Conformally Covariant Operators on Riemann Surfaces (with applications to conformal and integrable models)

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Abstract

Following the standard procedure for gauging in Yang-Mills and gravitational theories, we introduce projective connections to covariantize differential operators on Riemann surfaces. We present applications in integrable models (Lax pairs, Poisson operators) and conformal models (conformal Ward identity, diffeomorphism anomaly, Krichever-Novikov algebra, Virasoro algebra and its representations, Kac-Moody-algebras, W-algebras, WZW-model, twistor theory). The generalization to higher dimensions is indicated and the whole discussion is generalized to the supersymmetric case (both in superspace and in component field formalism).

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Chapter 1

Introduction

In this paper, we discuss the covariantization of the linear differential operators $\partial^n \equiv (\partial/\partial z)^n$ (with $n = 0, 1, 2, \dots$) on compact Riemann surfaces (as well as the supersymmetric extension of this subject). Here and in the following, 'covariant' means 'conformally covariant', i.e. conformal fields are mapped into conformal fields. Apart from section 2.9 and some applications, we will concentrate on the covariant version of ∂^n which is *holomorphic*, i.e. is a polynomial in ∂ with holomorphic coefficients. Thus, the whole discussion boils down to a one-dimensional problem and one could equally well consider real manifolds ($z \in \mathbf{R}$) and require covariance with respect to diffeomorphisms. As we will see, the problem we just formulated - as well as the more general question of classifying all holomorphic and conformally covariant differential operators - amounts to the study of operators which are covariant with respect to Möbius (projective) transformations.

One of the simplest and most important examples is the covariantization of ∂^3 . It has the form $\partial^3 + 2R\partial + (\partial R)$ where $R(z)$ is the component of a projective connection. An operator of this type occurs in the Poisson brackets for the Korteweg-de Vries equation and for the classical Virasoro algebra; it also appears in various contexts of conformal field theory, although this is not always realized. In complex analysis, the covariant version of the operator ∂^n (which is referred to as '*Bol operator*' [1]) has been studied in recent years and was found to play an important rôle in a wide range of mathematical applications, e.g. to invariants of differential equations, to the theory of Hankel operators [1] and to twistor theory [2].

Although we focus our attention on holomorphic operators, it should be noted that the operators that we construct preserve their covariance properties, if their coefficients are smooth functions of $(z, \bar{z})$ rather than holomorphic ones. This generalization is commented upon in section 2.9. Since we do not allow for operators which are polynomials in both ∂ and $\bar{\partial}$, we can - and will - always assume that our operators act on smooth, rather than holomorphic fields (which is essential for the applications in local conformal field theory).

The supersymmetric extension of this subject consists of the covariantization of $D^{2n+1} \equiv (\partial/\partial\theta + \theta\partial/\partial z)^{2n+1}$ (with $n = 0, 1, 2, \dots$) on compact super Riemann surfaces. To our knowledge, this problem and none of the mathematical applications we just mentioned have been addressed in the literature so far. Again, we will concentrate on the holomorphic case apart from section 3.8 and some applications.

As for the organization of the text, we will successively consider the bosonic and the

$N = 1$ supersymmetric theories. First, we briefly recall the basic notions which are needed in the sequel, i.e. the concepts of Schwarzian derivative and connection. Then, we illustrate the covariantization procedure by the example of ∂^3 . Some mathematical properties and various physical applications are presented thereafter. In chapter 3, we define the necessary concepts for the supersymmetric theory and discuss the covariantization of D^{2n+1} . The corresponding results are formulated in terms of superfields on super Riemann surfaces and their projection to component field formalism is described. In conclusion, a number of open problems is mentioned.

While working out these notes, we gradually became aware of various recent articles in both the Physics and Mathematics literature in which the Bol operators have been studied from other viewpoints [1]-[8]. In particular, we would like to mention the work of Gustafsson and Peetre [1] and of Bauer, di Francesco, Itzykson and Zuber [8] whose results are summarized in sections 2.8 and 2.14.1 (together with some new results). We have attempted a clear and simple presentation of the bosonic theory so as to have a solid basis for the supersymmetric extension developed thereafter.

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Chapter 2

Bosonic theory

The arena we will work in is a compact Riemann surface Σ of genus $g \geq 0$. We recall [9] [10] that a Riemann surface is a connected topological manifold of real dimension 2 which is equipped with a *complex structure*: this means that any two local systems of coordinates, say $(z, \bar{z})$ and $(z', \bar{z}')$, are related by a conformal (=biholomorphic) transformation, $z' = z'(z), \bar{z}' = \bar{z}'(\bar{z})$. The natural tangent space basis will be denoted by

$$\partial \equiv \partial_z = \frac{\partial}{\partial z}, \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}.$$

Any compact Riemann surface Σ is diffeomorphic to a 2-sphere with a certain number of handles, the so-called genus of Σ .

Besides the complex structure, we will need a further concept on Σ , namely a projective structure or, equivalently, a projective connection. These notions will be introduced in the following sections after a short reminder of the Schwarzian derivative.

2.1 Schwarzian derivative

Consider a smooth (i.e. C^∞) complex-valued function $Z(z, \bar{z})$ of the local coordinates on the Riemann surface. For all $(z, \bar{z})$ with $(\partial Z)(z, \bar{z}) \neq 0$, the *Schwarzian derivative* of Z is defined by [11]

$$\begin{aligned} S(Z; z, \bar{z}) &\equiv \frac{\partial^3 Z}{\partial \bar{z}} - \frac{3}{2} \left(\frac{\partial^2 Z}{\partial \bar{z}} \right)^2 \\ &= \partial^2 \ln \partial Z - \frac{1}{2} (\partial \ln \partial Z)^2 \\ &= -2 \sqrt{\partial \bar{z}} \partial^2 \frac{1}{\sqrt{\partial \bar{z}}}. \end{aligned} \quad (2.1)$$

If the local coordinates are changed by a biholomorphic mapping $z \mapsto z'(z)$, the derivative S transforms according to the 'chain rule'

$$S(Z; z) = S(Z; z') (\partial z')^2 + S(z'; z). \quad (2.2)$$

By choosing $Z = z$ in this equation, we see that the Schwarzian derivative of a conformal change of coordinates $z \mapsto z'(z)$ has the property

$$S(z'; z) = -(\partial z')^2 S(z; z'). \quad (2.3)$$

2.2 Projective connections

A *projective* (or *Schwarzian*) *connection* [11] on the Riemann surface Σ is a collection $\{R_{zz}(z, \bar{z})\}$ of local functions (one for each coordinate system $(z, \bar{z})$) with the properties

(i) R_{zz} is locally holomorphic, i.e. $\partial_{\bar{z}} R_{zz} = 0$,

(ii) R_{zz} transforms with the Schwarzian derivative under a conformal change of coordinates $z \mapsto z'(z)$:

$$R_{z'z'}(z') = (\partial z')^{-2} [R_{zz}(z) - S(z'; z)] \quad (2.4)$$

or, equivalently,

$$R_{z'z'}(z') (dz')^2 = [R_{zz}(z) - S(z'; z)] dz^2.$$

A simple constructive proof for the global existence of such connections (on compact Riemann surfaces of any genus) is presented in Appendix A.1.

Obviously, the difference of two projective connections represents a holomorphic quadratic differential. Thereby, the set of all projective connections on a compact Riemann surface of genus g is an affine space of dimension $3g - 3$.

To simplify the notation, we will usually write R for R_{zz} . We note that it is also possible to define projective connections which are not holomorphic [1] or to assume that they are only holomorphic up to some poles [12], but in the body of these notes we will only work with the definition we just formulated (unless otherwise stated).

2.3 Covariant differential operators: a simple example

Consider two smooth tensor fields on the Riemann surface which are locally given by $c^z(z, \bar{z}) \partial_z$ and $\mu_{\bar{z}}^z(z, \bar{z}) dz \otimes \partial_{\bar{z}}$. Under a conformal change of coordinates, their components transform according to

$$c^z = c'^z \partial' z, \quad \mu_{\bar{z}}^z = \mu_{\bar{z}'}^z (\partial' z) (\bar{\partial} \bar{z}'). \quad (2.5)$$

A concrete example for the tensor field μ is provided by the Beltrami differentials which parametrize complex structures on a Riemann surface, see Appendix B; this also yields an example for the other tensor, since the combination $\xi^z + \bar{\xi}^{\bar{z}} \mu_{\bar{z}}^z$ of the (diffeomorphism generating) vector field components $\xi^z, \bar{\xi}^{\bar{z}}$ transform as the c -field.

Now, consider the integral [13]

$$\int d^2 z \mathcal{A}(c; \mu) \equiv \int dz d\bar{z} c \partial^3 \mu = - \int dz d\bar{z} (\partial^3 c) \mu$$

involving the third-order differential operator ∂^3 . This expression is not well defined on a generic compact Riemann surface due to the fact that the integrand does not transform with

the Jacobian upon passage from one coordinate chart to another. To be more precise, under a conformal change of coordinates $z \rightarrow z'(z)$, the quantity $\partial^3 c$ transforms according to

$$(\partial^3 c)' = (\partial' z)^2 [\partial^3 + 2S\partial + (\partial S)] c, \quad (2.6)$$

where $S \equiv S(z', z)$ denotes the Schwarzian derivative introduced in eq.(2.1).

The S -dependent terms in the transformation law (2.6) can be eliminated [13] by the introduction of a projective projection. In fact, the modified expression

$$\begin{aligned} L_3 c &\equiv [\partial^3 + 2R\partial + (\partial R)] c \\ &= [\partial^3 + (R\partial + \partial R)] c \end{aligned} \quad (2.7)$$

transforms tensorially,

$$(L_3 c)' = (\partial' z)^2 L_3 c, \quad (2.8)$$

and thereby the modified integrand $\tilde{A}(c; \mu) \equiv -\mu (L_3 c)$ transforms with the Jacobian:

$$\tilde{A}(c; \mu)' = (\partial' z) (\tilde{\partial} \bar{z}) \tilde{A}(c; \mu). \quad (2.9)$$

The differential operator L_3 represents the holomorphic and conformally covariant version of ∂^3 [1]-[8] and transforms as

$$L_3' = (\partial' z)^2 L_3 (\partial' z). \quad (2.10)$$

It commutes with the operator ∂ and has the same symmetry property as ∂^3 ,

$$\int_{\Sigma} d^2 z c (L_3 \mu) = - \int_{\Sigma} d^2 z (L_3 c) \mu. \quad (2.11)$$

Remarks : (1) Note that the operator L_3 is conformally covariant, even if the projective connection R is not holomorphic. However, in that case, it does not commute with ∂ . We will only consider this case in section 2.9 and in some applications and we will restrict ourselves to the holomorphic situation in the remainder of the text.

(2) We emphasize that L_3 is only covariant when acting on a conformal field of specific weight in the z -sector (namely weight -1). Quite generally, a specific weight is relevant for the covariant version of ∂^n studied below. In our approach, the weight is simply determined by the symmetric integral under consideration.

(3) The $\bar{z}$ -index only played a passive rôle in the previous discussion. In fact, the same argumentation could be applied to the line integral $\int dz b^* \partial_z^3 c^*$ where b^* transforms as c^* and where the integration goes over a 1-cycle on the Riemann surface. This last reasoning also makes sense in the real setting ($z \in \mathbb{R}$): in that case, a change of coordinates $z \rightarrow z'(z)$ represents a diffeomorphism (of type C^∞) while $b^*(z)$ and $c^*(z)$ are to be viewed as real-valued and covariantly transforming fields.

(4) Our passage from ∂^3 to L_3 is very reminiscent of (and indeed motivated by) the gauging procedure in Yang-Mills and gravitational theories. We would have an even closer analogy by replacing the first-order derivative ∂ by some kind of covariant derivative. However, it is only for genus $g = 1$ that such derivatives can be given a global sense, in which case they lead to equivalent results, see theorem 2.8.5 below. (If the projective connection is not assumed to be holomorphic, it is not clear whether the results are equivalent on surfaces of arbitrary genus, see section 2.9.)

2.4 Projective transformations and structures

Before generalizing the example discussed in the previous section, we would like to have another look at it, namely in a special coordinate system that we introduce now [14].

A change of local coordinates $z \rightarrow z'(z)$ which has the form

$$z' = \frac{az + b}{cz + d} \quad \text{with} \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc \neq 0, \quad (2.12)$$

is called a *projective* (or *Möbius* or *fractional linear*) *transformation*. By some rescalings, one can always achieve $ad - bc = 1$.

A *projective structure* on Σ is an atlas of local coordinates for which all coordinate transformations are projective. It can be shown [11] that every Riemann surface admits such a structure.

Projective coordinates are characterized by the following [14]

Lemma 2.4.1 $S(z', z) = 0$ iff the mapping $z \rightarrow z'(z)$ is a projective transformation.

From this result, it immediately follows that a projective connection transforms tensorially in a projective atlas and therefore may locally be set to zero in this case. In fact, we will see in the next section that the operator ∂^n transforms homogeneously in a projective atlas and therefore does not require the introduction of a connection in such coordinate systems.

2.5 Covariant differential operators in projective coordinates

In this section, coordinates belonging to a projective atlas are denoted by capital letters, $Z, Z', \dots$. We recall that they are related to each other by the transformations

$$Z' = \frac{aZ + b}{cZ + d} \quad (a, b, c, d \in \mathbb{C}, \quad ad - bc = 1) \quad (2.13)$$

and we note that the associated Jacobian is given by

$$\partial_Z Z' = (cZ + d)^{-2}. \quad (2.14)$$

The projective coordinates Z are related to the generic holomorphic coordinates z by a conformal mapping; under this map, a conformal field $c^*(z, \bar{z})$ becomes a field

$$C^Z(Z, \bar{Z}) = c^*(z, \bar{z}) \partial_Z \quad (2.15)$$

transforming as

$$C^Z = C^{Z'} (cZ + d)^2. \quad (2.16)$$

Let us now look at the transformation law of $\partial_Z^2 C^Z$ under a projective coordinate transformation given by eq.(2.13). Obviously, the calculation is the same as the one which led to eq.(2.6) and from $S(Z', Z) = 0$ for projective transformations, we conclude that

$$(\partial_Z^2 C^Z)' = (cZ + d)^4 \partial_Z^2 C^Z. \quad (2.17)$$

To summarize sections 2.3 and 2.5, the operator ∂^3 is covariant in a projective atlas, but not in an ordinary one where it has to be supplemented by terms depending on a projective connection. In fact, if we pass from holomorphic coordinates z to projective coordinates Z by a conformal transformation, we can again use eq.(2.6) to conclude that

$$\begin{aligned}\partial_Z^2 C^Z &= (\partial Z)^{-2} \left[\partial^3 + 2S(Z, z) \partial + (\partial S(Z, z)) \right] c^z \\ &\equiv (\partial Z)^{-2} \left[\partial^3 + 2R_{zz}(z) \partial + (\partial R_{zz}(z)) \right] c^z.\end{aligned}\quad (2.18)$$

Here, we defined

$$R_{zz}(z) \equiv S(Z, z), \quad (2.19)$$

which expression has the correct transformation properties by virtue of the chain rule for the Schwarzian derivative. From this chain rule, it also follows that the definition (2.19) is not affected by a projective transformation of Z . Since the change of coordinates $z \rightarrow Z(z)$ is holomorphic, the quantity (2.19) also has this property. Eq.(2.19) expresses the *one-to-one correspondence between projective structures and projective connections* [11].

Comparison with eq.(2.1) shows that the quantity (2.19) can be rewritten in terms of the vector field component $f \equiv f^z \equiv (\partial_z Z)^{-1}$ as

$$R_{zz}(z) = - \left[\frac{\partial^2 f}{f} - \frac{1}{2} \left(\frac{\partial f}{f} \right)^2 \right] \quad (2.20)$$

or in terms of the tensor field component $\rho \equiv (\partial_z Z)^{-\frac{1}{2}}$ as

$$R_{zz}(z) = -2 \frac{\partial^2 \rho}{\rho}. \quad (2.21)$$

Expressions of this form have previously been considered for the Schwarzian connection by Bonora and Matone [5] and by others.

2.6 Dependence on the complex structure

This section has its origin in discussions with A.Bilal and R.Stora which were motivated by reference [15]. Note that the interpretation of the coordinates ' Z ' in this section differs from the one in the last section.

Suppose the system of holomorphic coordinates Z represents the complex structure parametrized by the Beltrami coefficient μ (see Appendix A.2): locally, Z and $\bar{Z}$ then correspond to smooth invertible functions of some reference coordinates $(z, \bar{z})$,

$$(z, \bar{z}) \longrightarrow (Z(z, \bar{z}), \bar{Z}(z, \bar{z})) \quad (2.22)$$

Under the inverse of this mapping, a conformal field $\hat{c}^Z(Z, \bar{Z})$ becomes a field $c^z(z, \bar{z})$, both variables being related by

$$\hat{c}^Z(Z, \bar{Z}) = c^z(z, \bar{z}) \partial Z. \quad (2.23)$$

Obviously, the conformally covariant operator $\partial_Z^3 + 2R_{ZZ}\partial_Z + (\partial_Z R_{ZZ})$ acting on $\hat{c}^Z(Z, \bar{Z})$ does not preserve its form under the mapping (2.22) which mixes z and $\bar{z}$. This problem can be overcome by assuming that $\hat{c}^Z$ is locally holomorphic: if

$$\partial_{\bar{Z}} \hat{c}^Z = 0, \quad (2.24)$$

then c^z has the holomorphy property

$$\left[\bar{\partial} - \mu \partial + (\partial \mu) \right] c^z = 0. \quad (2.25)$$

Moreover, we have the covariant transformation law

$$\left[\partial_Z^3 + 2R_{ZZ}\partial_Z + (\partial_Z R_{ZZ}) \right] \hat{c}^Z = (\partial Z)^{-2} \left[\partial_z^3 + 2R_{zz}\partial_z + (\partial_z R_{zz}) \right] c^z, \quad (2.26)$$

where R_{zz} is defined by

$$R_{ZZ}(Z) = (\partial Z)^{-2} [R_{zz}(z, \bar{z}) - S(Z, z)] \quad (2.27)$$

The holomorphy condition $\partial_{\bar{Z}} R_{ZZ} = 0$ implies

$$\begin{aligned}\left[\bar{\partial} - \mu \partial - 2(\partial \mu) \right] R_{zz} &= \left[\bar{\partial} - \mu \partial - 2(\partial \mu) \right] S(Z, z) \\ &= \partial^3 \mu.\end{aligned}\quad (2.28)$$

Here, eq.(2.25) follows from

$$\partial_Z = \frac{1}{\lambda(1 - \mu\bar{\mu})} (\bar{\partial} - \mu \partial), \quad \partial_{\bar{Z}} = \frac{1}{\lambda(1 - \mu\bar{\mu})} (\partial - \bar{\mu} \bar{\partial}) \quad (2.29)$$

(which relations are equivalent to eqs.(A.7)) and from the differential equation (A.8) satisfied by the factor $\lambda \equiv \partial Z$. To derive eq.(2.26), one starts from (2.29) and eliminates $\bar{\partial} c^z$ and $\bar{\partial} \lambda$ by virtue of eqs.(2.25) and (A.8), respectively. The relations (2.28) are a consequence of the definition (2.27) and eq.(A.8).

2.7 Bol operators: the general case

To generalize the discussion of section 2.3 to the differential operator ∂^n , we first introduce some notation.

Under conformal changes of coordinates, the quantities dz and ∂ transform with inverse factors :

$$dz' = (\partial z')^{-1} dz, \quad \partial' = (\partial z')^{-1} \partial.$$

Therefore, one can view vector fields as forms of negative degree and rewrite, for instance, $c^z \partial_z$ as $c_{-1} (dz)^{-1}$. Furthermore, on a 2-dimensional surface, it is possible to consider forms of half-integer degree [11] (also known as densities [6] [2] or primary fields). Globally, these are obtained by taking the (appropriate power of the) square root of the canonical sheaf¹ on

¹Following the usual mathematical practice, we do not make a clear distinction between a vector bundle λ and the sheaf $C^\infty(\lambda)$ of sections of this bundle. We note that sections of the canonical sheaf are of the form $\omega_z(z, \bar{z}) dz$.

