \frac{\gamma_i}{L} + \alpha_i x + \beta_i y + \tilde{A}_i(x, y) \quad (2.16)$$

$$\phi(x, y) = \phi_0 + \phi_1 x^2 + \phi_2 y^2 + \phi_3 xy + \tilde{\phi}(x, y) \quad (2.17)$$

with $B_0 = \alpha_y - \beta_x$ and $\tilde{A}_i(\vec{x})$, $\tilde{\phi}(\vec{x})$ are periodic with a discrete Fourier expansion

$$\tilde{A}_i(\vec{x}) = \frac{1}{L} \sum_{\vec{k} \neq 0} e^{-i\vec{k} \cdot \vec{x}} a_i(\vec{k}) \quad (2.18)$$

$$\tilde{\phi}(\vec{x}) = \frac{1}{L} \sum_{\vec{k} \neq 0} e^{-i\vec{k} \cdot \vec{x}} \varphi(\vec{k}) \quad (2.19)$$

$$\vec{k} = \frac{2\pi}{L}(n, m) \quad (2.20)$$

The commutation relations become

$$[\tilde{A}_i(\vec{x}), \tilde{A}_j(\vec{y})] = \frac{i}{\theta} \epsilon_{ij} (\delta_p(\vec{x} - \vec{y}) - \frac{1}{L^2}) ; \quad [\gamma_i, \gamma_j] = \frac{i}{\theta} \epsilon_{ij}$$

$$[\tilde{\pi}_\phi(\vec{x}), \tilde{\phi}(\vec{y})] = -i (\delta_p(\vec{x} - \vec{y}) - \frac{1}{L^2}) ; \quad [Q, \phi_0] = -i$$

where $\delta_p(\vec{x} - \vec{y})$ is the periodic delta function with period L in both directions, and all other commutators being zero.

The coefficients of the quadratic terms for ϕ and linear terms for A_i are related by the boundary conditions (2.12-2.15). It is straightforward to see that these boundary conditions require that

$$\alpha_y = \beta_x$$

or equivalently that $B_0 = 0$, that is *a constant magnetic field is not compatible with toroidal boundary conditions*. This result is general and easy to understand, if there is a constant magnetic field, the gauge field will necessarily have a term that is linear in the coordinates, although the form of this term depends on the gauge, its presence is gauge independent. This linear dependence on the coordinates is not compatible with toroidal boundary conditions. It is, however, straightforward to show that the following boundary conditions admit a constant magnetic field (mean field): i) free boundary conditions along the two directions (x,y); ii) free boundary conditions along the y-direction and the subset (2.12,2.14); iii) free boundary conditions along

the x-direction and the subset (2.13,2.15), these boundary conditions will be referred to as (FXY), (FY), (FX) respectively. There may certainly be other types of boundary conditions compatible with a constant magnetic field but we will illustrate the resulting physical effects by concentrating on these.

We now turn our attention to Gauss' law constraint (2.7). For this purpose it proves convenient to split the gauge field into a transverse and a longitudinal component as

$$A_i(\vec{x}) = \partial_i \eta(\vec{x}) + \epsilon_{ij} \partial_j \chi(\vec{x}) \quad (2.21)$$

$$B(\vec{x}) = -\nabla^2 \chi(\vec{x}) \quad ; \quad \partial_i A_i = \nabla^2 \eta(\vec{x}) \quad (2.22)$$

With the expansions for A_i and B given above, it is a simple exercise to show that

$$[B(\vec{x}), \eta(\vec{y})] = \frac{i}{\theta} \delta_P^2(\vec{x} - \vec{y}) \quad (2.23)$$

Physical states are annihilated by Gauss' law constraint

$$\mathcal{G}(\vec{x}) |Phys\rangle = 0 \quad (2.24)$$

The operators that create physical excitations out of the physical vacuum must commute with Gauss' law constraint

$$[\mathcal{G}(\vec{x}), \mathcal{O}(\vec{y})] = 0$$

With (2.23) it is straightforward to see that

$$\Phi(\vec{x}) = \phi(\vec{x}) - e\eta(\vec{x}) \quad (2.25)$$

$$\Pi(\vec{x}) = \frac{1}{2e} (e\pi_\phi(\vec{x}) + \theta B(\vec{x})) \quad (2.26)$$

are physical operators and that they form a canonical conjugate pair. We see that the combination $\phi - e\eta$ is gauge invariant and the condition that Gauss' law annihilates

the physical states implies that these states (and wave functionals) do not depend on the gauge-variant combination $\phi + e\eta$.

