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QED_3 and two-dimensional superconductivity without parity violation

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Abstract

We investigate a recently proposed parity-invariant model of two-dimensional superconductivity. The model consists of two-species of fermions coupled with opposite sign to an abelian gauge field and is closely related to QED_3 . The first part of the paper is devoted to a detailed analysis of the relevant properties of QED_3 . In particular we study the dynamical generation of a parity-conserving fermion mass and the finite-temperature symmetry restoration transition. We show how the parity-invariant model arises as an effective long-wavelength theory of the dynamics of holes in a two-dimensional quantum antiferromagnet. The model exhibits type-II superconductivity without parity or time-reversal symmetry violation, a high value of $2\Delta/k_B T_c$, flux quantization with quantum $hc/2e$ and a two-dimensional Meissner effect. We discuss the possible relevance of this model to the high- T_c superconductors.

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1 Introduction

Ever since the discovery of the quasi-planar high- T_c oxides in 1986 [1], there has been considerable theoretical interest in two-dimensional superconductivity. In two dimensions, particles are no longer limited to Bose and Fermi statistics but can acquire an arbitrary interchange phase; such particles with fractional statistics are known as anyons. Laughlin [2] suggested that a gas of anyons may exhibit superconductivity at low temperature and, subsequently, this idea was supported by the results of calculations in the random phase approximation [3] which demonstrated that a perfect gas of charged anyons with certain values of the statistics parameter is indeed a superconductor at zero temperature. This ‘anyonic superconductivity’ is an entirely novel phenomenon which has no analog in three-dimensional systems. Motivated by the role of anyonic quasi-particles in the Fractional Quantum Hall Effect, Laughlin went on to suggest that the charge carriers in the copper oxide planes of materials such as La_2CuO_4 and YBa_2CuO_6 might also have fractional statistics and that superconductivity in these materials may be well described by the anyonic model.

There are, however, several serious problems with the anyonic model as a candidate theory of high- T_c superconductivity. A field theoretic realisation of anyonic matter consists of fermions interacting with an abelian ‘statistical’ gauge field whose dynamics is governed by a Chern-Simons term. As discussed in [4], this term leads to observable parity violation in an anyonic superconductor for which there is, as yet, no conclusive experimental evidence. In addition, if the anyonic model is physically relevant, the Chern-Simons action should arise as a long-wavelength limit of a more realistic microscopic model of these materials. Although an abelian gauge field occurs naturally in such models, there is no evidence for the appearance of a Chern-Simons term at large length-scales [5].

There is also a problem of a more fundamental nature with the anyonic scenario. The occurrence of superconductivity in the anyonic model depends critically on a delicate cancellation of the bare Chern-Simons term against the Chern-Simons term radiatively generated by one-loop vacuum polarisation so that the statistical gauge field becomes massless. Only when this cancellation occurs does the pole required for superconductivity appear in the current-current correlator. However, the coefficient of the radiatively generated term is temperature dependent and the cancellation no longer occurs at finite temperature [6]. Hence, unless the cancellation is restored by higher order effects, the conventional anyonic model is only a true superconductor at exactly zero temperature. We note that the unconventional scenario proposed in [7] appears to be the only variant of the anyonic model to avoid this problem.

The necessary condition for superconductivity is the existence of a massless scalar mode ϕ which couples to the electromagnetic field A_μ in the standard London action,

$$\mathcal{L} = K(\partial_\mu \phi - A_\mu)^2 \quad (1.1)$$

where K is a constant. In (2+1)-dimensions a massless gauge field has only one degree of freedom and the London action can be derived exactly from a theory of a massless gauge field a_μ , defined by $\epsilon^{\mu\nu\rho}\partial_\nu a_\rho = \partial_\mu \phi$, which couples to the electromagnetic field via a mixed Chern-Simons term [8],

$$\mathcal{L} = -\frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu} + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho \quad (1.2)$$

The presence of a parity-violating Chern-Simons term which provides a topological mass for the a_μ field spoils this equivalence and hence this term must be cancelled exactly in the anyonic model. On the other hand, the mixed Chern-Simons term which couples the electromagnetic and statistical gauge fields is essential for superconductivity. From this point of view an obvious question to ask is whether there exists a theory which includes the latter term but not the former.

Recently, several authors [9, 10, 11] have proposed a simple gauge-theory model which has exactly this property, and thereby exhibits two-dimensional superconductivity without parity violation. In its most general form, the model consists of two species of massive fermions coupled with opposite signs to an abelian gauge field (which we will continue to refer to as the statistical gauge field to avoid confusion with the electromagnetic gauge field). The two species have equal and opposite masses and hence parity is conserved overall. Although this model has no bare Chern-Simons term and the charge carriers do not acquire fractional statistics, a mixed Chern-Simons term is radiatively generated at one-loop and the effective action takes on the form (1.2) exactly as in the anyonic case. Unlike anyonic superconductivity, the parity-conserving case invokes no artificial cancellations and superconductivity persists at finite temperature. In this paper we will examine the phenomenology of

this model in detail and discuss its possible relevance to the high- T_c superconductors. In general [9, 11], this model has been introduced at the continuum level and discussed without reference to a specific physical system. However, in [10], we showed that a special case of the parity-conserving theory arises as an approximate long-wavelength limit of an idealised model of the dynamics of the charge carriers in the copper oxide layers of the high- T_c materials. In this special case the fermionic terms in the action have a Dirac-like structure. Neglecting the coupling to the three-dimensional electromagnetic field, the resulting effective theory of the charge carriers interacting with the statistical gauge field is closely related to QED in (2+1)-dimensional space-time (QED₃).

The overall layout of the paper requires some further explanation. Three-dimensional QED provides the simplest possible realisation of the parity-conserving model. Variants of the model with non-relativistic fermions and a finite particle density exhibit the same mechanism of superconductivity and only introduce added calculational complexity. In addition, as mentioned above, the case of Dirac fermions at zero density appears to be a physically relevant one. For these reasons Section 2 is devoted to a self-contained discussion of the relevant properties of QED₃. We review the symmetries of the massless theory and the patterns of symmetry breaking which occur when different mass terms are introduced. In particular, we identify the relevant form of QED₃ in which the introduction of a parity-conserving mass term leads to the spontaneous breaking of the symmetry of the system under global phase rotations, denoted $U_E(1)$. This pattern of symmetry breaking leads directly to a mixed Chern-Simons term and hence superconductivity when the system is coupled to external (three-dimensional) electromagnetic fields.

As in the conventional BCS theory, superconductivity occurs only when there is a mass gap in the fermion spectrum and, in the more general case of the parity-conserving model the origin of this mass is not specified (in other words, it is regarded as a bare mass). With the physical application in mind, we will be particularly interested in the case in which the fermion mass is dynamically generated by the gauge attraction between particles of opposite statistical charge. This scenario allows a direct comparison with the phononic pairing mechanism of the BCS theory. In the later parts of Section 2 we discuss the phenomenon of dynamical mass generation in QED₃, reviewing previous analytical and numerical work at zero temperature and providing an approximate treatment of the finite temperature case. We find a non-zero critical temperature above which the dynamically generated fermion mass vanishes identically. In the context of the parity-conserving model this has an obvious interpretation as the phase transition separating the superconducting and normal states. We find that this transition has no local order parameter and thus is not subject to the Mermin-Wagner theorem [12].

In Section 3 we discuss a microscopic model of the charge carriers in the copper oxide layers of the high- T_c materials which yields the parity-conserving model in the long-wavelength limit. In the absence of doping impurities these materials are antiferromagnetic (AF) insulators and are well described by the two-dimensional Heisenberg antiferromagnet. We review the derivation of the (2+1)-dimensional nonlinear σ -model as a long-wavelength effective action in the undoped case. Following the ideas of Shankar [13], Weigmann [14] and Lee [15], we model dynamics of charged holes in the antiferromagnet by coupling two-species of Grassman fields, corresponding to the two sublattices defined by the AF order, to the σ -model action. The long-wavelength limit of this system is a quasi-relativistic quantum field theory of Dirac fermions coupled with opposite sign to an abelian gauge field. We also discuss an alternative approach [16] which leads to a similar effective theory as a long-wavelength limit of the t - J model.

In Section 4 we apply the results obtained in Section 2 for QED₃ to this long-wavelength theory. We find that a parity-conserving fermion mass term is dynamically generated at low temperature which spontaneously breaks the global $U_E(1)$ symmetry of the model. We then couple the theory to a physical three-dimensional electromagnetic field and demonstrate the occurrence of superconductivity explicitly. In particular, we demonstrate infinite conductivity, flux quantisation and a two-dimensional Meissner effect. We also discuss the role of topologically non-trivial gauge field configurations and argue that superconductivity in the model survives these non-perturbative effects. We end Section 4 with a comparison of the properties of the parity-conserving theory with those of the anyonic model.

Finally, in Section 5, we discuss the possible relevance of this model to the high- T_c materials. We find that the parity-conserving model overcomes many of the phenomenological obstacles confronting the anyonic theory. The model exhibits parity-invariant superconductivity with a non-zero transition temperature and the occurrence of two species of fermions with opposite masses and gauge couplings has a natural interpretation in terms of the antiferromagnetic structure of the copper oxide layers. Although a quantitative comparison with experiment is not yet possible, the special case of the model in which the fermion mass is dynamically generated seems to

capture several qualitative features of high- T_c superconductivity. These include an extremely large value of the Landau-Ginsburg constant (high- T_c superconductors are strongly type-II) and a high value for the characteristic ratio $2\Delta/k_B T_c$.

2 Three-dimensional QED

2.1 The action and its symmetries

The Lagrangian for massless Euclidean QED₃ with N flavours of fermions is given by¹

$$\mathcal{L} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \bar{\psi}_i(i\not{D} - e\not{A})\psi_i \quad (2.1)$$

where $i = 1, 2, \dots, N$. In (2+1)-dimensional spacetime the lowest rank irreducible representation of the Dirac algebra is two-dimensional. In this representation the γ -matrices may be chosen as $\gamma_\mu = i(\sigma_3, \sigma_1, \sigma_2)$ where σ_i are the usual Pauli matrices and Dirac fermions are described by two-component spinors, ψ_α . As the three Pauli matrices are a complete set of mutually-anticommuting (2×2) matrices, it is impossible to define a γ_5 in this representation and consequently the massless Lagrangian (2.1) has no chiral symmetry. However, the mass term for two-component fermions, $M\bar{\psi}\psi$, breaks parity and time-reversal symmetries of (2.1). In three-dimensional spacetime the effect of a parity transformation is simply to reverse the orientation of one spatial axis. For example, in a particular basis the parity operator P takes the three-vector $x_\mu = (x_0, x_1, x_2)$ to $(x_0, -x_1, x_2)$. The action of P on two-component spinors is given in this basis by $\psi \rightarrow P_2\psi = \sigma_1\psi$, hence $\bar{\psi}\psi \rightarrow -\bar{\psi}\psi$ and the two-component mass term is odd under parity.

At one-loop the fermion mass contributes to the vacuum polarisation (see Figure 1) a term antisymmetric in Lorentz indices;

$$\Pi_{\mu\nu}^{odd}(p) = -e^2 m N \int \frac{d^3k}{(2\pi)^3} \frac{\text{Tr}(\gamma_\mu \not{k} \gamma_\nu)}{(k^2 + m^2)((p+k)^2 + m^2)} \quad (2.2)$$

This term involves the trace of three γ -matrices and would thus vanish in an even number of space-time dimension. However, in three dimensions with the two dimensional representation of the Dirac algebra shown above, we have the identity;

$$\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho) = 2\varepsilon_{\mu\nu\rho} \quad (2.3)$$

which leads to the result,

$$\Pi_{\mu\nu}^{odd}(p) = \frac{e^2 m N}{4\pi} \varepsilon_{\mu\nu\rho} p^\rho \int_0^1 dx \frac{1}{\sqrt{m^2 + x(1-x)p^2}} \quad (2.4)$$

At low momenta, the corresponding term in the effective action is given by [17];

$$L^{CS} = \frac{\text{sign}[m]N}{8\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \quad (2.5)$$

where $\text{sign}[m] = m/|m|$. This is a Chern-Simons term for the gauge field which breaks parity and time-reversal symmetry explicitly. In addition, the presence of a Chern-Simons term is known to alter the ‘bare’ statistics of the spinor particles turning the fermions into ‘anyons’ [19]. From (2.5), the coefficient of the radiatively generated Chern-Simons term depends only on the sign of the mass term not its magnitude. This property plays a key role in models of anyonic superconductivity (see Section 4.8). For a theory with N_+ species of fermions with $m > 0$ and N_- species with $m < 0$, the coefficient is $(N_+ - N_-)/4\pi$.

Alternatively, one may consider a reducible four-dimensional representation of the Dirac algebra [20, 21]. A possible choice of (4×4) γ -matrices is,

$$\gamma_0 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \quad (2.6)$$

In this representation it is possible to define two further matrices, γ_3 and γ_5 , which anticommute with γ_0, γ_1 and γ_2 ;

$$\gamma_3 = i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \gamma_5 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (2.7)$$

¹In this section we adopt the conventional notation for QED₃ and denote the dimensionful gauge coupling e . We reiterate that, for the physical application, a_μ is the statistical gauge field rather than the electromagnetic one and that e has nothing to do with the charge of the electron.

where I is the (2×2) unit matrix. Fermions are now described by four-component spinors, ψ , which can be written as vectors of two-component spinors, $\psi = (\psi_1, \psi_2)$. In terms of four-component spinors, the massless Lagrangian (2.1), now has two continuous global symmetries generated by γ_3 and γ_5 respectively. The four component mass term, $M\bar{\psi}\psi = M\bar{\psi}_1\psi_1 - M\bar{\psi}_2\psi_2$, coincides with that of two species of two-component fermions with equal and opposite masses. Examining the vacuum polarisation tensor (2.2), we see that Chern-Simons terms with equal and opposite coefficients are generated ($N_+ = N_-$), which automatically cancel in the effective action. The action of the parity operator on four-component spinors is given by

$$P_4\psi = \begin{pmatrix} 0 & P_2 \\ P_2 & 0 \end{pmatrix} \psi \quad (2.8)$$

and so interchanges the labels 1 and 2. The four-component mass term is parity invariant but breaks the ‘chiral’ symmetries generated by γ_3 and γ_5 .

In the four-component case, the Lagrangian (2.1) also has a continuous global symmetry generated by the matrix $\tau_3 = \text{diag}(I, -I)$. This symmetry is sometimes referred to as ‘chiral’ but will be referred to here as τ_3 -symmetry to avoid any confusion with the chiral symmetries mentioned above. Under a τ_3 -transformation, the upper and lower components of spinors receive equal and opposite phase rotations:

$$\psi_1 \rightarrow \exp(ig\theta)\psi_1 \quad \psi_2 \rightarrow \exp(-ig\theta)\psi_2 \quad (2.9)$$

Because τ_3 commutes with the three block-diagonal gamma matrices, a mass term $M\bar{\psi}\psi$ does not break the τ_3 -symmetry of the massless Lagrangian. However, a non-vanishing mass term leads to a vacuum state which is not τ_3 invariant. The corresponding conserved current $J_\mu^\tau = \bar{\psi}\tau_3\gamma_\mu\psi$ acquires a non-zero matrix element between the vacuum and a one-photon state. The lowest order contribution to this matrix element is shown in Figure 2 and gives [22]

$$\langle 0 | J_\mu^\tau | 1, \mathbf{p} \rangle = -\text{sign}[m] \frac{ie}{2\pi^2 \sqrt{2p_0}} \epsilon_{\mu\nu\rho} p^\nu \epsilon^\rho \quad (2.10)$$

where ϵ^ρ is the photon polarisation vector. Again the non-vanishing of the diagram shown in Figure 2 depends on the non-zero trace of three γ -matrices (2.3) and is special to three-dimensional spacetime. In Ref [22] it is shown that the non-vanishing of this matrix element implies the non-invariance of the vacuum under τ_3 transformations. Although this constitutes the spontaneous breaking of τ_3 -symmetry, there is no local order parameter for the symmetry breaking (ie no τ_3 -non-invariant operator acquires a non-zero vacuum expectation value). In this respect, the spontaneous breaking of τ_3 -symmetry is similar to the Kosterlitz-Thouless transition of the two-dimensional XY-model [23].

In the four-component representation, each species of fermion is equivalent to two species of two-component fermions with the same gauge coupling. As mentioned in Section 1, the relevant case for the parity-conserving superconductivity model has two species coupled with opposite sign. This case is realised in four-component notation by replacing the gauge coupling in (2.1) by $e\tau_3$. This variant will be referred to as ‘ τ_3 -QED₃’. Because τ_3 anticommutes with γ_5 and γ_3 , τ_3 -QED₃ has no chiral symmetry. In addition, the presence of a τ_3 in the gauge coupling means that the roles of τ_3 phase rotations and ordinary phase rotations are interchanged in τ_3 -QED₃. The τ_3 -symmetry now becomes the $U(1)$ gauge symmetry of the model and remains unbroken by a mass term. Transformation under ordinary phase rotations is now an additional global symmetry, which will be called $U_E(1)$. Once the system is coupled to external three-dimensional electromagnetic fields this symmetry is just the $U(1)$ symmetry associated with the conservation of physical electric charge. When a parity-conserving mass term is introduced the vacuum is no longer invariant under this symmetry and the system becomes a superconductor. In this paper we are primarily interested in the dynamical generation of a parity-conserving mass term. For τ_3 -QED₃ this corresponds to the dynamical breaking of $U_E(1)$. We note that for τ_3 -QED₃ with more than one flavour of four-component fermion it is possible to construct parity-conserving mass terms which break global flavour symmetry. However, even if the dynamically generated mass has a flavour symmetry violating component, the vacuum is still non-invariant under $U_E(1)$.

The symmetry properties of QED₃ and τ_3 -QED₃ and the corresponding patterns of symmetry breaking induced by a parity-conserving mass term are summarised in Table 1. The entry ‘KT’ in the last two rows of this table indicates that the symmetry in question is realised in the Kosterlitz-Thouless mode in the presence of the parity-conserving mass term as explained above. For the remainder of Section 2 we will be primarily concerned with dynamical mass generation. The analysis is identical for QED₃ and τ_3 -QED₃ and for simplicity we will restrict our attention to the former.

2.2 The photon propagator for $T = 0$

Although the ordinary perturbation expansion in the dimensionful coupling e is plagued with infrared divergences, the situation is dramatically improved by considering instead an expansion in powers of $1/N$ with $\alpha = e^2 N$ kept fixed. The leading order contribution to the photon propagator, $\Delta_{\mu\nu}$, in this expansion is given by the infinite sum of diagrams shown in Figure 3. Evaluating this sum in the Landau gauge gives [20],

$$\Delta_{\mu\nu}(p) = \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^2(1 + \Pi(p))} \quad (2.11)$$

where $\Pi(p) = \alpha/8p$. The low momentum behaviour of the photon propagator is therefore modified from the $1/p^2$ behaviour of the bare propagator to a softer $1/p$ behaviour. A simple power-counting analysis shows that no IR divergences occur at higher order in the $1/N$ expansion [21].