Σ . Locally, the coefficients of half-integer forms transform with half-integer powers of $\partial z'$ under conformal changes of coordinates.

The method used in section 2.3 can now be applied to the symmetric integral $\int d^2 z \mu_{\frac{1-n}{2}} \partial^n c_{\frac{1-n}{2}}$ (with $n = 0, 1, 2, \dots$); here, c has conformal weight $\frac{1-n}{2}$, i.e. it transforms as

$$\left(c_{\frac{1-n}{2}}\right)' = c_{\frac{1-n}{2}} (\partial' z)^{\frac{1-n}{2}}, \quad (2.30)$$

whereas μ transforms as

$$\left(\mu_{\frac{1-n}{2}}\right)' = \mu_{\frac{1-n}{2}} (\partial' z)^{\frac{1-n}{2}} (\bar{\partial}' \bar{z}). \quad (2.31)$$

(For $n = 3$ and $c_{-1} = c'$, we recover the example studied above.)

Proceeding as before, we find the following expressions for the lowest values of n :

$$\begin{aligned} L_0 &= 1 \\ L_1 &= \partial \\ L_2 &= \partial^2 + \frac{1}{2} R \\ L_3 &= \partial^3 + 2R\partial + (\partial R) \\ L_4 &= \partial^4 + 5R\partial^2 + 5(\partial R)\partial + \frac{3}{2} \left[(\partial^2 R) + \frac{3}{2} R^2 \right] \\ L_5 &= \partial^5 + 10R\partial^3 + 15(\partial R)\partial^2 + [9(\partial^2 R) + 16R^2]\partial + 2[(\partial^3 R) + 8R(\partial R)] \end{aligned} \quad (2.32)$$

These results are in agreement with the explicit expressions given in some of the references [1]-[8]. Following the classic work of Bol [16], Gustafsson and Peetre refer to the L_n as the *Bol operators* [1].

Obviously, the coefficient of the second term of L_n is proportional to R and the other coefficients are polynomial functions of R and its derivatives. Up to numerical factors, these coefficients are determined by their z -index structure. The operator L_2 is sometimes referred to as *Hill's operator* and $L_2 c = 0$ as *Hill's equation* [6].

One might wonder whether the Bol operators are related to each other by composition or by simple differentiation rules. This does not seem to be the case, but in the next section we will show that they are related to another set of differential operators satisfying simple recurrence relations.

According to the fundamental theorems on differential equations, the n -th order equation $F_n = L_n f_n$ (where F_n and f_n are functions of z and $\bar{z}$) is equivalent to a system of n first-order equations. This system can be cast into matrix form, e.g. for $n = 2$,

$$\begin{pmatrix} F_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial & \frac{1}{2} R \\ -1 & \partial \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

For general n , the resulting matrix is more complicated, but it can be given a simple canonical form [8], which provides the most efficient way for evaluating the Bol operators (see next section).

2.8 Mathematical properties

Mostly following references [1] and [8], we summarize some general properties of the Bol operators. Moreover, we establish some extra results from which explicit expressions can easily be determined.

Throughout this section, we assume that the Riemann surface Σ is equipped with a projective structure or, equivalently, supplemented with a projective connection R (both being related by eq.(2.19)).

In a projective atlas, the operator ∂^n transforms homogeneously, which fact can be expressed as [1]

$$\left(c_{\frac{1-n}{2}}\right)' = c_{\frac{1-n}{2}} (cZ + d)^{1-n} \quad (n = 0, 1, 2, \dots), \quad (2.33)$$

then $\partial_2^n c_{\frac{1-n}{2}}$ transforms according to

$$\left(\partial_2^n c_{\frac{1-n}{2}}\right)' = (cZ + d)^{1+n} \partial_2^n c_{\frac{1-n}{2}}. \quad (2.34)$$

Different proofs of this statement are presented in reference [1].

When passing from the projective coordinates Z to the holomorphic coordinates z by a conformal transformation, the operator ∂_2^n becomes the Bol operator L_n :

$$\partial_2^n c_{\frac{1-n}{2}} = (\partial Z)^{-\frac{1+n}{2}} L_n c_{\frac{1-n}{2}}, \quad (2.35)$$

where

$$c_{\frac{1-n}{2}} \equiv c_{\frac{1-n}{2}} (\partial Z)^{-\frac{1-n}{2}}. \quad (2.36)$$

After substituting eq.(2.36) into eq.(2.35), we get the operatorial relation

$$L_n = (\partial Z)^{\frac{1+n}{2}} \left(\frac{1}{\partial Z}\right)^n (\partial Z)^{-\frac{1-n}{2}}. \quad (2.37)$$

By abuse of notation, we define the action of L_n on (half-integer) forms by

$$L_n \left[c_{\frac{1-n}{2}} (dz)^{\frac{1-n}{2}} \right] \equiv \left[L_n c_{\frac{1-n}{2}} \right] (dz)^{\frac{1-n}{2}} \quad (n = 0, 1, 2, \dots).$$

Denoting the space of forms of degree p by λ^p ($p \in \mathbb{Z}/2$), we have the following

Theorem 2.8.1 For $n = 0, 1, 2, \dots$, the Bol operator L_n (locally defined by eq.(2.35)) is a linear differential operator between spaces of half-integer forms² on Σ :

$$L_n : \lambda^{\frac{1-n}{2}} \longrightarrow \lambda^{\frac{1+n}{2}}. \quad (2.38)$$

²Cf. footnote in section 2.7.

With respect to local coordinates z , it has the form

$$L_n = \partial^n + \sum_{j=2}^n b_j^{(n)} \partial^{n-j}, \quad (2.39)$$

where the holomorphic coefficients $b_j^{(n)}(z)$ ($j = 2, \dots, n$) are certain universal polynomials in $R, \partial R, \dots, \partial^{j-2} R$. Thus, the L_n depend on a projective connection R except for L_0 and L_1 .

For a given n , the transformation laws of the $b_j^{(n)}(z)$ under a conformal change of coordinates $z \rightarrow z'(z)$ are such that L_n transforms covariantly,

$$L'_n = (\partial z')^{-\frac{1+n}{2}} L_n (\partial z')^{\frac{1+n}{2}}. \quad (2.40)$$

The coefficient $b_2^{(n)}$ is explicitly given by [8]

$$b_2^{(n)} = k_n R \quad \text{with} \quad k_n = \frac{n(n^2-1)}{12} \quad (2.41)$$

and transforms as

$$(b_2^{(n)})' = (\partial' z')^2 [b_2^{(n)} - k_n S(z'; z)]. \quad (2.42)$$

The operator L_n commutes with $\bar{\partial}$ and has the same symmetry properties as ∂^n ,

$$\int_{\Sigma} d^2 z \, c_{\frac{1-n}{2}} \left(L_n \mu_{\frac{1-n}{2}} \right) = (-1)^n \int_{\Sigma} d^2 z \left(L_n c_{\frac{1-n}{2}} \right) \mu_{\frac{1-n}{2}}, \quad (2.43)$$

where μ transforms according to eq.(2.31).

We note that the number k_n appearing in eq.(2.41) is exactly the one occurring in the central term for the Virasoro algebra, see eq.(2.85). (The latter is the central extension of the Lie algebra of infinitesimal diffeomorphisms of the circle [17] and its generators are given by $L_n = -z^{n+1} \partial$.) It would be interesting to find an explanation for this relationship.

A simple way of understanding why the coefficient $b_1^{(n)}$ does not occur in the expansion (2.39) is the following. In a special coordinate system, L_n is nothing but ∂^n and therefore (2.39) should have the same symmetry properties as ∂^n , i.e. satisfy eq.(2.43); this immediately implies the vanishing of $b_1^{(n)}$.

Based on the more general study [8], we now present the most efficient way for determining explicit expressions for the L_n .

Theorem 2.8.2 (Matrix representation) For $n = 1, 2, \dots$, the Bol operator L_n is given by the scalar relation

$$F_n = L_n f_n \quad (f_n \in \lambda^{\frac{1-n}{2}}) \quad (2.44)$$

which is obtained after elimination of $f_1, \dots, f_{n-1}$ from the matrix equation

$$\begin{pmatrix} F_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \partial & 1(n-1)\frac{R}{2} & 0 & \dots & \dots & 0 \\ -1 & \partial & 2(n-2)\frac{R}{2} & 0 & \dots & 0 \\ 0 & -1 & \partial & 3(n-3)\frac{R}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & -1 & \partial \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{pmatrix}. \quad (2.45)$$

Proof: For $f_n \in \lambda^{\frac{1-n}{2}}$, set

$$f_{n-1} \equiv \partial f_n, \quad f_{n-2} \equiv \partial f_{n-1} = \partial^2 f_n, \quad \dots, \quad f_1 \equiv \partial f_2 = \partial^{n-1} f_n$$

and recall that L_n is given by the expression (2.39). Then, the n -th order scalar equation (2.44) is equivalent to the first-order matrix equation

$$\bar{F} = \hat{Q}_n \bar{f} \quad (2.46)$$

with $\bar{F}^T = (F_n, 0, \dots, 0)$, $\bar{f}^T = (f_1, f_2, \dots, f_n)$ and an $n \times n$ matrix $\hat{Q}_n = 1_n \partial - J + b$ where 1_n is the unit matrix and

$$J = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & b_2^{(n)} & b_3^{(n)} & \dots & b_n^{(n)} \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Following the classic work of Drinfel'd and Sokolov [18], one transforms eq.(2.46) by upper triangular matrices

$$N(z, \bar{z}) = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \dots & * \\ 0 & \dots & & 1 \end{pmatrix},$$

which leads to the equation

$$\bar{F}' = \hat{Q}'_n \bar{f}' \quad (2.47)$$

with $\bar{F}' \equiv N \bar{F}$, $\bar{f}' \equiv N \bar{f}$ and $\hat{Q}'_n \equiv N \hat{Q}_n N^{-1}$. Since $\bar{F}' = \bar{F}$ and $\bar{f}' = f_n$, the matrix equation (2.47) is still equivalent to the scalar equation (2.44), i.e. describes the same n -th order covariant operator L_n . By an appropriate choice of N (motivated by Lie algebraic considerations), the matrix $\hat{Q}'_n$ can be cast into the form given in eq.(2.45), see the second of references [8].

We note that the matrix representation (2.45) is not unique. An equivalent and very similar representation was given by Baston and Eastwood [2] who used ideas from twistor theory: in that approach, the vector $(f_1, \dots, f_n)^T$ corresponds to a section of a certain twistor bundle and the matrix (which acts on it) is interpreted as a local twistor connection.

The following result has its origin in a very brief remark made in the second of references [1].

Theorem 2.8.3 (Recurrence relation) Let F_1, F_2 be a basis of the space of solutions of the differential equation $L_2 F = 0$:

$$\left(\partial^2 + \frac{R}{2} \right) F_i(z) = 0 \quad (i = 1, 2) \quad (2.48)$$

Using these solutions, we define a homogeneous polynomial of degree $k \geq 0$ by

$$f^{(k)}(z) \equiv (\xi \cdot F(z))^k \equiv \left(\sum_{i=1}^2 \xi_i F_i(z) \right)^k \quad (\xi_i \in \mathbb{C}) \quad (2.49)$$

Furthermore, we introduce a set of differential operators $P_n^{(k)}$ ($n, k \in \{0, 1, 2, \dots\}$) by

$$\begin{aligned} P_0^{(k)} f^{(k)} &= f^{(k)} \\ P_n^{(k)} f^{(k)} &= k(k-1) \dots (k-(n-1)) (\xi \cdot F)^{k-n} (\xi \cdot \partial F)^n \quad (n \geq 1, k \neq 0, 1, \dots, n-1) \end{aligned} \quad (2.50)$$

Then, the statement of the theorem is as follows. $P^{(k)}$ is a linear differential operator of order n whose coefficients only depend on $R, \partial R, \dots, \partial^{n-2} R$. Moreover, it satisfies the recurrence relation

$$P_{n+1}^{(k)} = \partial P_n^{(k)} + n(k-(n-1)) \frac{R}{2} P_{n-1}^{(k)} \quad (k \text{ fixed}, k \neq 0, 1, \dots, n) \quad (2.51)$$

The Bol operators are obtained from the solutions of this relation by setting

$$L_{n+1} = P_{n+1}^{(k=n)} \quad (2.52)$$

(Note that this operator acts on $f^{(n)}$ and that $f^{(n)} \in \lambda^{-\frac{n}{2}}$ for $F_i \in \lambda^{-\frac{1}{2}}$.)

Proof: From the definition (2.49), it follows that

$$\begin{aligned} \partial f^{(k)} &= k(\xi \cdot F)^{k-1} (\xi \cdot \partial F) \\ \partial^2 f^{(k)} &= k(k-1) (\xi \cdot F)^{k-2} (\xi \cdot \partial F)^2 - k \frac{R}{2} (\xi \cdot F)^k \end{aligned}$$

where eq.(2.48) was used for the last term. Henceforth,

$$\left(\partial^2 + k \frac{R}{2} \right) f^{(k)} = k(k-1) (\xi \cdot F)^{k-2} (\xi \cdot \partial F)^2, \quad (2.53)$$

i.e. the complicated expression on the r.h.s. corresponds to the action of a linear differential operator of second order (whose coefficients only depend on $R(z)$) on the polynomial $f^{(k)}$.

A similar equation of third order follows by differentiating eq.(2.53) and using again eq.(2.48):

$$\partial \left(\partial^2 + k \frac{R}{2} \right) f^{(k)} = k(k-1)(k-2) (\xi \cdot F)^{k-3} (\xi \cdot \partial F)^3 - 2k(k-1) \frac{R}{2} (\xi \cdot F)^{k-1} (\xi \cdot \partial F),$$

whence

$$\left[\partial \left(\partial^2 + k \frac{R}{2} \right) + 2(k-1) \frac{R}{2} \partial \right] f^{(k)} = k(k-1)(k-2) (\xi \cdot F)^{k-3} (\xi \cdot \partial F)^3. \quad (2.54)$$

Obviously, the right-hand sides of eqs.(2.53) and (2.54) generalize to the expression (2.50). The operatorial relation (2.51) is deduced by direct computation and proven by recurrence

with respect to n . Finally, the statement (2.52) follows by considering forms of the correct degree, see eq.(2.38).

q.e.d.

Another method³ for recursively determining an explicit expression for L_n consists of evaluating $\partial^n \psi^{n-1}$ where ψ satisfies $\partial^2 \psi = -\frac{1}{2} R \psi$:

Theorem 2.8.4 If $L_2 \psi = 0$, then $L_n \psi^{n-1} = 0$ for $n = 2, 3, \dots$.

For example, $\partial^3 \psi^2 = 6(\partial \psi)(\partial^2 \psi) + 2\psi \partial(\partial^2 \psi)$ and substitution of $\partial^2 \psi = -\frac{1}{2} R \psi$ immediately yields the known expression for the operator L_3 .

Locally, a Riemann surface Σ looks like $\mathbb{C}$, for which space the conformal transformations are generated by the Virasoro algebra. Thus, one should expect a close relationship between the theory of conformally covariant differential operators on Σ and the representation theory of the Virasoro algebra. This connection was first discussed by A.I.B.Zamolodchikov who also indicated how to construct the covariant operators from this point of view [4]. Since these results involve further knowledge of representation theory, we refer to the original work (and also to the second of references [8]) and we only note that the Bol operators are related to the null- or singular vectors in the Verma module of the Virasoro algebra.

The operator L_n is a product of n operators in a projective atlas and it can be factorized in a similar way in an arbitrary system of holomorphic coordinates [1] [8]. This product consists of covariant derivatives involving an affine connection that we introduce now [11].

An affine connection is a collection $\{\gamma_z(z, \bar{z})\}$ of local functions (one for each coordinate system $(z, \bar{z})$) which are locally holomorphic (i.e. $\partial \gamma_z = 0$) and which transform under a conformal change of coordinates as

$$\gamma_{z'}(z') = (\partial z')^{-1} [\gamma_z(z) - \partial \ln(\partial z')] \quad (2.55)$$

The only compact Riemann surfaces which admit a globally defined affine connection are those of genus one [11].

If γ_z represents an affine connection, then

$$R_{zz} \equiv \partial_z \gamma_z - \frac{1}{2} \gamma_z^2 \quad (2.56)$$

is a projective connection. We note that the last equation has the same form as the Riccati equation or Miura transformation [19] which occur in the theory of integrable systems, see section 2.12.2.

Given an affine connection (which is not necessarily holomorphic), one can define a covariant derivative which maps p -forms into $(p+1)$ -forms:

$$\begin{aligned} \nabla_{(p)} : \lambda^p &\longrightarrow \lambda^{p+1} \\ c_p &\longrightarrow \nabla_{(p)} c_p \equiv (\partial_z - p \gamma_z) c_p \end{aligned} \quad (2.57)$$

In the theory of integrable models, the derivative $\nabla_{(1)} = \partial - \gamma$ is also known as the Fréchet Jacobian of R with respect to γ , see reference [20] or section 2.12.2.

Using these notions, we can now formulate the following result [1] [8].

³I wish to thank A.Bilal for pointing out this procedure.

Theorem 2.8.5 (Factorisation of Bol operators) Consider a compact Riemann surface of genus one and suppose the projective connection R comes from an affine connection γ as in equation (2.56). Writing $n = 2l + 1$ with $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, the Bol operators L_n are then given by

$$\begin{aligned} L_n &= \nabla_{(l)} \nabla_{(l-1)} \dots \nabla_{(-l)} \\ &= [\partial - l\gamma_z][\partial - (l-1)\gamma_z] \dots [\partial + l\gamma_z] \end{aligned} \quad (2.58)$$

On a generic compact Riemann surface, this decomposition can only exist locally.

The proof consists of going to projective coordinates (where $L_n = \partial^{2l+1}$ and $R = 0$) and using induction with respect to l . As a matter of fact, it is not at all clear whether all projective connections arise from affine connections by the construction (2.56).

For concreteness, we give the simplest examples of the factorization (2.58) :

$$\begin{aligned} L_1 &= \partial \\ L_2 &= (\partial - \frac{1}{2}\gamma_z)(\partial + \frac{1}{2}\gamma_z) \\ L_3 &= (\partial - \gamma_z)\partial(\partial + \gamma_z) \end{aligned} \quad (2.59)$$

Comparison of the expressions (2.58) and (2.37) suggests rewriting the covariant derivative as [8]

$$\nabla_{(p)} = (\partial Z)^p \partial (\partial Z)^{-p} = \partial - p(\partial \ln \partial Z) \quad (2.60)$$

The right-hand side of this equation transforms properly under a conformal change of coordinates $z \rightarrow z'(z)$, but it is not invariant under projective transformations of Z .

In conclusion, we generalize the results on the action of smooth transformations discussed in section 2.6 :

Theorem 2.8.6 (Dependence on the complex structure) Let $(z, \bar{z})$ denote a system of reference coordinates and suppose the coordinates $(Z, \bar{Z})$ represent the complex structure parametrized by the Beltrami differential $\mu_z^{\bar{z}}(z, \bar{z}) dz \otimes \partial_{\bar{z}}$ (i.e. the Z satisfy eqs. (A.7) and (A.8)). Under the smooth invertible mapping which relates the coordinates $(Z, \bar{Z})$ and $(z, \bar{z})$, the form $\hat{c}_{1-\frac{n}{2}}(Z, \bar{Z})$ becomes

$$\hat{c}_{1-\frac{n}{2}}(Z, \bar{Z}) \equiv c_{1-\frac{n}{2}}(z, \bar{z}) (\partial Z)^{-\frac{1-n}{2}} \quad (2.61)$$

If the form $\hat{c}_{1-\frac{n}{2}}$ is locally holomorphic,

$$\partial_{\bar{z}} \hat{c}_{1-\frac{n}{2}} = 0, \quad (2.62)$$

then $c_{1-\frac{n}{2}}(z, \bar{z})$ has the holomorphy property

$$\left[\bar{\partial} - \mu \partial - \frac{1-n}{2} (\partial \mu) \right] c_{1-\frac{n}{2}}(z, \bar{z}) = 0 \quad (2.63)$$

and we have the covariant transformation law

$$(L_n \hat{c}_{1-\frac{n}{2}})(Z, \bar{Z}) = (\partial Z)^{-\frac{1-n}{2}} (L_n c_{1-\frac{n}{2}})(z, \bar{z}) \quad (2.64)$$

Here, the L_n denote the Bol operators in terms of the coordinates Z and $\bar{z}$, respectively, and the projective connections $R_{Z\bar{Z}}$ and $R_{z\bar{z}}$ occurring in the L_n are related by eq. (2.27). These connections satisfy $\partial_{\bar{z}} R_{Z\bar{Z}} = 0$ and eq. (2.28), respectively.