Using the relation

$$\pi_\phi = \Pi - \mathcal{G}/2e \quad (2.27)$$

the Hamiltonian density (2.2) becomes in the physical subspace

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\Phi - e\vec{A}^T)^2 \quad (2.28)$$

with $\vec{A}^T$ the transverse part of the gauge connection given by (2.21,2.22). In the physical subspace we use Gauss' law to write χ in terms of Π , however, to do this we need to invert the relation (2.22) and for this we need the two-dimensional Coulomb Green's function.

The two dimensional Coulomb Green's function has an infrared ambiguity associated with the zero momentum component in its Fourier expansion. This ambiguity is resolved in a finite volume by carefully accounting for the boundary conditions.

For the present geometry the Green's function can be represented as

$$G(\vec{x}, \vec{x}') = g(\vec{x}, \vec{x}') + \tilde{G}(\vec{x} - \vec{x}') \quad (2.29)$$

$$\tilde{G}(\vec{x} - \vec{x}') = -\frac{1}{L^2} \sum_{\vec{k} \neq 0} \frac{e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')}}{\vec{k}^2} \quad (2.30)$$

$$g(\vec{x}, \vec{x}') = \frac{a(x - x')^2 + b(y - y')^2}{2L^2} \quad (2.31)$$

with $a + b = 1$, this Green's function satisfies

$$\nabla^2 G(\vec{x}, \vec{x}') = \delta_p^2(\vec{x} - \vec{x}'); \quad \nabla^2 \tilde{G}(\vec{x} - \vec{x}') = \delta_p^2(\vec{x} - \vec{x}') - \frac{1}{L^2}$$

The constants in $g(\vec{x}, \vec{x}')$ are determined by the boundary conditions, and in turn, these constants determine the linear terms in $\vec{A}_T$. Thus it becomes clear that boundary conditions determine the linear dependence on the coordinates of the gauge connection through the infrared behavior of the two dimensional Green's function.

In particular, for the three types of boundary conditions referred to above: (FXY), (FY), (FX) we find:

$$(FXY): A_i^T(\vec{x}) = -\frac{eQ}{2\theta L^2} \epsilon_{ij} x_j + \mathcal{A}_i(\vec{x}) \quad (2.32)$$

$$(FY): A_i^T(\vec{x}) = \frac{eQ}{\theta L^2} x \delta_{i,2} + \mathcal{A}_i(\vec{x}) \quad (2.33)$$

$$(FX): A_i^T(\vec{x}) = -\frac{eQ}{\theta L^2} y \delta_{i,1} + \mathcal{A}_i(\vec{x}) \quad (2.34)$$

$$\mathcal{A}_i(\vec{x}) = -\frac{e}{\theta} \epsilon_{ij} \partial_j \int_{\Omega} d^2 x' \tilde{G}(\vec{x} - \vec{x}') \tilde{\Pi}(\vec{x}') \quad (2.35)$$

where $\tilde{\Pi}$ is periodic with a Fourier expansion similar to those in equations (2.18,2.19), without the zero mode. We write

$$\Phi(\vec{x}) = \Phi_0 + \tilde{\Phi}(\vec{x}); \quad \Pi(\vec{x}) = \frac{Q}{L^2} + \tilde{\Pi}(\vec{x})$$

where we separated the zero modes and imposed periodic boundary conditions on these gauge invariant fields.

Let us now consider in detail the case of (FXY) boundary conditions. The Hamiltonian becomes (up to constant terms)

$$H = \frac{1}{2} \int_{\Omega} d^2 x \left\{ \tilde{\Pi}^2 + (\vec{\nabla} \tilde{\Phi})^2 + e^2 \tilde{\mathcal{A}}^2 + \frac{Qe^4}{\theta^2 L^2} (\vec{x} \cdot \vec{\nabla}) \int d^2 x' \tilde{G}(\vec{x} - \vec{x}') \tilde{\Pi}(\vec{x}') \right\} \quad (2.36)$$

with $\tilde{\mathcal{A}}$ determined by (2.35). The last term in the above Hamiltonian explicitly exhibits the breakdown of translational invariance as a consequence of the homogeneous component of the magnetic field. The different boundary conditions change this last term appropriately. This Hamiltonian becomes much simpler in terms of Fourier transforms