At leading order in $1/N$, the effective static potential between charges is given by the one-photon-exchange diagram shown in Figure 4. The fermion world lines in Figure 4 are taken as those of static infinitely massive particles of opposite charge and the corresponding (position-space) amplitude is;

$$V(R) = -e^2 \int_{-\infty}^{\infty} dt \Delta_{00}(t, R) \quad (2.12)$$

rewriting in terms of the momentum space propagator gives

$$\begin{aligned} V(R) &= -e^2 \int \frac{d^3 k}{(2\pi)^3} \exp(ik \cdot x) \delta(k_0) \Delta_{00}(k) \\ &= -\frac{e^2}{2\pi} \int K dK J_0(KR) \Delta_{00}(k_0 = 0, K) \end{aligned} \quad (2.13)$$

The characteristic lengthscale of the theory is $(\alpha)^{-1}$ and substituting expression (2.11) for $\Delta_{\mu\nu}$ in (2.14) gives the following behaviour in the limits where R is small or large compared to this lengthscale;

$$\begin{aligned} V(R) &= (\alpha/2\pi N) \log(R\alpha/8) & R \ll (\alpha)^{-1} \\ &= -4/\pi N R & R \gg (\alpha)^{-1} \end{aligned} \quad (2.14)$$

At short distance the effective static potential between charges retains the familiar logarithmic form given by the Fourier transform of a $1/p^2$ propagator in two spatial dimensions. However at large distances the logarithmic potential is screened by vacuum polarisation effects and becomes a Coulombic ($1/R$) attraction which is weak in the limit of large- N .

A particularly interesting case is the strong-coupling limit of the theory $\alpha \rightarrow \infty$. By rescaling the fields, this is simply equivalent to studying the action;

$$\mathcal{L}_\infty = \bar{\psi}_i (i\hat{\partial} - \phi) \psi_i \quad (2.15)$$

In this case the gauge field has no bare Maxwell term and is therefore non-dynamical. However the leading order contribution to the rescaled photon propagator is non-trivial and is given in the Landau gauge by

$$\Delta_{\mu\nu}^\infty(p) = \frac{8}{p} (\delta_{\mu\nu} - p_\mu p_\nu / p^2) \quad (2.16)$$

The corresponding effective potential is simply $V(R) = -4/\pi N R$. Hence, at leading order in $1/N$, quantum corrections restore the dynamics of the gauge field and lead to a long-range attraction between opposite charges. This effect will be essential for the long-wavelength effective theory of superconductivity presented in Section 4. In that case the microscopic dynamics gives rise to a short-range attraction in the form of a non-dynamical gauge field and this is promoted to a long-range attraction in a similar fashion.

2.3 The photon propagator for $T > 0$

The calculation of the $O(1/N)$ photon propagator (2.11) can be repeated using the finite temperature Feynman rules given in Ref[25]. Each integral over the temporal component of a fermion loop momentum is replaced by an infinite sum over Matsubara frequencies according to the prescription

$$\int \frac{dk_0}{(2\pi)} \rightarrow \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \quad k_0 \rightarrow (2n+1)\frac{\pi}{\beta} \quad (2.17)$$

In contrast, bosonic loops are evaluated by summing over ‘even’ Matsubara frequencies, $k_0 \rightarrow 2n\pi/\beta$. To leading order in $1/N$, the inverse Landau gauge photon propagator for frequency $p_0 = 2m\pi/\beta$ and spatial momentum $P = |\mathbf{p}|$ is given by;

$$\Delta_{\mu\nu}^{-1}(p_0, P) = \Delta_{\mu\nu}^{(0)-1}(p_0, P) + \Pi_{\mu\nu}^m(P, \beta) \quad (2.18)$$

where $\Delta_{\mu\nu}^{(0)}$ is the bare photon propagator and the finite temperature vacuum polarisation is given by the evaluation of the one-loop diagram shown in Figure 1;

$$\Pi_{\mu\nu}^m(P, \beta) = -\frac{\alpha}{\beta} \sum_{-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \frac{\text{Tr}(\gamma_\mu \not{k} \gamma_\nu (\not{p} + \not{k}))}{k^2(p+k)^2} \quad (2.19)$$

The vacuum polarisation tensor $\Pi_{\mu\nu}$ is transverse and is conveniently written in terms of the tensors,

$$A_{\mu\nu} = \left(\delta_{\mu 0} - \frac{p_\mu p_0}{p^2} \right) \frac{p^2}{P^2} \left(\delta_{0\nu} - \frac{p_0 p_\nu}{p^2} \right) \quad B_{\mu\nu} = \delta_{\mu i} \left(\delta_{ij} - \frac{p_i p_j}{P^2} \right) \delta_{j\nu} \quad (2.20)$$

as

$$\Pi_{\mu\nu}^m(P, \beta) = \Pi_A^m(P, \beta) A_{\mu\nu} + \Pi_B^m(P, \beta) B_{\mu\nu} \quad (2.21)$$

An explicit calculation of the components of Π_A^m and Π_B^m is presented in Appendix A. The effective static potential, $V(R)$, depends only on the photon propagator at zero frequency and so only $\Pi_{\mu\nu}^0$ will be given here. The result of setting $m = 0$ in formula (A.28) of the Appendix is

$$\Pi_A^0(P, \beta) = \frac{2\alpha}{\pi\beta} \int_0^1 dx \log \left(2 \cosh \left[\beta P \sqrt{x(1-x)}/2 \right] \right) \quad (2.22)$$

$$\Pi_B^0(P, \beta) = \frac{\alpha P}{\pi} \int_0^1 dx \sqrt{x(1-x)} \tanh \left[\beta P \sqrt{x(1-x)}/2 \right] \quad (2.23)$$

In either of the limits $P \rightarrow \infty$ or $\beta \rightarrow \infty$, both components tend to $\alpha P/8$ and the photon propagator is given by the zero temperature result (2.11). However, as $P \rightarrow 0$, $\Pi_A^0(P, \beta)$ tends to a constant limit, $M_{00}^2(\beta)$ giving

$$\Delta_{\mu\nu}(p_0 = 0, P \rightarrow 0, \beta) = \frac{\delta_{\mu 0} \delta_{0\nu}}{P^2 + M_{00}^2} + \frac{\delta_{\mu i} (\delta_{ij} - P_i P_j / P^2) \delta_{j\nu}}{P^2 (1 + \alpha\beta/12\pi)} \quad (2.24)$$

where the temporal component of the propagator, sometimes referred to as the plasmon, has acquired a mass given by,

$$M_{00}(\beta) = \sqrt{\frac{2\alpha \log 2}{\pi\beta}} \quad (2.25)$$

These results are entirely analogous to the corresponding situation in four-dimensional gauge theory (see, for example [26] and references therein). The mass of the temporal photon implies that external electric fields are screened by thermal excitations. Transverse photons, however, remain massless and so external magnetic fields are unscreened. The massless pole in the spatial part of the photon propagator (2.24) potentially leads to IR divergences at higher order.

The effect of thermal fluctuations on the static potential can be examined by substituting the finite temperature result (2.18) for $\Delta_{\mu\nu}$ in (2.14). The resulting integral was performed numerically for a several values of β and the corresponding graphs of $V(R)$ are shown in Figure 5. Using the low-momentum propagator (2.24), the behaviour of the potential for $R \gg (M_{00})^{-1}$ can be extracted analytically;

$$\begin{aligned} V(R, \beta) &\simeq -\frac{e^2}{2\pi} \int_0^\infty \frac{K dK J_0(KR)}{K^2 + M_{00}^2} \\ &= -\frac{\alpha}{2\pi N} K_0(M_{00}R) \\ &\simeq -\frac{\alpha}{N} \sqrt{\frac{1}{8\pi R M_{00}}} \exp(-M_{00}R) \end{aligned} \quad (2.26)$$

using the asymptotic expansion of the Bessel function $K_0(M_{00}R)$ for large R [27]. In contrast, the high-momentum behaviour of the propagator shows that $V(R)$ retains its zero temperature behaviour (2.14) for $R \ll (M_{00})^{-1}$. Thus, the main effect of thermal corrections is to reduce the effective range of the static interaction between fermions, which is infinite at zero temperature, to the screening length, $\xi = (M_{00})^{-1}$.

2.4 Fermion bound-state spectrum

As demonstrated above, at zero temperature fermion-antifermion pairs are subject to a long-range $1/NR$ attraction which becomes strong as N is decreased. Hence, it is natural to expect that, for N sufficiently small, the perturbative vacuum of QED₃ becomes unstable and the true vacuum of the system consists of fermion-antifermion bound states. For QED₃, a simple quantum mechanical model of this phenomenon can easily be constructed [28]. Consider a fermion-antifermion pair, with spatial separation $R \gg (\alpha)^{-1}$, moving in the effective static potential $V(R) \simeq -4/\pi NR$. In the centre-of-mass frame the bound state wavefunction $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \psi^\dagger(\mathbf{r})\psi(-\mathbf{r})$ where $2\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the single particle wavefunction $\psi(\mathbf{r})$ obeys the two-dimensional Dirac equation;

$$(\sigma \cdot \partial + V(R))\psi = E\psi \quad (2.27)$$

The problem is now identical to that of a two-dimensional Dirac fermion moving in a $1/R$ potential well. In three dimensions, it is well known that if the coefficient of $1/R$ is sufficiently large, the attraction overwhelms the centrifugal force and the particle falls into the center. This phenomenon is frequently referred to as ‘collapse of the wavefunction’ and arises in the case of an electron moving in the field of a superheavy nucleus with $Z > 137$ (so that $Z\alpha > 1$). This scenario was originally studied by Popov [29] and its relevance to the problem of dynamical mass generation in (3+1)-dimensional QED was pointed out by Miransky et al [30]. The treatment given here is simply an adaptation of the approach of Ref [30] to the two-dimensional Dirac problem which arises in the case of QED₃.

After separating the equation in polar coordinates and identifying the appropriate eigenstate of orbital angular momentum zero, the radial component of the wavefunction, $\psi(R)$, is found to obey a differential equation of Whittaker form [31];

$$\frac{d^2\psi}{d\rho^2} + \left[-\frac{1}{4} + \frac{\kappa}{\rho} + \frac{(\frac{1}{4} - \mu^2)}{\rho^2} \right] \psi = 0 \quad (2.28)$$

where $\rho = 2iER$, $\kappa = -4i/N\pi$ and $\mu = \sqrt{(N_c^2/N^2 - 1)}/2$ with $N_c = 8/\pi$. The appropriate boundary condition at large R is that for an outgoing plane wave [29]

$$\psi(R) \rightarrow \exp(i|E|R) \quad R \gg (\alpha)^{-1} \quad (2.29)$$

The corresponding solution with the above asymptotic behaviour for large R is proportional to the Whittaker function $W_{\kappa,\mu}(2iER)$.

For $N < N_c$ the parameter μ is real and the wavefunction has the following asymptotic form for small R ;

$$\psi(R) \rightarrow \frac{\Gamma(-2i\mu)}{\Gamma(\frac{1}{2} - \kappa - i\mu)} (2iER)^{\frac{1}{2} + i\mu} + \frac{\Gamma(2i\mu)}{\Gamma(\frac{1}{2} - \kappa + i\mu)} (2iER)^{\frac{1}{2} - i\mu} \quad (2.30)$$

which oscillates pathologically like $\sin(\mu \log ER)$ as $R \rightarrow 0$. The wavefunction has an infinite number of nodes and the spectrum of the system is unbounded below. This corresponds to the fall of the particle into the center. In the case of QED₄, considered in [30], this is remedied by introducing a short-distance cutoff. This can only be removed at the end of the calculation by renormalising the coupling in a particular way. However, in the (2+1)-dimensional case, no such artificial device is needed. From (2.14) we see that the effective potential is proportional to $1/R$ only down to length-scales of order $(\alpha)^{-1}$. As usual in a superrenormalizable theory, the dimensionful coupling constant, α , provides a natural short-distance cutoff for QED₃ and this is incorporated in the quantum mechanical model by choosing the UV boundary condition $\psi((\alpha)^{-1}) = 0$. Using the expansion of the Whittaker function for small arguments [31] this condition gives the relation,

$$\frac{\Gamma(-2i\mu)}{\Gamma(\frac{1}{2} - \kappa - i\mu)} \left(\frac{2iE}{\alpha} \right)^{\frac{1}{2} + i\mu} + \frac{\Gamma(2i\mu)}{\Gamma(\frac{1}{2} - \kappa + i\mu)} \left(\frac{2iE}{\alpha} \right)^{\frac{1}{2} - i\mu} = 0 \quad (2.31)$$

When $N > N_c$, the energy eigenvalues calculated from (2.31) are real and correspond to the usual stable energy levels of the $1/R$ potential well. For $N < N_c$ the eigenvalues acquire an imaginary part and the energy levels become *quasi-stationary* [29]. An explicit calculation in the case $N = 2$, yields a groundstate eigenvalue,

$$E(0) \simeq 0.49\alpha(\cos(\gamma) + i\sin(\gamma)) \exp\left(-\frac{\pi}{\mu}\right) \quad (2.32)$$

where $\gamma = (1.05)\pi$. Hence the energy level is given by $\text{Re}E(0) \simeq -0.16 \times 10^{-3}\alpha$ with a lifetime $\tau \simeq 1/|\text{Im}E(0)| \simeq 6 \times 10^5/\alpha$.

In the context of QED₃, the occurrence of a quasi-stationary groundstate for the effective two-body problem signals the instability of the perturbative vacuum. The true vacuum for $N < N_c$ is one in which a mass for the fermion is dynamically generated and chiral symmetry is spontaneously broken. This will be confirmed by the approximate treatment of the Schwinger-Dyson equations presented in the next section. In fact, if a mass term for the fermion is introduced by hand in (2.27) it is easy to show that this tends to cancel the imaginary part of $E(0)$ and thus stabilise the groundstate. In Section 4, τ_3 -QED₃ will be used as an effective description of the interactions of the charged holes in a doped antiferromagnet. In this context the new groundstate is the superconducting state.

It is also of interest to examine the effect of finite temperature corrections in this simplified picture. At non-zero temperature the effective two-body potential dies off rapidly for $R > (M_{00})^{-1}$. According to the above analysis, the bound-state spectrum depends only the boundary condition at $R = (\alpha)^{-1}$ and is relatively insensitive to the behaviour of $V(R)$ for large R . Hence, to a first approximation, the spectrum remains unchanged for $M_{00} \ll \alpha$. For $M_{00} \gg \alpha$ the potential is almost zero for $R \simeq (\alpha)^{-1}$ and the quasi-stationary levels no longer exist. This suggests that for sufficiently high temperature the stability of the perturbative vacuum is restored. Hence, setting $M_{00} = \alpha$ and using the expression (2.25) for the plasmon mass provides a crude estimate of the critical temperature;

$$T_c \simeq \frac{\pi\alpha}{2\log 2k_B} \quad (2.33)$$

In conclusion, the simple two-body model presented here suggests that, for N less than a critical value $N_c = 8/\pi$, the attractive $1/R$ potential between opposite charges is sufficiently strong to destabilise the massless vacuum and dynamically generate a fermion mass. As the temperature is increased thermal screening of the potential eventually destroys this effect. In the more realistic treatment given in Section 2.6 we consider a large number of Dirac fermions rather than a single pair. In that case, the entropy of the fermions at finite temperature tends to favour the high temperature phase and reduce the transition temperature. Hence the expression (2.33) is certainly an upper bound for T_c .

2.5 Dynamical mass generation

The chiral symmetry of the massless Lagrangian (2.1) is preserved order by order both in the $1/N$ expansion and in the ordinary weak coupling expansion. Hence the phenomenon of dynamical mass generation requires a nonperturbative approach. One such approach is the numerical simulation of the lattice regulated theory. Monte-Carlo simulations of non-compact lattice QED₃ [32] suggest that, for N sufficiently small, the chiral condensate $\langle \bar{\psi}\psi \rangle$ is non-zero, indicating that a fermion mass has been generated. Dynamical mass generation in QED₃ can be studied directly using an approximate treatment of the Schwinger-Dyson equation for the fermion propagator, $S_F(p)$,

$$S_F^{-1}(p) = S_F^{(0)-1}(p) - e \int \frac{d^3k}{(2\pi)^3} \gamma^\mu S_F(k) \Delta_{\mu\nu}(q) \Gamma^\nu(k, p) \quad (2.34)$$

where $\Gamma^\nu(k, p)$ is the fermion-photon vertex and $q = k - p$. This relation is shown graphically in Figure 6. This equation is part of an infinite hierarchy of integral equations for the Green's functions of the theory and some approximation is required before progress can be made. The simplest possible truncation of this equation is to replace $\Delta_{\mu\nu}$ by the expression (2.11) and replace Γ^ν by the bare vertex, $e\gamma^\nu$. Both these substitutions are correct to $O(1/N)$. Writing the fermion propagator as $S_F^{-1}(p) = (1 + A(p))\not{p} + \Sigma(p)$ and separating eqn(2.34) into scalar and spinor parts yields coupled integral equations for $A(p)$ and $\Sigma(p)$.

$$p^2 A(p) = \frac{2\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{(p \cdot q)(k \cdot q)}{q^4(1 + \Pi(q))} \frac{1 + A(k)}{k^2(1 + A(k))^2 + \Sigma^2(k)} \quad (2.35)$$

$$\Sigma(p) = \frac{2\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{1}{q^2(1 + \Pi(q))} \frac{\Sigma(k)}{k^2(1 + A(k))^2 + \Sigma^2(k)} \quad (2.36)$$

The wave-function renormalization $A(p)$ is suppressed by an explicit power of $1/N$ in the self-energy equation (2.36) and, following [21], we assume that it may be safely neglected. The angular integrals may then be performed giving the following closed integral equation for $\Sigma(p)$.

$$\Sigma(p) = \frac{\alpha}{2\pi^2 N p} \int_0^\Lambda dk \frac{k \Sigma(k)}{k^2 + \Sigma^2(k)} \log \left(\frac{8(k+p) + \alpha}{8|k-p| + \alpha} \right) \quad (2.37)$$

where we have introduced an explicit momentum-space UV cutoff Λ . In the application to high- T_c superconductivity considered below, QED₃ arises a long-wavelength limit of an underlying lattice model and the UV cutoff has an obvious interpretation as the inter-site lattice spacing.