2.9 The non-holomorphic case

Most of the results presented in the last section also hold if the projective connection $R_{z\bar{z}}(z, \bar{z})$ is smooth rather than holomorphic. The corresponding Bol operators are still covariant, but they do not commute with $\bar{\partial}$ and they are not related to the Möbius covariant operators ∂_z^2 by a conformal change of coordinates. A result which does not hold is theorem 2.8.3, since it relies on a basis of solutions for a differential equation in one variable. For surfaces of arbitrary genus, it is not known whether or not all non-holomorphic projective connections arise from affine connections by virtue of eq. (2.56); therefore it is not clear whether the decomposition (2.58) globally exists in the smooth case for any compact Riemann surface.

2.10 The real case ($z \in \mathbb{R}$)

In the real setting, the Schwarzian derivative and connection are defined in analogy to the complex framework. In this case, there is yet another way to write or evaluate L_n [6]:

Theorem 2.10.1 Let $\psi(dz)^{-\frac{1}{2}} \in \lambda^{-\frac{1}{2}}$ be a solution of

$$0 = L_2 \psi \equiv \left(\partial^2 + \frac{1}{2} R \right) \psi \quad (2.65)$$

and assume $\psi > 0$ in some interval $I \subset \mathbb{R}$. For $z \in I$ and $n = 0, 1, 2, \dots$, the operator

$$L_n \equiv \psi^{-(1+n)} (\psi^2 \partial)^n \psi^{1-n} \quad (2.66)$$

is then well defined and it depends only on the projective connection R ; it coincides with the Bol operator L_n .

Proof: The fact that L_n only depends on R follows from a variational argument (for details, see reference [8]). Now, the quantity $\gamma \equiv \partial \ln \psi^{-2}$ locally represents an affine connection and eq. (2.65) implies the relation (2.56). By substituting

$$\begin{aligned} \nabla_{(p)} &\equiv \psi^{-2p} \partial \psi^{2p} \\ &= \partial - p\gamma \end{aligned} \quad (2.67)$$

into eq. (2.66), we recover the factorized expression for the Bol operator L_n , eq. (2.58). q.e.d.

In the real context, the transformation law of a projective connection can be (and originally was [6]) introduced in a different way. For this purpose, we consider the group $G = \text{Diff}_+(S^1)$ of orientation-preserving diffeomorphisms of the unit circle or rather its central extension $\hat{G}$ [17]. The Lie algebra of the latter, $\text{Lie}(\hat{G}) \cong \text{Vect}(S^1) \oplus \mathbb{R}$, represents the Virasoro algebra and its dual is given by $(\text{Lie}(\hat{G}))^* \cong \lambda^2(S^1) \oplus \mathbb{R}$ where $\lambda^2(S^1)$ denotes the space of quadratic differentials $q_{zz}(z)(dz)^2$. If the circle is parametrized by real coordinates z , the coadjoint action of $Z \in G$ on $(q, 1) \in (\text{Lie}(\hat{G}))^*$ is defined by [21] :

$$\begin{aligned} \text{Ad}_{Z^{-1}}^*(q, 1) &= ((\partial Z)^2 q \circ Z + S(Z; z), 1) \\ &= ((\partial Z)^2 [q_{zz} \circ S(z; Z)], 1) \end{aligned} \quad (2.68)$$

Here, we recognize the transformation law of a projective connection R_{zz} under a diffeomorphism $z \rightarrow Z(z)$.

2.11 Generalization to n -manifolds

As noted by R. Gover [3], the Bol operators induce an infinite series of conformally covariant operators on conformal n -manifolds, i.e. n -dimensional manifolds which are equipped with a conformal class of metrics. The procedure works for real as well as for complex manifolds (see [2] and references therein). It is due to the fact that null geodesics on conformal n -manifolds inherit a natural projective structure. (Note that the Bol operators can act along these geodesics which correspond to one-dimensional submanifolds.)

To illustrate the result, we consider a real conformal n -manifold M with local coordinates $(x^i)_{i=1,\dots,n}$ and we pick a metric (g_{ij}) in the conformal class. Let (∇_i) denote the covariant derivative with respect to the Levi-Civita connection associated to (g_{ij}) and define

$$P_{ij} \equiv \frac{1}{2-n} \left(R_{ij} + \frac{1}{2(1-n)} g_{ij} r \right) .$$

Here, R_{ij} is the Ricci tensor and r the scalar curvature of the connection ∇_i . Then, the Bol operators $L_2 = \partial^2 + \frac{1}{2}R$ and $L_3 = \partial^3 + 2R\partial + (\partial R)$, respectively, give rise to the conformally covariant operators

$$\text{tff } L_{(ij)} \quad \text{and} \quad \text{tff } L_{(ijk)} ,$$

where 'tff' denotes the 'totally trace free part' and

$$\begin{aligned} L_{ij} &= \nabla_i \nabla_j + P_{ij} \\ L_{ijk} &= \nabla_i \nabla_j \nabla_k + 4P_{ij} \nabla_k + 2(\nabla_i P_{jk}) . \end{aligned} \quad (2.69)$$

It should be clear how this procedure generalizes to arbitrary L_n . For further details and elaborations, we refer to [2] and to the work cited therein.

2.12 Applications in integrable models

2.12.1 Lax operators

The Bol operator $L_z = \partial^2 + \frac{1}{2}R(z)$ has the same form as a one-dimensional Schrödinger operator with scalar potential. An operator of this type, $l \equiv \partial^2 + \frac{1}{2}u(z, \bar{z})$ generates the Lax equations for the *Korteweg-de Vries (KdV) hierarchy* [19] :

$$\partial l = [l_+^{\frac{2k+1}{2}}, l] \quad (k = 0, 1, 2, \dots) . \quad (2.70)$$

In this context, z and $\bar{z}$ correspond to (real) space and time variables, respectively. The pseudo-differential operator $l^{\frac{1}{2}}$ is to be understood as a power series in ∂^{-1} [19],

$$l^{\frac{1}{2}} = \partial + \sum_{j=1}^{\infty} a_{-j} \partial^{-j} ,$$

and the 'plus' means that only non-negative powers of ∂ are considered. For instance, for $k = 1$,

$$l_+^{\frac{3}{2}} = \partial^3 + \frac{3}{4}u\partial + \frac{3}{8}(\partial u)$$

and the corresponding Lax equation coincides with the ordinary KdV equation,

$$4\bar{\partial}u = \partial^3 u + 3u\partial u . \quad (2.71)$$

Although $l_+^{\frac{3}{2}}$ and, more generally, $l_+^{\frac{2k+1}{2}}$ have the same form as the Bol operators L_3 and L_{2k+1} , respectively, the numerical factors in these expressions are not identical. (However, the operators $l_+^{\frac{3}{2}}$ and L_3 do coincide after a rescaling of u .)

We note that the operator L_3 explicitly intervenes if the KdV equation is written on a Riemann surface [5].

If one replaces the Schrödinger operator l in the Lax equations (2.70) by operators of higher order, one obtains other sequences of integrable models, see section 2.14.2.

2.12.2 Poisson operators

There are two equivalent ways for writing the KdV equation in Hamiltonian form [19]. The first is given by the Poisson brackets

$$\begin{aligned} \{u(z'), u(z)\}_1 &= \partial \delta(z - z') \\ &\equiv (u, u)_1 \delta(z - z') \end{aligned} \quad (z \in \mathbf{R}) \quad (2.72)$$

and the Hamiltonian $H^{(1)}[u] = \frac{1}{8} \int_{-\infty}^{+\infty} dz \{u^3 - (\partial u)^2\}$. The second is defined by

$$\begin{aligned} \{u(z'), u(z)\}_2 &= \left[\partial^3 + 2u\partial + (\partial u) \right] \delta(z - z') \\ &\equiv (u, u)_2 \delta(z - z') \end{aligned} \quad (z \in \mathbf{R}) \quad (2.73)$$

and $H^{(2)}[u] = \frac{1}{8} \int_{-\infty}^{+\infty} dz u^2$. In each case, the KdV equation is equivalent to the Hamiltonian equations of motion

$$\begin{aligned} \partial u(z, \bar{z}) &= \{u(z, \bar{z}), H[u]\} \\ &= (u, u) \frac{\delta H}{\delta u(z, \bar{z})} . \end{aligned} \quad (2.74)$$

The differential operators determining the Poisson brackets for the basic field u are sometimes referred to as *Poisson operators*. Obviously, those occurring in equations (2.72) and (2.73) have exactly the same form as the first and third order Bol operators, respectively.

The existence of the second Hamiltonian structure for the KdV equation can be traced back [20] to the existence of both the *Miura transformation* (cf. eq.(2.56)),

$$u = \partial v - \frac{1}{2}v^2 , \quad (2.75)$$

and the natural Hamiltonian structure for the *modified KdV (mKdV) equation*,

$$4\bar{\partial}v = \partial^3 v - \frac{3}{2}v^2 \partial v . \quad (2.76)$$

This natural Hamiltonian structure is defined by the Poisson brackets

$$\{v(z'), v(z)\} = -\partial \delta(z - z') \quad (2.77)$$

and the Hamiltonian $H[v] = H^{(2)}[u = \partial v - \frac{1}{2}v^2]$ (i.e. $H^{(2)}[u]$ with u replaced by virtue of eq.(2.75)). Now, the existence of the second Hamiltonian structure is a consequence of the identity

$$\partial^3 + 2u\partial + (\partial u) = -\mathcal{D}\partial\mathcal{D}^\dagger, \quad (2.78)$$

where u is given by eq.(2.75) and where

$$\mathcal{D} \equiv \partial - v, \quad -\mathcal{D}^\dagger = \partial + v.$$

The identity (2.78) coincides with the factorisation formula for the Bol operator L_3 , see eq.(2.59).

2.13 Applications in conformal models

2.13.1 Anomalies and central extensions

The integral $\int d^2z c \partial^3 \mu$ (plus its complex conjugate) that we considered as the starting point of our investigation represents the chirally split form of the *diffeomorphism anomaly* in two-dimensional conformal field theory when defined in the complex plane [22]. The modified integral $\int_\Sigma d^2z c L_3 \mu$ provides an expression for this anomaly on a compact Riemann surface Σ without boundary [13]. In this context, the field c is to be viewed as a cohomology generator for the BRS differential algebra⁴

$$\begin{aligned} s\mu &= \{\bar{\partial} - \mu\partial + (\partial\mu)\}c \\ sc &= c\partial c, \quad sR = 0. \end{aligned} \quad (2.79)$$

The anomaly represents a solution of the Wess-Zumino (WZ) consistency condition

$$s \int_\Sigma \mathcal{A}_2^1(c, \mu) = 0.$$

It is convenient to anti-symmetrize the integrand and to consider the associated 2-form

$$\begin{aligned} \hat{\mathcal{A}}_2^1(c, \mu) &\equiv -\frac{1}{2} [c(L_3\mu) - \mu(L_3c)] dz \wedge d\bar{z} \\ &= -\frac{1}{2} [(c\partial^3\mu - \mu\partial^3c) + 2R(c\partial\mu - \mu\partial c)] dz \wedge d\bar{z}, \end{aligned} \quad (2.80)$$

whose integral still solves the WZ integrability condition. Obviously, this expression has the same structure as the 2-cocycle

$$\hat{\chi}(f, g) \equiv -\frac{1}{2} [(f\partial^3g - g\partial^3f) + 2R(f\partial g - g\partial f)] dz, \quad (2.81)$$

which defines the *central extension of the Krichever-Novikov algebra* [12]. The latter is the Lie algebra of those vector fields on a Riemann surface which are holomorphic outside two

⁴Cf. eq.(A.10).

points "in general position" (i.e. a "2-punctured Riemann surface"); it represents the generalization of the Virasoro algebra (which is only defined locally) to Riemann surfaces of higher genus. The Krichever-Novikov algebra can be obtained by an oscillator construction from the *generalized Heisenberg algebra* [12] [23], i.e. the algebra of meromorphic functions on punctured Riemann surfaces; it can also be defined on Riemann surfaces with more than two punctures [24].

2.13.2 Ward identities

The Lagrangian conformal models we just mentioned provide a further example where the covariantization of ∂^3 occurs [4].

Consider the connected Green's functional $Z_c[\mu]$ as a functional of the Beltrami coefficient μ (cf. appendix A.2). Infinitesimal diffeomorphisms generated by a vector field $\xi \cdot \partial \equiv \xi\partial + \bar{\xi}\bar{\partial}$ act on $Z_c[\mu]$ by the Ward operator $W(\xi \cdot \partial) \equiv \int_\Sigma d^2z (\delta_{\xi\partial}\mu) \delta/\delta\mu$ where $\delta_{\xi\partial}\mu$ is given by eq.(A.10). The anomalous Ward identity in the complex plane $\mathbb{C}$ reads [25]

$$W(\xi \cdot \partial) Z_c[\mu] = -k \int_{\mathbb{C}} d^2z c \partial^3 \mu \quad (k \in \mathbb{R}) \quad \text{and c.c.},$$

or, equivalently,

$$\left[\bar{\partial} - \mu\partial - 2(\partial\mu) \right] \frac{\delta Z_c[\mu]}{\delta\mu} = k \partial^3 \mu \quad (k \in \mathbb{R}) \quad \text{and c.c.} \quad (2.82)$$

We note that this equation is solved by $\delta Z_c/\delta\mu = kS(Z; z)$ where S represents the Schwarzian derivative and Z a projective coordinate system (see references [13] or eq.(2.28)).

By a Legendre transformation

$$Z_c[\mu] = \Gamma[T] + \int_{\mathbb{C}} d^2z T \mu, \quad T \equiv T_{zz} = \frac{\delta Z_c[\mu]}{\delta \mu_z},$$

the *Ward identity* (2.82) becomes

$$\left[\partial^3 + 2T\partial + (\partial T) \right] \frac{\delta \Gamma[T]}{\delta T} = -\bar{\partial} T, \quad (2.83)$$

where we chose $k=1$. The operator on the l.h.s. has the same form as the covariant operator L_3 . It would be nice if one could further exploit the occurrence of this operator in the Ward identity, but the r.h.s. of eq.(2.83) seems to represent an obstruction for such applications.

2.13.3 The Virasoro algebra

The components T_{zz} and $T_{z\bar{z}}$ of the *energy-momentum tensor* in two-dimensional conformal field theory transform in the same way as the components of a projective connection (or as the coefficient $b_2^{(s)} = k_s R$ of L_n , see eq.(2.42)). They generate the *classical Virasoro algebra* through Poisson brackets involving the operator L_3 ,

$$\{u(z'), u(z)\} \propto \left[\partial^3 + 2u\partial + (\partial u) \right] \delta(z - z') \quad (z \in \mathbb{R}). \quad (2.84)$$

$$u(z) = -\frac{12}{c} \sum_{n=-\infty}^{+\infty} l_n e^{-inz} + \frac{1}{2},$$

we recover the graded Virasoro algebra relation

$$i \{l_n, l_m\} = (n-m) l_{n+m} + c \frac{n(n^2-1)}{12} \delta_{n,-m}. \quad (2.85)$$

Note that the brackets (2.84) coincide with the second Poisson brackets for the KdV equation up to a numerical factor [26].

In the present context, the Miura transformation, $R = \partial\gamma - \frac{1}{2}\gamma^2$, may be interpreted as a realization of R by a field γ of weight one in the presence of a background charge [7].

The deep relationship between conformal invariants and the representation theory of the Virasoro algebra has already been advocated in section 2.8: the Bol operators (and their generalizations to be discussed in section 2.14) are associated with the *highest weight representations of the Virasoro algebra*, see [4] and the second of references [8].

2.13.4 Liouville field theory

In isothermal coordinates $(Z, \bar{Z})$, the Liouville equation reads

$$\partial_Z \partial_{\bar{Z}} \varphi = -\frac{1}{2} e^{2\varphi}. \quad (2.86)$$

Here, $e^{2\varphi}(Z, \bar{Z}) \equiv \rho_{ZZ}(Z, \bar{Z})$ represents a conformal field of dimension $(1, 1)$:

$$\rho_{ZZ'} = \rho_{ZZ} (\partial_Z \bar{Z}')^{-1} (\partial_{\bar{Z}} \bar{Z}')^{-1}. \quad (2.87)$$

Defining [27]

$$T_{ZZ} \equiv -2e^\varphi (\partial_Z^2 e^{-\varphi}), \quad T_{\bar{Z}\bar{Z}} \equiv -2e^\varphi (\partial_{\bar{Z}}^2 e^{-\varphi}), \quad (2.88)$$

one finds that

$$\partial_{\bar{Z}} T_{ZZ} = 0 = \partial_Z T_{\bar{Z}\bar{Z}}$$

by virtue of eq.(2.86). The quantities (2.88) are the components of the *stress-energy tensor* and they can be rewritten in the more familiar form

$$T_{ZZ} = -2 \left[\partial_Z^2 \varphi - (\partial_Z \varphi)^2 \right], \quad T_{\bar{Z}\bar{Z}} = -2 \left[\partial_{\bar{Z}}^2 \varphi - (\partial_{\bar{Z}} \varphi)^2 \right]. \quad (2.89)$$

As a consequence of eq.(2.87), they transform like the components of a projective connection. Their defining relations (2.88) are equivalent to

$$\left[\partial_Z^2 + \frac{1}{2} T_{ZZ} \right] e^{-\varphi} = 0 = \left[\partial_{\bar{Z}}^2 + \frac{1}{2} T_{\bar{Z}\bar{Z}} \right] e^{-\varphi}, \quad (2.90)$$

where we recognize the Bol operator L_2 and its complex conjugate.

2.13.5 $SU(2)$ WZW model

Consider the $SU(2)$ Wess-Zumino-Witten (WZW) model on the torus. In this theory, the two-point correlators of primary fields satisfy simple differential equations, see [28] and references therein. For fields of highest isospin, the corresponding differential operators have the same form as the Bol operators - as was noted by Jatkar [29].

Let us be a bit more explicit. In a $SU(2)$ WZW model at level k , there are $k+1$ primary fields of isospin $j = 0, \frac{1}{2}, \dots, \frac{k}{2}$, respectively. Now, for the case of a torus and a fixed-level k , consider the two-point correlators for fields of highest isospin, i.e. isospin $\frac{k}{2}$. They satisfy a differential equation of order $k+1$, e.g. [28]

$$\begin{aligned} k=1 : 0 &= \left[\partial^2 - \frac{3}{4} \mathcal{P} \right] f \\ k=2 : 0 &= \left[\partial^3 - 3\mathcal{P}\partial - \frac{3}{2}(\partial\mathcal{P}) \right] f \\ k=3 : 0 &= \left[\partial^4 - \frac{15}{2}\mathcal{P}\partial^2 - \frac{15}{2}(\partial\mathcal{P})\partial - \frac{9}{16}(4\partial^2\mathcal{P} - 9\mathcal{P}^2) \right] f \\ k=4 : 0 &= \left[\partial^5 - 15\mathcal{P}\partial^3 - \frac{45}{2}(\partial\mathcal{P})\partial^2 - \frac{9}{2}(3\partial^2\mathcal{P} - 8\mathcal{P}^2)\partial - 3(\partial^3\mathcal{P} - 12\mathcal{P}\partial\mathcal{P}) \right] f \end{aligned} \quad (2.91)$$

Here, $\mathcal{P}$ denotes the Weierstrass function and the coefficient $\partial^3\mathcal{P} - 12\mathcal{P}\partial\mathcal{P}$ in the last term could also be dropped, since it vanishes due to the basic properties of the $\mathcal{P}$ -function [14]. The differential operators (2.91) coincide with the Bol operators $L_2, \dots, L_5$ modulo the substitution

$$\mathcal{P}(z) \longrightarrow -\frac{2}{3} R(z). \quad (2.92)$$

This relation between the Bol operators and the differential operators for correlation functions is also expected to hold for higher values of k , but it does not exist for fields of lower isospin.