$$H = \frac{1}{2} \sum_{\vec{k} \neq 0} \left\{ \tilde{\Pi}(\vec{k}) \tilde{\Pi}(-\vec{k}) \left[1 + \frac{e^4}{k^2 \theta^2} \right] + k^2 \tilde{\Phi}(\vec{k}) \tilde{\Phi}(-\vec{k}) + 2 \tilde{\Pi}(\vec{k}) h(-\vec{k}) \right\} \quad (2.37)$$

with

$$h(-\vec{k}) = h(n, m) = -\frac{e^4 L Q}{8\pi\theta^2(n^2 + m^2)} \{ \delta_{m,0} [(-1)^n - \delta_{n,0}] + \delta_{n,0} [(-1)^m - \delta_{m,0}] \} \quad (2.38)$$

The Coulomb term is evident in the first term in (2.37), and arises from the $\vec{A}^2$ term in (2.36). The Hamiltonian may now be diagonalized by the Bogoliubov (canonical) transformation

$$\mathcal{P}(\vec{k}) = \left[1 + \frac{e^4}{k^2\theta^2} \right]^{\frac{1}{2}} \tilde{\Pi}(\vec{k}) + \frac{h(\vec{k})}{\left[1 + \frac{e^4}{k^2\theta^2} \right]^{\frac{1}{2}}} \quad (2.39)$$

$$\mathcal{Q}(\vec{k}) = \frac{\tilde{\Phi}(\vec{k})}{\left[1 + \frac{e^4}{k^2\theta^2} \right]^{\frac{1}{2}}} \quad (2.40)$$

and becomes (up to constant terms)

$$H = \frac{1}{2} \sum_{\vec{k} \neq 0} \left\{ \mathcal{P}(\vec{k}) \mathcal{P}(-\vec{k}) + \left[k^2 + \frac{e^4}{\theta^2} \right] \mathcal{Q}(\vec{k}) \mathcal{Q}(-\vec{k}) \right\} \quad (2.41)$$

The expectation value of the charge density in an *arbitrary* energy eigenstate is

$$\langle \pi_\phi(\vec{x}) \rangle = \langle \Pi(\vec{x}) \rangle$$

where we used (2.27, 2.24). This expectation value may now be computed easily by retracing the transformations, we find

$$\langle \Pi(\vec{x}) \rangle = \frac{Q}{L^2} - \frac{1}{L} \sum_{\vec{k} \neq 0} \frac{h(\vec{k})}{\left[1 + \frac{e^4}{k^2\theta^2} \right]} e^{-i\vec{k} \cdot \vec{x}} \quad (2.42)$$

carrying out the sums explicitly the final result becomes ($-L/2 \leq x, y \leq L/2$)

$$\langle \Pi(\vec{x}) \rangle = \frac{Q e^2}{4L\theta} \frac{\cosh(e^2 x/\theta) + \cosh(e^2 y/\theta)}{\sinh(e^2 L/2\theta)} \quad (2.43)$$

This is one of our main results. The charge density is *redistributed* in the system by becoming *localized near the boundaries*, in the limit $L \rightarrow \infty$ the charge escapes to infinity with the boundaries and locally the charge density goes to zero pointwise.

The physics behind this phenomenon is very clear, the repulsive Coulomb interaction forces the charge to become localized near the boundaries, unstabilizing the mean field (homogeneous) component of the charge. This is the situation in a metal in which the charge is not neutralized by an external background. The excitations of the theory are massive, the massless (gauge-variant) scalar field ϕ gave a screening (Debye) mass to the gauge field $M = e^2/\theta$, the only physical excitations are plasma-like density fluctuations with a gap. The charge is found within a screening length from the boundaries and in the large L limit no local observation will detect the presence of charges in the bulk because these are completely screened.

This is also the situation in the Schwinger model in one space dimension[16].

Translational invariance, broken by the mean field density and constant magnetic field, is restored in the $L \rightarrow \infty$ limit because the charge density, localized within a screening length from the boundaries escapes to infinity and goes exponentially fast to zero in the bulk.