The above equation has been solved numerically by several authors, but the essential behaviour of the solution can be seen in an approximate analytical treatment. The integrand on the RHS of (2.37) is dominated by momenta in the range $k \ll \alpha$. Expanding the logarithm and keeping only the leading term for small k and p gives.

$$\Sigma(p) = \frac{8}{N\pi^2} \int_0^\alpha dk \frac{1}{\max\{k, p\}} \frac{k^2 \Sigma(k)}{k^2 + \Sigma^2(k)} \quad (2.38)$$

The simplest possible analysis of the above equation, due to Pisarski [20], is to assume that $\Sigma(p) \simeq \Sigma(0) = M$ throughout equation (2.38). This leads to the approximate solution

$$M \simeq \alpha \exp(-N\pi^2/8) \quad (2.39)$$

A more accurate approach is to use the method of Maskawa and Nakajima [33] to transform (2.38) into a boundary value problem. Differentiating (2.38) twice with respect to p gives a second order non-linear differential equation for $\Sigma(p)$,

$$\frac{d}{dp} \left(p^2 \frac{d\Sigma}{dp} \right) = -\frac{8}{N\pi^2} \frac{p^2 \Sigma(p)}{p^2 + \Sigma^2(p)} \quad (2.40)$$

The boundary conditions at $p = 0$ and $p = \alpha$ are

$$\left(p^2 \frac{d\Sigma}{dp} \right)_{p=0} = 0 \quad (2.41)$$

$$\left(\Sigma(p) + p \frac{d\Sigma(p)}{dp} \right)_{p=\alpha} = 0 \quad (2.42)$$

For momenta in the range $\Sigma(0) \ll p \ll \alpha$, the differential equation (2.40) may be linearised by making the replacement $\Sigma(p) \simeq \Sigma(0) = M$ in the denominator of the RHS. Again the reliability of this approximation has been confirmed by the numerical solution of the full equation (2.40). Writing $\Sigma(p) = M f(p/M)$ where $f(0) = 1$, the resulting linear differential equation is of standard hypergeometric form,

$$z(1-z) \frac{d^2 f(-z)}{dz^2} + \frac{3}{2}(1-z) \frac{df(-z)}{dz} - \frac{1}{4}f(-z) = 0 \quad (2.43)$$

with $z = -p^2/M^2$. The solution which obeys the IR boundary condition (2.41) is proportional to the hypergeometric function $F(a, b; c; z)$ with parameters;

$$a = -\frac{1}{4}(1 + \sqrt{1 - 32/N\pi^2}) \quad b = -\frac{1}{4}(1 - \sqrt{1 - 32/N\pi^2}) \quad c = \frac{3}{2} \quad (2.44)$$

Hence the UV asymptotic form of the self energy for $p \rightarrow \infty$ is

$$\Sigma(p) \rightarrow M \left(\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} \left(\frac{p}{M} \right)^{\frac{1}{2}+i\frac{\gamma}{2}} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \left(\frac{p}{M} \right)^{\frac{1}{2}-i\frac{\gamma}{2}} \right) \quad (2.45)$$

where $\gamma = \sqrt{32/N\pi^2 - 1}$. For $N < \bar{N}_c = 32/\pi^2 \simeq 3.2$, γ is real and Σ oscillates like $\sin(\gamma \log(p/m))$. This is the same pathological behaviour found for the Dirac wavefunction $\psi(R)$ in the previous section, but as in that case the dimensionful coupling constant α provides a high-momentum/short-distance cutoff for the problem. The UV boundary condition (2.42) gives a relation between M , N and α which determines the allowed values of M . For $0 < M \ll \alpha$ the asymptotic solution (2.45) may be substituted for Σ in (2.42) to give,

$$\sin(\gamma \log(\alpha/M) + \delta) = 0 \quad (2.46)$$

For $N > \bar{N}_c$, γ is pure imaginary and the UV boundary condition only has the trivial solution $M = 0$. However, for $N < \bar{N}_c$, (2.46) gives an infinity of solutions labelled by the integer $n > 0$,

$$M = A\alpha \exp \left(\frac{-2n\pi}{\sqrt{N_c/N - 1}} \right) \quad (2.47)$$

where A is a constant of order unity. An examination of the vacuum energy density shows that it decreases monotonically as a function of M , implying that the physical solution is given by $n = 1$ [21].

This approximate solution for the SD equation indicates that for $N < \bar{N}_c$ a fermion mass is generated dynamically and chiral symmetry is spontaneously broken. This confirms the results and interpretation of the simple quantum mechanical model given in the previous section. As in the four-dimensional case, the relation between dynamical mass generation and the corresponding bound-state problem can be illuminated further by considering the Bethe-Salpeter equation [30]. The Bethe-Salpeter wavefunction for the pseudo-scalar fermion-antifermion bound-state obeys an equation almost identical to (2.36) and is non-zero only for $N < \bar{N}_c$.

As mentioned above, the results of the linearised analysis presented here have been confirmed by a full numerical solution of eqn(2.37) [21]. However the approximations leading to a closed integral equation for $\Sigma(p)$ are still the subject of some controversy. In particular the validity of neglecting the wave-function renormalisation, $A(p)$, has been criticised by Pennington et al [34]. A study of the full coupled equations, (2.35, 2.36), calls into question the existence of a critical number of flavours, indicating instead that chiral symmetry is broken for all N by a fermion mass which depends on N as $M \sim \alpha \exp(-CN)$ for some constant C , much like the simplified solution of Pisarski (2.39). Unfortunately, lattice simulations have so far been unable to distinguish between the two possible behaviours. This is because the fermion mass decays exponentially with N and the corresponding length scales quickly become much larger than the size of the lattice.

2.6 Symmetry restoration transition

Both numerical simulations [32] of non-compact lattice QED₃ and the approximate analytic approach of the previous section indicate that, for $T = 0$ and N sufficiently small, a fermion mass is generated dynamically. As stated above, the dynamical generation of a fermion mass spontaneously breaks chiral symmetry and leads to the existence of a massless Goldstone boson analogous to the pion in QCD. Frequently in quantum field theory, a symmetry which is spontaneously broken at zero temperature is restored above some critical temperature. The aim of this section is to present an approximate analysis of the Schwinger-Dyson equation for the fermion self-energy at non-zero temperature and provide a simple model of chiral symmetry restoration in QED₃. In three-dimensional spacetime, conventional spontaneous symmetry breaking cannot occur at non-zero temperature due to the Coleman-Mermin-Wagner theorem [35, 12]. The existence of massless particles in two dimensions leads to severe IR divergences and consequently ordinary Goldstone bosons cannot occur. The same arguments also apply in (2+1)-dimensions for $T > 0$ because of the effectively two-dimensional nature of loop integrals like (A.1) [38]. In fact this theorem does not preclude dynamical mass generation as is illustrated by the chirally-symmetric Gross-Neveu model [36, 37] (see also [38] for the (2+1)-dimensional case at finite temperature). This subtlety can be ignored if we consider instead τ_3 -QED₃ which, as stated above, has no chiral symmetry. The following analysis applies equally to both variants.

The Schwinger-Dyson equation for the fermion propagator at non-zero temperature is given by,

$$S_F^{-1}(p_0, P, \beta) = S_F^{(0)-1}(p) - \frac{e}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \gamma^\mu S_F(k_0, K, \beta) \Delta_{\mu\nu}(q_0, Q, \beta) \Gamma_\nu^\beta \quad (2.48)$$

where

$$\begin{aligned} p &= (p_0, \mathbf{p}) & P &= |\mathbf{p}| & p_0 &= \frac{(2m+1)\pi}{\beta} \\ k &= (k_0, \mathbf{k}) & K &= |\mathbf{k}| & k_0 &= \frac{(2n+1)\pi}{\beta} \\ q &= (q_0, \mathbf{q}) & Q &= |\mathbf{q}| = |\mathbf{p} - \mathbf{k}| & q_0 &= \frac{2(m-n)\pi}{\beta} \end{aligned} \quad (2.49)$$

As in the zero temperature case, the equation is truncated at leading order in $1/N$ by replacing Γ^ν by its bare value $e\gamma^\nu$, and $\Delta_{\mu\nu}$ by the $O(1/N)$ propagator shown in Figure 2.3. Neglecting wavefunction renormalisation, the inverse fermion propagator is written as $S_F^{-1} = \not{p} + \Sigma_m(P, \beta)$ and the trace of equation (2.48) yields a closed integral equation for Σ_m ,

$$\Sigma_m(P, \beta) = \frac{\alpha}{N\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \Delta_{\mu\mu}(q_0, Q, \beta) \frac{\Sigma_n(K, \beta)}{k^2 + \Sigma_n^2(K, \beta)} \quad (2.50)$$

Using the full finite temperature photon propagator (2.18), (2.50) becomes

$$\Sigma_m(P, \beta) = \frac{\alpha}{N\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \left[\frac{1}{q^2 + \Pi_A^{m-n}(Q, \beta)} \right]$$

$$+ \frac{1}{q^2 + \Pi_B^{m-n}(Q, \beta)} \left] \frac{\Sigma_n(K, \beta)}{k^2 + \Sigma_n^2(K, \beta)} \quad (2.51)$$

The temporal and spatial components of the vacuum polarisation Π_A and Π_B are calculated in Appendix A and are given by

$$\Pi_A^m(P, \beta) = \frac{\alpha}{\pi\beta} \int_0^1 dx \log(4 \cosh^2(p\beta\sqrt{x(1-x)}/2) - 4 \sin^2(xm\pi)) \quad (2.52)$$

$$\Pi_B^m(P, \beta) = \frac{\alpha}{2\pi\beta} \int_0^1 dx \frac{p\beta\sqrt{x(1-x)} \sinh(p\beta\sqrt{x(1-x)}/2) + m\pi(1-2x) \sin(2xm\pi)}{\cosh^2(p\beta\sqrt{x(1-x)}/2) - \sin^2(xm\pi)} \quad (2.53)$$

Equation (2.51) is the generalisation of the integral equation (2.37) to finite temperature and it is easily checked that as $\beta \rightarrow \infty$ the correct zero temperature form is recovered. As at $T = 0$, (2.51) can be solved numerically, although a full solution would require significantly more computing power than the zero temperature case due to the large number of integrations required at each iteration. As in the zero temperature case, particular simplifications occur if the approximation $\Sigma_m(P) \simeq \Sigma_0(0) = m(\beta)$ is adopted. In this case equation (2.21) becomes,

$$1 = \frac{\alpha}{N\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \left[\frac{1}{4n^2\pi^2/\beta^2 + K^2 + \Pi_A^{-n}(K, \beta)} + \frac{1}{4n^2\pi^2/\beta^2 + K^2 + \Pi_B^{-n}(K, \beta)} \right] \frac{1}{4(2n+1)^2\pi^2/\beta^2 + K^2 + m^2(\beta)} \quad (2.54)$$

The solution of the above equation coincides with (2.39) at $T = 0$ and shows how this result is modified for $T > 0$. However, even (2.54) presents a considerable numerical challenge and will not be solved directly here.

The term in the brackets in equation (2.54) represents the contribution of the interaction generated by one exchanged photon to the dynamical mass. This kernel includes parts corresponding to an instantaneous interaction ($n^2 = 0$) and to a retarded contribution ($n^2 > 0$) from both the spatial and temporal components of the photon propagator. To examine a simplified model of chiral symmetry restoration at non-zero temperature we approximate (2.54) by retaining only the part of the kernel that corresponds to the static interaction $V(R)$. A similar approximation has been used to study chiral symmetry restoration in QCD [39]. As shown by equation (2.14), the effective static interaction between fermions depends only on the temporal component of the propagator at zero frequency, $\Delta_{00}(k_0 = 0, K, \beta)$, so the corresponding approximate Schwinger-Dyson equation becomes,

$$1 = \frac{\alpha}{N\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \frac{1}{K^2 + \Pi_A^0(K, \beta)} \frac{1}{4(2n+1)^2\pi^2/\beta^2 + K^2 + m^2(\beta)} \quad (2.55)$$

The Matsubara frequency sum can now be performed analytically to give,

$$1 = \frac{\alpha}{4N\pi} \int_0^\Lambda K dK \frac{\tanh[\beta\sqrt{K^2 + m^2(\beta)}/2]}{(K^2 + \Pi_A^0(K, \beta))\sqrt{K^2 + m^2(\beta)}} \quad (2.56)$$

Equation (2.56) was solved numerically for several values of N and α/Λ . The results show clearly that the fermion mass is a monotone decreasing function of temperature up to a critical temperature, T_c , above which m vanishes identically. For convenience we give the results as functions of the equivalent number of two-component flavours $n = 2N$. The graph of m against $k_B T$ for $n = 1$ is shown in Figure 7. The values of $m(T = 0)$ and $k_B T_c$ for a range of n at intermediate and strong coupling ($\alpha/\Lambda = 1$ and ∞) are given in Tables 2 and 3 respectively. These quantities are also given at $n = 1$ for a range of α/Λ in Table 4 and, as expected, both grow as the coupling is increased. For $\alpha < \Lambda$ the fermion mass is proportional to α only and is thus insensitive to Λ . This confirms the expectation that the theory is naturally cut-off in the UV by α , unless, of course, a lower cutoff Λ is imposed. For completeness, the equation was also solved without a cutoff [24] and the results were found to confirm the stability of the numerical solution presented here. Even in the instantaneous-exchange approximation, the zero-temperature fermion mass retains its characteristic exponential dependence on n . In the strong-coupling limit, for $n > 1$, the mass is given by

$$m(T = 0) \simeq 2\Lambda \exp(-n\pi/4) \quad (2.57)$$

²The formulae of Ref [24] are incorrect by a factor of 2. This can be corrected by replacing N by $n = 2N$ throughout. This does not affect the conclusions reached in that paper.

A quantity of interest for condensed matter applications is the ratio of twice the zero-temperature mass to the critical temperature, $r = 2m(T = 0)/k_B T_c$. We find that r varies only slowly with n and α/Λ . In particular, for $1 < n < 5$ and $1 \leq \alpha/\Lambda \leq \infty$, r is in the range 9.36-9.70. This ratio has also been calculated recently for the Gross-Neveu model in (2+1)-dimensions [41] where the result 2.77 was found. Another relevant comparison is with the value $r = 3.54$ which characterizes BCS superconductors. In both of these cases the effective two-body potential $V(R)$ is largely independent of temperature and for comparison we have repeated our calculation for QED₃ replacing $\Pi_A^0(K, \beta)$ by the zero-temperature kernel $\alpha K/8$ in (2.56). The resulting values of the ratio are shown as r^* in Table 4 and are comparable in magnitude with the BCS value. Hence the comparatively large values of r (ie low T_c in units of $m(T = 0)$) shown in Tables 2 and 3 are directly due to thermal screening of the two-body potential $V(R, \beta)$. A discussion of symmetry restoration in finite-temperature QED₃ has also been given in [42] using real-time formalism and a different approximation for the gap equation in which the entropy of the fermions is not fully included.

3 A long-wavelength theory of the two-dimensional doped antiferromagnet

3.1 The high- T_c materials

The high- T_c materials La_2CuO_4 , YBa_2CuO_6 and several related compounds have a layered structure, consisting of planes of CuO_2 separated by distances much greater than the average inter-site distance within the plane. Many properties of these substances exhibit a strong anisotropy in the direction perpendicular to these strata. In particular, the copper oxide layers are thought to be the essential structural feature for superconductivity which occurs when these materials are doped. Recent experiments have succeeded in effectively isolating superconductivity in a single layer [43, 44]. Although interplanar effects may play an important role in determining the exact transition temperature, we will assume that superconductivity in the doped materials is an essentially two-dimensional phenomenon. As discussed in Chapter 1, superconductivity is the direct consequence of the spontaneous breaking of the global $U(1)$ symmetry of electromagnetism. The Mermin-Wagner theorem [12] states that a continuous symmetry of a two-dimensional system cannot be spontaneously broken by a local order parameter at non-zero temperature. Hence, the mere fact of two-dimensionality suggests that the conventional BCS theory, which involves a local condensate, cannot adequately describe this phenomenon.

The addition of Strontium impurities to pure La_2CuO_4 creates holes at both copper and oxygen sites in the CuO_2 layers. The doping of YBa_2CuO_6 proceeds by increasing the oxygen content of this substance which also leads indirectly to the creation of holes. Below a critical temperature, the doped substances have zero electrical resistance and exhibit the usual phenomenology of superconductivity. There is an energy gap Δ in the quasi-particle spectrum, a Meissner effect and flux quantization in units of $h/2e$. This value for the flux quantum and importance of doping suggest a pairing of holes analogous to Cooper pairing in BCS theory. In the BCS case this pairing is mediated by the exchange of phonons between electrons near the Fermi surface and the important role played by vibrations of the lattice is manifested in the isotope effect. Experimental investigations of the high- T_c materials have found that the isotope effect is either not present at all or highly suppressed [45]. In addition, the BCS gap equation relates the mass gap and critical temperature to the Debye energy which characterises phonon interactions. This leads to rigorous upper bounds on the transition temperatures possible for phonon-mediated pairing. The fact that these bounds are clearly violated for high- T_c superconductors together with the apparent absence of an isotope effect for these materials suggests that the pairing mechanism is non-phononic. In Section 3.3 we examine a possible candidate for such a mechanism. One of the most striking features of the BCS theory was the successful prediction of the universal ratio $2\Delta/k_B T_c = 3.54$ which characterises the traditional superconductors. The quasi-planar materials exhibit values of this ratio much greater than the BCS value, typically as large as 8 [46], and any successful theory of high- T_c superconductivity must explain this important deviation from BCS theory. High- T_c superconductors are also distinguished by their exceptionally small coherence lengths [47].

In the absence of doping impurities, the quasi-planar materials are antiferromagnetic insulators. They are believed to belong to the class of solids known as Mott insulators, for which the strong Coulomb repulsion between valence electrons at neighbouring sites effectively prevents charge transport. The potential importance of antiferromagnetic correlations for high-temperature superconductivity was first noted by Anderson [48] who suggested that the correct model for the dynamics of electrons in the copper oxide layers was the single-band, large- U Hubbard model. The two-dimensional Hubbard Hamiltonian is written in terms of operators, $C_{i,\sigma}$ and

$C_{i,\sigma}^\dagger$, which annihilate and create electrons in the $d_{x^2-y^2}$ orbital at each copper site,

$$H = -t \sum_{\langle ij \rangle, \sigma} C_{i,\sigma}^\dagger C_{j,\sigma} + U \sum_i n_{i,\uparrow} n_{i,\downarrow} \quad (3.1)$$

where t is the electron-hopping matrix element, U is the strong Coulomb repulsion and $n_{i\sigma} = C_{i\sigma}^\dagger C_{i\sigma}$ is the occupation number at each site. In the limit $U \rightarrow \infty$, a single-occupancy constraint is rigidly imposed. The undoped case is described by the Hubbard model with half-filled band and hence the spins are the only degrees of freedom in this limit. To leading order in large- U perturbation theory, the half-filled Hubbard model is simply equivalent to the two-dimensional Heisenberg antiferromagnet [49];

$$H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \quad (3.2)$$

where $J = 4t^2/U$ and $\mathbf{S}_i$ is the electron spin at site i . Antiferromagnetic correlations in pure La_2CuO_4 have been measured in neutron scattering experiments [50] and the results are reasonably well described by the 2D Heisenberg model with $J \simeq 1500\text{K}$ [51]. This Hamiltonian will be the starting point for the long-wavelength effective theory of the undoped case reviewed in the next section.

Clearly, the effect of adding holes to (ie removing spins from) the background antiferromagnetic order is the key issue for superconductivity. Unfortunately, little is known about the Hubbard model away from half-filling and no rigorous results are available. Numerical simulation indicates that the addition of holes destroys the long-range antiferromagnetic order, although short-range correlations may survive [52]. We should also mention that the physical relevance of any simple model of interacting holes and spins is limited by the recent experimental evidence that up to 70% of the charge carriers in the CuO_2 layers of $\text{La}_2(\text{Sr})\text{CuO}_4$ and YBa_2CuO_6 are holes in the oxygen bonds rather than at the copper sites that carry the spins [53]. The problem of constructing a realistic model involving both types of hole is a formidable one. However, it is certainly possible that all mobile holes are coupled to some extent to the spins and the effect of the oxygen holes can be included by renormalizing the parameters of the Hubbard Hamiltonian. In Section 3.3 we will study a simplified model of the dynamics of holes in a 2D antiferromagnet and examine the possible mechanism of hole-pairing. In Section 3.4 we introduce a long-wavelength effective theory describing the interactions of holes and spins and in the Section 4 we demonstrate the occurrence of superconductivity in this model and its interpretation.

3.2 A long-wavelength action: the undoped case

As discussed in the previous section, the large- U limit of the Hubbard Hamiltonian (3.1) is just the two-dimensional, spin- $\frac{1}{2}$ Heisenberg quantum antiferromagnet. Under certain assumptions, which will be discussed below, the effective long-wavelength degrees of freedom of the antiferromagnet can be described by a ‘relativistic’ quantum field theory in (2+1)-dimensional spacetime. In particular, the large- S limit of the spin- S Heisenberg antiferromagnet is equivalent, at large length-scales, to the quantum nonlinear σ -model [54, 55]. The relativistic covariance of the effective action arises from the linear dispersion relation for long-wavelength magnons and the spin-wave velocity plays the role of the velocity of light in this formulation. In this section we will review the derivation of the non-linear σ -model from the Heisenberg Hamiltonian and, in particular, derive the CP^1 formulation of the σ -model which leads to an equivalent description of the effective theory in terms of an abelian gauge field.