2.14 The most general covariant operators

The operator L_n only depends on a projective connection or, equivalently, a projective structure on the Riemann surface. In the following, we will introduce some covariant operators which depend on such a structure and on some conformal fields. In conclusion, we point out that these parametrize the most general holomorphic operators which are 'normalized' and covariant. The introduction of the new operators starts again from Möbius covariant ones and therefore, we can conclude that essentially all holomorphic and covariant operators come from Möbius covariant ones.

The most general holomorphic differential operator of order n has the local form

$$L = a_0^{(n)} \partial^n + a_1^{(n)} \partial^{n-1} + a_2^{(n)} \partial^{n-2} + \dots + a_n^{(n)},$$

where the coefficients $a_j^{(n)}(z)$ are holomorphic functions. If the leading coefficient $a_0^{(n)}$ does not have any zeros, one can achieve $a_0^{(n)} \equiv 1$ by dividing by this coefficient. Then, one can

restrict oneself to $a_1^{(n)} \equiv 0$ by performing the rescaling [8]

$$L \longrightarrow g^{-1} L g \quad \text{with} \quad g(z) = \exp \left\{ -\frac{1}{n} \int^z dz' a_1^{(n)}(z') \right\} . \quad (2.93)$$

As a result, one obtains the 'normalized operator'

$$L^{(n)} = \partial^n + \sum_{j=2}^n a_j^{(n)} \partial^{n-j} , \quad (2.93)$$

which is to be studied in the sequel from the point of view of conformal transformations.

2.14.1 Mathematical aspects

Consider a fixed $n \in \mathbb{N}$. Using holomorphic k -forms w_k (with $1 \leq k \leq n$), we will construct n covariant operators $M_{w_k}^{(n)}$ of order $n-k$, respectively, which can be added to the Bol operator L_n :

$$\begin{aligned} M_{w_k}^{(n)} : \lambda^{\frac{1-n}{2}} &\longrightarrow \lambda^{\frac{1-n}{2}} \\ c_{\frac{1-n}{2}} &\longmapsto M_{w_k}^{(n)} c_{\frac{1-n}{2}} . \end{aligned} \quad (2.94)$$

Following references [1] and [8], we first introduce these operators in a projective coordinate system Z and subsequently pass over to generic analytic coordinates z .

Theorem 2.14.1 Suppose that $C_{\frac{1-n}{2}}(Z, \bar{Z})$ behaves under Möbius transformations as in eq.(2.93) and that $\mathcal{W}_k(Z)$ transforms analogously for $1 \leq k \leq n$:

$$(\mathcal{W}_k)' = \mathcal{W}_k (cZ + d)^{2k} .$$

For each $\mathcal{W}_k$, define the action of the operator $M_{w_k}^{(n)}$ on $C_{\frac{1-n}{2}}$ by

$$M_{w_k}^{(n)} C_{\frac{1-n}{2}} \equiv \sum_{l=0}^{n-k} \alpha_{kl}^{(n)} (\partial_Z^l \mathcal{W}_k) (\partial_Z^{n-k-l} C_{\frac{1-n}{2}}) , \quad (2.95)$$

where⁵

$$\alpha_{kl}^{(n)} = \frac{\binom{k+l-1}{l} \binom{n-k}{l}}{\binom{2k+l-1}{l}} . \quad (2.96)$$

Then, the differential operator (2.95) transforms covariantly under a projective change of coordinates:

$$(M_{w_k}^{(n)} C_{\frac{1-n}{2}})' = (cZ + d)^{1+n} M_{w_k}^{(n)} C_{\frac{1-n}{2}} . \quad (2.97)$$

⁵The leading coefficient in the expression (2.95) has been normalized by $\alpha_{k0}^{(n)} = 1$ for all k .

We note that $M_{w_k}^{(n)}$ is linear in $\mathcal{W}_k$ and its derivatives up to order $n-k$. For instance,

$$\begin{aligned} M_{w_n}^{(n)} &= \mathcal{W}_n , \quad M_{w_{n-1}}^{(n)} = \mathcal{W}_{n-1} \partial_Z + \frac{1}{2} (\partial_Z \mathcal{W}_{n-1}) \\ M_{w_{n-2}}^{(n)} &= \mathcal{W}_{n-2} \partial_Z^2 + (\partial_Z \mathcal{W}_{n-2}) \partial_Z + \frac{n-1}{2(2n-3)} (\partial_Z^2 \mathcal{W}_{n-2}) , \dots \end{aligned} \quad (2.98)$$

For the sake of motivating the definition (2.95), we remark [1] that this expression can be viewed as the result of a bilinear and covariant map J , namely

$$M_{w_k}^{(n)} C_{\frac{1-n}{2}} \propto J(C_{\frac{1-n}{2}}, \mathcal{W}_k) ,$$

where J is known as *Gordan's transvectant*. Alternatively, the coefficients $\alpha_{kl}^{(n)}$ in eq.(2.95) can be determined by choosing $\alpha_{k0}^{(n)} = 1$ and requiring the lemma below to hold [8].

Next, we pass over to arbitrary holomorphic coordinates z by virtue of eqs.(2.36) and

$$\mathcal{W}_k(Z) = w_k(z) (\partial_Z)^{-k} . \quad (2.99)$$

This leads to the differential operators (2.94) defined by

$$M_{w_k}^{(n)} C_{\frac{1-n}{2}} = (\partial_Z)^{-\frac{1-n}{2}} \left(M_{w_k}^{(n)} c_{\frac{1-n}{2}} \right) , \quad (2.100)$$

or, in operatorial form,

$$\begin{aligned} M_{w_k}^{(n)} &= (\partial_Z)^{\frac{1-n}{2}} M_{w_k}^{(n)} (\partial_Z)^{-\frac{1-n}{2}} \\ &= (\partial_Z)^{\frac{1-n}{2}} \sum_{l=0}^{n-k} \alpha_{kl}^{(n)} \left[\left(\frac{1}{\partial_Z} \partial \right)^l (\partial_Z)^{-k} w_k \right] \left(\frac{1}{\partial_Z} \partial \right)^{n-k-l} (\partial_Z)^{-\frac{1-n}{2}} \\ &= \sum_{l=0}^{n-k} \alpha_{kl}^{(n)} \left[\nabla^l w_k \right] \nabla^{n-k-l} . \end{aligned} \quad (2.101)$$

In the last line, ∇^l denotes the l -th power of the covariant derivative defined by eq.(2.60). By varying $b \equiv \partial \ln \partial Z$ while keeping the Schwarzian $S(Z, z) = \partial b - \frac{1}{2} b^2$ fixed [8], one can show the

Lemma 2.14.1 The operator $M_{w_k}^{(n)}$ defined by eq.(2.101) depends on ∂Z and its derivatives only through $S(Z, z)$, i.e. through a projective connection $R_{zz}(z) \equiv S(Z, z)$.

For example, for the operators (2.98), one finds the explicit expressions

$$\begin{aligned} M_{w_n}^{(n)} &= w_n , \quad M_{w_{n-1}}^{(n)} = w_{n-1} \partial + \frac{1}{2} (\partial w_{n-1}) \\ M_{w_{n-2}}^{(n)} &= w_{n-2} (\partial^2 - \frac{1-n}{2} R) + (\partial w_{n-2}) \partial + \frac{n-1}{2(2n-3)} (\partial^2 - (n-2)R) w_{n-2} \end{aligned} \quad (2.102)$$

By adding the operators $M_{w_k}^{(n)}$ to L_n , we obtain an n -th order operator

$$L \equiv L_n + M_{w_1}^{(n)} + \dots + M_{w_n}^{(n)} , \quad (2.103)$$

for which L_n is the leading term. All terms in the expression (2.103) are separately covariant and the $M_{w_k}^{(n)}$ are linear in w_k and its derivatives up to order $n - k$.

Further covariant operators of order n can be constructed by composition with holomorphic forms. For instance, if $v_{\frac{n-k}{2}}$ and $\tilde{v}_{\frac{n-k}{2}}$ are forms of order $\frac{n-k}{2}$ ($0 \leq k \leq n$) and if $L^{(k)}$ is a covariant operator of order k , then

$$\tilde{v}_{\frac{n-k}{2}} L^{(k)} v_{\frac{n-k}{2}} : \lambda^{\frac{n-k}{2}} \rightarrow \lambda^{\frac{n-k}{2}} \quad (0 \leq k \leq n) \quad (2.104)$$

On surfaces of genus one where covariant derivatives are globally defined, the latter can be used to construct further covariant operators of order n . For example [7], if $L^{(n-2)}$ is covariant and of order $n - 2$, then

$$\nabla_{\frac{1}{2}L} L^{(n-2)} \nabla_{\frac{1}{2}L} : \lambda^{\frac{n-2}{2}} \rightarrow \lambda^{\frac{n-2}{2}} \quad (2.105)$$

(Note that this expression depends on γ and $\partial\gamma$ and not just on the projective connection $\partial\gamma - \frac{1}{2}\gamma^2$.)

As pointed out in reference [1] and explicitly proven in [8], the procedure leading to the expression (2.103) can be reversed to show that essentially all holomorphic covariant operators of order n can be cast into the form (2.103):

Theorem 2.14.2 (Classification of holomorphic, conformally covar. differ. oper.)
Let $L^{(n)}$ be a normalized, holomorphic differential operator of order n (locally given by the expression (2.93)). Then, $L^{(n)}$ maps forms into forms or, more precisely,

$$L^{(n)} : \lambda^{\frac{n}{2}} \rightarrow \lambda^{\frac{n}{2}} ,$$

if the coefficients $a_j^{(n)}$ transform according to the following rule under infinitesimal changes of coordinates $z \rightarrow z' = z - \epsilon(z)$ [8]:

$$\delta a_j^{(n)} = \epsilon \partial a_j^{(n)} + j a_j^{(n)} \partial \epsilon + \frac{1}{2} \binom{n+1}{n-j} (j-1) \partial^{j+1} \epsilon - \sum_{i=2}^{j-1} \left\{ \binom{n-i}{j-i+1} - \frac{n-1}{2} \binom{n-i}{j-i} \right\} a_i^{(n)} \partial^{j-i+1} \epsilon \quad (2.106)$$

The operator $L^{(n)}$ then transforms as in eq.(2.40).

Furthermore, in this case, one can find a reparametrization of the coefficients $a_j^{(n)}$ in terms of holomorphic forms $w_3, \dots, w_n$ of degree 3, ..., n , respectively, such that $L^{(n)}$ is given by the expression (2.103) with $w_1 = 0 = w_2$. More specifically, the coefficient $a_2^{(n)}$ of $L^{(n)}$ coincides with the coefficient $b_2^{(n)}$ of the Bol operator L_n as given by equation (2.41):

$$a_2^{(n)} \equiv b_2^{(n)} = k_n R \quad \text{with} \quad k_n = \frac{n(n^2-1)}{12} \quad (2.107)$$

Moreover, for any k with $3 \leq k \leq n$, the k -form w_k is a linear combination of $a_2^{(n)}, \dots, a_k^{(n)}$ and their derivatives, with coefficients that are polynomials in $a_2^{(n)}$ and its derivatives. The relations between the w_k and $a_2^{(n)}, \dots, a_k^{(n)}$ are invertible and they allow us to express $a_k^{(n)}$

linearly in terms of $a_2^{(n)}, w_3, \dots, w_k$ and their derivatives, with coefficients that are differential polynomials of $a_2^{(n)}$. (For some explicit formulae, we refer to the first of references [8].)

To summarize, any holomorphic and covariant operator of the form (2.93) can be parametrized in terms of a projective connection R and $n-2$ holomorphic forms $w_3, \dots, w_n$ (which are differential polynomials in the $a_j^{(n)}$).

This result seems to go back to the pioneering work of Gordan and Laguerre-Forsyth, see references [30] and [1]. Concerning its proof, we note that the transformation law (2.106) can be derived by virtue of the relation [8]

$$\delta_L L^{(n)} = X_{\frac{1}{2}L} L^{(n)} - L^{(n)} X_{\frac{1}{2}L} , \quad (2.108)$$

where $X_h \equiv \epsilon \partial + h(\partial\epsilon)$ denotes the vector field that generates conformal transformations on fields of weight h . The relation (2.108) simply follows by varying the equation $f_{\frac{1}{2}L} = L^{(n)} f_{\frac{1}{2}L}$ and subsequently substituting $\delta_L f_h = X_h f_h$ in the so-obtained formula.

Examples : (1) In order to illustrate the last theorem, we consider the covariant operator (2.104) for $k=2$ and $L^{(2)} = L_2 \equiv \partial^2 + \frac{1}{2}R$. It can be represented as

$$\tilde{v}_{\frac{n-2}{2}} L_2 v_{\frac{n-2}{2}} = M_{w_{n-2}}^{(n)} + M_{w_{n-1}}^{(n)} + M_{w_n}^{(n)}$$

with holomorphic forms

$$w_{n-2} = \tilde{v}_{\frac{n-2}{2}} v_{\frac{n-2}{2}} , \quad w_{n-1} = \tilde{v}_{\frac{n-2}{2}} (\partial v_{\frac{n-2}{2}}) - v_{\frac{n-2}{2}} (\partial \tilde{v}_{\frac{n-2}{2}}) \\ w_n = \frac{n-2}{2(2n-3)} [\partial^2 - (n-2)R] (\tilde{v}_{\frac{n-2}{2}} v_{\frac{n-2}{2}}) - (\partial \tilde{v}_{\frac{n-2}{2}}) (\partial v_{\frac{n-2}{2}}) .$$

(2) As a second example, let us go to a surface of genus one and consider the fifth order operator defined by eq.(2.105) with $n=5$ and $L^{(n-2)} \equiv v_3$ (a holomorphic 3-form). This covariant operator can be rewritten as $M_{w_3}^{(5)} + M_{w_4}^{(5)} + M_{w_5}^{(5)}$ with

$$w_3 = v_3 , \quad w_4 = 0 \\ w_5 = -\frac{2}{7} (\nabla^2 v_3) - \frac{8}{7} (R-h) v_3 \quad (h \equiv \partial\gamma - \frac{1}{2}\gamma^2) .$$

As stressed in the second of references [8], the parametrization presented in the last theorem is by no means unique. Instead of the w_k , one can consider another set of generators for the ring of those differential polynomials in $a_j^{(n)}$ which are k -forms ($3 \leq k \leq n$). Such a parametrization is provided by the canonical matrix representation of the differential operators (2.93) [8]. The canonical generators W_k are related to the generators w_k by non-linear differential polynomials, all relations being invertible.

In conclusion, we note that the remarks made in section 2.9 concerning non-holomorphic Bol operators also apply to the more general operators studied in this section.

2.14.2 Applications in integrable models

Lax operators

For fixed $n=2, 3, \dots$, the Lax equations (2.70) can be generalized to [7]

$$\bar{\partial} t^{(n)} = [(t^{(n)})^{\frac{k}{n}}_+ , t^{(n)}] \quad (k=1, 2, 3, \dots) \quad (2.109)$$

where $\{^{(n)}\}$ represents a covariant operator of order n with smooth coefficients. The KdV hierarchy corresponds to $n = 2$. By considering a covariant, third-order operator of the type discussed in section 2.14.1,

$$\{^{(3)}\} \equiv \partial^3 + 2u\partial + (\partial u) + w_3, \quad (2.110)$$

one obtains the *Boussinesq sequence* [31] [7]. For instance, from

$$\left(\{^{(3)}\}\right)^{\frac{1}{2}} = \partial + \frac{2}{3}u\frac{1}{\partial} + \frac{1}{3}(w_3 - \partial u)\frac{1}{\partial^2} + \mathcal{O}\left(\frac{1}{\partial^3}\right),$$

we find that the $k = 2$ relation in eqs.(2.109) is equivalent to the set of differential equations

$$\begin{aligned} \partial u &= \partial w_3 \\ -3\partial w_3 &= \partial^3 u + 8u\partial u. \end{aligned} \quad (2.111)$$

Substitution of the first equation into the second yields the ordinary *Boussinesq equation*,

$$-3\partial^2 u = \partial[\partial^3 u + 8u\partial u]. \quad (2.112)$$

Poisson operators

The Hamiltonian equations of motion (2.74) can be generalized to

$$\bar{\partial} \begin{pmatrix} u \\ w_n \end{pmatrix} = \mathcal{D} \begin{pmatrix} \delta H / \delta u \\ \delta H / \delta w_n \end{pmatrix}. \quad (2.113)$$

Here, w_n is a smooth conformal field of weight n and the Poisson operator $\mathcal{D}$ represents a matrix of differential operators [7] [29]. For instance, for $n = 3$ and

$$H = \int_{-\infty}^{+\infty} dz \left\{ \frac{1}{2}w_3^2 - \frac{4}{9}u^3 + \frac{1}{6}(\partial u)^2 \right\}, \quad \mathcal{D} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix},$$

we recover the $k = 2$ equations of the Boussinesq sequence, i.e. eqs.(2.111). For $n = 3$ and H as in the last equation, but

$$\mathcal{D} = \begin{pmatrix} (u, u) & (u, w_3) \\ (w_3, u) & (w_3, w_3) \end{pmatrix} \equiv \begin{pmatrix} L_3 & 3w_3\partial + 2(\partial w_3) \\ 3w_3\partial + (\partial w_3) & L_5 \end{pmatrix}$$

(where L_3, L_5 denote the *Bol* operators with $R(z)$ substituted by $u(z, \bar{z})$), one obtains the $k = 5$ relations of the Boussinesq sequence [7].

2.14.3 W -algebras

The variables $a_2^{(n)}$ and w_k (with $3 \leq k \leq n$) which parametrize covariant operators of order n according to theorem 2.14.2 can be viewed as generators of the so-called *W_n -algebra* [32] [7] [8]. As pointed out in section 2.13.3, the coefficient $a_2^{(n)} = \bar{b}_2^{(n)}$ which is a multiple of a Schwarzian connection generates the classical *Virasoro algebra* through the Poisson brackets

$$\{a_2^{(n)}(z'), a_2^{(n)}(z)\} = \left[k_n \partial^3 + 2a_2^{(n)}\partial + (\partial a_2^{(n)}) \right] \delta(z - z'). \quad (2.114)$$

The bracket of $a_2^{(n)}$ with w_k expresses the fact that w_k is a form of degree k :

$$\{a_2^{(n)}(z'), w_k(z)\} = [(\partial w_k) + k w_k \partial] \delta(z - z') \quad (3 \leq k \leq n). \quad (2.115)$$

For the brackets between the w_k and for the corresponding central term, we refer to the expressions derived in reference [8]. Here, we only quote the result for $n = 3 = k$ [7] [8],

$$\{w_3(z'), w_3(z)\} = -\frac{1}{6} L_5 \delta(z - z'), \quad (2.116)$$

in which equation L_5 denotes the fifth-order *Bol* operator. This relation coincides with the second Poisson brackets for the Boussinesq equation [7].

The reader interested in the wide range of applications of W_n -algebras may consult the review article [33].