For (FY) boundary conditions, A_i^T is given by equation (2.32), by following the steps indicated in the previous case, we see that the only change in the Hamiltonian (2.35) is in the last term that is now obtained by the replacement $(\vec{x} \cdot \vec{\nabla}) \rightarrow 2x\partial_x$. Repeating the steps involved for the case of (FXY) conditions, we now find

$$h(n, m) = -\frac{e^4 L Q}{4\pi\theta^2 n^2} \delta_{m,0} [(-1)^n - \delta_{n,0}] \quad (2.44)$$

the expectation value of the charge density in an arbitrary energy eigenstate is now given by

$$\langle \Pi(\vec{x}) \rangle = \frac{Q e^2 \cosh(\frac{e^2 x}{\theta})}{2L\theta \sinh(\frac{e^2 L}{2\theta})} \quad (2.45)$$

We now see that the charge is localized near the boundaries *in the x direction*, but there is translational invariance along the y direction. This is a consequence of the boundary conditions. Repeating the calculation for the case (FY) one finds a similar

situation but now the charge is localized near the boundaries *in the y direction*. In any case in the limit $L \rightarrow \infty$ translational invariance is restored because the charge density goes to zero in the bulk, escaping to infinity with the boundaries.

It is instructive to consider the case in which there is an *external* constant magnetic field b . The Hamiltonian is now obtained by replacing $e\vec{A} \rightarrow e\vec{A} + \vec{a}_{ext}$ in (2.2). With (FXY) boundary conditions $a_{ext,i} = -b\epsilon_{ij}x_j/2$ and (2.32) now becomes $A_i^T \rightarrow A_i^T + a_{ext,i}$. Alternatively this external magnetic field may be viewed as an external background charge because it effectively replaces $Q \rightarrow Q_{eff} = Q + Q_b$, with $Q_b = b\theta L^2/e$ in (2.32). In the last term of the Hamiltonian (2.36) and in (2.38) Q becomes replaced by Q_{eff} . The expectation value of the charge density in an arbitrary energy eigenstate now becomes

$$\langle \Pi(\vec{x}) \rangle = \frac{-Q_b}{L^2} + \frac{e^2 Q_{eff}}{4L\theta} \frac{\cosh(e^2 x/\theta) + \cosh(e^2 y/\theta)}{\sinh(e^2 L/2\theta)} \quad (2.46)$$

We then see that the homogeneous contribution to the charge density, arises solely from the “external” charge associated with the external magnetic field. In particular when $Q_b = -Q$ the system becomes charge neutral and the second term in the charge density above vanishes. The only contribution to the expectation value of the charge density is the homogeneous term. The physics is clear, through Gauss’ law the external magnetic field polarizes charge, this charge is coming from the boundary (or going to the boundary depending on the sign of the external magnetic field) and when the external magnetic field corresponds exactly to the value such that $Q_b = -Q$ the total charge localized near the boundary in the absence of the external magnetic field becomes distributed homogeneously throughout the system. This value of the external magnetic field acts as a neutralizing background charge, the zero momentum component of the charge density is effectively cancelled in the Coulomb interaction, the infrared singularities are no longer present and the system becomes translational

invariant. The case of the other boundary conditions may be analyzed in a similar manner, and the same mechanism takes place. Within the setting of the (FQHE) this particular value of the external magnetic field at which the system becomes translational invariant was found to minimize the effective free energy[10] and used in reference ([19]) within the context of anyon superfluidity.

III. LARGE AND SMALL GAUGE TRANSFORMATIONS

In the previous section we have understood how boundary conditions influence the physics of the distribution of the charge density. However there is a puzzling aspect in this analysis, one realizes that the boundary conditions (2.32,2.33,2.34) differ by a *gauge transformation* with a gauge function of the form

$$\Lambda(x, y) \propto \frac{eQ}{\theta L^2} xy \quad (3.1)$$

Our gauge invariant quantization, i. e. imposing the first class constraints onto the states seemed to guarantee that our states and hence the physics will be insensitive to gauge transformations. However it seems that by performing gauge transformations we are changing the physics, that is the charge distribution. In order to resolve this puzzle, we must understand the issue of gauge transformations. The operators that implement the gauge transformations for the gauge and matter fields are

$$\mathcal{U}_\Lambda = \exp \left[i \int d^2 y \{ \theta \epsilon_{kj} \partial_k \Lambda(\vec{y}) A_j(\vec{y}) + e \pi_\phi(\vec{y}) \Lambda(\vec{y}) \} \right] \quad (3.2)$$

By integrating by parts these operators may be written as

$$\mathcal{U}_\Lambda = \exp \left[i \theta \int d^2 y \epsilon_{kj} \partial_k (\Lambda(\vec{y}) A_j(\vec{y})) \right] \exp \left[i \int d^2 y \mathcal{G}(\vec{y}) \Lambda(\vec{y}) \right] \quad (3.3)$$