The two-dimensional, undoped, spin- S , Heisenberg antiferromagnet is described by the Hamiltonian,

$$H = J \sum_{ij} \mathbf{S}_i \cdot \mathbf{S}_j \quad (3.3)$$

where i and j are nearest-neighbour sites on a square lattice. The spin operators $\mathbf{S}_i = (\hat{S}^a)_i$ obey independent $O(3)$ commutation relations at each site:

$$[\hat{S}_i^a, \hat{S}_j^b] = i\epsilon^{abc} \hat{S}_i^c \delta_{ij} \quad (3.4)$$

At zero-temperature, the classical ground state of the antiferromagnet is just the Néel ordered state defined by $\mathbf{S}_i = S(-1)^I \hat{\mathbf{z}}$ where $(-1)^I$, with $I = i_x + i_y$, is the parity of the site i and $\hat{\mathbf{z}}$ is a unit vector.

Following Haldane [54], the Hilbert space of the system is conveniently described in terms of a basis of coherent states, $|\Omega\rangle$, defined at each site by

$$|\Omega\rangle = |\theta, \phi\rangle = \exp(-i\phi(\hat{S}_z - S)) \exp(-i\theta\hat{S}_y)|S\rangle \quad (3.5)$$

where Ω is a unit vector with angular coordinates (θ, ϕ) and $|S\rangle$ is the eigenstate of $\hat{S}_z$ with largest eigenvalue S . The coherent state $|\Omega\rangle$ can be expressed in the eigenbasis of $\hat{S}_z$ as

$$|\Omega\rangle = \sum_{m=-S}^S C_m(\theta, \phi) |m\rangle \quad (3.6)$$

where $|m\rangle$ is the eigenstate of $\hat{S}_z$ with eigenvalue m and the coefficient $C_m(\theta, \phi)$ can be expressed as,

$$C_m(\theta, \phi) = \exp(-im\theta) \sin^{S-m} \left(\frac{\theta}{2}\right) \cos^{S+m} \left(\frac{\theta}{2}\right) \frac{(2S)!}{(S+m)!(S-m)!} \quad (3.7)$$

Using (3.7), we find the inner product of two coherent states $|\Omega\rangle$ and $|\Omega'\rangle$ is given by;

$$\langle\Omega|\Omega'\rangle = \left(\cos \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta'}{2}\right) + \exp i(\phi - \phi') \sin \left(\frac{\theta}{2}\right) \sin \left(\frac{\theta'}{2}\right) \right)^{2S} \quad (3.8)$$

and the expectation of the spin operator, $\mathbf{S}$, in a coherent state is

$$\langle\Omega|\mathbf{S}|\Omega\rangle = S\Omega \quad (3.9)$$

From (3.8) it can be seen that the overlap of neighbouring coherent states decays rapidly with angular separation in the large- S limit. A state of the system is then described in the obvious way as a direct product of coherent states $|\Omega_i\rangle$ at each site and, using (3.3) and (3.9), the expected energy of the system in such a state is given by,

$$E = \langle\{\Omega_i\}|\hat{H}|\{\Omega_i\}\rangle = S^2 J \sum_{\langle ij \rangle} \Omega_i \cdot \Omega_j \quad (3.10)$$

Using the standard arguments, the partition function of the system can be written as a path integral over the unit-vector field Ω_i . Introducing a series of discretised time-slices τ_i with $i = 1, 2 \dots M$ and $\tau_{i+1} - \tau_i = \epsilon$, the time evolution operator is given by

$$\exp(-\beta\hat{H}) = \prod_{i=1}^M (1 - \epsilon\hat{H}) + \dots \quad (3.11)$$

where $\beta = M\epsilon$. Resolving the identity operator at each site as

$$I = \int \frac{d\Omega}{4\pi} |\Omega\rangle\langle\Omega| \quad (3.12)$$

the partition function of the system becomes

$$\mathcal{Z} = \int \mathcal{D}\Omega \prod_{i=1}^M \langle\{\Omega_i(\tau_i + \epsilon)\} | 1 - \epsilon\hat{H} | \{\Omega_i(\tau_i)\}\rangle \quad (3.13)$$

The overlap of the coherent states at neighbouring time-slices is written as

$$\langle\Omega(\tau + \epsilon)|\Omega(\tau)\rangle = |\langle\Omega(\tau + \epsilon)|\Omega(\tau)\rangle| \exp(-i\Phi) \quad (3.14)$$

Assuming that the time evolution of Ω is sufficiently smooth, then, using (3.8), the phase Φ can be expanded for small ϵ as

$$\Phi = (1 - \cos \theta) \Delta\phi + \dots \quad (3.15)$$

where $\Delta\phi = \phi(\tau + \epsilon) - \phi(\tau)$. Hence the lowest order contribution to Φ is just the area swept out on the unit sphere, in the time interval $[\tau, \tau + \epsilon]$, by the great circle arc which joins $\Omega(\tau)$ and the north pole. The modulus of the overlap is

$$\begin{aligned} |\langle\Omega(\tau + \epsilon)|\Omega(\tau)\rangle| &= \exp \left(1 - \frac{1}{4} |\Omega(\tau + \epsilon) - \Omega(\tau)|^2 \right) \\ &\simeq \exp \left(-\frac{S}{4} \left[\frac{\partial\Omega}{\partial\tau} \right]^2 \epsilon^2 \right) \end{aligned} \quad (3.16)$$

Hence the partition function (3.13) is given by

$$Z = \int \mathcal{D}\Omega \exp \left[-iS \sum_i \oint_{C_i} (1 - \cos \theta_i) d\phi_i - S^2 J \int_0^\beta \sum_{\langle ij \rangle} \Omega_i \cdot \Omega_j d\tau \right] \quad (3.17)$$

where the C_i is the closed path on the unit sphere given by $\{\Omega_i(\tau) : \tau \in [0, \beta]\}$.

The integral in the first term in (3.17) is simply equal to the solid angle α_i subtended by the path C_i and using Stoke's theorem this can be written in terms of a vector potential $\mathbf{A}$ at each site as,

$$\alpha = \int_A \Omega \cdot \Omega d\sigma = \oint_C \mathbf{A} \cdot d\Omega \quad (3.18)$$

where A is the area enclosed by C and the vector potential obeys,

$$\nabla_\Omega \times \mathbf{A}(\Omega) = \Omega \quad (3.19)$$

As usual, the potential $\mathbf{A}$ is not uniquely defined by (3.19) and there is a residual gauge invariance $\mathbf{A} \rightarrow \mathbf{A} + \nabla_\Omega \Omega$. In particular, a gauge can be chosen in which $\mathbf{A}(\Omega) = \mathbf{A}(-\Omega)$. The phase factor $iS\alpha_i$ which occurs in the action is known as the *Berry phase* [56] for the adiabatic motion of the quantum spin at site i and has important physical consequences when holes are introduced.

Haldane has derived an effective continuum field theory from the path integral (3.17) by expressing the vector field $\Omega_i(\tau)$ in terms of a staggered order parameter $\mathbf{n}_i(\tau)$ [54],

$$\Omega_i(\tau) \simeq (-1)^I \left[1 - \left(\frac{a^2 \mathbf{L}_i(\tau)}{S} \right)^2 \right]^{\frac{1}{2}} \mathbf{n}_i(\tau) + \frac{a^2}{S} \mathbf{L}_i(\tau) \quad (3.20)$$

where the field $\mathbf{L}_i(\tau)$ describes fluctuations in the local antiferromagnetic spin density. In the limit $S \rightarrow \infty$ the partition function for long-wavelength fluctuations can be written as a path integral over the effective continuum field $\mathbf{n}(\mathbf{x}, \tau)$ as

$$Z = \int \mathcal{D}\mathbf{n} \exp \left[-\frac{JS^2}{2} \int_0^\beta d\tau \int d^2x \frac{1}{v_S^2} |\partial_\tau \mathbf{n}|^2 + |\nabla \mathbf{n}|^2 \right] \quad (3.21)$$

This effective action describes long-wavelength spin-waves with a linear dispersion relation $E(\mathbf{k}) = v_S |\mathbf{k}|$ where the spin wave velocity is given by $v_S = 2\sqrt{2}SJ a$. The action has the familiar form of the $O(3)$ nonlinear σ -model in (2+1)-dimensional spacetime and can be written in a manifestly 'relativistic' form by choosing natural units in which $v_S = 1$,

$$S_\sigma = \int d^3x \frac{1}{\gamma} (\partial_\mu \mathbf{n}) (\partial^\mu \mathbf{n}) \quad (3.22)$$

where $\gamma = 2/JS^2$. The σ -model action may be rewritten in terms of CP^1 variables in the usual way by making the substitution $\mathbf{n} = \bar{z} z$ with $\bar{z} z = 1$. This gives

$$S_{CP^1} = \int d^3x \frac{1}{\gamma} |(\partial_\mu + iS a_\mu) z|^2 \quad (3.23)$$

where the $U(1)$ gauge field a_μ is determined by its equation of motion; $a_\mu = i\bar{z} \partial_\mu z$. Although this abelian gauge field is non-dynamical, quantum corrections ensure that the gauge propagator acquires a pole at zero momentum and a Maxwell term for a_μ is radiatively generated [57, 58]. The CP^1 form of the nonlinear σ -model will be particularly convenient for describing the dynamics of the 'spin sector' in the doped case.

In conclusion, we have reviewed the large- S derivation of the nonlinear σ -model as an effective action for long-wavelength degrees of freedom of the Heisenberg antiferromagnet. Although the relevance of the large- S limit to the physical case $S = \frac{1}{2}$ is not clear, there are more general grounds for introducing the σ -model. The massless spin-waves are the Goldstone bosons of the spontaneously broken $O(3)$ symmetry of the antiferromagnet, and general symmetry arguments dictate that the interaction of these Goldstone modes be described by the σ -model action (3.21) [59]. The appearance of an accidental relativistic invariance motivates the application of methods and notation which are more usually associated with high-energy physics.

3.3 The dynamics of holes

The long-wavelength limit of the half-filled Hubbard Hamiltonian is relatively well understood in terms of the action derived in the previous section. Numerical simulations confirm the existence of long-range Néel order at low temperature and agree with predictions for the correlation length derived from the $O(3)$ nonlinear σ -model [51]. Unfortunately, no similar theoretical understanding exists away from half-filling. No rigorous results exist for the Hubbard model with a finite density of holes and, like lattice simulation of finite-density QCD, direct numerical simulation of the doped case is hindered by the occurrence of a complex fermion determinant [60]. Doping introduces mobile charges which hop from site to site against the antiferromagnetic background of the spins. The coupled dynamics of holes and spins in the doped system is highly non-trivial. The hopping of holes tends to disorder the spins reducing the antiferromagnetic correlation length and the spins also mediate interactions between the holes. Roughly speaking there is competition between the influence of the spins which favour a Néel-ordered ground-state and that of the holes which tend to form a Fermi liquid. A general conjecture is that a superconducting pairing of holes arises out of this competition.

Returning to the materials themselves, the Hubbard model suggested by Anderson is a purely phenomenological description of electrons in the CuO_2 layers and there are several alternative approaches which may be equally relevant. The approach which will be adopted here is originally due to Shankar [13] and focuses on the effect of introducing ‘holes’ in the coherent-state path-integral representation of the Heisenberg antiferromagnet given in equation (3.17). The aim will be to examine the effect of doping on the long-wavelength σ -model description derived in the previous section and construct a modified effective action describing both charge and spin degrees of freedom. Rewriting the first term in the partition function (3.17) in terms of the gauge potential $\mathbf{A}$,

$$\mathcal{Z} = \int \mathcal{D}\Omega \exp \left[\int_0^\beta \left(iS \sum_i \mathbf{A}(\Omega_i) \frac{d\Omega_i}{d\tau} - S^2 J \sum_{\langle ij \rangle} \Omega_i \cdot \Omega_j \right) d\tau \right] \quad (3.24)$$

In the large- S limit, the relation (3.20) becomes

$$\Omega_i = (-1)^I \mathbf{n}_i + O(a) \quad (3.25)$$

and we may rewrite (3.24) as

$$\mathcal{Z} = \int \mathcal{D}\Omega \exp \left[\int_0^\beta \left(iS \sum_i (-1)^I a_0(i, \tau) - S^2 J \sum_{\langle ij \rangle} \Omega_i \cdot \Omega_j \right) d\tau \right] \quad (3.26)$$

where a_0 is the zero component of an abelian gauge field defined by $a_\mu = \mathbf{A} \cdot \partial_\mu \mathbf{n}$. The origin of this gauge field is best described in differential geometric terms. The order parameter field $\mathbf{n}(\mathbf{x}, \tau)$ defines a mapping from three-dimensional spacetime to the unit sphere. The spacetime gauge field a_μ is the *pullback* of the one-form gauge potential $\mathbf{A}$ defined on the unit sphere. In particular, if the gauge for the vector potential $\mathbf{A}$ is chosen appropriately then the above definition of a_μ coincides with that of the auxiliary gauge field occurring in the CP^1 formulation of the nonlinear σ -model (3.23).

Each spin contributes a Berry phase factor $iS\alpha_i$ to the action where α_i is the solid angle subtended by the closed path C_i as discussed in Section 3.2. Using (3.25) the Berry phase for the spin at site i can be written as $i(-1)^I S \Sigma_i$ where Σ is the solid angle subtended by the closed path $\tilde{C}_i = \{\mathbf{n}_i(\tau) : \tau \in [0, \beta]\}$. As the order parameter varies only slowly in space this means that the Berry phase contributions of spins at neighbouring sites interfere destructively. Now consider the effect of removing the spin at site i . If the spin is removed at time τ_1 and replaced at τ_2 the Berry phase contribution it would have made in this time interval, $B_i(\tau_1, \tau_2)$, must be subtracted from the action. Hence the contribution of a static hole is given by the Wilson line,

$$-B_i(\tau_1, \tau_2) = -i(-1)^I S \int_0^\beta d\tau a_0(i, \tau) \quad (3.27)$$

To implement this idea, Shankar introduces annihilation and creation operators for holes at each site, ψ_i and $\psi_i^\dagger$. These operators obey fermionic anticommutation relations

$$\{\psi_i, \psi_j^\dagger\}_+ = \delta_{ij} \quad \{\psi_i, \psi_j\}_+ = \{\psi_i^\dagger, \psi_j^\dagger\}_+ = 0 \quad (3.28)$$

and the corresponding contribution to the action for static holes is

$$S_{static} = -iS \int_0^\beta d\tau \sum_i (-1)^I \psi_i^\dagger \psi_i a_0(i, \tau). \quad (3.29)$$

This term clearly has the required property of subtracting the appropriate Wilson line whenever a hole is present at a particular site. As the sign of this term alternates from site to site it is convenient to divide the hole-field into two species, labeled A and B corresponding to the two sublattices defined by the Néel order (see Fig 8). As we shall see below, particle numbers for A and B type holes are conserved independently in the large- S limit. The two species are coupled with opposite sign to the gauge field a_0 , and the static action is written as

$$S_{static} = -iS \int_0^\beta d\tau \sum_{i \in A} \psi_i^\dagger \psi_i a_0(i, \tau) + (A \rightarrow B, S \rightarrow -S) \quad (3.30)$$

This form of the static action suggests an attractive gauge interaction between holes of the two species. In fact this is just a consequence of the interference of Berry phases described above. The creation of an isolated hole adds an oscillating Wilson line (3.29) to the path-integral. The integration over the gauge field a_μ at that site averages to $\exp(-\beta E)$ where E is the energy of the hole. Hence the creation of two uncorrelated holes costs energy $2E$. However if two holes are created on nearest neighbour sites the two phase factors interfere destructively and the energy cost is zero [13]. This implies that the bare effective potential between two holes in their common centre of mass is just a square well of depth $2E$ and width of the order of one lattice spacing. The idea that the interference of Berry phase contributions leads to an attraction between holes is originally due to Wiegmann [14] and has been developed in an alternative approach by Lee [15]. The existence of such an attraction between holes on neighbouring sites has recently been confirmed by numerical simulation of the non-linear σ -model [61].

In a realistic model, holes are not static but hop from site to site, and a hopping term must be added to the action. The corresponding term in an extended Hubbard Hamiltonian is

$$H_{hop} = - \sum_{(ij), \sigma} t_{ij} C_{i, \sigma}^\dagger C_{j, \sigma} \quad (3.31)$$

where the sum is understood to run over both nearest-neighbour (NN) and next-to-nearest neighbour (NNN) sites i and j . Such a term is responsible for the transport of an electron from site i to site j . The electron possesses both spin and charge degrees of freedom and so this transition is equivalent to the coherent transport of the electron spin from i to j accompanied by the transport of a charged hole in the opposite direction. Hence, in terms of the variables of Shankar's model the appropriate matrix element for this process is;

$$\begin{aligned} M_{ij} &= \langle \Omega_i, \text{hole at } j | H_{hop} | \text{hole at } i, \Omega_j \rangle \\ &= -t_{ij} \langle \Omega_i | \Omega_j \rangle \end{aligned} \quad (3.32)$$

If i and j are on opposite sublattices, i.e. if they are NN, then the spin states $|\Omega_i\rangle$ and $|\Omega_j\rangle$ are almost orthogonal by virtue of the short-range antiferromagnetic order (SRAFO). This clearly implies that intersublattice hopping is highly suppressed as long as SRAFO persists and hence the integrity of the two species of holes introduced above is preserved. In contrast, if i and j are NNN and so belong to the same sublattice, then the corresponding spin states are almost parallel and, from (3.8), $M_{ij} = -t \exp[iS(-1)^I \mathbf{A} \cdot (\mathbf{n}_i - \mathbf{n}_j)]$ [13]. Hence the final form of the hole-hopping action which reproduces this factor for each hole transition can be written in terms of the spatial gauge field, $a_k(i) = \mathbf{A} \cdot \partial_k \mathbf{n}_i$ ($k = 1, 2$), as

$$S_{hop} = -t \int_0^\beta d\tau \sum_{i \in A, \delta} \psi_i^\dagger \exp(iS \mathbf{a}(i) \cdot \delta) \psi_{i+\delta} + h.c. + (A \rightarrow B, S \rightarrow -S) \quad (3.33)$$

where δ is the vector separating sites i and j . As in the static case, the holes of the two different species couple to the spatial components of the gauge field with opposite sign. A finite density of holes is introduced by adding a chemical potential term to the action

$$S_\mu = \mu \int_0^\beta d\tau \sum_i \psi_i^\dagger \psi_i \quad (3.34)$$

where the sum is taken over both sublattices. Finally, the hole terms must be added to the σ -model action which describes the long-wavelength limit of the half-filled case. For the purposes of the next section, it will be convenient to use the CP^1 form of the σ -model action given in (3.23). In the presence of dynamical holes, the identification of the gauge field a_μ of Shankar's model with the auxiliary gauge field of the CP^1 action is no longer exact but rather must be thought of as a kind of Born-Oppenheimer approximation.