2.14.4 Coupling to Yang-Mills fields

A natural generalization of the conformally covariant operators discussed so far consists of the coupling to a non-Abelian gauge field [34]. Locally, the latter is represented by a vector field with components

$$A \equiv A^a T_a, \quad \bar{A} \equiv \bar{A}^a T_a,$$

where T_a are the generators of a compact Lie group G . Under conformal changes of coordinates, A and $\bar{A}$ transform like conformal fields of weight one (or, more precisely, $(1,0)$ and $(0,1)$, which means that they transform with $\partial z'$ and $\bar{\partial} z'$, respectively). Differential operators which are *covariant with respect to both conformal and gauge transformations* are obtained from the covariant operator $L^{(n)}$ by the substitution

$$\partial \longrightarrow \partial + A. \quad (2.117)$$

They act on forms belonging to a linear representation of the group G .

More specifically, the coupling of $\tilde{l}^{(2)} \equiv \partial^2 + \frac{1}{2}u$ and $\tilde{l}^{(3)} \equiv \partial^3 + 2u\partial + (\partial u) + w_3$ to Yang-Mills fields was discussed by Das and Roy [34]. If $\tilde{l}^{(2)}$ and $\tilde{l}^{(3)}$ denote the operators obtained by the substitution (2.117), then one can postulate the Lax equations

$$(\bar{\partial} + \bar{A}) \tilde{l}^{(2)} = \{(\tilde{l}^{(2)})^{\frac{2k+1}{2}}\}_+, \tilde{l}^{(2)} \quad (k = 0, 1, 2, \dots) \quad (2.118)$$

$$(\bar{\partial} + \bar{A}) \tilde{l}^{(3)} = \{(\tilde{l}^{(3)})^{\frac{k}{3}}\}_+, \tilde{l}^{(3)} \quad (k = 1, 2, 3, \dots). \quad (2.119)$$

As was shown for all values of k in the first equation and for $k = 1, 2$ in the second, these relations are equivalent to the evolution equations without Yang-Mills fields, if the latter satisfy $\bar{A} = 0$ and $\bar{\partial} A = 0$ [34]. The corresponding Schrödinger equations,

$$\tilde{l}^{(2)} c_{-\frac{1}{2}} = 0, \quad \tilde{l}^{(3)} c_{-1} = 0, \quad (2.119)$$

admit as symmetry algebras the *Virasoro-Kac-Moody algebra* and the *spin 3-Kac-Moody algebra*, respectively.

Chapter 3

Supersymmetric theory

For brevity, we have suppressed the analytic subtleties in the supersymmetric (SUSY) case; the mathematically minded reader should have no problem appending them once he/she has chosen his/her favourite category of supermanifolds, see e.g. [35] and references therein.

The arena we will work on in this chapter is a compact super Riemann surface (SRS) of genus $g \geq 0$ [36] [37] [35]. By definition, a SRS is a supermanifold of real dimension $2|2$ which is equipped with a *superconformal structure*. Roughly speaking, this means that it admits local coordinates $(z, \bar{z}, \theta, \bar{\theta})$ and that any two sets of such coordinates, $(z, \bar{z}, \theta, \bar{\theta})$ and $(z', \bar{z}', \theta', \bar{\theta}')$, are related by a superconformal transformation, i.e. a smooth⁶ mapping

$$(z, \bar{z}, \theta, \bar{\theta}) \longrightarrow (z', \bar{z}', \theta', \bar{\theta}')$$

with the properties

$$z' = z'(z, \theta), \quad \theta' = \theta'(z, \theta) \quad \text{and c.c.}$$

$$Dz' = \theta' D\theta' \quad \text{and c.c.} \quad (3.1)$$

where $D \equiv \partial/\partial\theta + \theta \partial/\partial z$. (An explicit solution of the last equation can easily be given in terms of ordinary space-time fields [37].)

In order to avoid an excessively cumbersome notation, we will often drop the prefix ‘super’ and refer to the superconformal coordinates as holomorphic ones and so on.

For later reference, we remark that application of D to the last equation leads to the relation

$$(D\theta')^2 = \partial z' + \theta' \partial\theta' \quad (3.2)$$

Using the notation

$$e^{-w} \equiv D\theta' \quad (3.3)$$

the induced variations of D and ∂ take the form

$$D' = e^w D \\ \partial' = e^{2w} [\partial + (Dw)D]$$

⁶By “smooth”, we mean the appropriate generalization of C^∞ to the category of supermanifolds one is working in.

3.1 Super Schwarzian derivative

The *super Schwarzian derivative* [36] of an odd, smooth function $\Theta(z, \bar{z}, \theta, \bar{\theta})$ satisfying $D\Theta \neq 0$ is defined by

$$\begin{aligned} S(\Theta; z, \bar{z}, \theta, \bar{\theta}) &\equiv \frac{\partial^2 \Theta}{\partial \Theta} - 2 \frac{(\partial \Theta)(\partial D\Theta)}{(D\Theta)^2} \\ &= -[\partial DW + (\partial W)(DW)] \\ &= -e^{-w} \partial D e^w, \end{aligned} \quad (3.4)$$

where W is given by $e^{-W} \equiv D\Theta$. If one considers a superconformal change of local coordinates from $(z, \bar{z}, \theta, \bar{\theta})$ to $(z', \bar{z}', \theta', \bar{\theta}')$, the super Schwarzian can be verified to satisfy the ‘chain rule’ [36]

$$S(\Theta; z, \theta) = S(\Theta; z', \theta') e^{-3w} + S(\theta'; z, \theta) \quad (3.5)$$

with w defined by eq.(3.3).

In the remainder of the text, the smooth function $\Theta(z, \bar{z}, \theta, \bar{\theta})$ will always be part of a smooth mapping

$$(z, \bar{z}, \theta, \bar{\theta}) \longrightarrow (Z(z, \bar{z}, \theta, \bar{\theta}), \bar{Z}(z, \bar{z}, \theta, \bar{\theta}), \Theta(z, \bar{z}, \theta, \bar{\theta}), \bar{\Theta}(z, \bar{z}, \theta, \bar{\theta})) \quad (3.6)$$

where $(z, \bar{z}, \theta, \bar{\theta})$ are superconformal coordinates; the super Schwarzian derivative (3.4) will be denoted by $S(Z, \bar{Z}; z, \theta)$. Choosing $Z = z$ and $\bar{Z} = \bar{z}$, the chain rule (3.5) then implies the following relation for a superconformal change of coordinates:

$$S(z', \theta'; z, \theta) = -e^{-3w} S(z, \theta; z', \theta') \quad (3.7)$$

If the smooth mapping (3.6) satisfies the ‘superconformal condition’ $DZ = \Theta D\Theta$, it also satisfies the relation $(D\Theta)^2 = \partial Z + \Theta \partial\Theta$ by virtue of which the super Schwarzian derivative contains the ordinary one: $DS(Z, \bar{Z}; z, \theta) = 1/2 S(\bar{Z}; z)$ plus SUSY contributions⁷.

3.2 Superprojective connections

We make the following definition: a *superprojective connection* is a collection of odd superfields $\{\mathcal{R}_{z\theta}(z, \bar{z}, \theta, \bar{\theta})\}$ (one for each coordinate system $(z, \bar{z}, \theta, \bar{\theta})$) which are locally superanalytic (i.e. $D\mathcal{R}_{z\theta} = 0$) and which transform with the super Schwarzian derivative under a superconformal change of coordinates:

$$\mathcal{R}_{z\theta'}(z', \theta') = e^{3w} [\mathcal{R}_{z\theta}(z, \theta) - S(z', \theta'; z, \theta)] \quad (3.8)$$

From $\bar{D}^2 = \bar{\partial}$ and $\bar{D}\mathcal{R}_{z\theta} = 0$, it follows that $\bar{\partial}\mathcal{R}_{z\theta} = 0$. Thus, the superfield $\mathcal{R}_{z\theta}$ admits a θ -expansion of the form

$$\mathcal{R}_{z\theta}(z, \theta) = \frac{i}{2} \rho_{z\theta}(z) + \theta \left[\frac{1}{2} R_{zz}(z) \right]$$

⁷Here and in the following, the bar denotes the projection of the superfield onto its lowest component.

and eq.(3.8) implies that the component R_{zz} transforms like an ordinary projective connection up to supersymmetric contributions. A constructive proof for the existence of superprojective connections on compact SRS's is presented in Appendix B.1.

The difference of two superprojective connections is a superanalytic quadratic differential and therefore these connections define a superaffine space of dimension $3g - 3 | 2g - 2$.

3.3 Covariant differential operators: a simple example

We proceed along the lines of the bosonic theory (section 2.3). Consider two smooth tensor fields on the SRS Σ whose components transform according to

$$\begin{aligned} C^z &= e^{-2w} C^z \\ H_{\theta'}^z &= e^w e^{-2w} H_{\theta'}^z \end{aligned} \quad (3.9)$$

Concrete examples for such tensors are provided by the geometry of SRS's: the supercomplex structures on these surfaces are parametrized by two odd superfields, $H_{\theta'}^z$ and H_{θ}^z , and the complex conjugate variables (cf. Appendix B.2). If the geometry is restricted by $H_{\theta}^z = 0$ (and c.c.), then $H_{\theta'}^z$ transforms as in eq.(3.9) under superconformal changes of coordinates; furthermore, by combining the components of a (superdiffeomorphism generating) vector field in a natural way, one obtains a tensor transforming as C^z , see Appendix B.2.

From

$$\begin{aligned} C^z &\equiv c^z, & (\bar{D} H_{\theta'}^z) &\equiv \mu_{\bar{z}}^z \\ (DC^z) &\equiv i \epsilon^{\theta} & (D \bar{D} H_{\theta'}^z) &\equiv i \alpha_{\bar{z}}^{\theta} \end{aligned} \quad (3.10)$$

it follows that the Berezinian integral

$$\int d^4 z \mathcal{A}(C^z, H_{\theta'}^z) \equiv \int d^4 z C^z D^5 H_{\theta'}^z = \int d^4 z C^z \partial^2 D H_{\theta'}^z = - \int d^4 z (\partial^2 D C^z) H_{\theta'}^z \quad (3.11)$$

contains the ordinary space integral $\int d^2 z c \partial^3 \mu$ considered in section 2.3. Here, the problem is similar: the expression (3.11) is not well defined on a compact SRS, since the integrand does not transform with the Jacobian upon passage from one coordinate chart to another.

One finds

$$(\partial^2 D C^z)' = e^{3w} \left[\partial^2 D + 3S \partial + (DS) D + 2(\partial S) \right] C^z,$$

where $S \equiv S(z', \theta'; z, \theta)$ denotes the super Schwarzian derivative defined in eq.(3.4).

As suggested by the gauging principle, the differential operator $\partial^2 D$ can be modified by the inclusion of a superprojective connection so as to transform tensorially when acting on C^z . In fact, the modified expression

$$\mathcal{L}_2 C^z \equiv \left[\partial^2 D + 3\mathcal{R} \partial + (D\mathcal{R}) D + 2(\partial\mathcal{R}) \right] C^z \quad (3.12)$$

transforms properly,

$$(\mathcal{L}_2 C^z)' = e^{3w} \mathcal{L}_2 C^z, \quad (3.13)$$

and so does the modified integrand $\tilde{\mathcal{A}}(C^z; H_{\theta'}^z) \equiv -(\mathcal{L}_2 C^z) H_{\theta'}^z$. The differential operator $\mathcal{L}_2$ represents the covariantization of $\partial^2 D$. It anticommutes with $\bar{D}$ and it is Hermitian in the sense that

$$\int_{S\Sigma} d^4 z C^z (\mathcal{L}_2 H_{\theta'}^z) = \int_{S\Sigma} d^4 z (\mathcal{L}_2 C^z) H_{\theta'}^z. \quad (3.14)$$

By substituting the component field expressions (3.10), one finds

$$\begin{aligned} \int_{S\Sigma} d^4 z \tilde{\mathcal{A}}(C^z; H_{\theta'}^z) &= \int_{\Sigma} d^2 z (D \bar{D} \tilde{\mathcal{A}}) \\ &= \int_{\Sigma} d^2 z \left\{ \epsilon \left[L_2 \alpha + \left(\frac{3}{2} \rho \partial + \partial \rho \right) \mu \right] \right. \\ &\quad \left. - c \left[L_3 \mu + \left(\frac{3}{2} \rho \partial + \frac{1}{2} \partial \rho \right) \alpha \right] \right. \\ &\quad \left. + \text{terms in } (\bar{D} C^z), (D \bar{D} C^z) \right\}, \end{aligned} \quad (3.15)$$

where L_2 and L_3 are the familiar Bol operators, see eq.(2.32).

3.4 Covariant differential operators continued

Superconformal fields with n upper θ -indices transform as

$$(C_{-n})' = e^{-nw} C_{-n}, \quad (3.16)$$

a z -index being identified with two θ -indices [36]. They can be viewed as components of the invariant expressions $(E^{\theta})^n C_n$ where E^{θ} is a generator transforming as

$$(E^{\theta})' = e^{-w} E^{\theta}. \quad (3.17)$$

(In the terminology of reference [35], C_n and $\bar{C}_n$ represent $(\frac{n}{2}, 0)$ and $(0, \frac{n}{2})$ superdifferentials, respectively; E^{θ} corresponds to a smooth section of the so-called Berezinian bundle which has transition functions $e^{-w} = D\theta'$.) We suppose that C_{-n} is even (odd) for n even (odd).

Following the procedure of the last section, we can determine the super Bol operator $\mathcal{L}_n$, i.e. the superconformally covariant version of D^{2n+1} . We find, for instance,

$$\begin{aligned} \mathcal{L}_0 &= D \\ \mathcal{L}_1 &= \partial D + \mathcal{R} \\ \mathcal{L}_2 &= \partial^2 D + 3\mathcal{R} \partial + (D\mathcal{R}) D + 2(\partial\mathcal{R}) \\ \mathcal{L}_3 &= \partial^3 D + 6\mathcal{R} \partial^2 + 4(D\mathcal{R}) \partial D + 8(\partial\mathcal{R}) \partial + 2(\partial D\mathcal{R}) D + 3(\partial^2 \mathcal{R} + 3\mathcal{R} D\mathcal{R}) \\ \mathcal{L}_4 &= \partial^4 D + 10\mathcal{R} \partial^3 + 10(D\mathcal{R}) \partial^2 D + 20(\partial\mathcal{R}) \partial^2 + 10(\partial D\mathcal{R}) \partial D + 5(3\partial^2 \mathcal{R} + 11\mathcal{R} D\mathcal{R}) \partial \\ &\quad + (3\partial^2 D\mathcal{R} + 9(D\mathcal{R})^2 + (\partial\mathcal{R})\mathcal{R}) D + 4(\partial^3 \mathcal{R} + 7\mathcal{R} \partial D\mathcal{R} + 9(D\mathcal{R})(\partial\mathcal{R})) \end{aligned} \quad (3.18)$$

Thanks to their covariance, these operators give a well-defined meaning to the superspace integrals

$$\int_{S\Sigma} d^4 z C_{-n} (\mathcal{L}_n H_{-n}) = - \int_{S\Sigma} d^4 z (\mathcal{L}_n C_{-n}) H_{-n}, \quad (3.19)$$

where H_{-n} transforms according to

$$(H_{-n})' = e^{\bar{\psi}} e^{-n\psi} H_{-n} .$$

(Since H_{-n} has an extra $\bar{\psi}$ -index, it is odd (even) for n even (odd).) For example, for $n = 1$, we are dealing with an expression of the form $\int_{\Sigma} d^4 z C^{\theta} (L_1 H_{\theta}^{\theta})$. In this case, we define component fields by

$$\begin{aligned} C^{\theta} &\equiv i\gamma^{\theta} , & (\bar{D} H_{\theta}^{\theta}) &\equiv i\beta_z^{\theta} \\ (DC^{\theta}) &\equiv a , & (\bar{D} D H_{\theta}^{\theta}) &\equiv v_z \end{aligned} \quad (3.20)$$

and we obtain

$$\begin{aligned} \int_{\Sigma} d^4 z C^{\theta} (L_1 H_{\theta}^{\theta}) &= - \int_{\Sigma} d^2 z \left\{ a \left[L_1 v_z + \frac{1}{2} \rho \beta_z^{\theta} \right] - \gamma^{\theta} \left[L_2 \beta_z^{\theta} + \frac{1}{2} \rho v_z \right] \right. \\ &\quad \left. + \text{terms in } (\bar{D} C^{\theta}) , (\bar{D} \bar{D} C^{\theta}) \right\} \end{aligned} \quad (3.21)$$

where L_1 and L_2 are the ordinary Bôl operators given by eq.(2.32).

Analogously, for $n = 3$, we introduce fields in ordinary space by

$$\begin{aligned} C^{z\theta} &\equiv i\eta^{z\theta} , & (\bar{D} H_{\theta}^{z\theta}) &\equiv i\tau_z^{z\theta} \\ (DC^{z\theta}) &\equiv k^z , & (\bar{D} D H_{\theta}^{z\theta}) &\equiv v_z^z \end{aligned} \quad (3.22)$$

and we get

$$\begin{aligned} \int_{\Sigma} d^4 z C^{z\theta} (L_3 H_{\theta}^{z\theta}) &= - \int_{\Sigma} d^2 z \left\{ k^z \left[L_3 v + \left(3\rho\partial^2 + 4(\partial\rho)\partial + \frac{3}{2}(\partial^2\rho + \frac{3}{2}\rho R) \right) \tau \right] \right. \\ &\quad \left. - \eta^{z\theta} \left[L_4 \tau + \frac{9}{4}\rho(\partial\rho)\tau + \left(3\rho\partial^2 + 2(\partial\rho)\partial + \frac{1}{2}(\partial^2\rho + 3\rho R) \right) v \right] \right. \\ &\quad \left. + \text{terms in } (\bar{D} C^{z\theta}) , (\bar{D} \bar{D} C^{z\theta}) \right\} . \end{aligned} \quad (3.23)$$

Quite generally, the scalar equation $F_n = L_n f_n$ (where f_n transforms like C_{-n}) is equivalent to a set of equations of first order in D and ∂ . This system can be cast into matrix form, e.g., for $n = 1$,

$$\begin{pmatrix} F_1 \\ 0 \end{pmatrix} = \begin{pmatrix} D & \mathcal{R} \\ -1 & \partial \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} .$$

The general case will be discussed in section 3.8 and provides an extremely efficient way for determining explicit expressions for the operators L_n .

3.5 Superprojective structures

Superprojective mappings are defined by generalizing lemma 2.4.1 to the SUSY case [36]: by definition, a superconformal change of coordinates $(Z, \Theta) \rightarrow (Z'(Z, \Theta), \Theta'(Z, \Theta))$ is a

^aWe note that the transformation laws of C^{θ} and H_{θ}^{θ} differ from those of the variables C^{θ} and H_{θ}^{θ} considered in Appendix B.2.

superprojective transformation, if $S(Z', \Theta'; Z, \Theta) = 0$. By virtue of eq.(3.4), this condition is equivalent to

$$\partial_Z D_{\Theta} (D_{\Theta} \Theta')^{-1} = 0 ,$$

which equation has a general solution

$$D_{\Theta} \Theta' = (cZ + d + \Theta\gamma)^{-1} \quad (3.24)$$

with $c, d \in \mathbb{C}$ and γ an odd Grassmann number.

3.6 Covariant operators in superprojective coordinates

The superprojective coordinates (Z, Θ) are related to the holomorphic coordinates (z, θ) by a superconformal transformation upon which the conformal field C^z becomes a field

$$C^z = C^z (\partial Z + \Theta\partial\Theta) = C^z (D\Theta)^2 \quad (3.25)$$

transforming as

$$C^{z'} = C^z (cZ + d + \Theta\gamma)^{-2} . \quad (3.26)$$

The transformation law of $\partial_z^2 D_{\Theta} C^z$ under superprojective changes of coordinates is homogeneous, but upon passage to arbitrary holomorphic coordinates (z, θ) by a superconformal transformation, one has

$$\partial_z^2 D_{\Theta} C^z = (D\Theta)^{-3} \left[\partial^2 D + 3\mathcal{R}\partial + (D\mathcal{R})D + 2(\partial\mathcal{R}) \right] C^z . \quad (3.27)$$

Here, the superanalytic expression

$$\mathcal{R}_{z\theta}(z, \theta) \equiv \mathcal{S}(Z, \Theta; z, \theta) \quad (3.28)$$

has the correct transformation properties by virtue of the chain rule for the super Schwarzian derivative; it can eventually be rewritten in terms of a supervector field component $F \equiv F^{\theta} \equiv (D_{\theta}\Theta)^{-1}$ as

$$\mathcal{R}_{z\theta}(z, \theta) = -F^{-1} \partial D F . \quad (3.29)$$

3.7 Dependence on the supercomplex structure

In this section, the interpretation of Z and Θ differs from the one considered in the last two sections.