In the above expression, the second exponential (involving Gauss' law operator) leaves the physical states invariant because Gauss' law annihilates the physical states. The first exponential is a total surface term. The gauge functions ($\Lambda(\vec{y})$) for which this

surface term vanishes are “small” gauge transformations, and the physical states are invariant under these “small” gauge transformations. In particular these “small” transformations maintain the boundary conditions imposed on the fields. There are, however, gauge functions for which this surface term does not vanish, these functions induce “large” gauge transformations. Hosotani[18] has studied a subset of this “large” gauge transformations corresponding to gauge functions linear in the coordinates. The physical states are not invariant under these “large” gauge transformations, in particular it now becomes evident that the gauge functions that change the boundary conditions in (2.32-2.34) $\Lambda \propto eQxy/\theta L^2$, *do not allow this surface term to vanish* as may be seen from the expansion of $A_j(\vec{y})$.

Furthermore, for these gauge functions that change the boundary conditions we find that the commutator of the surface integral in the first exponential in (3.3) and Gauss’ law operator is given by

$$[\int d^2y \{ \epsilon_{kj} \partial_k^y (\Lambda(\vec{y}) A_j(\vec{y})) \}, \mathcal{G}(\vec{z})] \propto \frac{eQ}{\theta L} [\delta_p^1(\frac{L}{2} - z_x) - \delta_p^2(\frac{L}{2} - z_y)] \quad (3.4)$$

where the superscripts in the periodic delta functions refer to the directions.

We then see that the operators that implement these “large” gauge transformations are *not physical* since they do not commute with Gauss’ law. From the above expression we see that the obstruction to the commutativity is a boundary effect in complete agreement with the fact that these transformations change the boundary conditions. In the limit when the boundaries go to infinity, the charge density escapes with the boundaries and these transformations become acceptable everywhere. This is the resolution to the puzzle presented in the beginning of this section. The “large” gauge transformations change the boundary conditions, they also change the Hamiltonian and the distribution of the charge density. However, these transformations are *not physical*, physical states do change under these transformations and opera-

tors implementing these gauge transformations are not allowed to act on the physical Hilbert space because they do not commute with the constraints. These transformations will redistribute the charge near the boundaries but as the boundaries are taken to infinity in the infinity volume limit, these transformations will have no effect in the local physics in the bulk.

IV. SUMMARY

The model that we studied in this note, a scalar field coupled to a Chern-Simons statistical gauge field allowed us to understand the physics of the mean field ansatz. We studied this model in a finite geometry and found that the presence of an homogeneous component of the charge density and magnetic field requires a careful examination of boundary conditions that are compatible with this mean field ansatz. The homogeneous component of the magnetic field leads to a breakdown of the translational symmetries as is the case in the more familiar anyon theories. We found that fluctuations around this mean field induce a repulsive Coulomb interaction again as in the usual anyon theories. This long range interaction forces the charge to become localized near the boundaries of the system. When the boundaries are taken to infinity, translational invariance in the bulk is restored because the charge density is now localized at infinity. We also showed how in the finite geometry boundary conditions are very important and that the charge density distribution depends on the particular boundary conditions imposed, but in all situations the charge becomes localized near the boundaries. The spectrum of the theory consists of bosonic plasmon-like modes with a gap. This gap determines a Debye screening length which is the localization length of the charge density near the boundaries. An external magnetic field polarizes statistical charge to or from the boundary (depending on the sign), for certain value of the external magnetic field the system becomes translational invariant and

the statistical charge homogeneously distributed in the system.

We clarified the role of “large and small” gauge transformations, and showed that “large” gauge transformations that change the boundary conditions have the physical effect of redistributing the charge density. However these transformations are not allowed to act on physical states since they do not commute with Gauss’ law in a finite system.

Although the model that we studied is a simplified version of more familiar models, we believe that many of the features revealed by our study are robust and will be present in the more realistic situations. In particular the redistribution of the charge density we believe to be a general feature. However, this model does not yield to a better understanding of the Goldstone pole found in the fermionic models. Chen et. al.[14] suggested that boundary effects are important in order to understand the mechanism of restoration of translational invariance by this Goldstone mode. We believe that these boundary effects are precisely the ones described by our simple model. It would be very illuminating to try and understand if the mechanism of statistical charge redistribution takes place in the more familiar fermionic theories of anyon superconductivity, and the relationship between this mechanism and the Goldstone pole.

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