In fact, in the presence of holes, the gauge field of Shankar's model appears not as an independent degree of freedom but rather as a function of the magnon field. This introduces complications in the continuum language. Such problems can be overcome if one starts from a t - J model [16] on a square lattice and includes next-to-nearest neighbor interactions and hopping. The spin-charge separation is achieved by representing the electron operators $C_{i,\sigma}$ using a 'slave-fermion' ansatz [16, 62],

$$C_{i,\sigma}^\dagger = \psi_i z_{i,\sigma}^\dagger \quad (3.35)$$

where ψ is a Grassmann field representing the absence of a spin at a given site (hole) which carries the electric charge and $z_{i,\sigma}$ is the spin degree of freedom, which can be identified [16] with the magnon field of the CP^1 σ -model. The ansatz (3.35) carries essentially all the information about local gauge invariance of the model as one can perform simultaneous local phase rotations on ψ_i and $z_{i,\sigma}$ fields with opposite couplings without changing $C_{i,\sigma}$. It is this symmetry that is responsible for the gauge nature of the interactions between holes. The physical reason for such a symmetry is the restriction of having at most one electron per lattice site. This redundancy of degrees of freedom is the characteristic feature of gauge models. In a mean-field treatment [16] this local symmetry leads to gauge fields which can be represented as phases of bilocal fields stemming from linearisation of the contact terms arising upon substitution of the ansatz (3.35) in the interaction terms of the extended t - J hamiltonian considered in [16]. We shall not give the details here to avoid repetition. We only mention the basic features. As in Shankar's model the suppression of intrasublattice hopping for holes defines two colours of fermions coupled to a single gauge field but with opposite couplings between sublattices. The physical origin of the attraction between holes on opposite sublattices is again the overlap of the Berry-phase, and one recovers Shankar's results in the static limit. The novel feature of this approach is that the gauge field arises as an independent degree of freedom. The effect of the mobile holes is to modify the form of the CP^1 constraint $\bar{z}z = 1$ by terms which are of order $O(J')$, where J' is the strength of the Heisenberg interactions between next-to-nearest neighbors. In the realistic cases one is interested in the limit $J' \rightarrow 0$ and, hence, the modifications to the usual CP^1 -constraint are negligible [16].

The full partition function of the model is given as a path-integral over the Grassmann fields ψ_i and $\psi_i^\dagger$ as well as the CP^1 variables $z, \bar{z}$ and a_μ .

$$\mathcal{Z} = \int \mathcal{D}\bar{z} \mathcal{D}z \mathcal{D}a_\mu \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp(-S_{CP^1} - S_{static} - S_{hop} - S_\mu) \quad (3.36)$$

This formulation seems rather odd as the spin degrees of freedom are described as continuous fields while the holes are confined to a spatial lattice. In the next section we will take a long-wavelength limit for the hole terms in the action and arrive at an effective continuum Lagrangian in (2+1)-dimensional spacetime which describes the coupled dynamics of both holes and spins.

3.4 A long-wavelength action: the doped case

The aim of this section is to find a suitable model for the electrically charged degrees of freedom of the doped antiferromagnet. At zero temperature, a finite density of fermionic holes will occupy all available states in momentum space up to some Fermi surface. The long-wavelength excitations of this system are 'quasi-hole' states, in which one hole is excited to a state above the Fermi surface leaving an unoccupied state behind. As we will see, the electrical properties of the system depend crucially on the nature of the quasi-hole excitation spectrum and, in particular, on the occurrence of an energy gap in this spectrum. Although the electrons themselves must be thought of as ordinary non-relativistic fermions, the long-wavelength quasi-holes, like the spin-waves discussed previously, have a linear, 'relativistic' dispersion relation. The limiting velocity, corresponding to the light-velocity in a conventional relativistic formulation, is v_F the Fermi velocity for holes. As in the previous section, A and B type holes are treated entirely separately: a quasi-hole has the same quantum numbers as an ordinary hole; electrical charge $+e$ and 'statistical' charge $\pm S$ depending on sublattice label. Hence, the resulting effective action describes two species of linear-spectrum fermions coupled with opposite sign to the gauge field of the CP^1 model but with the same sign to an external electromagnetic field. In this section, as in the previous ones, we will neglect the ordinary electric charge of the holes.

In order to derive a long-wavelength effective theory of the fermionic sector of the model it will be useful to work in a Hamiltonian formalism. The relevant terms are

$$H_F = -t \sum_{i \in A, \delta} \psi_i^\dagger \exp(iS a(i, \delta)) \psi_{i+\delta} + h.c. + \mu \sum_{i \in A} \psi_i^\dagger \psi_i + (A \rightarrow B, S \rightarrow -S) \quad (3.37)$$

The above Hamiltonian describes Grassmann fields coupled to an abelian gauge field and is reminiscent of compact $U(1)$ lattice gauge theory. Related gauge-invariant descriptions of the Hubbard model have been proposed by several authors [58, 63, 64]. Affleck and Marston linearise the quartic interactions in the Hubbard Hamiltonian by introducing a bilocal auxiliary field $\chi_{ij} \sim \langle C_i^\dagger C_j \rangle$. In the formalism of Ref [63], χ_{ij} is a gauge non-invariant quantity and so must have a zero expectation value by Elitzur's theorem [65]. In a mean-field approach, it is natural to consider the phase fluctuations of χ_{ij} which enforce this vanishing. Writing

$$\chi_{ij} \simeq |\chi_{ij}| \exp i(\theta_j - \theta_i) \quad (3.38)$$

the phase differences $\theta_{ij} = \theta_i - \theta_j$ can be directly related to the gauge field $a_{ij} = \mathbf{a}(i) \cdot \delta$ of Shankar's model. In all such treatments, the natural gauge-invariant parameter which characterises configurations of the gauge field is the product of group elements around an elementary plaquette, $\Pi = \chi_{12}\chi_{23}\chi_{34}\chi_{41}$, where 1, 2...4 are the corners of a unit square. Affleck and Marston [63] performed a variational calculation to determine the true ground-state of the system for a range of the Hubbard parameters and found that, for light doping and U sufficiently large, the free energy is minimised by a gauge field configuration with $\Pi = -1$. This is equivalent to a gauge flux of π passing through each plaquette and the corresponding phase of the the Hubbard model characterised by this ground-state is referred to as the 'flux phase'.

For the present purposes, we will assume that the same energetics arguments apply equally to give a flux phase for our model and initially consider a background gauge configuration with $\Pi = -1$ where Π is evaluated around a sublattice placquette. Fluctuations of the gauge field around this background value will be restored after the long-wavelength limit has been taken. Remarkably, the background flux-phase configuration naturally supplies a γ -matrix structure and leads directly to a theory of Dirac fermions. In fact, the flux-phase Hamiltonian on each sublattice has exactly the correct form for the construction of staggered lattice fermions [66]. In the continuum, the usual doubling leads to two separate flavours of two component spinors. Representing an arbitrary site in sublattice A as $i = n_x \delta_x + n_y \delta_y$, where δ_x and δ_y are orthogonal unit vectors of sublattice A (see Fig 3.1), spinor components are defined according to the parity of the integers n_x and n_y as $\psi^1 = (\text{even}, \text{even})$, $\psi^2 = (\text{even}, \text{odd})$, $\psi^3 = (\text{odd}, \text{even})$, $\psi^4 = (\text{odd}, \text{odd})$ [67]. Restricting our attention to sublattice A only, the Hamiltonian (3.37) is given in momentum space by,

$$H_A = -t \int d^2 k \psi^\dagger(k) (M(k))_{ij} \psi(k)^j \quad (3.39)$$

where the k integration is restricted to the appropriate Brillouin zone and the 4×4 matrix $M(k)$ is given by

$$M(k) = \begin{pmatrix} -M & S_y & -iS_x & 0 \\ S_y & -M & 0 & iS_x \\ iS_x & 0 & -M & S_y \\ 0 & -iS_x & S_y & -M \end{pmatrix} \quad (3.40)$$

here $M = \mu/t$ and $S_i = -\sin(k_i/t)$ with $k_i = \mathbf{k} \cdot \delta_i$ and $i = x, y$. The Hamiltonian (3.39) can be recast in block diagonal form by a suitable change of basis in spinor space:

$$H_A = -t \int d^2 k \chi_a^\dagger(\tilde{M}(k))_{ab} \chi_b \quad (3.41)$$

where χ_a , $a = 1, 2$, is now a vector of two-component spinors given by $\chi_1 = \frac{1}{\sqrt{2}}(\psi_1 + \psi_4, \psi_2 + \psi_3)$ and $\chi_2 = \frac{1}{\sqrt{2}}(\psi_2 - \psi_3, \psi_1 - \psi_4)$ and

$$\tilde{M}(k) = \begin{pmatrix} S_x \sigma_x + S_y \sigma_y + M\mathcal{I} & 0 \\ 0 & S_x \sigma_x + S_y \sigma_y + M\mathcal{I} \end{pmatrix} \quad (3.42)$$

where σ_i are the usual Pauli matrices and $\mathcal{I}$ is the 2×2 unit matrix.

The energy spectrum of the hopping Hamiltonian is obtained by solving the eigenvalue equation of the matrix $\tilde{M}(k)$;

$$E(k) = \mu \pm t[S_x^2 + S_y^2]^{\frac{1}{2}} = \mu \pm \epsilon(k) \quad (3.43)$$

The long-wavelength continuum limit is taken by expanding the Hamiltonian in powers of momenta around the Fermi surface and retaining only linear terms. In particular, the Fermi surface is defined by the relation $\epsilon(k) = \mu$ and the momenta may be split as $\mathbf{k} = \mathbf{k}_F + \mathbf{k}'$ where $\mathbf{k}_F$ is a particular expansion point on the Fermi

surface. New spinors $\hat{\chi}_a$ and $\hat{\chi}_a^\dagger$ which describe ‘quasi-hole’ excitations about the Fermi surface are then defined as zero modes of the matrix $\hat{M}(k_F)$. The Hamiltonian can be written, to leading order in $\mathbf{k}'$, as

$$H_A = -t \int d^2 k' \hat{\chi}_a^\dagger (C_i^F \sigma_i k'_i) \hat{\chi}_a \quad (3.44)$$

where $C_i^F = \cos(k_{Fi}/t)$ and summation over $i = x, y$ is implied. In the one-dimensional case the dependence of this result on the Fermi momentum k_F could be absorbed by simply rescaling the coupling t by a factor C_1^F . In this case, there is also dependence on an arbitrary choice of direction in the plane and an additional rescaling of momentum component k_2^F by a factor C_2^F/C_1^F is required to absorb this. Although such a rescaling spoils the rotational symmetry of the model, the relationship $J = 4t^2/U$ implies that $C_i \simeq 1 + O(\mu^2/UJ)$ and consequently the anisotropic terms are suppressed in the physical regime of large- U and light doping [10]³. In this approximation, all dependence on the expansion point chosen disappears and the Hamiltonian (3.44) takes on the standard Dirac form. Taking the Fourier transform and restoring the fluctuations of the gauge field around its background value gives

$$H_A = -t \int d^2 x \hat{\chi}_a^\dagger (\sigma^i \partial_i - S \sigma^i a_i - S \sigma^z a_0) \hat{\chi}_a \quad (3.45)$$

Hence the long-wavelength limit of the hopping term for sublattice A describes a theory of two ‘flavours’ (corresponding to the index $a=1,2$) of two-component Dirac fermions minimally coupled to a $U(1)$ gauge field, with coupling S . The whole procedure is easily repeated for the Grassmann fields inhabiting sublattice B, this yields two more ‘flavours’ of Dirac fermions coupled to the gauge field with coupling $-S$. Combining A and B type fermions in a new four-component spinor, Ψ , the complete Hamiltonian for the fermionic sector of the model becomes

$$H_F = H_A + H_B = \int d^2 x \Psi_a^\dagger (\sigma^i \partial_i - S \sigma^i \tau_3 a_i - S \sigma^z \tau_3 a_0) \Psi_a \quad (3.46)$$

where the matrix τ_3 is given by $\text{diag}(1, -1)$ acting on the internal index of Ψ which labels the sublattices A and B. Thus the sublattice structure naturally provides a reducible four-component representation of the Dirac algebra, like the one discussed in Section 2.1.

Returning to a Lagrangian formalism, the effective action for quasi-holes becomes,

$$S_F = \int_0^\beta d\tau \int d^2 x \frac{1}{v_F} \Psi_a^\dagger \partial_\tau \Psi_a - H_F \quad (3.47)$$

The Fermi velocity for holes is approximately given by $v_F \simeq n^{\frac{1}{2}} t a$, where n is the number density of holes, and is comparable in magnitude or less than the spin-wave velocity $v_S = 2\sqrt{2}Ja$ [15]. In the general case we can set only one of these physical velocities equal to unity and should retain the ratio v_F/v_S as a parameter of the long-wavelength action. However, the analysis of Appendix B indicates that observable quantities relevant to superconductivity are almost independent of v_S . Hence we restrict our attention to the special case $v_F = v_S$ and work in natural units in which both velocities have unit value. Exploiting the freedom to redefine path integral variables as $\Psi^\dagger \rightarrow \Psi^\dagger \gamma_0 = \bar{\Psi}$, defining four-component γ -matrices appropriately, the fermionic or ‘charge’ sector of the model is described by a relativistic Lagrangian density in $(2+1)$ -dimensional spacetime:

$$\mathcal{L}_F = \bar{\Psi}_a (i\not{\partial} - S\tau_3\not{a}) \Psi_a \quad (3.48)$$

Comparing with (15), (3.48) is just the Lagrangian density for the strong-coupling limit of τ_3 -QED₃ in the case $N = 2$ and the theoretical analysis of Section 2 will be applied directly in the next section. The full effective action describing the long-wavelength degrees of freedom of the doped antiferromagnet is obtained by adding the above fermionic terms to the CP^1 non-linear σ -model. In the physical case, $S = \frac{1}{2}$, the result is

$$S_{eff} = \int_0^\beta d\tau \int d^2 x \frac{1}{\gamma} |(\partial_\mu + i a_\mu) z|^2 + \bar{\Psi}_a (i\not{\partial} - \tau_3\not{a}) \Psi_a \quad (3.49)$$

³It should be noticed that in the approach of [16] the same result is reached in the limit of vanishing field strength (J') for the Heisenberg interactions between next-to-nearest neighbors, independently of the doping concentration.

In conclusion, we find that the addition of charged holes to the antiferromagnet is correctly modelled by coupling two flavours of four-component Dirac fermions to the gauge field of the CP^1 action which describes the half-filled case. A reducible four-component representation of the Dirac algebra, follows directly from the local sublattice structure defined by the antiferromagnetic order and the opposite charges of A and B type holes leads to a τ_3 coupling in four-component formalism. A similar conclusion is reached in Ref [16] using a mean-field analysis of the t - J model although, as mentioned above, the familiar constraint on the CP^1 variables, $\bar{z}z = 1$, is also modified by doping.

In the next section, we will consider the effect of coupling the system described by (4.54) to an external electromagnetic field. Consequently, we wish to focus our attention to the effective action for the electrically charged degrees of freedom; the ‘quasi-hole’ fields, Ψ and $\bar{\Psi}$. To this end, it is convenient to integrate out the neutral magnon fields and keep only the leading terms in a derivative expansion. As is well known [57, 58], this leads to a radiatively generated Maxwell term for the auxiliary gauge field a_μ . The resulting effective Lagrangian at low momenta is;

$$\mathcal{L} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \bar{\Psi}_a (i\not{\partial} - S\tau_3\not{\partial}) \Psi_a. \quad (3.50)$$

The dimensionful gauge coupling g^2 is proportional to $(\gamma)^{-1} \sim J$ and, as mentioned above, the equivalent temperature, $T = g^2/k_B$, is about 1500K.

4 Phenomenology of the parity-conserving model

4.1 A mechanism for superconductivity

In the previous section we saw that the dynamics of holes and spins in a two-dimensional antiferromagnet could be described by an effective theory of Dirac fermions coupled to an abelian gauge field in (2+1)-dimensional spacetime. Our aim in this section is to demonstrate the occurrence of superconductivity in this continuum model. The starting point for the description of superconductivity is the effective Lagrangian,

$$\mathcal{L}_{eff} = \mathcal{L}_{kin} + \bar{\Psi}_a (i\not{\partial} - \tau_3\gamma_0 a^0 - \frac{\tau_3}{v_S} \gamma_i a^i) \Psi_a \quad (4.1)$$

$$\mathcal{L}_{kin} = -\frac{1}{4g^2} \left[\left(\frac{\partial_0}{v_S} a_i - \partial_i a_0 \right) \left(\frac{\partial^0}{v_S} a^i - \partial^i a^0 \right) + (\partial_i a_j - \partial_j a_i) (\partial^i a^j - \partial^j a^i) \right] \quad (4.2)$$

where the (4×4) γ -matrices are those given in (7). To avoid confusion with the ordinary electromagnetic gauge field, we will refer to the vector field a_μ as the statistical gauge field. We have adopted natural units in which the Fermi velocity for holes, v_F , is set to unity, so that the spin-wave velocity, v_S , is less than or equal to one depending on the parameters of the microscopic theory discussed in Section 2. To illustrate the properties of this model it will be convenient to restrict our attention to the special case $v_F = v_S$. The analysis of the general case given in Appendix B indicates that the physical quantities of interest are almost independent of v_S . When the two characteristic velocities of the system are equal the effective Lagrangian has a ‘relativistic’ form and coincides with that of τ_3 -QED₃ with $N = 2$ (see Section 2.1),

$$\mathcal{L} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \bar{\Psi}_a (i\not{\partial} - \tau_3\not{\partial}) \Psi_a. \quad (4.3)$$

The gauge coupling, g^2 , has the dimensions of mass and is, in general, a function of the Hubbard model parameters t , U and δ . As we shall see, g^2 sets the overall energy scale of the gap and the transition temperature for the model in much the same way as the Debye energy does for the BCS theory. Unlike the Debye energy which is related to the phonon spectrum, the energy scale of this model is set by the characteristic energy of electronic correlations which is typically of the order of a few electron-Volts. To demonstrate the occurrence of superconductivity rigorously, the Dirac fermions which describe the charged quasi-holes should be coupled to a ‘real’ external electromagnetic field A_μ , as well as the ‘statistical’ gauge field a_μ . This will be done explicitly in Section 4.2. In this section we will ignore the electric charge of the holes and simply demonstrate the breaking of the global $U_E(1)$ symmetry of ordinary phase rotations and the appearance of a pole in the corresponding current-current correlator. Once the system is coupled to an external electromagnetic field this leads directly to superconductivity and the Meissner effect.

The properties of QED₃ were discussed in detail in Section 2 and much of the analysis can be applied directly to this case. In particular we will study the behaviour of Green's functions in the large- N expansion and hope that the conclusions remain reliable in the realistic case $N = 2$. For the quantities of interest in this section, the leading order of the $1/N$ expansion coincides with the first order in one-loop perturbation theory and, unless otherwise stated, the term 'leading order' is used below to indicate both of these. The leading order contribution to the statistical photon propagator is given by an infinite sum of diagrams, similar to those shown in Figure 8. In the Landau gauge, the result is

$$\Delta_{\mu\nu}(p) = g^2 \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^2(1 + \Pi(p))} \quad (4.4)$$

where $\Pi(p) = g^2 N / 8p$. The two-component fermions describing holes on the two sublattices, A and B, have opposite statistical charges $\pm g$. In four-component notation, this is incorporated by the τ_3 coupling in (4.3). Hence, from (14), the long-range attractive static potential between an A type hole and a B type hole is the same as that between a hole and its antiparticle and is given by

$$V_{AB}(R) = V_{A\bar{A}}(R) = V_{B\bar{B}}(R) = -4/\pi NR \quad (4.5)$$

This result needs some explanation in the context of the microscopic model. As discussed in Section 3, holes have a tendency to sit on NN sites due to the antiferromagnetic order. In Shankar's model this short-range interaction leads to the opposite couplings of A and B type holes to an auxiliary gauge field with no bare Maxwell term. In the effective continuum theory of Dirac fermions coupled to the CP^1 σ -model, the gauge field acquires dynamics due to vacuum polarisation by the magnon fields z and $\bar{z}$, and also by the quasi-hole fields Ψ and $\bar{\Psi}$. The magnon vacuum polarisation generates the bare Maxwell term which appears in (4.3) and this is modified at leading order by the fermion vacuum polarisation, $\Pi(p)$. In fact the latter effect is the dominant one at low momenta and consequently, at least in the context of the large- N expansion, the resulting long-range interaction given in (4.5) is independent of the bare gauge coupling g^2 [28]. In microscopic language, fermion vacuum polarisation corresponds to virtual processes in which one hole is excited, for a short time, to a state above the Fermi surface leaving an unoccupied state (corresponding to the 'antifermion') behind. Naturally a state that is 'unoccupied by a hole' is filled by an electron, however it is convenient to continue to regard the holes as particles and the filled states as their antiparticles. The short-distance attraction between holes is enhanced by these quantum fluctuations in the 'hole gas' and becomes long-range.