Following Appendix B.2, we consider holomorphic coordinates Z, Θ which represent the supercomplex structure parametrized by the Beltrami superfields H_{θ}^z and $H_{\theta}^z = 0$. Locally, $(Z, \bar{Z}, \Theta, \bar{\Theta})$ are related to the reference coordinates $(z, \bar{z}, \theta, \bar{\theta})$ by a smooth invertible mapping satisfying the 'superconformal condition' $DZ = \Theta D\Theta$ (and c.c.): the latter is equivalent to $H_{\theta}^z = 0$ and it implies $(D\Theta)^2 = \partial Z + \Theta\partial\Theta$.

Now, suppose the superconformal field $C^{\theta}(z, \bar{z}, \theta, \bar{\theta})$ is related to $\hat{C}^{\theta}(Z, \bar{Z}, \Theta, \bar{\Theta})$ by

$$\hat{C}^{\theta} \equiv C^{\theta} (D\Theta) \quad (3.30)$$

and $\hat{C}^\theta$ is locally superanalytic, i.e.

$$D_\theta \hat{C}^\theta = 0. \quad (3.31)$$

Then, C^θ satisfies the holomorphy condition

$$\left[\bar{D} - H_\theta^z \partial + \frac{1}{2} (DH_\theta^z) D + \frac{1}{2} (\partial H_\theta^z) \right] C^\theta = 0 \quad (3.32)$$

and we have the covariant transformation law

$$[\partial_Z D_\theta + \mathcal{R}_{Z\theta}] \hat{C}^\theta = (D\Theta)^{-2} [\partial D + \mathcal{R}_{\theta\theta}] C^\theta. \quad (3.33)$$

Here, $\mathcal{R}_{\theta\theta}$ is defined by

$$\mathcal{R}_{Z\theta}(Z, \Theta) = (D\Theta)^{-3} [\mathcal{R}_{\theta\theta}(z, \bar{z}, \theta, \bar{\theta}) - S(Z, \Theta; z, \theta)] \quad (3.34)$$

and $D_\theta \mathcal{R}_{Z\theta} = 0$ implies

$$\begin{aligned} \left[\bar{D} - H_\theta^z \partial + \frac{1}{2} (DH_\theta^z) D - \frac{3}{2} (\partial H_\theta^z) \right] \mathcal{R}_{\theta\theta} &= \left[\bar{D} - H_\theta^z \partial + \frac{1}{2} (DH_\theta^z) D - \frac{3}{2} (\partial H_\theta^z) \right] S \\ &= -\frac{1}{2} \partial^2 D H_\theta^z, \end{aligned} \quad (3.35)$$

where $S \equiv S(Z, \Theta; z, \theta)$. The relations (3.32), (3.33) and (3.35) can be derived in a straightforward way by using eq.(B.17) for $\Lambda \equiv (D\Theta)^2$ and expressing the derivatives ∂_Z, D_θ and D_θ in terms of the so-called tilde coordinates, see eqs.(3.61)-(3.64) of reference [38].

3.8 Mathematical properties

We assume that the compact SRS $\Sigma\Sigma$ is equipped with a superprojective structure or, equivalently, supplemented with a superprojective connection $\mathcal{R}$ (both being related by eq.(3.28)).

In an atlas of superprojective coordinates (denoted by capital letters, $(Z, \Theta), (Z', \Theta'), \dots$), the operator $D_\theta^{2n+1} \equiv (\partial/\partial\Theta + \Theta\partial/\partial Z)^{2n+1}$ transforms homogeneously since $S(Z', \Theta'; Z, \Theta) = 0$:

Lemma 3.8.1 (Super Bol lemma) *Consider a superprojective atlas on $\Sigma\Sigma$ with coordinate transformations (3.24). If the function $C_{-n}(Z, \Theta)$ transforms according to the rule*

$$(C_{-n})' = C_{-n} (cZ + d + \Theta\gamma)^{-n} \quad (n = 0, 1, 2, \dots), \quad (3.36)$$

then $D_\theta^{2n+1} C_{-n}$ transforms according to

$$(D_\theta^{2n+1} C_{-n})' = (cZ + d + \Theta\gamma)^{n+1} D_\theta^{2n+1} C_{-n}. \quad (3.37)$$

Upon passage to an arbitrary system of superconformal coordinates, the operator D_θ^{2n+1} becomes the super Bol operator $\mathcal{L}_n$:

$$D_\theta^{2n+1} C_{-n} = (D\Theta)^{-(n+1)} \mathcal{L}_n C_{-n} \quad \text{where} \quad C_{-n} \equiv C_{-n}(D\Theta)^n. \quad (3.38)$$

In operatorial form, we have :

$$\mathcal{L}_n = (D\Theta)^{n+1} \left(\frac{1}{D\Theta} D \right)^{2n+1} (D\Theta)^n. \quad (3.39)$$

Denoting the space of superconformal fields transforming as in eq.(3.16) by Λ^{-n} , we have the following

Theorem 3.8.1 *For $n = 0, 1, 2, \dots$, the super Bol operator $\mathcal{L}_n$ (defined by eq.(3.39)) is a linear differential operator between spaces of superconformal fields on $\Sigma\Sigma$:*

$$\mathcal{L}_n : \Lambda^{-n} \longrightarrow \Lambda^{n+1}. \quad (3.40)$$

Locally, it has the form

$$\mathcal{L}_n = D^{2n+1} + \sum_{j=3}^{2n+1} B_j^{(n)} D^{2n+1-j}, \quad (3.41)$$

where the coefficients $B_j^{(n)}$ ($j = 3, \dots, 2n+1$) are certain universal polynomials in $\mathcal{R}, D\mathcal{R}, D^2\mathcal{R}, \dots, D^j\mathcal{R}$ (i.e. the $\mathcal{L}_n$ depend on a superprojective connection $\mathcal{R}$ on $\Sigma\Sigma$ except for $\mathcal{L}_0 = D$).

For a given n , the transformation laws of the $B_j^{(n)}$ are such that $\mathcal{L}_n$ transforms covariantly, i.e. maps superconformal fields into superconformal fields :

$$\mathcal{L}'_n = e^{(n+1)\omega} \mathcal{L}_n e^{n\omega}. \quad (3.42)$$

The coefficients $B_3^{(n)}$ and $B_4^{(n)}$ are explicitly given by

$$\begin{aligned} B_3^{(n)} &= \iota_n \mathcal{R} & \text{with} & \quad \iota_n = \frac{n(n+1)}{2} \\ B_4^{(n)} &= \kappa_n D\mathcal{R} & \text{with} & \quad \kappa_n = \frac{n(n^2-1)}{6}. \end{aligned} \quad (3.43)$$

The factors ι_n and κ_n add up to the factor (2.41) of the bosonic theory :

$$\frac{1}{2} (\iota_n + \kappa_n) = \kappa_{n+1}. \quad (3.44)$$

The transformation laws of $B_3^{(n)}$ and $B_4^{(n)}$ follow from the one of $\mathcal{R}$, e.g.

$$(B_3^{(n)})' = e^{3\omega} [B_3^{(n)} - \iota_n S(z', \theta'; z, \theta)]. \quad (3.45)$$

The operator $\mathcal{L}_n$ commutes with $\bar{D}$ and has the same symmetry properties as D^{2n+1} , see eq.(3.19).

We now present an elegant and efficient way for evaluating the operators $\mathcal{L}_n$.

Theorem 3.8.2 (Matrix representation) For $n = 0, 1, 2, \dots$, the super Bol operator $\mathcal{L}_n$ is given by the scalar equation

$$F_n = \mathcal{L}_n f_n \quad (f_n \in \Lambda^{-n}) \quad (3.46)$$

which is obtained after elimination of $f_0, \dots, f_{n-1}$ from the matrix equation

$$(F_n, 0, \dots, 0)^T = \hat{Q}_n (f_0, f_1, \dots, f_n)^T \quad (3.47)$$

where

$$\hat{Q}_n \equiv \begin{pmatrix} D & n\mathcal{R} & 0 & \dots & \dots & 0 \\ -1 & \partial & 2(n-1)(D\mathcal{R}) - (n-1)\mathcal{R}D & 0 & \dots & \vdots \\ 0 & -1 & \partial & 3(n-2)(D\mathcal{R}) - (n-2)\mathcal{R}D & \dots & \vdots \\ \vdots & 0 & -1 & \partial & \dots & 0 \\ \vdots & \vdots & 0 & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & -1 & \partial \end{pmatrix} \quad (3.48)$$

The proof of this result proceeds along the lines of the bosonic theory.

Theorem 3.8.3 (Recurrence relation) Let F_1, F_2 be a basis of the space of Grassmann-odd solutions of the differential equation $\mathcal{L}_1 F = 0$:

$$(\partial D + \mathcal{R}) F_i(z, \theta) = 0 \quad (i = 1, 2) \quad (3.49)$$

(By application of D , it follows that

$$(\partial^2 - \mathcal{R}D + (D\mathcal{R})) F_i = 0 \quad (i = 1, 2) \quad (3.50)$$

Introducing odd parameters ξ_i , we define a homogeneous polynomial of degree $k \geq 0$ by

$$f^{(k)}(z, \theta) \equiv (\xi \cdot F(z, \theta))^k \equiv \left(\sum_{i=1}^2 \xi_i F_i(z, \theta) \right)^k \quad (3.51)$$

Furthermore, for $n, k \in \{0, 1, 2, \dots\}$, we define differential operators $\mathcal{P}_n^{(k)}$ by

$$\mathcal{P}_n^{(k)} f^{(k)} = -k(k-1) \dots (k-(n-1)) (\xi \cdot F)^{k-n} (\xi \cdot \partial F)^{n-1} (\xi \cdot D F) \quad (n \geq 1, k \neq 0, 1, \dots, n-1) \quad (3.52)$$

Then, $\mathcal{P}_n^{(k)} = D^{2n-1} + \dots$ is a linear differential operator whose coefficients only depend on $\mathcal{R}, D\mathcal{R}, \dots, D^{2n-1}\mathcal{R}$. For $n \geq 2$, it satisfies the recurrence relation

$$\mathcal{P}_{n+1}^{(k)} = \partial \mathcal{P}_n^{(k)} + (k-(n-1)) [\mathcal{R}D\mathcal{P}_{n-1}^{(k)} + (n-1)(D\mathcal{R})\mathcal{P}_{n-1}^{(k)}] \quad (k \text{ fixed}, k \neq 0, 1, \dots, n) \quad (3.53)$$

The super Bol operators are obtained from the solutions of this relation by setting

$$\mathcal{L}_n = \mathcal{P}_{n+1}^{(k=n)} \quad (3.54)$$

(Note that this operator acts on $f^{(n)}$ and that $f^{(n)} \in \Lambda^{-n}$ for $F_i \in \Lambda^{-1}$.)

Proof: The argumentation is analogous to the one for theorem 2.8.3. It makes use of the relations (3.49) and (3.50) and of the fact that $\mathcal{R}$ and $\xi \cdot DF$ are odd variables. q.e.d.

Starting from $\partial D\psi = -\mathcal{R}\psi$, one can determine all the operators $\mathcal{L}_n$ by evaluating $\partial^n D\psi^n$.

Theorem 3.8.4 Suppose ψ is even and satisfies $\mathcal{L}_1\psi = 0$. Then, $\mathcal{L}_n\psi^n = 0$ for $n = 1, 2, \dots$

The operator $\mathcal{L}_n$ which is a product of $2n+1$ operators in a projective atlas can be factorized in a similar way in an arbitrary system of superanalytic coordinates. In order to describe this decomposition, we first introduce the supersymmetric generalization of the concepts of covariant derivative and affine connection.

A *superaffine connection* is a collection $\{\Gamma_\theta(z, \bar{z}, \theta, \bar{\theta})\}$ of odd superfields which are locally superanalytic and which transform under a superconformal change of coordinates as

$$\Gamma_\theta(z', \theta') = e^w [\Gamma_\theta(z, \theta) - D_\theta w] \quad (3.55)$$

If Γ_θ is a superaffine connection, then

$$\mathcal{R}_{z\theta} \equiv -[\partial\Gamma_\theta + \Gamma_\theta(D\Gamma_\theta)] \quad (3.56)$$

represents a superprojective connection. The last equation is formally identical to the *super Miura transformation* which is relevant for the study of super KdV equations [39].

Given a superaffine connection, one can define a *covariant derivative* which maps p -tensors into $(p+1)$ -tensors:

$$\begin{aligned} \nabla_{(p)} : \Lambda^p &\longrightarrow \Lambda^{p+1} \\ C_p &\longrightarrow \nabla_{(p)} C_p \equiv (D_\theta + p\Gamma_\theta) C_p \end{aligned} \quad (3.57)$$

Theorem 3.8.5 (Factorisation of super Bol operators) Suppose the projective connection $\mathcal{R}$ comes from an affine connection Γ_θ as in equation (3.56). Then, we have

$$\begin{aligned} \mathcal{L}_n &= \nabla_{(n)} \nabla_{(n-1)} \dots \nabla_{(-n)} \\ &= [D + n\Gamma_\theta] [D + (n-1)\Gamma_\theta] \dots [D - n\Gamma_\theta] \end{aligned} \quad (3.58)$$

For instance, $\mathcal{L}_1 = (D + \Gamma_\theta) D (D - \Gamma_\theta)$. From eqs. (3.39) and (3.58), we get the following formal expression for the covariant derivative:

$$\nabla_{(p)} = (D\Theta)^p D (D\Theta)^{-p} = D - p(D \ln D\Theta) \quad (3.59)$$

Theorem 3.8.6 (Dependence on the complex structure) Consider the holomorphic coordinates $Z, \bar{Z}$ which represent the supercomplex structure parametrized by the Beltrami superfields H_θ^z and $H_\theta^{\bar{z}} = 0$. Locally, the coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$ are related to the reference coordinates $(z, \bar{z}, \theta, \bar{\theta})$ by a smooth invertible mapping satisfying $DZ = \Theta D\Theta$ and c.c.; using this mapping, superconformal fields C_{-n} are defined by

$$\hat{C}_{-n}(Z, \bar{Z}, \Theta, \bar{\Theta}) \equiv C_{-n}(z, \bar{z}, \theta, \bar{\theta}) (D\Theta)^n \quad (3.60)$$

If $\hat{C}_{-n}$ is locally superanalytic,

$$D_{\theta} \hat{C}_{-n} = 0, \quad (3.61)$$

then

$$\left[\bar{D} - H_{\theta}^z \partial + \frac{1}{2} (DH_{\theta}^z) D + \frac{n}{2} (\partial H_{\theta}^z) \right] C_{-n}(z, \bar{z}, \theta, \bar{\theta}) = 0 \quad (3.62)$$

and we have

$$(\mathcal{L}_n \hat{C}_{-n})(Z, \bar{Z}, \Theta, \bar{\Theta}) = (D\Theta)^{-(n+1)} (\mathcal{L}_n C_{-n})(z, \bar{z}, \theta, \bar{\theta}). \quad (3.63)$$

Here, the super Bol operators $\mathcal{L}_n$ on the l.h.s. and r.h.s. involve the superprojective connections $\mathcal{R}_{Z\theta}(Z, \bar{Z})$ and $\mathcal{R}_{z\theta}(z, \bar{z}, \theta, \bar{\theta})$, respectively, both being related by eq.(3.34). These connections satisfy the differential equations $D_{\theta} \mathcal{R}_{Z\theta} = 0$ and (3.35), respectively.

3.9 The non-holomorphic case

The remarks made in section 2.9 carry over almost verbatim to the SUSY case ($\bar{z}$ and $\bar{\theta}$ become $(\bar{z}, \bar{\theta})$ and $(\bar{\partial}, \bar{D})$, respectively).

3.10 The real case

In the real framework, the supermanifold is parametrized by real coordinates (z, θ) .

Theorem 3.10.1 Suppose the superfield $\Psi(z, \theta) \equiv \psi_0(z) + \theta \psi_1(z)$ transforms as $\Psi' = e^{-w} \Psi$ and satisfies

$$0 = \mathcal{L}_1 \Psi \equiv (\partial D + \mathcal{R}) \Psi. \quad (3.64)$$

Furthermore, assume that Ψ is even and that $\psi_0 > 0$ in some interval $I \subset \mathbb{R}$ (which implies that $\ln \Psi = \ln \psi_0 + \theta \psi_1/\psi_0$ is well defined on the interval I). Then, $\Gamma_{\theta} \equiv D \ln \Psi$ represents a superaffine connection for $z \in I$ and the associated covariant derivative is given by

$$\nabla_{(p)} = \Psi^{-p} D \Psi^p = D + p \Gamma_{\theta}. \quad (3.65)$$

Moreover, eq.(3.64) leads to the relation (3.56) for the superprojective connection $\mathcal{R}$ and the factorisation formula (3.58) yields the following representation for the super Bol operator:

$$\mathcal{L}_n = \Psi^{-(n+1)} (\Psi D)^{2n+1} \Psi^{-n} \quad \text{for } z \in I. \quad (3.66)$$

As in the bosonic theory, the transformation law of a (super) projective connection has the same form as the coadjoint action of the (super) Virasoro group. To be more precise, we consider the supercircle $\mathbb{S}^1$ parametrized by (z, θ) with $z \in \mathbb{S}^1$ and a smooth invertible mapping [36] [40]

$$(z, \theta) \longrightarrow (Z(z, \theta) = Z_0(z) + \theta Z_1(z), \Theta(z, \theta) = \Theta_0(z) + \theta \Theta_1(z))$$

with $DZ = \Theta D\Theta$ and $Z_0(z) \in \mathbb{S}^1$. The group G of these transformations admits a central extension $\hat{G}$ whose Lie algebra, $\text{Lie}(\hat{G})$, corresponds to the super Virasoro algebra. Its dual

has the form $(\text{Lie}(\hat{G}))^* \cong \Lambda^3(\mathbb{S}\mathbb{S}^1) \oplus \mathbb{R}$ where $\Lambda^3(\mathbb{S}\mathbb{S}^1)$ denotes the space of superquadratic differentials $(E^{\theta})^3 Q_{z\theta}$ on $\mathbb{S}\mathbb{S}^1$. The coadjoint action of $(Z, \Theta) \in G$ on $(Q, 1) \in (\text{Lie}(\hat{G}))^*$ is defined by [40]:

$$\begin{aligned} \text{Ad}_{(Z, \Theta)^{-1}}^*(Q, 1) &= ((D\Theta)^3 Q \circ (Z, \Theta) + \mathcal{S}(Z, \Theta; z, \theta), 1) \\ &= ((D\Theta)^3 [Q_{Z\theta} - \mathcal{S}(z, \theta; Z, \Theta)], 1). \end{aligned} \quad (3.67)$$

3.11 Applications in superintegrable models

3.11.1 Super Lax operators

According to the ‘‘theorem on fractional powers’’ of Manin and Radul [41], it is possible to take the N -th root (N odd) of an odd operator of the form

$$l = D^N + \sum_{n \leq N-1} u_n D^n.$$

For instance, for $l = \partial D + 3\mathcal{R}$, we have

$$\begin{aligned} l_+^{\frac{1}{3}} &= D \\ l_+^{\frac{2}{3}} &= \partial^2 D + 3\mathcal{R} \partial + (D\mathcal{R})D + 2(\partial\mathcal{R}), \end{aligned}$$

where the ‘plus’ means that we have only considered non-negative powers of D . Obviously, $l_+^{\frac{1}{3}}$ and $l_+^{\frac{2}{3}}$ coincide with the super Bol operators $\mathcal{L}_0$ and $\mathcal{L}_2$, respectively. We note that odd operators occur in the *super KP hierarchy* [41], but the Lax operator for the super-KdV equation is even ($l = \partial^2 + \mathcal{U}$), see [41] and references therein. Thus, it appears that the Bol operators do not play the same rôle for the Lax representation of superintegrable models as they do for the non-supersymmetric theory (see sections 2.12.1 and 2.14.2).