The existence of a long-range attraction between holes immediately suggests an analogy with the BCS theory, where electrons near the Fermi surface are subject to attractive forces mediated by phonon exchange. In the BCS theory this leads to a new ground-state in which electrons form s -wave bound-states or Cooper pairs. In the present case, the quantum mechanical model developed in Section 2.4 is applicable and indicates that a similar pairing between A and B type holes occurs in the physical case $N = 2$ ($< N_c = 8/\pi$). As both types of holes carry electric charge $+e$, the hole-pairs carry charge $2e$. The analogy with BCS theory suggests these bosonic hole-pairs condense at low temperature giving a superconducting ground state. However, because of the Mermin-Wagner theorem [12], this interpretation must be treated with some caution in a two-dimensional system. In four-component formalism, the A and B type holes correspond to the upper and lower components of the spinor field Ψ . As the A and B type holes correspond to spin-up and spin-down sublattices respectively, we note in passing a remarkable similarity to the quasi-particle formalism of Bogoliubov [68]. From (7), the relevant condensate for A/B pairing is $\Psi\gamma_5\Psi$. This operator is doubly charged and is therefore a local order parameter for $U_E(1)$ symmetry breaking. The Mermin-Wagner theorem states that $\langle \Psi\gamma_5\Psi \rangle = 0$ at any non-zero temperature which would suggest that the superconducting transition temperature is exactly zero. In fact the occurrence of superconductivity in the model does not require an order parameter and can be exhibited whenever the electrically neutral condensate $\langle \bar{\Psi}\Psi \rangle$ is non-zero. This operator corresponds to a Peierls-like pairing between quasi-holes and their 'antiparticles' [69]. An analysis of the Bethe-Salpeter equation shows that bound-states poles appear in each of the channels, $\Psi\gamma_5\Psi$, $\bar{\Psi}\Psi$ and $\bar{\Psi}\gamma_5\Psi$, corresponding to pairs of charge $+2e$, 0 and $-2e$ respectively. By analogy with the Gross-Neveu model [37], we expect that the phase of the wavefunction of charged pairs varies rapidly in space causing the expectation value of the corresponding charged operators to vanish in accord with the Mermin-Wagner theorem.

The transition of the system to a new ground-state consisting of fermion bound-states is signaled by the opening of a gap, Δ , in the quasi-hole spectrum. Roughly speaking, the magnitude of the gap is the energy required to break up a pair and thereby create a free quasi-hole. In terms of the effective continuum theory (4.3) the opening of a gap corresponds to the dynamical generation of a fermion mass. As discussed in Section 2.5, this phenomenon has been investigated both by Monte-Carlo lattice simulation [32] and by several different

approximate treatments of the Schwinger-Dyson equation for the fermion propagator [20, 21, 34]. In the physical case, $N = 2$, all these investigations agree that a fermion mass is generated. The analysis given in Section 2.5 yields the formula (47) for the dynamical mass. Setting $N = 2$ this gives $\Delta \simeq 6.9 \times 10^{-4} g^2$ corresponding to an energy of the order of 1 meV which is comparable to the gap occurring in BCS superconductors. We note that this value is at least one order of magnitude less than the typical observed gap in high- T_c superconductors. However, the exact gap equation is sensitive to many other factors and so this estimate is extremely crude. As usual for theories of superconductivity, the gap itself cannot be calculated with any degree of accuracy and the only reliable quantitative predictions of the model are for dimensionless ratios such as $2\Delta/k_B T_c$. For the moment we will assume only that a non-zero gap is dynamically generated for $N = 2$ and examine the qualitative consequences.

Before proceeding, there is an obvious question which should be asked concerning the role of the two flavours which correspond to lattice doubles in the microscopic model. Although there are energetics arguments for the dynamical generation of a mass term which is overall parity-invariant [21] which implies that two two-component species acquire positive mass and two negative, these arguments do not tell us how these signs are distributed among the 'flavours' and 'colours'. However, the analysis of the quantum mechanical model of [28] indicates that the gap must appear at the level of the microscopic model prior to the consideration of lattice doubling. In the continuum limit there may also be an additional gap term where signs alternate between flavours (doubles). However such a term does not affect the occurrence of superconductivity.

The massless Lagrangian (4.3) is invariant under statistical $U_\tau(1)$ gauge transformations,

$$\Psi \rightarrow \exp(i\tau_3 \Lambda(x)) \Psi \quad a_\mu \rightarrow a_\mu + \partial_\mu \Lambda(x) \quad (4.6)$$

as well as ordinary global phase rotations $\Psi \rightarrow \exp(i\theta)\Psi$, denoted $U_E(1)$, and discrete parity transformations P_4 (see Section 2.1). However, because of the τ_3 coupling, (4.3) has no chiral symmetry in the sense of [21]. The dynamically generated four-component mass term, $\Delta \bar{\Psi} \Psi$, is fully invariant under $U_\tau(1) \times U_E(1) \times P_4$ and so is not an order parameter for the spontaneous breaking of any of these symmetries. However, exactly as discussed in Section 2.1, the matrix element of the conserved current, $J_\mu = \bar{\Psi} \gamma_\mu \Psi$, between the vacuum state and one statistical photon becomes non-zero in the presence of a non-zero gap, Δ [22]. The lowest order diagram contributing to this matrix element is identical, up to factors of τ_3 , to that shown in Figure 2 and gives the result,

$$\langle 0 | J_\mu | 1, \mathbf{p} \rangle = -\text{sign}[\Delta] \frac{ig}{\pi^2 \sqrt{2p_0}} \epsilon_{\mu\nu\rho} p^\nu \epsilon^\rho \quad (4.7)$$

where ϵ^ρ is the polarisation vector of the statistical photon. The non-vanishing of this matrix element implies that the vacuum state is not invariant under $U_E(1)$ transformations in the massive phase and, as we will demonstrate explicitly in the next section, this leads to superconductivity when the system is coupled to an external electromagnetic field. The 'square' of this matrix element provides a contribution to the current-current correlation function which is shown diagrammatically in Figure 9. The result is

$$\langle 0 | J_\mu(p) J_\nu(-p) | 0 \rangle = \frac{g^2}{\pi^2} \frac{(p_\mu p_\nu / p^2 - \delta_{\mu\nu})}{1 + \Pi(p)} \quad (4.8)$$

Once an electromagnetic coupling is introduced, the global symmetry $U_E(1)$ is promoted to a local gauge invariance and J_μ is the corresponding conserved current. Hence, from (4.8), the Minkowski space correlator of the electromagnetic current at two separate points has a massless pole on the light-cone. This is the usual criterion for superconductivity [3].

The above arguments demonstrate that the long-wavelength model of a two-dimensional doped antiferromagnet developed in Section 3 is superconducting if and only if there is a non-zero gap in the quasi-hole spectrum. Because the corresponding condensate $\langle \bar{\Psi} \Psi \rangle$ does not violate any symmetry of the massless action, spontaneous breaking of $U_E(1)$ symmetry can occur without an order parameter. Thus the Mermin-Wagner theorem [12] does not apply and there is no theoretical obstacle to a non-zero transition temperature. In fact the simple treatment of the finite-temperature gap equation given in Section 2.6 indicates that a non-zero gap persists up to some non-zero critical temperature T_c . Above T_c the gap vanishes and superconductivity is extinguished. In the present case, the results of Section 2.6 suggest that the dimensionless ratio $r = 2\Delta/k_B T_c$ is about 9.58, much larger than the BCS value. This is an encouraging result in the light of the large values of r measured for the quasi-planar high- T_c materials [46]. The large value of the ratio for this model occurs because the strength of the long-range attraction between holes decreases rapidly with temperature [24]. In contrast, the phonon-mediated attraction of BCS theory is assumed to be temperature independent.

4.2 Electromagnetic effects in the plane

So far we have discussed the physical mechanism of hole-pairing and the resulting pattern of symmetry breaking in the context of a purely two-dimensional effective model. In physical systems the holes carry a 'real' electric charge $+e$ and couple to external three-dimensional electromagnetic fields. In this section we consider the two-dimensional system discussed above embedded in three-dimensional space and coupled to a physical electromagnetic field which is described by the usual Maxwell action

$$S_{EM} = - \int d^3x dz \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (4.9)$$

where the Lorentz indices $\mu, \nu = 0, 1, \dots, 3$ and $z = x_3$ is the spatial coordinate perpendicular to the plane of the antiferromagnet. In the static case, the electric and magnetic fields, $\mathbf{E}(x, y, z)$ and $\mathbf{B}(x, y, z)$, obey the time-independent Maxwell equations,

$$\begin{aligned} \nabla \times \mathbf{E} &= 0 & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \frac{\mathbf{J}}{c} & \nabla \cdot \mathbf{E} &= \rho \end{aligned} \quad (4.10)$$

The static fields can be expressed in the usual way in terms of scalar and vector potentials, ϕ and $\mathbf{A}$, as

$$\mathbf{E} = -\nabla\phi \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (4.11)$$

In discussing the electromagnetic properties of the two-dimensional superconductor we are primarily interested in the behaviour of the electromagnetic fields on the (x, y) plane. It is convenient to consider the two-dimensional potentials, $\phi_{2D}(x, y) = \phi(x, y, 0)$ and $\mathbf{A}_{2D}(x, y) = (A_x(x, y, 0), A_y(x, y, 0), 0)$ and the corresponding electric and magnetic fields

$$\mathbf{E}_{2D} = -\nabla\phi_{2D} \quad \mathbf{B}_{2D} = \nabla \times \mathbf{A}_{2D} \quad (4.12)$$

As the holes are confined to move in the plane, the charge and current densities, ρ and $\mathbf{J}$, are non-zero only for $z = 0$ and can be written in terms of two-dimensional sources as

$$\rho(x, y, z) = \delta(z)\rho_{2D}(x, y) \quad \mathbf{J}(x, y, z) = \delta(z)\mathbf{J}_{2D}(x, y) \quad (4.13)$$

where $\mathbf{J}_{2D} = (J_x, J_y, 0)$. Solving the three-dimensional Maxwell equations (4.10) for the two-dimensional potentials gives,

$$\phi_{2D} = \int dx' dy' \frac{\rho_{2D}(x', y')}{\sqrt{(x-x')^2 + (y-y')^2}} \quad \mathbf{A}_{2D} = \frac{1}{c} \int dx' dy' \frac{\mathbf{J}_{2D}(x', y')}{\sqrt{(x-x')^2 + (y-y')^2}} \quad (4.14)$$

The static fields are uniquely defined in the plane by (4.12) and (4.14) and it is natural to consider the *two-dimensional* equation of motion that gives rise to this solution. The factor $1/|\mathbf{x} - \mathbf{x}'|$, which occurs in the integrands of (4.14), is the Green's function of the three-dimensional Laplacian operator ∇^2 . However, in two dimensions this expression is the Green's function not of the Laplacian but rather of its 'square-root', written formally as $\sqrt{\nabla^2}$ (this has an obvious meaning in Fourier space). It follows that the two-dimensional fields, $\mathbf{E}_{2D}$ and $\mathbf{B}_{2D}$, obey the equations of motion

$$\frac{1}{\sqrt{\nabla^2}} \nabla \times \mathbf{B} = \frac{1}{c} \mathbf{J} \quad \frac{1}{\sqrt{\nabla^2}} \nabla \cdot \mathbf{E} = \rho \quad (4.15)$$

where we have suppressed the label $2D$. These field equations follow, in the static case, from the covariant (Euclidean) action,

$$S_{EM} = - \int d^3x \frac{1}{4\sqrt{\partial^2}} F_{\mu\nu} F^{\mu\nu} \quad (4.16)$$

where $\partial^2 = \nabla^2 + (1/c^2)\partial^2/\partial t^2$. The arguments leading to this result have a straightforward generalisation to the non-static case, although for the present purposes all we shall require are the two-dimensional equations of motion (4.15). The action (4.16) has also been used by Rosenstein and Kovner to describe the dynamics of an electromagnetic field coupled to a planar superconductor [9]. Working in units in which $v_F = 1$, the 'real' velocity of light, c , must be retained explicitly. The corresponding bare effective propagator for the photon is

$$D_{\mu\nu}^{(0)}(p) = \frac{c}{\sqrt{p_0^2 + c^2|\mathbf{p}|^2}} \left(\delta_{\mu\nu} - \frac{c^2 \tilde{p}_\mu \tilde{p}_\nu}{p_0^2 + c^2|\mathbf{p}|^2} \right) \quad (4.17)$$

with $\tilde{p}_\mu = (p_0/c, \mathbf{p})$. The propagator has a $1/p$ behaviour, and by construction reproduces the static Coulomb repulsion between two charged holes.

$$V_{EM}(R) = \frac{e^2}{4\pi c R} \quad (4.18)$$

The $1/R$ form of the potential allows a direct comparison between the electromagnetic repulsion and the statistical attraction (4.5). As $c \gg 1$ in the system of units we have chosen, the statistical attraction is the stronger effect by many orders of magnitude. Hence the electrostatic repulsion between holes has a negligible effect on the magnitude of the gap.

4.3 The photon mass and superconductivity

The formalism developed above allows us to describe the combined system of matter coupled to the relevant components of the electromagnetic field by a Lagrangian density in two spatial dimensions.

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_{matter} \quad (4.19)$$

$$\mathcal{L}_{EM} = -\frac{1}{4\sqrt{\partial^2}} \left(\left(\frac{\partial_0}{c} A_i - \partial_i A_0 \right) \left(\frac{\partial^0}{c} A^i - \partial^i A^0 \right) + (\partial_i A_j - \partial_j A_i) (\partial^i A^j - \partial^j A^i) \right) \quad (4.20)$$

$$\mathcal{L}_{matter} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \bar{\Psi}_a (i\not{\partial} - \tau_3 \not{\partial} - e\gamma_0 A^0 - \frac{e}{c} \gamma_i A^i - \Delta) \Psi_a \quad (4.21)$$

where we have explicitly introduced the dynamically generated gap, Δ . As discussed in the previous section, the value of $\Delta = \Sigma(0)$ is determined by the self consistent solution of the Schwinger-Dyson equation for the fermion self-energy $\Sigma(p)$.

We proceed by evaluating the leading order quantum corrections to the quadratic part of this action. The kinetic term for the statistical gauge field is given at low momentum by

$$\mathcal{L}_{kinetic} = -\frac{1 + \Pi(0)}{4g^2} f_{\mu\nu} f^{\mu\nu} \quad (4.22)$$

For a non-zero gap, the vacuum polarisation $\Pi(p)$ is given at leading order by [20],

$$\Pi(p) = \frac{g^2}{2\pi p^2} \left[2\Delta + \frac{(p^2 - 4\Delta^2)}{p} \sin^{-1} \left(\frac{p}{\sqrt{p^2 - 4\Delta^2}} \right) \right] \quad (4.23)$$

so that $\Pi(0) = g^2/3\pi\Delta$. From the discussion of Section 2.5, we expect the gap, Δ , to be exponentially smaller than the dimensionful coupling g^2 . Hence $\Pi(0) \gg 1$, and the coefficient of the Maxwell term is approximately $-1/12\pi\Delta$. Thus the effective action is approximately independent of the coupling constant g^2 . At leading order the Maxwell term for the electromagnetic field receives a similar correction,

$$\mathcal{L}_A = \mathcal{L}_{EM} - \frac{e^2 \Pi(0)}{g^2} F_{\mu\nu} F^{\mu\nu} \quad (4.24)$$

$\mathcal{L}_{EM}$ corresponds to an inverse propagator proportional to p . The correction due to vacuum polarisation provides corrections of order p^2 which are negligible at low momentum.

As discussed in Section 2.1, the four-component mass term, $\Delta \bar{\Psi} \Psi$ is parity preserving and no Chern-Simons terms are radiatively generated, either for the statistical gauge field or for the two-dimensional electromagnetic field. However, at leading order, a ‘mixed’ Chern-Simons term is generated which provides a coupling between the two gauge fields. The graph which contributes to this term is shown in Figure 10, where a wavy line denotes the ‘real’ electromagnetic photon propagator and a curly line the statistical photon propagator. In the limit $p \rightarrow 0$, the corresponding term in the effective action is

$$\mathcal{L}_{CS} = \frac{\text{sign}[\Delta]e}{2\pi} \epsilon^{\mu\nu\rho} \tilde{A}_\mu f_{\nu\rho} \quad (4.25)$$

where $\tilde{A}_\mu = (A_0, \mathbf{A}/c)$. Due to the τ_3 structure of its coupling, the statistical gauge field a_μ is odd under parity transformations, while the ordinary electromagnetic field A_μ is even. Consequently, as expected, the mixed Chern-Simons term does not violate parity or time-reversal symmetry. The arguments of Coleman and Hill [70] may be applied to show that the coefficient of the mixed term is not renormalised by higher order contributions and so the result (4.25) is exact to all orders.

The mixed Chern-Simons term provides an effective coupling between the electromagnetic and statistical gauge fields when the gap is non-zero. It is then straightforward to see that this coupling contributes a mass for the electromagnetic photon and leads to the spontaneous breaking of the $U_E(1)$ symmetry of electric charge. The leading order contribution to the photon propagator is

$$D_{\mu\nu}^{-1}(p) = D_{\mu\nu}^{(0)-1}(p) - \Gamma_{\mu\alpha}(p)\Delta_{\alpha\beta}(p)\Gamma_{\beta\nu}(-p) \quad (4.26)$$

where $\Gamma_{\mu\nu}$ is the effective two-point A - a vertex corresponding to $\mathcal{L}_{CS}$ and $D_{\mu\nu}^{(0)}$ is given by (4.17). This relation is shown graphically in Figure 11. Evaluating the expression (4.26), we find that the transverse spatial part of the propagator is given by,

$$D_{ij}^T(p) = \frac{c}{\sqrt{p_0^2 + c^2|\mathbf{p}|^2} + 3e^2\Delta/c\pi} \left(\delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2} \right) \quad (4.27)$$

Thus the spatial part of electromagnetic photon propagator has acquired a transverse mass $\mu = 3e^2\Delta/c^2\pi$, by ‘eating’ the massless statistical photon. In the language of high-energy physics, this is just a novel version of the Anderson-Higgs mechanism [71]. The unconventional feature is that the massless pole which generates the photon mass corresponds to a vector particle rather than a scalar. However, in three-dimensional spacetime a massless vector particle has only one transverse polarisation state and, in this respect, is equivalent to a scalar. In fact, as we will see below, the effective action can be expressed in terms of a scalar field making the correspondence with the Anderson-Higgs mechanism exact. The photon mass implies that a static magnetic field inside the superconductor decays exponentially with distance from the boundary. For an externally applied magnetic field, this is the famous Meissner effect, with a penetration depth $\lambda = c^2\pi/3e^2\Delta$.

4.4 The London Action

The behaviour of the photon propagator described demonstrates the occurrence of superconductivity at leading order. In fact, once the mixed Chern-Simons term $\mathcal{L}_{CS}$ is included, the matter action can be rewritten exactly in the standard London form and the phenomenology of the superconducting state can be exhibited without further approximation. The partition function of the system is defined as a functional integral over the fields

$$\mathcal{Z} = \int \mathcal{D}a_\mu \mathcal{D}A_\mu \exp \left[- \int d^3x \mathcal{L}_{EM} + \mathcal{L}_{matter}(f_{\mu\nu}) \right] \quad (4.28)$$

where, including the effects of vacuum polarisation to leading order, the matter Lagrangian is given by

$$\mathcal{L}_{matter}(f_{\mu\nu}) = -\frac{1}{12\pi\Delta} f_{\mu\nu} f^{\mu\nu} - \frac{e}{2\pi} \varepsilon^{\mu\nu\rho} \tilde{A}_\mu f_{\nu\rho} \quad (4.29)$$

for $\Delta > 0$. The London form of the matter action can then be derived by performing the a_μ integral exactly following the method of Banks and Lykken [8]. As the action depends on a_μ only through the field strength $f_{\mu\nu}$ it is convenient to introduce an antisymmetric tensor field $Z_{\mu\nu}$ via the relation

$$1 = \int \mathcal{D}Z_{\mu\nu} \delta(Z_{\mu\nu} - f_{\mu\nu}) \quad (4.30)$$

Using the functional identity

$$\delta(T_{\mu\nu}) = \int \mathcal{D}L_\mu \exp \left[- \int d^3x L_\mu \varepsilon^{\mu\alpha\beta} T_{\alpha\beta} \right] \quad (4.31)$$

the δ -function may be written as an integral over an auxiliary field, L_μ , and the a_μ integral becomes

$$\mathcal{I} = \int \mathcal{D}a_\mu \mathcal{D}Z_{\mu\nu} \mathcal{D}L_\mu \exp \left[- \int d^3x \mathcal{L}_{matter}(Z_{\mu\nu}) + L_\mu \varepsilon^{\mu\alpha\beta} (Z_{\alpha\beta} - f_{\alpha\beta}) \right] \quad (4.32)$$

The integration over a_μ can be performed trivially giving a δ -function

$$\mathcal{I} = \int \mathcal{D}Z_{\mu\nu} \mathcal{D}L_\mu \delta(\varepsilon^{\mu\alpha\beta} \partial_\alpha L_\beta) \exp \left[- \int d^3x \mathcal{L}_{matter}(Z_{\mu\nu}) + L_\mu \varepsilon^{\mu\alpha\beta} Z_{\alpha\beta} \right] \quad (4.33)$$

The constraint $\varepsilon^{\mu\alpha\beta} \partial_\alpha L_\beta = 0$ can be implemented exactly in the path integral by setting $L_\mu = \partial_\mu \phi$ where ϕ is a scalar field.

$$\mathcal{I} = \int \mathcal{D}Z_{\mu\nu} \mathcal{D}\phi \exp \left[\int d^3x \frac{1}{12\pi\Delta} Z_{\mu\nu} Z^{\mu\nu} - \left(\partial_\mu \phi - \frac{e}{2\pi} \tilde{A}_\mu \right) \varepsilon^{\mu\nu\rho} Z_{\nu\rho} \right] \quad (4.34)$$

The functional integral over $Z_{\mu\nu}$ can then be performed by completing the square to write it in Gaussian form. The final result for the partition function is

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\phi \exp \left[- \int d^3x \mathcal{L}_{EM} + \mathcal{L}_\phi \right] \quad (4.35)$$

where $\mathcal{L}_\phi$ is an Lagrangian density of London type

$$\mathcal{L}_\phi = 6\pi\Delta \left(\partial_\mu \phi - \frac{e}{2\pi} \tilde{A}_\mu \right)^2 \quad (4.36)$$

This action includes a spatial photon mass term $\kappa^2 A_i A_i / 2$ with $\kappa^2 = 3e^2 \Delta / c^2 \pi$ which agrees with the result (4.26) given above. Note that the true photon mass is κ^2 rather than κ because the form of the modified equations of motion (4.15). The theory is now expressed in terms a massless scalar field ϕ which is formally related to the statistical photon by $\partial_\mu \phi = \varepsilon_{\mu\alpha\beta} \partial^\alpha a^\beta$. In conventional BCS theory ϕ is identified as the massless Goldstone mode corresponding to the spontaneous breaking of $U_E(1)$ symmetry. The interpretation is slightly different in this case; the $U_E(1)$ symmetry of the vacuum is broken in the low temperature phase for which $\Delta > 0$, but there is no local order parameter. The statistical photon is massless in both phases but only couples to the real electromagnetic photon in the low-temperature phase where it plays the same role as a Goldstone mode and provides an electromagnetic photon mass.

4.5 Infinite conductivity

Having rewritten the matter sector of the theory in terms of the massless mode ϕ , the various phenomenological features of the superconducting state can be demonstrated using standard arguments [77]. The two-dimensional electromagnetic charge and current densities are given by,

$$\rho = \frac{\delta \mathcal{L}_\phi}{\delta A_0} \quad \mathbf{J} = \frac{\delta \mathcal{L}_\phi}{\delta \mathbf{A}} \quad (4.37)$$

As $\mathcal{L}_\phi$ depends on the fields in the combination $(\partial_\mu \phi - (e/2\pi)\tilde{A}_\mu)$, the charge density can be written as,

$$\rho = -\frac{e}{2\pi} \frac{\delta \mathcal{L}_\phi}{\delta(\partial_0 \phi)} \quad (4.38)$$

From the above relation, the charge density ρ is the canonical momentum conjugate to the field ϕ . Hamilton's equation of motion then gives,

$$\partial_0 \phi = \frac{e}{2\pi} \frac{\delta \mathcal{H}_\phi}{\delta \rho} \quad (4.39)$$

where $\mathcal{H}_\phi$ is the Hamiltonian density for the matter sector. Following the approach of Weinberg [77], the electric voltage $V(\mathbf{x})$ at a point $\mathbf{x}$ inside the sample is defined as the response of the energy density of matter, $\delta \mathcal{H}_\phi$, to a change of the charge density, $\delta \rho(\mathbf{x})$, at that point. From (4.39), the voltage is given by

$$V(\mathbf{x}, t) = \frac{2\pi}{e} \frac{\partial \phi(\mathbf{x}, t)}{\partial t} \quad (4.40)$$

Hence, if the system remains in a steady state with a time independent electric current I , the voltage vanishes at all points inside the sample. The electrical resistance of the sample, R , is defined by Ohm's law, $V = IR$, and is clearly zero. The vanishing of electrical resistance is the most obvious property of the superconducting state and, as demonstrated above, is a direct consequence of functional form of the matter Lagrangian, $\mathcal{L}_\phi$.

4.6 The two-dimensional Meissner effect

From (4.15) and (4.37), the two-dimensional equation of motion for a static magnetic field inside the sample is given by

$$\frac{1}{\sqrt{\nabla^2}} \nabla \times \mathbf{B} = -\frac{3e^2 \Delta}{\pi c^2} \left(\mathbf{A} - \frac{2\pi c}{e} \nabla \phi \right) \quad (4.41)$$

Taking the curl of both sides and using the Maxwell equation $\nabla \cdot \mathbf{B} = 0$,

$$\frac{1}{\sqrt{\nabla^2}} \nabla^2 \mathbf{B} = -\frac{3e^2 \Delta}{\pi c^2} \mathbf{B} \quad (4.42)$$

To exhibit the Meissner effect we will restrict our attention to the case of a simple, effectively one-dimensional, geometry. Consider a homogeneous sample occupying the half-plane ($z = 0, x > 0$) with an externally applied magnetic field $\mathbf{B} = (0, 0, B_z)$ which takes the constant value $(0, 0, B_0)$ outside the sample. Inside the sample the field clearly depends on the x -coordinate only and the field equation, (4.42), becomes

$$\frac{dB_z}{dx} = -\frac{3e^2\Delta}{\pi c^2}B_z \quad (4.43)$$

Applying the boundary condition, $B_z(0) = B_0$, the solution is,

$$B_z(x) = B_0 \exp\left(-\frac{x}{\lambda}\right) \quad (4.44)$$

where $\lambda = \pi c^2/3e^2\Delta$. Thus the magnetic field strength perpendicular to the plane is exponentially small in the interior of the sample. As the superconductor is cooled below the transition temperature, magnetic flux is expelled and only penetrates a distance of order λ from the edge. This is a two-dimensional version of the ordinary Meissner effect which occurs in all known superconductors. The penetration depth agrees with the leading order result from the photon propagator (4.27) and, even in the general case $v_S \neq v_F$ (see Appendix B), only depends on the parameters of the microscopic model through the gap, Δ . Thus the model provides a numerical prediction for the product $P = \Delta\lambda$. Restoring factors of $\hbar$ the result is $P = 2.25 \times 10^{-6} \text{eVm}$. The coherence length, ξ , of the superconductor is approximately $\hbar v_F/\Delta$. Thus the corresponding Landau-Ginzburg constant is given by, $\kappa = \lambda/\xi = \pi \hbar c^2/3e^2 v_F \simeq 10(c/v_F) \gg 1$, implying that the superconductor described by this model is strongly type-II. We note that the quasi-planar high- T_c materials are all type-II superconductors by virtue of their exceptionally short coherence lengths.

4.7 Flux quantization

As in traditional superconductors, the phenomenon of infinite conductivity and the Meissner effect follow directly from the London form of the action. In contrast, we will demonstrate that the quantisation of magnetic flux in the model considered here has a quite different origin to that occurring in the conventional theory. At any finite temperature, $\beta = (k_B T)^{-1} < \infty$, statistical gauge transformations are divided into topological classes according to their winding number, n , around the compactified time direction.

$$a_\mu \rightarrow a_\mu + \partial_\mu \Lambda \quad \Lambda(\beta) = \Lambda(0) + 2n\pi i \quad (4.45)$$

where Λ is taken to be constant in two-dimensional space. The action corresponding to the mixed Chern-Simons term is

$$S_{CS} = \frac{ie}{\pi} \int_0^\beta dt \int d^2x \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu \tilde{A}_\rho \quad (4.46)$$

This term is not invariant under topologically non-trivial ($n > 0$) statistical gauge transformations and has variation

$$\delta S_{CS} = \frac{ie}{\pi} \int_0^\beta dt \partial_0 \Lambda \int d^2x \varepsilon^{0\nu\rho} \partial_\nu \tilde{A}_\rho = 2in\pi e \int d^2x \varepsilon_{ij} \partial^i \tilde{A}^j \quad (4.47)$$

As every other term in the action is fully gauge-invariant, δS_{CS} must be an integer multiple of $2\pi i$ to ensure the invariance of the partition function. Now consider a superconducting annulus in the plane centred at the origin, with the difference between the inner and outer radii much larger than the penetration depth. Let S be a region bounded by a closed path C which winds round the origin once in the interior of the sample. Because of the Meissner effect, the total magnetic flux through S is equal to the total flux, Φ , passing through the non-superconducting region bounded by the inner radius,

$$\Phi = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \quad (4.48)$$

Taking S to be the whole sample, equation (4.47) implies that $\Phi = (c/2in\pi e)\delta S_{CS}$. Restoring factors of $\hbar = h/2\pi$, the invariance of the partition function under 'large' gauge transformations implies that the magnetic flux trapped in the annulus obeys the quantisation condition,

$$\Phi = m \frac{hc}{2e} \quad (4.49)$$

where m is an integer. This coincides with the experimentally observed flux quantum both for traditional superconductors and the high- T_c materials. In the conventional Landau-Ginzburg theory flux quantization

follows when the scalar field ϕ is identified with the phase of the charge $2e$ order parameter. From the field equation (4.41) the quantity $\nabla\phi - (e/2c\pi)\mathbf{A}$ is vanishingly small in the interior of the sample. Hence, using Stoke's theorem,

$$\Phi = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{l} = \frac{2\pi c}{e} \int_C \nabla\phi \cdot d\mathbf{l} = \frac{2\pi c}{e} \delta\phi \quad (4.50)$$

where $\delta\phi$ is the change in ϕ on traversing C . Reversing the usual argument, the quantisation condition implies that the field

$$\chi = |\chi| \exp\left(\frac{2ie}{c}\phi\right) \quad (4.51)$$

is single-valued on C . Hence, for our model, ϕ may be identified as the phase of a charge $2e$ condensate, χ . The obvious candidate is $\chi = \langle \Psi \gamma_5 \Psi \rangle$ consistent the pairing of A and B type holes conjectured in the Section 4.1. However, by analogy with the Kosterlitz-Thouless transition [23], we expect that rapid spatial fluctuations in ϕ enforce the vanishing of the order parameter at finite temperature as demanded by the Mermin-Wagner theorem.

4.8 Non-perturbative effects

So far we have essentially restricted ourselves to perturbation theory. The symmetry breaking patterns that lead to superconductivity have been discussed in the context of Feynman graphs and are inherently perturbative. However superconductivity is an *exact* phenomenon which should be valid beyond perturbation theory. If our model has a possibility of describing high T_c superconductivity it should yield exact results. Notice that due to the fact that our gauge theory model is obtained as a long wavelength limit of a lattice model, the statistical gauge field is a *compact* dynamical variable. As such it can have non-perturbative configurations, such as monopoles etc. Indeed the zero doped limit of the theory is described by CP^1 σ -models which are known to have hedgehog configurations [72] that essentially correspond to monopole-like configurations for the CP^1 gauge field. Of course due to infrared infinities such configurations can only appear in pairs of monopole-antimonopole. Due to spin alignment at infinity the configuration space of the σ model is compactified to a two sphere, S^2 . The monopoles can live in the center of that sphere, thereby not belonging to the real space. However they are connected to the antimonopole that lies outside S^2 through a Dirac string that carries the magnetic flux of the monopole. We assume that the monopole configurations carry units of flux, although this is not essential for the subsequent arguments. The intersection point of the Dirac string with S^2 defines a so called baby skyrmion [73] (the name is attributed to the unit of flux it carries). The effect of these skyrmions has been argued to be an induced anomaly in the topological statistical current $j_\mu \equiv \epsilon_{\mu\nu\rho} \partial_\nu a_\rho$, which can no longer be conserved but it yields a divergence proportional to the monopole density. The arguments of [73] were based on lattice simulations. These simulations have been questioned recently by Bitar and Manousakis [61] on the basis of the incorrect algorithm used in [73] to detect the presence of monopoles. The reason is that the monopole has a long range structure, while the lattice cubes used in [73] are very small in this respect. Refining the analysis the authors of [61] have observed that monopole configurations have no effect on the dynamics of the CP^1 model. This is a nice result since the flux symmetry has been argued to be responsible for the masslessness of the statistical photon [74], which is viewed as the Goldstone boson pertaining to the spontaneous breaking of the topological symmetry (in the sense of [74]). If the arguments of [73] were correct monopole configurations could in this way destroy superconductivity of the model, even at zero temperature, by giving the statistical photon a mass.

However is this the end of the story? In other words is it only the masslessness of the statistical photon that is important for superconductivity? Certainly not! There are other exact properties such as Meissner effects and flux quantisation whose validity could be affected by non perturbative effects even in the case where the latter induce no anomaly in the topological current. Below we shall re-examine the issue of the effects of monopole configurations of the statistical gauge field in the case of interest, i.e. in the presence of electrically charged fermions coupled, in addition to statistical photons, to an *external* electromagnetic field.

Let us concentrate in the term that couples the external field to the statistical photon in the low energy effective action obtained from a derivative expansion of the Dirac determinant that arises from fermion integration. The term has the form:

$$\int dt d^2x \frac{ie}{\pi} \epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho \quad (4.52)$$

If the effect of Hwang et al were correct then the statistical current could not be globally represented by $\epsilon_{\mu\nu\rho} \partial_\nu a_\rho$ since it would obey the divergence equation $\partial^\mu j_\mu = (\text{const.})\rho$, where ρ is the monopole density. From the definition of the electromagnetic current $J_\mu(A)$ it follows naively that $J_\mu = j_\mu$ at low energies. Hence an anomaly in the statistical current would induce an anomaly in the external electromagnetic current! This of course is unacceptable since the external field simply obeys Maxwell equations. To remedy this one should couple

to the external photon only the conserved quantity constructed out of the anomalous topological j -current. This means that the term (4.52) should be replaced by

$$\int dt d^2x (\epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho - A^\mu \frac{\partial_\mu}{\partial^2} \rho(x, t)) \quad (4.53)$$

Consider now the case, relevant to superconductivity, where the space is a thick annulus (or rather a cylindrical annulus in realistic cases). In this case, as discussed in Section 4.7, the electromagnetic flux is quantised as in (4.49). We assume for simplicity that the annulus is extended to spatial infinity. If we make a gauge transformation on the monopole $a_\mu \rightarrow a_\mu + \partial_\mu \Lambda$ then Λ does not vanish at infinity but rather winds around m times. In section 4 on the basis of this we argued that invariance of the action under large gauge transformations is consistent with a flux quantum $2e$. Here we shall examine whether such configurations are compatible in a quantum treatment. Our approach will be that of [75] where it has been argued that monopole configurations are not compatible with fixed boundary conditions at infinity, but require free ones. In a path integral language this implies an integration over all possible ‘phases’ Λ . The mixed term (4.53) changes under large statistical gauge transformations as (in units $\hbar = 2\pi$, $c = 1$)

$$\delta S_{CS} = \frac{ie}{\pi} \Lambda \frac{m\pi}{e} = im\Lambda \quad (4.54)$$

We observe that upon integrating over Λ ,

$$\int_0^{2\pi} d\Lambda e^{im\Lambda} \quad (4.55)$$

any non trivial flux configuration for the electromagnetic field is washed out since the result is $\delta_{m,0}$. Thus, a non-trivial flux for the A -field is compatible only with zero Λ . This means that only monopole-antimonopole pairs confined inside the sample are allowed in the case of non-trivial electromagnetic fluxes. Thus the potentially dangerous configurations of [73] are ruled out. A similar conclusion can be reached by considering the model in finite temperatures. As we will discuss in Section 4.9, temperature dependent corrections to the coefficient of the term (4.52) occur, which means that in case of topologically non-trivial configurations of the a -field with fixed boundary conditions the exponentiated action is not invariant under large gauge transformations, unless the electromagnetic flux quantum is temperature dependent, which of course cannot happen since it would violate Dirac’s quantisation condition.

The conclusion of this section therefore is that if monopoles were not confined in pairs with their anti-monopoles, the electromagnetic flux penetrating a superconducting annulus-shaped sample would be necessarily vanishing! It is therefore essentially the effect of fermions that ensures the confinement of the monopole configurations, and therefore the absence of any would-be anomaly associated with them. Thus the exactness of our superconductivity is established, on the basis of quite general arguments. We note in passing that the scenario of fermions ensuring masslessness of photons in (2+1)-dimensions has been known for some time in the context of three-dimensional Georgi-Glashow model [76]. In compact QED_3 gauge theory instanton effects were responsible for an explicit breaking of the fermion number symmetry, whose spontaneous breaking in perturbation theory ensured masslessness of photon, the latter being the pertinent Goldstone boson. The effect of instantons was therefore to give a non perturbative mass to the photon, which however in the *particular* model of [76] was cancelled exactly by the effects of fermions. In our case the role of fermions would be similar if Hwang et al were right.