3.11.2 Super Poisson operators

The *supersymmetric KdV (sKdV) equation* is defined by

$$4\bar{\partial}U = \partial [\partial^2 U + 3U(DU)]. \quad (3.68)$$

Here, $\mathcal{U}(z, \bar{z}, \theta) \equiv \zeta(z, \bar{z}) + \theta [\frac{1}{2} u(z, \bar{z})]$ represents an odd superfield depending on both the real space-time coordinates $(z, \bar{z})$ and the real Grassmannian variable θ (see [39] and references therein). The superfield equation (3.68) is equivalent to the component field equations

$$\begin{aligned} 4\bar{\partial}\zeta &= \partial^2 \zeta + \frac{3}{2} \partial(u\zeta) \\ 4\bar{\partial}u &= \partial^3 u + 3u(\partial u) - 6\zeta(\partial^2 \zeta), \end{aligned}$$

which reduce to the ordinary KdV equation for the component u , if ζ is set to zero.

The so-called 'second' Hamiltonian structure for the sKdV equation is given by the Hamiltonian $H^{(2)}[\mathcal{U}] = \frac{1}{8} \int dz \int d\theta \mathcal{U} (D\mathcal{U})$ (which is to be understood as a Berezinian integral) and by the Poisson brackets

$$\begin{aligned} \{\mathcal{U}(\mathcal{Z}'), \mathcal{U}(\mathcal{Z})\}_2 &= [\partial^2 D + 3\mathcal{U}\partial + (D\mathcal{U})D + 2(\partial\mathcal{U})] \delta(\mathcal{Z} - \mathcal{Z}') \\ &\equiv (\mathcal{U}, \mathcal{U})_2 \delta(\mathcal{Z} - \mathcal{Z}'), \end{aligned} \quad (3.69)$$

where $\mathcal{Z} \equiv (z, \bar{z}, \theta)$ and $\delta(\mathcal{Z} - \mathcal{Z}') \equiv (\theta - \theta') \delta(z - z')$. The evolution equation reads

$$\begin{aligned} \bar{\partial}\mathcal{U}(\mathcal{Z}) &= \{\mathcal{U}(\mathcal{Z}), H^{(2)}[\mathcal{U}]\}_2 \\ &\equiv (\mathcal{U}, \mathcal{U})_2 \frac{\delta H^{(2)}}{\delta \mathcal{U}(\mathcal{Z})}. \end{aligned} \quad (3.70)$$

Note that the *super Poisson operator* in eq.(3.69) coincides with the covariant operator $\mathcal{L}_2$ after substituting $\mathcal{U}(z, \bar{z}, \theta)$ by $\mathcal{R}_{-\theta}(z, \theta)$.

In view of the known results for the bosonic theory, one would expect the existence of another ('first') Hamiltonian structure described by the super Bol operator $\mathcal{L}_0 = D$. However, this is only true in a first approximation, since the corresponding Poisson operator is non-local [42],

$$\{\mathcal{U}(\mathcal{Z}'), \mathcal{U}(\mathcal{Z})\}_1 = \partial \frac{1}{\partial D + \mathcal{U}} \partial \delta(\mathcal{Z} - \mathcal{Z}'), \quad (3.71)$$

although the corresponding Hamiltonian, $H^{(1)}[\mathcal{U}] \equiv -\frac{1}{8} \int dz \int d\theta \{(\partial\mathcal{U})(\partial D\mathcal{U}) - 2\mathcal{U}(D\mathcal{U})^2\}$, and the evolution equation are not. In fact,

$$\partial \frac{1}{\partial D + \mathcal{U}} \partial = D + \mathcal{O}\left(\frac{1}{\partial}\right)$$

coincides with D to lowest order and we note that it involves an operator of the form $\mathcal{L}_1 = \partial D + \mathcal{R}$.

The *super Miura transformation* [39] is defined by (cf. eq.(3.56))

$$-\mathcal{U} = \partial\mathcal{V} + \mathcal{V}D\mathcal{V}, \quad (3.72)$$

where $\mathcal{V}(z, \bar{z}, \theta) \equiv \eta(z, \bar{z}) + \theta \left[\frac{1}{2} v(z, \bar{z})\right]$ denotes an odd superfield. This transformation relates the sKdV equation (3.68) to the *modified sKdV (msKdV) equation*

$$4\bar{\partial}\mathcal{V} = \partial^3\mathcal{V} - 3(D\mathcal{V})\partial(\mathcal{V}D\mathcal{V}), \quad (3.73)$$

which contains the mKdV equation [39]. The natural Hamiltonian structure for this evolution equation is given by

$$\{\mathcal{V}(\mathcal{Z}'), \mathcal{V}(\mathcal{Z})\} = -D \delta(\mathcal{Z} - \mathcal{Z}') \quad (3.74)$$

and $H[\mathcal{V}] \equiv H^{(2)}[-\mathcal{U} + \partial\mathcal{V} + \mathcal{V}D\mathcal{V}]$ [39].

Given the super Miura transformation and the natural Hamiltonian structure for the msKdV equation, the mere existence of a bi-Hamiltonian structure for the sKdV equation should follow from the identity

$$\partial^2 D + 3\mathcal{U}\partial + (D\mathcal{U})D + 2(\partial\mathcal{U}) = -\mathcal{D}D\mathcal{D}^\dagger, \quad (3.75)$$

where

$$\begin{aligned} D &\equiv \partial + \mathcal{V}D + (D\mathcal{V}) = (D + 2\mathcal{V})(D + \mathcal{V}) \\ -\mathcal{D}^\dagger &= \partial + \mathcal{V}D - 2(D\mathcal{V}) = (D - \mathcal{V})(D - 2\mathcal{V}). \end{aligned} \quad (3.76)$$

The relation (3.75) is nothing but the factorisation formula for the super Bol operator $\mathcal{L}_2$. In the literature, the derivative $\mathcal{D}$ is referred to as the *Fréchet derivative* of $\mathcal{U}$ with respect to $\mathcal{V}$ [39]. By contrast to the bosonic theory, it does not correspond to the covariant derivative with respect to an affine connection, but to the square of the latter.

3.12 Applications in superconformal models

3.12.1 Anomalies and cocycles

The expressions (3.11) and (3.14) occur in superconformal models defined on SRS's in the particular case where the Beltrami superfield H_θ^z vanishes [38] [43]. In this framework, these quantities (plus their complex conjugate expressions) represent, respectively, the factorized form of the *superdiffeomorphism anomaly* on the superplane and on a compact SRS. The component field contributions that we have explicitly written down in eq.(3.15) define this anomaly in the so-called WZ-gauge.

To establish a relation with the Krichever-Novikov algebra, we symmetrize the expression for the superdiffeomorphism anomaly:

$$\begin{aligned} \hat{A}(C^z; H_\theta^z) &\equiv \frac{1}{2} [C^z \mathcal{L}_2 H_\theta^z + H_\theta^z \mathcal{L}_2 C^z] \\ &= \frac{1}{2} \left[(C^z \partial^2 D H_\theta^z + H_\theta^z \partial^2 D C^z) + 3\mathcal{R}(C^z \partial H_\theta^z - H_\theta^z \partial C^z) \right. \\ &\quad \left. + (D\mathcal{R})(C^z D H_\theta^z + H_\theta^z D C^z) \right]. \end{aligned} \quad (3.77)$$

The Wess-Zumino consistency relation reads $s \int_{\Sigma} d^4 z \hat{A}(C^z; H_\theta^z) = 0$ where C^z is to be interpreted as the cohomology generator and s as the BRS differential (cf. eq.(B.16)):

$$\begin{aligned} s H_\theta^z &= \left[\bar{D} - H_\theta^z \partial + \frac{1}{2} (D H_\theta^z) D + (\partial H_\theta^z) \right] C^z \\ s C^z &= - \left[C^z \partial C^z + \frac{1}{4} (D C^z)(D C^z) \right], \quad s \mathcal{R} = 0. \end{aligned}$$

In terms of component fields, we have the expression

$$\begin{aligned} \int_{\Sigma} d^4 z \hat{A}(C^z; H_\theta^z) &= -\frac{1}{2} \int_{\Sigma} d^2 z \left\{ (c\partial^3 \mu - \mu\partial^3 c) + 2R(c\partial\mu - \mu\partial c) \right. \\ &\quad \left. + \frac{3}{2} \rho(c\partial\alpha - \alpha\partial c) - \frac{1}{2} (\partial\rho)c\alpha \right\} \\ &\quad + \frac{1}{2} \int_{\Sigma} d^2 z \left\{ (c\partial^2 \alpha - \alpha\partial^2 c) + R c\alpha - \frac{3}{2} \rho(c\partial\mu - \mu\partial c) - \frac{1}{2} (\partial\rho)\mu c \right\}. \end{aligned} \quad (3.78)$$

For $\rho = 0$, we recognize the central extension of the so-called *graded Krichever-Novikov algebra* discussed by P.Bryant (following earlier work of Bonora, Martellini, Rinaldi and

Russo) [44]: the corresponding bosonic cocycle, $\hat{\chi}(f, g)$, is the same as the one in the bosonic theory, eq.(2.81), while the fermionic cocycle is given by

$$\hat{\varphi}(\sigma, \tau) \equiv - \left[(\partial\sigma)(\partial\tau) + \frac{1}{2} R \sigma \tau \right] dz . \quad (3.79)$$

Here, σ and τ are the components of σD_θ and τD_θ , respectively. The more general expressions (3.77) and (3.78) should provide a basis for defining the super Krichever-Novikov algebra on a general superconformal manifold - a problem raised in reference [44].

3.12.2 Ward identities

In the framework of superconformal field theory on surfaces with $H_\theta^z = 0$, the covariant version of $\partial^2 D$ is also relevant for the Ward identity associated to superdiffeomorphisms. In the superplane SC, the superdiffeomorphisms which leave the condition $H_\theta^z = 0$ invariant generate the anomalous Ward identity

$$\left[D - H_\theta^z \partial + \frac{1}{2} (D H_\theta^z) D - \frac{3}{2} (\partial H_\theta^z) \right] \frac{\delta Z_c}{\delta H_\theta^z} = -k \partial^2 D H_\theta^z \quad \text{and c.c.} , \quad (3.80)$$

where k denotes a constant and $Z_c[H_\theta^z]$ the connected Green's functional [43]. A solution of this equation is given by $\delta Z_c / \delta H_\theta^z = 2k S(Z, \bar{\theta}; z, \bar{\theta})$, i.e. the Schwarzian derivative of some superprojective coordinates [43], see eq.(3.35).

By performing a Legendre transformation,

$$\Gamma[T_\theta] = Z_c[H_\theta^z] - \int_{\text{SC}} d^4 z H_\theta^z T_\theta^z$$

$$T_\theta^z = \frac{\delta Z_c}{\delta H_\theta^z} , \quad H_\theta^z = \frac{\delta \Gamma}{\delta T_\theta^z} ,$$

the $k = 1/2$ Ward identity takes the form

$$\left[\partial^2 D + 3 T_\theta^z \partial + (D T_\theta^z) D + 2 (\partial T_\theta^z) \right] \frac{\delta \Gamma}{\delta T_\theta^z} = -2 D T_\theta^z , \quad (3.81)$$

where we recognize the covariant version of $\partial^2 D$.

In the WZ-gauge, eq.(3.80) reduces to the the anomalous Ward identities generated by ordinary diffeomorphisms and local SUSY transformations [43]

$$0 = \left[\bar{\partial} - \mu \partial - 2 \partial \mu \right] \frac{\delta Z_c}{\delta \mu} - \left[\frac{1}{2} \alpha \partial + \frac{3}{2} \partial \alpha \right] \frac{\delta Z_c}{\delta \alpha} + k \partial^3 \mu \quad \text{and c.c.}$$

$$0 = \left[\bar{\partial} - \mu \partial - \frac{3}{2} \partial \mu \right] \frac{\delta Z_c}{\delta \alpha} + \frac{1}{2} \alpha \frac{\delta Z_c}{\delta \mu} - k \partial^2 \alpha \quad \text{and c.c.} \quad (3.82)$$

and a Legendre transformation⁹

$$\Gamma[T, Q] = Z_c[\mu, \alpha] - \int_{\text{C}} d^2 z \mu T - \int_{\text{C}} d^2 z \alpha Q$$

⁹In order to avoid a confusion with the Schwarzian derivative, we denote the supercurrent by Q rather than the usual S .

$$T \equiv T_{zz} = \frac{\delta Z_c}{\delta \mu_z^z} , \quad Q \equiv Q_{\theta z} = \frac{\delta Z_c}{\delta \alpha_z^\theta}$$

leads to

$$\left[\partial^3 + 2 T \partial + \partial T \right] \frac{\delta \Gamma}{\delta T} + \left[\frac{3}{2} Q \partial + \frac{1}{2} \partial Q \right] \frac{\delta \Gamma}{\delta Q} = -\partial T$$

$$\left[\partial^2 + \frac{1}{2} T \right] \frac{\delta \Gamma}{\delta Q} + \left[\frac{3}{2} Q \partial + \partial Q \right] \frac{\delta \Gamma}{\delta T} = -\partial Q . \quad (3.83)$$

Here, the component field expressions on the l.h.s. have exactly the same form as those found above in eq.(3.15).

3.12.3 Super Liouville theory

In super-isothermal coordinates $(Z, \bar{Z}, \bar{\theta}, \bar{\theta})$, the superfield Liouville equation [45] has the form

$$D_\theta D_{\bar{\theta}} \Phi = -\beta e^\Phi \quad (\beta \in \mathbb{R}) , \quad (3.84)$$

where $e^{*(Z, \bar{Z}, \bar{\theta}, \bar{\theta})} \equiv \rho_{\theta\bar{\theta}}(Z, \bar{Z}, \bar{\theta}, \bar{\theta})$ transforms according to :

$$\rho_{\theta'\bar{\theta}'} = \rho_{\theta\bar{\theta}} (D_\theta \theta')^{-1} (D_{\bar{\theta}} \bar{\theta}')^{-1} . \quad (3.85)$$

Defining

$$T_{Z\theta} \equiv -e^\Phi (\partial_Z D_\theta e^{-\Phi})$$

$$= \partial_Z D_\theta \Phi - (\partial_Z \Phi) (D_\theta \Phi)$$

(and c.c.), we conclude that

$$0 = D_{\bar{\theta}} T_{Z\theta}$$

$$0 = [\partial_Z D_\theta + T_{Z\theta}] e^{-\Phi} . \quad (3.87)$$

The superfield (3.86) and its complex conjugate represent the components of the *super stress tensor*. By virtue of eq.(3.85), they transform like the components of a superprojective connection and the differential operator in the last equation coincides with the super Bol operator $\mathcal{L}_1$.

3.12.4 Super Virasoro and W-algebras

Up to a rescaling, the Poisson brackets for the *classical super Virasoro algebra* coincide [39] with the second Poisson brackets for the sKdV equation, i.e. the brackets (3.69) involving the covariant operator $\mathcal{L}_2$.

The *classical super W-algebras* which extend the super Virasoro algebra are expected to be determined by Poisson brackets involving the most general, normalized and covariant differential operators; the leading terms of these are given by the super Bol operators $\mathcal{L}_n = D^{2n+1} + \dots$ (So far, it was shown [46] that super W-algebras arise from differential operators of the form $D^{2n+1} + \dots$). Explicit expressions should follow from the supersymmetric extension of the results presented in section 2.14.1 and in references [8]. However, a further elaboration on this program is beyond the scope of the present work.

Chapter 4

Concluding remarks

Obviously, the normalized and covariant differential operators and, in particular, their leading terms, the Bol operators, lie at the very heart of many mathematical and physical theories. In most cases, their appearance can simply be traced back to the underlying conformal symmetry (e.g. in conformal models), but in other situations, their manifestation is not so well understood (e.g. in integrable models).

Most definitions and results concerning ordinary Bol operators could be extended in a fairly direct way to the supersymmetric setting following our previous presentation of the bosonic theory. Others - like the classification of all covariant operators and their application to classical W -algebras - are more subtle and their SUSY generalization requires further work. Besides, it should be worthwhile to get a better understanding of the fact that the super Bol operators do not play the same rôle for the Lax representation of superintegrable models as the ordinary Bol operators do for ordinary integrable models. A study of this point might also be of interest for understanding the relations between superintegrable models and super W -algebras [46].

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Appendix A

Bosonic theory

A.1 Existence and construction of projective connections

In this appendix, we give a simple and constructive proof for the existence of projective connections on compact Riemann surfaces. We learned this proof from R.Stora.

First, we recall [10] that the exterior differential on a Riemann surface can be decomposed according to

$$d = \delta + \bar{\delta} \quad .$$

The action of δ and $\bar{\delta}$ on smooth 0-forms f and on smooth 1-forms $\omega \equiv \omega_z dz + \omega_{\bar{z}} d\bar{z}$ is defined by

$$\begin{aligned} \delta f &= (\partial f) dz & , & & \bar{\delta} f &= (\bar{\partial} f) d\bar{z} \\ \delta \omega &= (\delta \omega_z) \wedge dz & , & & \bar{\delta} \omega &= (\bar{\delta} \omega_z) \wedge dz \quad . \end{aligned}$$

It follows that $\delta^2 = \bar{\delta}^2 = d^2 = 0$.

On the compact Riemann surface Σ , we now introduce a metric and express it locally with respect to isothermal coordinates $(z, \bar{z})$:

$$ds^2 = \rho_{z\bar{z}}(z, \bar{z}) dz d\bar{z} \quad . \quad (\text{A.1})$$

Using $\rho_{z\bar{z}}$, we define a smooth $(1,0)$ -tensor field,

$$\gamma(z, \bar{z}) \equiv \gamma_z(z, \bar{z}) dz \equiv \delta \ln \rho_{z\bar{z}} \quad ,$$

transforming like an affine connection under a conformal change of coordinates $z \rightarrow z'(z)$:

$$\gamma' = \gamma - \delta \ln \partial z' \quad .$$

Then, the component of the $(2,0)$ -tensor

$$r(z, \bar{z}) \equiv r_{z\bar{z}}(z, \bar{z}) dz^2 \equiv \left[\partial \gamma_z - \frac{1}{2} \gamma_z^2 \right] dz \otimes dz \quad (\text{A.2})$$

transforms like a projective connection,

$$\tau_{z'z'}(z', \bar{z}') = (\partial z')^{-2} [\tau_{zz}(z, \bar{z}) - S(z'; z)] \quad (A.3)$$

Both γ and τ are smooth, rather than holomorphic. In the following, we will show that it is possible to construct a locally holomorphic field R out of τ which has the same transformation law.

For this purpose, we first remark that one obtains a globally well-defined tensor field by acting on τ with the differential $\bar{\partial}$:

$$\begin{aligned} \bar{\partial}\tau &= (\bar{\partial}\tau_{zz}) d\bar{z} \otimes dz^2 = (\bar{\partial}\tau_{z'z'}) d\bar{z}' \otimes (dz')^2 \\ &\equiv a_{zz}(z, \bar{z}) d\bar{z} \otimes dz^2 \in C^\infty(\Omega^{(0,1)} \otimes \lambda^2) \end{aligned} \quad (A.4)$$

Here, $\Omega^{(0,1)}$ and λ denote, respectively, the vector bundles of smooth $(0,1)$ -forms and smooth $(1,0)$ -tensors. Thus, $\Omega^{(0,1)} \otimes \lambda^2$ is the bundle of $(1,0)$ -forms with values in the $(2,0)$ -tensors and $C^\infty(\cdot)$ represents the smooth global sections of this bundle.

Since $a \equiv a_{zz} d\bar{z} \otimes dz^2$ is annihilated by the differential $\bar{\partial}$, it is ' $\bar{\partial}$ -closed'. Henceforth, it can be written as

$$a = a_0 + \bar{\partial}b \quad (A.5)$$

with $b \equiv b_{zz}(z, \bar{z}) dz^2 \in C^\infty(\lambda^2)$ and a_0 representing a cohomology class $[a_0] \in H^1(\Sigma, \lambda^2)$ (i.e. the cohomology group of holomorphic 1-forms on Σ with values in λ^2).

We now apply the Serre-duality for compact Riemann surfaces [10],

$$H^k(\Sigma, \mathcal{F}^* \otimes \lambda) \cong (H^{1-k}(\Sigma, \mathcal{F}))^* \quad (\mathcal{F} = \text{locally free sheaf}) ,$$

to the case where $\mathcal{F}$ is the sheaf of vector fields, $\mathcal{F} = \lambda^{-1}$:

$$H^1(\Sigma, \lambda^2) \cong (H^0(\Sigma, \lambda^{-1}))^* \quad (A.6)$$

(Note that this duality simply reflects the pairing $\int_\Sigma dz \wedge d\bar{z} a_{zz} c^z$.) Accordingly, $[a_0]$ can be viewed as an element of the dual of $H^0(\Sigma, \lambda^{-1})$. For genus $g > 1$, the latter only contains the trivial class [0], because there are no holomorphic vector fields which are globally defined on Riemann surfaces of genus $g > 1$. Thus, in this case, eq.(A.5) reduces to $a = \bar{\partial}b$ and, by the defining relation (A.4) of a , this means that

$$0 = \bar{\partial}(\tau_{zz} - b_{zz}) = \bar{\partial}(\tau_{z'z'} - b_{z'z'})$$

Consequently, the quantity $R_{zz} \equiv \tau_{zz} - b_{zz}$ is locally holomorphic and, by construction, it transforms in the same way as τ_{zz} . Thus, it is the local representative of a globally defined projective connection.

For $g = 0$ and $g = 1$, we can always choose projective and affine¹⁰ coordinates, respectively, so that $S(z'; z) = 0$ [11]. Then, $R_{zz} = 0$ represents a solution.

¹⁰ Affine coordinates are related by affine transformations, $z' = az + b$ ($a, b \in \mathbb{C}, a \neq 0$).

A.2 Beltrami coefficients

In the following, we summarize a few results on the parametrization of supercomplex structures on Riemann surfaces [9] [13].

Let $(z, \bar{z})$ denote a reference system of holomorphic coordinates on the Riemann surface Σ . Then, the holomorphic coordinates $(Z, \bar{Z})$ corresponding to the complex structure parametrized by the Beltrami coefficient μ are given by the relation

$$dZ = \lambda [dz + \mu d\bar{z}] \quad \text{and c.c.} \quad (A.7)$$

Here, λ and μ are smooth complex-valued functions of $(z, \bar{z})$ and they satisfy

$$\begin{aligned} (\bar{\partial} - \mu \partial)Z &= 0 & \text{and c.c.} \\ (\bar{\partial} - \mu \partial)\lambda &= (\partial\mu)\lambda & \text{and c.c.} \end{aligned} \quad (A.8)$$

Locally, Z and $\bar{Z}$ can be given by smooth invertible functions $(z, \bar{z}) \rightarrow (Z(z, \bar{z}), \bar{Z}(z, \bar{z}))$ so that $\lambda = \partial Z$ and $\mu = (\bar{\partial}Z)/(\partial Z)$.

Now, consider infinitesimal diffeomorphisms generated by a vector field $\xi \cdot \partial \equiv \xi(z, \bar{z})\partial + \bar{\xi}(z, \bar{z})\bar{\partial}$. Let $L_{\xi \cdot \partial} = i_{\xi \cdot \partial} d + di_{\xi \cdot \partial}$ denote the Lie derivative along $\xi \cdot \partial$ (as acting on forms). Then, an infinitesimal variation of Z reads

$$\begin{aligned} \delta_{\xi \cdot \partial} Z &= L_{\xi \cdot \partial} Z = i_{\xi \cdot \partial} dZ = i_{\xi \cdot \partial} [\lambda (dz + \mu d\bar{z})] = \lambda (\xi + \mu \bar{\xi}) \\ &\equiv \lambda c \end{aligned} \quad (A.9)$$

and the induced variation of μ takes the form

$$\delta_{\xi \cdot \partial} \mu = [\bar{\partial} - \mu \partial + (\partial\mu)]c \quad (A.10)$$

The expression (A.9), i.e. $c^Z(Z, \bar{Z}) \equiv (c^z \lambda_z^Z)(z, \bar{z})$, reflects the complex analytic structure of $\text{Diff}_0(\Sigma)$, i.e. the group of diffeomorphisms on Σ which are homotopic to the identity [13].

Appendix B

SUSY theory

B.1 Existence and construction of superprojective connections

The proof (which holds for compact SRS's of genus $g \geq 0$) proceeds along the lines of appendix A. It involves an amusing interplay between differential forms and superconformal fields.

We decompose the exterior differential acting on smooth superfunctions according to

$$df = \delta f + \bar{\delta} f$$

with

$$\delta f = [e^z \partial + e^\theta D] f, \quad \bar{\delta} f = [e^{\bar{z}} \bar{\partial} + e^{\bar{\theta}} \bar{D}] f. \quad (\text{B.1})$$

The action of δ and $\bar{\delta}$ on the canonical 1-forms is defined by

$$\begin{aligned} \delta e^z &= -e^\theta e^\theta, & \delta e^\theta &= 0 \\ \bar{\delta} e^{\bar{z}} &= -e^{\bar{\theta}} e^{\bar{\theta}}, & \bar{\delta} e^{\bar{\theta}} &= 0 \end{aligned} \quad (\text{B.2})$$

and

$$\bar{\delta} e^z = 0 = \bar{\delta} e^\theta, \quad \delta e^{\bar{z}} = 0 = \delta e^{\bar{\theta}}. \quad (\text{B.3})$$

On superforms, the operators $\delta, \bar{\delta}$ and d are assumed to act from the right as antiderivations. They satisfy $\delta^2 = \bar{\delta}^2 = d^2 = 0$.

For later reference, we note that a smooth $(0, 1)$ -form,

$$\omega \equiv e^{\bar{z}} \omega_{\bar{z}}(z, \bar{z}, \theta, \bar{\theta}) + e^{\bar{\theta}} \omega_{\bar{\theta}}(z, \bar{z}, \theta, \bar{\theta}), \quad (\text{B.4})$$

which is $\bar{\delta}$ -closed (i.e. $\bar{\delta}\omega = 0$) is necessarily of the form $e^{\bar{z}}(\bar{D}\omega_{\bar{z}}) + e^{\bar{\theta}}\omega_{\bar{\theta}}$. In this expression, the independent superfield $\omega_{\bar{\theta}}$ transforms as $\omega_{\bar{\theta}} = e^{\bar{\theta}}\omega_{\bar{\theta}}$ under superconformal changes of coordinates; therefore, a $\bar{\delta}$ -closed $(0, 1)$ -form can be identified with a superconformal field $E^{\bar{\theta}}\omega_{\bar{\theta}}$ [35]. (Here, $E^{\bar{\theta}} \equiv \bar{E}^{\bar{\theta}}$ denotes the generator of superconformal fields, see eq.(3.17).)

On the compact SRS $S\Sigma$, we now introduce frame fields $(E_{++}, E_{+-}, E_{-+}, E_{--})$ and express them with respect to super-isothermal coordinates $(z, \bar{z}, \theta, \bar{\theta})$ [47]:

$$\begin{aligned} E_+ &= e^\theta \delta_+^\theta D_\theta & \text{and c.c.} \\ E_{++} &= e^{2\theta} [\delta_{++}^\theta \partial_z + 2\delta_{++}^\theta \delta_+^\theta (D_\theta \sigma) D_\theta] & \text{and c.c.} \end{aligned}$$

Here, $\sigma(z, \bar{z}, \theta, \bar{\theta})$ denotes an unconstrained superfield. Using this variable, we introduce the following fields:

$$\begin{aligned} \gamma &\equiv e^z \gamma_z + e^\theta \gamma_\theta \equiv e^z (\partial \sigma) + e^\theta (D \sigma) = \delta \sigma \\ -\tau_{z\theta} &\equiv \partial_z \gamma_\theta + \gamma_z \gamma_\theta. \end{aligned} \quad (\text{B.5})$$

Under a superconformal change of coordinates, σ transforms as $\sigma' = \sigma - w$ (with $e^{-w} \equiv D\theta'$) and the induced variations of γ and τ read

$$\begin{aligned} \gamma' &= \gamma - \delta w \\ \tau_{z'\theta'} &= e^{3w} [\tau_{z\theta} - S(z', \theta'; z, \theta)], \end{aligned} \quad (\text{B.6})$$

where S denotes the super Schwarzian derivative.

The field $\tau_{z\theta}$ represents the component of $\tau \equiv (E^\theta)^3 \tau_{z\theta}$ and we have

$$\begin{aligned} \bar{\delta} \tau &= (E^\theta)^3 \otimes [e^{\bar{z}} \bar{\partial} + e^{\bar{\theta}} \bar{D}] \tau_{z\theta} = (E^{\theta'})^3 \otimes [e^{\bar{z}'} \bar{\partial}' + e^{\bar{\theta}'} \bar{D}'] \tau_{z'\theta'} \\ &\equiv (E^\theta)^3 \otimes (e^{\bar{z}} a_{z\theta} + e^{\bar{\theta}} a_{\bar{z}\theta}) \\ &\equiv a \in C^\infty(\Omega^{(0,1)} \otimes \Lambda^3). \end{aligned}$$

Here, $\Omega^{(0,1)}$ denotes the space of smooth $(0, 1)$ -forms (locally represented by (B.4)) and Λ^3 is the space of smooth superconformal fields introduced in section 3.8.

Arguing as in appendix A, we have $a = a_0 + \delta b$ with $b \in C^\infty(\Lambda^3)$ and $\{a_0\} \in H^1(S\Sigma, \Lambda^3)$, i.e. a_0 represents a $\bar{\delta}$ -closed $(0, 1)$ -form with values in Λ^3 . According to the discussion presented after eq.(B.4), it can be identified with a superconformal field $(E^\theta)^3 \otimes E^{\bar{\theta}} a_{\bar{z}\theta}^{(0)}$.

Now, we can apply the super Serre-duality (see [35] and references therein),

$$H^1(S\Sigma, \Lambda^3) \cong (H^0(S\Sigma, \Lambda^{-2}))^*, \quad (\text{B.7})$$

which reflects the pairing $\int_{S\Sigma} d^4 z a_{\theta\bar{\theta}} C^z$. For genus $g > 1$, there are no globally defined superconformal fields which are superanalytic and transform as $C^{z'} = e^{-2w} C^z$ [35]. Therefore, for genus $g > 1$, we have $\{a_0\} = [0]$, i.e. $a = \delta b$ or, by definition of a ,

$$0 = \bar{\delta}(\tau_{z\theta} - b_{z\theta}) = \bar{D}(\tau_{z\theta} - b_{z\theta}).$$

Consequently, the superfield $\mathcal{R}_{z\theta} \equiv \tau_{z\theta} - b_{z\theta}$ represents the local expression for a superprojective connection that is globally defined on $S\Sigma$ for genus $g > 1$. For $g = 0$ and $g = 1$, one can choose $\mathcal{R}_{z\theta} \equiv 0$.

B.2 Super Beltrami differentials

The following is a summary of the basic facts which are referred to in the text; we follow the approach of reference [38] and remark that other super Beltrami parametrizations have been discussed earlier [48].

A SRS is locally parametrized by coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$. The super 1-forms $e^Z \equiv dZ + \Theta d\Theta$, $e^{\bar{\Theta}} \equiv d\bar{\Theta}$ (and c.c.) which span its cotangent space can be expressed with respect to a reference system $e^z \equiv dz + \theta d\theta$, $e^{\bar{\theta}} \equiv d\bar{\theta}$ (and c.c.) by

$$\begin{aligned} e^Z &= [e^z + e^{\bar{z}} H_z^z + e^{\bar{\theta}} H_{\bar{z}}^z + e^{\bar{\theta}} H_{\bar{\theta}}^z] \Lambda \\ e^{\bar{\Theta}} &= [e^{\bar{z}} + e^z H_z^{\bar{z}} + e^{\bar{\theta}} H_{\bar{z}}^{\bar{z}} + e^{\bar{\theta}} H_{\bar{\theta}}^{\bar{z}}] \tau + [e^{\bar{\theta}} H_{\bar{z}}^{\bar{\theta}} + e^z H_z^{\bar{\theta}} + e^{\bar{\theta}} H_{\bar{\theta}}^{\bar{\theta}}] \sqrt{\Lambda} \end{aligned} \quad (\text{B.8})$$

(and the c.c. expressions). The structure equations $de^Z + e^{\bar{\Theta}} e^{\Theta} = 0 = de^{\bar{\Theta}}$ (and c.c.) imply that the odd superfields $H_{\bar{z}}^z, H_{\bar{\theta}}^z$ (and c.c.) are the only independent Beltrami coefficients and that all the others depend on them and their derivatives:

$$\begin{aligned} (H_{\bar{\theta}}^{\bar{\theta}})^2 &= 1 - (D - H_{\bar{\theta}}^z \partial) H_{\bar{\theta}}^z \\ H_{\bar{\theta}}^{\bar{\theta}} &= -\frac{1}{2H_{\bar{\theta}}^z} [(\bar{D} - H_{\bar{\theta}}^z \partial) H_{\bar{\theta}}^z + (D - H_{\bar{\theta}}^z \partial) H_{\bar{\theta}}^z] \\ H_z^z &= (\bar{D} - H_{\bar{\theta}}^z \partial) H_{\bar{\theta}}^z + (H_{\bar{\theta}}^{\bar{\theta}})^2 \\ H_z^{\bar{\theta}} &= (\bar{D} - H_{\bar{\theta}}^z \partial) H_{\bar{\theta}}^{\bar{\theta}} + \frac{1}{2} H_{\bar{\theta}}^{\bar{\theta}} \partial H_{\bar{\theta}}^z \end{aligned} \quad (\text{B.9})$$

Furthermore, they imply that the integrating factor τ depends on the independent factor Λ through the relation

$$\tau = \frac{1}{(H_{\bar{\theta}}^{\bar{\theta}})^2} (D - H_{\bar{\theta}}^z \partial) (H_{\bar{\theta}}^{\bar{\theta}} \sqrt{\Lambda})$$

and that Λ satisfies the differential equation

$$\bar{D} \ln \Lambda = \partial H_{\bar{\theta}}^z - \frac{H_{\bar{\theta}}^{\bar{\theta}}}{H_{\bar{\theta}}^z} \partial H_{\bar{\theta}}^z \quad (\text{B.10})$$

Here, $\bar{D}$ is a linear differential operator whose coefficients are functions of $H_{\bar{z}}^z, H_{\bar{\theta}}^z$ and their derivatives [38].

The superfield $H_{\bar{\theta}}^z$ admits a component field expansion

$$H_{\bar{\theta}}^z = \sigma_z^z + \theta v^z + \bar{\theta} \mu_z^z + \bar{\theta} \bar{\theta} [-i\alpha_z^{\bar{\theta}}] \quad (\text{B.11})$$

where $\mu_z^z(z, \bar{z})$ and $\alpha_z^{\bar{\theta}}(z, \bar{z})$ denote the Beltrami coefficient and its fermionic partner, respectively.

Infinitesimal superdiffeomorphisms generated by a vector field

$$\Xi \cdot \partial \equiv \Xi^z(z, \bar{z}, \theta, \bar{\theta}) \partial + \Xi^{\bar{z}}(z, \bar{z}, \theta, \bar{\theta}) \bar{\partial} + \Xi^{\theta}(z, \bar{z}, \theta, \bar{\theta}) D + \Xi^{\bar{\theta}}(z, \bar{z}, \theta, \bar{\theta}) \bar{D}$$

act on $\Theta(z, \bar{z}, \theta, \bar{\theta})$ and $Z(z, \bar{z}, \theta, \bar{\theta})$ according to

$$\begin{aligned} \delta_{\Xi \cdot \partial} \Theta &= i_{\Xi \cdot \partial} \Theta = i_{\Xi \cdot \partial} e^{\bar{\Theta}} = C^z \tau + C^{\bar{\theta}} \sqrt{\Lambda} \\ \delta_{\Xi \cdot \partial} Z &= i_{\Xi \cdot \partial} dZ = i_{\Xi \cdot \partial} [e^z - \Theta e^{\bar{\Theta}}] = C^z \Lambda - \Theta (\delta_{\Xi \cdot \partial} \Theta) \end{aligned} \quad (\text{B.12})$$

where

$$\begin{aligned} C^z &\equiv \Xi^z + \Xi^{\bar{z}} H_z^z + \Xi^{\bar{\theta}} H_z^{\bar{\theta}} + \Xi^{\bar{\theta}} H_{\bar{\theta}}^z \\ C^{\bar{\theta}} &\equiv \Xi^{\bar{\theta}} H_{\bar{z}}^{\bar{\theta}} + \Xi^z H_z^{\bar{\theta}} + \Xi^{\bar{\theta}} H_{\bar{\theta}}^{\bar{\theta}} \end{aligned} \quad (\text{B.13})$$

The variations (B.12) induce the following transformation laws for the independent Beltrami superfields:

$$\begin{aligned} \delta_{\Xi \cdot \partial} H_{\bar{z}}^z &= [\bar{D} - H_{\bar{\theta}}^z \partial + (\partial H_{\bar{\theta}}^z)] C^z - 2 H_{\bar{\theta}}^{\bar{\theta}} C^{\bar{\theta}} \\ \delta_{\Xi \cdot \partial} H_{\bar{\theta}}^z &= [D - H_{\bar{\theta}}^z \partial + (\partial H_{\bar{\theta}}^z)] C^z - 2 H_{\bar{\theta}}^{\bar{\theta}} C^{\bar{\theta}} \end{aligned} \quad (\text{B.14})$$

Under a superconformal change of coordinates, the fields $H_{\bar{z}}^z, H_{\bar{\theta}}^z$ and $C^z, C^{\bar{\theta}}$ transform according to

$$\begin{aligned} H_{\bar{z}}^{z'} &= e^{-w} H_{\bar{z}}^z \\ H_{\bar{\theta}}^{z'} &= e^{\bar{w}} e^{-2w} Y^{-1} H_{\bar{\theta}}^z \\ C^{z'} &= e^{-2w} Y^{-1} C^z \\ C^{\bar{\theta}'} &= e^{-w} Y^{-1/2} [C^{\bar{\theta}} - C^z (Dw) H_{\bar{\theta}}^{\bar{\theta}}] \end{aligned} \quad (\text{B.15})$$

where

$$Y \equiv 1 + (Dw) H_{\bar{\theta}}^z$$

The restriction $H_{\bar{\theta}}^z = 0$ (and c.c.) is invariant under superconformal changes of coordinates (and the latter commute with superdiffeomorphisms). Thus, we can impose the condition $H_{\bar{\theta}}^z = 0$ together with

$$0 = \delta_{\Xi \cdot \partial} H_{\bar{\theta}}^z = D_{\bar{\theta}} C^z - 2 C^{\bar{\theta}}$$

The resulting variation of $H_{\bar{z}}^z$ reads

$$\delta_{\Xi \cdot \partial} H_{\bar{z}}^z = [\bar{D} - H_{\bar{\theta}}^z \partial + \frac{1}{2} (D H_{\bar{\theta}}^z) D + (\partial H_{\bar{\theta}}^z)] C^z \quad (\text{B.16})$$

and the integrating factor equation, eq.(B.10), takes the form

$$\left[\bar{D} - H_{\bar{\theta}}^z \partial + \frac{1}{2} (D H_{\bar{\theta}}^z) D \right] \ln \Lambda = \partial H_{\bar{\theta}}^z \quad (\text{B.17})$$

Furthermore, the superconformal transformation laws of $H_{\bar{z}}^z$ and C^z now reduce to the eqs.(3.9).

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