4.9 Comparison with the anyonic model

To illustrate the attractive phenomenological features of the long-wavelength theory developed above, it is instructive to compare it with the anyonic model. Like the model considered here, a relativistic anyonic superconductor can be described by four-component Dirac fermions coupled to an abelian gauge field in (2+1)-dimensional spacetime. When the anyonic theory is coupled to an external electromagnetic field the appropriate Lagrangian density is ⁴

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_{anyon} \quad (4.56)$$

where $\mathcal{L}_{EM}$ is given by (4.20) and

$$\mathcal{L}_{anyon} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \bar{\Psi}_a (i\not{\partial} - \not{A} - e\gamma_0 A^0 - \frac{e}{c} \gamma_i A^i - \Delta\tau_3) \Psi_a + \frac{\gamma}{8\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \quad (4.57)$$

⁴We work in the zero anyon density limit so as to make the comparison to our model direct. The conclusions remain the same if a chemical potential is introduced.

where $a = 1, \dots, N$. The anyonic Lagrangian differs from the long-wavelength Lagrangian $\mathcal{L}_{matter}$, given by (4.21), in three obvious respects. Firstly $\mathcal{L}_{anyon}$ has a bare Chern-Simons term which explicitly breaks parity and time-reversal symmetry. This term alters the bare statistics of the spin-1/2 fields, $\bar{\Psi}$ and Ψ , which become ‘anyons’ [19]. The wavefunction of two identical spin-1/2 particles acquires a phase $\pi i(1+2/\gamma)$ when the particles are interchanged. Secondly, the mass term is the four-component parity-violating term $\Delta\tau_3\bar{\Psi}\Psi$ discussed in Section 2.1. Lastly the gauge coupling omits the factor of τ_3 which occurs in (4.21) indicating that the upper and lower components of the spinors Ψ_a couple to the gauge field with the *same* sign.

The quantum corrections to $\mathcal{L}_{anyon}$ are similar to those occurring for $\mathcal{L}_{matter}$. At leading order, a mixed Chern-Simons term is generated and the bare Maxwell terms are corrected by vacuum polarisation. The key difference is that, because the mass term contains an extra factor of τ_3 , the antisymmetric part of the vacuum polarisation is non-vanishing and generates a Chern-Simons term for the electromagnetic gauge field,

$$\mathcal{L}_{CS}^A = \frac{N}{4\pi} \varepsilon^{\mu\nu\rho} \tilde{A}_\mu \partial_\nu \tilde{A}_\rho, \quad (4.58)$$

and also provides a radiative correction to the bare Chern-Simons term for the statistical gauge field,

$$\mathcal{L}_{CS}^a = \frac{(\gamma + 2N)}{8\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \quad (4.59)$$

In the special case, $\gamma = -2N$, the bare Chern-Simons term is cancelled by the radiatively generated one. Only when this cancellation occurs does the anyon model exhibit superconductivity [3]. In this case the one-loop effective action is formally identical to (4.29) and the London action (4.36) may be derived exactly as in Section 4.4 [8]. When $N = 2$, the value suggested by the Hubbard Model, the cancellation implies that matter particles have an interchange phase $i\pi/2$, half the value for fermions. This scenario is known as ‘semionic’ superconductivity [8].

There are several outstanding difficulties in applying the anyonic theory to the high- T_c materials. Firstly, the introduction of a bare Chern-Simons term has no motivation at the microscopic level. There is no evidence that such a term arises from the long-wavelength limit of the Hubbard model [5]. Also, the radiatively generated Chern-Simons term for the electromagnetic field leads to observable P and T violating effects [4]. All experimental searches for such effects have so far been either negative or inconclusive. However, the most serious difficulty with the anyonic scenario is a theoretical one. At finite temperature the coefficient of the radiatively generated Chern-Simons term for the statistical gauge field is no longer equal to $N/4\pi$ but is given by $N/4\pi \tanh(\Delta\beta/2)$ [78]. The cancellation of the bare term is then not exact and the residual Chern-Simons term provides a topological mass for the statistical gauge field which destroys superconductivity. Hence the choice $\gamma = -2N$ only ensures superconductivity at exactly zero temperature. We note in passing that the unconventional anyonic scenario considered in [7], involving only dynamical generation of parity violating effects, avoids this restriction and can give a non-zero transition temperature. The model developed in this section has none of the above difficulties. In particular, the exact cancellation of the Chern-Simons term for the statistical gauge field is an automatic consequence of the parity-conserving mass term and the transverse components of the statistical photon remain massless at finite temperature [24].

5 Conclusion

In Section 4 we presented a field theoretic model of a two-dimensional relativistic superconductor. In its most general form the model is described by the Lagrangian density,

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_{matter} \quad (5.1)$$

$$\mathcal{L}_{EM} = -\frac{1}{4\sqrt{\partial^2}} \left(\left(\frac{\partial_0}{c} A_i - \partial_i A_0 \right) \left(\frac{\partial^0}{c} A^i - \partial^i A^0 \right) + (\partial_i A_j - \partial_j A_i) (\partial^i A^j - \partial^j A^i) \right) \quad (5.2)$$

$$\mathcal{L}_{matter} = -\frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu} + \bar{\Psi}_a (i\not{\partial} - \tau_3 \not{\partial} - e\gamma_0 A^0 - \frac{e}{c} \gamma_i A^i - \Delta) \Psi_a \quad (5.3)$$

We have restricted our attention to the case where the gap Δ is dynamically generated by the gauge interactions, but in fact the above model can be considered in a more general context where the origin of the gap is not specified (in other words, Δ is considered as a bare mass). We note that Kovner and Rosenstein [9] have recently considered this model in the limit $\Delta \gg g^2$ and argued that it coincides with the two-dimensional Coulomb gas [23] which is known to exhibit a phase transition of Kosterlitz-Thouless type.

Irrespective of the origin and magnitude of the gap, the model exhibits genuinely two-dimensional superconductivity with a non-zero transition temperature and no violation of either parity or time-reversal symmetries. For these reasons alone, it is natural to consider the above model as a candidate theory of high-temperature superconductivity or, more precisely, as a candidate theory of superconductivity in an isolated copper oxide layer. Effective two-dimensionality is experimentally realised only for samples in which single layers of YBa_2CuO_6 are isolated from each other by many intervening layers of an insulating material in a ‘super-lattice’ structure [44]. Such materials are a very recent development and measurements of the parameters of the two-dimensional superconducting state have not yet appeared in the literature. Thus, for the time being, it is only possible to make a qualitative comparison between the predictions of the two-dimensional model and experimental results. Alternatively, the model could be generalised to include an explicit interplanar coupling allowing a direct comparison with experiments on multi-layer samples.

Perhaps the most encouraging phenomenological feature of the model is that, irrespective of the values of the microscopic parameters, it describes a type-II superconductor. A superconductor is characterised as type-I or type-II depending on whether the Landau-Ginzburg constant, $\kappa = \lambda/\xi$, is less than or greater than $1/\sqrt{2}$ respectively [79]. In addition to the superconducting and normal states, type-II superconductors exhibit an intermediate ‘mixed’ state when a sufficiently strong external magnetic field is applied. In BCS superconductors, the typical radius of a Cooper pair can be many thousands of lattice spacings and the coherence length is often as large as $10^4 \text{ \AA}$, comparable with the penetration depth. One of the most distinctive properties of the high- T_c materials is their extremely short coherence length, typically about $10 \text{ \AA}$ [47]. In contrast, the penetration depth for high- T_c superconductors is of the same order as the BCS value. Consequently the new materials are strongly type-II with $\kappa \gg 100$. As discussed in Section 4.6, this feature is correctly captured by the model. Starting from the Lagrangian (5.3), it should be possible to derive an effective description of the mixed state and predict the critical external field.

Section 3 was devoted to motivating the model as a long-wavelength limit of the doped antiferromagnet which is itself an idealised model of the dynamics of electrons in the CuO_2 layers of the high- T_c materials. Although the arguments leading from the doped antiferromagnet to the gauge theory considered here involve many assumptions and approximations, certain general features may survive a more rigorous analysis. The description of quasi-holes as Dirac fermions in the continuum limit assumes a flux-phase ground state for the antiferromagnet. However, like the anyonic model, we expect that the mechanism considered here can be generalised to include non-relativistic fermions. In general, antiferromagnetism leads to the attractive gauge interaction between holes on the two sublattices providing an analogy with the familiar, phononic attraction of BCS theory. Pursuing this analogy further, it is natural to think of the gauge attraction as dynamically generating the superconducting gap. The parity-conserving structure of the four-component gap term has an obvious interpretation in terms of the antiferromagnetic order of the microscopic model.

The gap equation of Section 2.6 relates the magnitudes of the gap and the transition temperature to that of the dimensionful gauge coupling. The coupling is naturally identified with the characteristic energy of antiferromagnetic correlations, typically equivalent to a temperature of about 1500 K . In traditional superconductors the energy scale of the gap and transition temperature is governed by the Debye temperature $\hbar\omega_D/k_B$ which is much lower. In particular, the BCS theory sets a theoretical upper bound on the transition temperature of about 30 K . Such an upper limit does not apply to the present model which potentially explains the high transition temperatures of the new materials. In fact, given that $J/k_B \simeq 1500 \text{ K}$, the relevant question is not why is T_c so high, but rather why is it so low! The analysis of the zero-temperature gap equation given in Section 2.5 suggests that the gap is related to the dimensionful coupling as,

$$\Delta \simeq g^2 \exp \left(\frac{-2n\pi}{\sqrt{N_c/N - 1}} \right) \quad (5.4)$$

For $N = 2$ the constant of proportionality, Δ/g^2 , is extremely small and, despite the large value of g^2 , the corresponding gap is smaller than the experimental one by two orders of magnitude. However, the formula (5.4) is the result of a very crude approximation to the Schwinger-Dyson equation for the fermion propagator and alternative treatments of these equations produce different results. The gap is non-perturbative in $1/N$ and the only reliable determination would be from Monte-Carlo simulations of non-compact lattice QED_3 . The gap cannot be measured directly on the lattice but could be inferred from the mass of the lowest lying scalar bound-state [80], while the critical temperature could be determined by simulation on an asymmetric lattice. In the absence of these results, the question of whether a model based on QED_3 can account for the experimentally measured values of Δ and T_c is still open.

As discussed in Section 2.6, the scenario of a dynamically generated superconducting gap suggests an explanation for the high values of the ratio $2\Delta/k_B T_c$ exhibited by the high- T_c materials. Despite the disparity between different approximate calculations of the gap and the critical temperature, the ratio of these quantities displays a remarkable stability [81]. Essentially, the difference between the ratio for QED₃ and the BCS value is due to the rapid decrease in the range of the gauge attraction between holes as the temperature is raised. We expect this qualitative feature to persist in a more accurate treatment. Again, this feature of the model could be tested by lattice simulation.

In conclusion, the model we have presented describes a new class of two-dimensional, parity-conserving superconductors which overcome many of the phenomenological difficulties of anyonic superconductivity. The model has a physical interpretation as the long-wavelength limit of a doped planar antiferromagnet and correctly describes certain qualitative features of high-temperature superconductivity. As we have outlined above, much more work, both theoretical and experimental, is required to determine whether or not this model is realised in nature.

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Appendices

A The finite-temperature photon propagator

The purpose of this Appendix is to evaluate explicitly the finite temperature one-loop vacuum polarisation tensor given by

$$\Pi_{\mu\nu}^m(P, \beta) = -\frac{\alpha}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \frac{\text{Tr}(\gamma_\mu \not{k} \gamma_\nu (\not{p} + \not{k}))}{k^2(p+k)^2} \quad (\text{A.1})$$

where,

$$\begin{aligned} p &= (p_0, \mathbf{p}) & P &= |\mathbf{p}| & p_0 &= \frac{2m\pi}{\beta} \\ k &= (k_0, \mathbf{k}) & K &= |\mathbf{k}| & k_0 &= \frac{(2n+1)\pi}{\beta} \end{aligned} \quad (\text{A.2})$$

In general, $\Pi_{\mu\nu}$ has four independent components corresponding to the four independent $O(2)$ -invariant tensors. However, as at $T = 0$, the explicit expression (A.1) satisfies the transversality condition, $p^\mu \Pi_{\mu\nu}(p) = 0$ and can be conveniently decomposed in terms of two independent transverse tensors,

$$\Pi_{\mu\nu}^m(P, \beta) = \Pi_A^m(P, \beta) A_{\mu\nu} + \Pi_B^m(P, \beta) B_{\mu\nu} \quad (\text{A.3})$$

where

$$A_{\mu\nu} = \left(\delta_{\mu 0} - \frac{p_\mu p_0}{p^2} \right) \frac{p^2}{P^2} \left(\delta_{0\nu} - \frac{p_0 p_\nu}{p^2} \right) \quad B_{\mu\nu} = \delta_{\mu i} \left(\delta_{ij} - \frac{p_i p_j}{P^2} \right) \delta_{j\nu} \quad (\text{A.4})$$

$A_{\mu\nu}$ and $B_{\mu\nu}$ are orthogonal and satisfy,

$$A_{\mu\nu} + B_{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \quad (\text{A.5})$$

so that the full finite temperature photon propagator $\Delta_{\mu\nu}(p_0, P, \beta)$, defined by (2.18), is given by;

$$\Delta_{\mu\nu}(p_0, P, \beta) = \frac{A_{\mu\nu}}{p^2 + \Pi_A^m(P, \beta)} + \frac{B_{\mu\nu}}{p^2 + \Pi_B^m(P, \beta)} \quad (\text{A.6})$$

The strategy adopted here will be to evaluate the temporal and spatial components of vacuum polarisation tensor Π_{00}^m and Π_{ii}^m and then use the relations

$$\Pi_A^m = \frac{p^2}{P^2} \Pi_{00}^m \quad \Pi_B^m = \Pi_{ii}^m - \frac{p_0^2}{P^2} \Pi_{00}^m \quad (\text{A.7})$$

As in the zero temperature case the loop integral in (A.1) can be evaluated by introducing Feynman parameters,

$$\Pi_{\mu\nu}^m = \frac{4\alpha}{\beta} \int_0^1 dx \sum_{n=-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \frac{2k_\mu k_\nu + k_\mu p_\nu + k_\nu p_\mu - p \cdot (p+k) \delta_{\mu\nu}}{((k+xp)^2 + x(1-x)p^2)^2} \quad (\text{A.8})$$

Defining a shifted three-momentum by $l = k + xp$ with,

$$l = (l_0, \mathbf{l}) \quad L = |\mathbf{l}| \quad l_0 = \frac{2\pi}{\beta} \left(n + xm + \frac{1}{2} \right) \quad (\text{A.9})$$

and changing variables in the momentum integral gives;

$$\begin{aligned} \Pi_{\mu\nu}^m = & \frac{4\alpha}{\beta} \int_0^1 dx \sum_{n=-\infty}^{+\infty} \int \frac{d^2 K}{(2\pi)^2} \left[\frac{l_\mu l_\nu}{(l^2 + x(1-x)p^2)^2} - \frac{\delta_{\mu\nu}}{(l^2 + x(1-x)p^2)} \right. \\ & \left. + \frac{2x(1-x)(p^2 \delta_{\mu\nu} - p_\mu p_\nu)}{(l^2 + x(1-x)p^2)^2} + \frac{(1-2x)(l_\mu p_\nu + l_\nu p_\mu - (p \cdot l) \delta_{\mu\nu})}{(l^2 + x(1-x)p^2)^2} \right] \end{aligned} \quad (\text{A.10})$$

Setting $\mu = i, \nu = j$ and using a gauge invariant regulator, the first two terms in (A.10) cancel and the remaining terms may be written as,

$$\Pi_{ij}^m(P, \beta) = \frac{4\alpha}{\beta} \int_0^1 dx \int \frac{d^2 L}{(2\pi)^2} [2x(1-x)(p^2 \delta_{ij} - p_i p_j) S_2 - (1-2x) \delta_{ij} p_0 S^*] \quad (\text{A.11})$$

while the temporal component is given by

$$\Pi_{00}^m(P, \beta) = \frac{4\alpha}{\beta} \int_0^1 dx \int \frac{d^2 L}{(2\pi)^2} [S_1 - 2(L^2 + x(1-x)p_0^2) S_2 + (1-2x) \delta_{ij} p_0 S^*] \quad (\text{A.12})$$

where,

$$S_i = \sum_{n=-\infty}^{+\infty} \frac{1}{(l_0^2 + L^2 + x(1-x)p^2)^i} \quad S^* = \sum_{n=-\infty}^{+\infty} \frac{l_0}{(l_0^2 + L^2 + x(1-x)p^2)^2} \quad (\text{A.13})$$

The sum S_1 may be evaluated using the result [27];

$$S(X, Y) = \sum_{n=0}^{\infty} \frac{1}{(n+X)^2 + Y^2} = \frac{1}{2Yi} (\psi(X+iY) - \psi(X-iY)) \quad (\text{A.14})$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and the parameters X and Y take the values

$$X = mx + \frac{1}{2} \quad Y = \frac{\beta}{2\pi} \sqrt{L^2 + x(1-x)p^2} \quad (\text{A.15})$$

Noticing that S_1 is proportional to $S(X, Y) + S(1-X, Y)$ and using the identity [27],

$$\psi(1-z) = \psi(z) + \pi \cot \pi z \quad (\text{A.16})$$

with $z = X + iY$ gives,

$$S_1 = \frac{\beta^2}{8i\pi Y} (\cot \pi(X+iY) - \cot \pi(X-iY)) \quad (\text{A.17})$$

The other two summations in (A.13) are easily obtained from (A.17) by parametric differentiation with respect to X and Y ;

$$S_2 = -\frac{\beta^2}{8\pi^2 Y} \frac{\partial S_1}{\partial Y} \quad S^* = -\frac{\beta}{4\pi} \frac{\partial S_1}{\partial X} \quad (\text{A.18})$$

The momentum integrals can now be performed term by term in (A.11) and (A.12) to give;

$$\Pi_{ij}^m(P, \beta) = \Pi_1^m \left(\delta_{ij} - \frac{p_i p_j}{p^2} \right) + \Pi_2^m \delta_{ij} \quad (\text{A.19})$$

$$\Pi_{00}^m(P, \beta) = \Pi_3^m - \frac{p_0^2}{p^2} \Pi_1^m - \Pi_2^m \quad (\text{A.20})$$

The components Π_i^m can be expressed as one-dimensional integrals over the Feynman parameter x ,

$$\Pi_1^m(P, \beta) = \frac{\alpha p}{2\pi} \int_0^1 dx \sqrt{x(1-x)} \frac{\sinh(p\beta\sqrt{x(1-x)})}{D_m(x, P, \beta)} \quad (\text{A.21})$$

$$\Pi_2^m(P, \beta) = \frac{\alpha m}{2\beta} \int_0^1 dx (1-2x) \frac{\sin(2xm\pi)}{D_m(x, P, \beta)} \quad (\text{A.22})$$

$$\Pi_3^m(P, \beta) = \frac{\alpha}{\pi\beta} \int_0^1 dx \log(4D_m(x, P, \beta)) \quad (\text{A.23})$$

where the denominator of each integrand is given by

$$D_m(x, P, \beta) = \cosh^2(p\beta\sqrt{x(1-x)}/2) - \sin^2(xm\pi) \quad (\text{A.24})$$

The three components Π_i^m obey a simple linear relation which can be derived by noticing that

$$\int_0^1 dx \int \frac{d^2 L}{(2\pi)^2} \frac{d(\cot \pi(X \pm iY))}{dx} = 0 \quad (\text{A.25})$$

Differentiating with respect to x using the formulae (A.15) and then performing the L integral, the above equation implies,

$$\Pi_3^m = \Pi_1^m + \frac{p^2}{p_0^2} \Pi_2^m \quad (\text{A.26})$$

This relation can also be derived directly from the expressions (A.21-A.23) by performing the integral on the RHS of (A.23) by parts and considering the transformation properties of each term under the change of variables $x \rightarrow 1-x$. The relation (A.26) shows that, as claimed above, the vacuum polarisation has only two independent components. Using (A.26) and (A.7) these are conveniently expressed as,

$$\Pi_A^m = \Pi_3^m \quad \Pi_B^m = \Pi_1^m + \Pi_2^m \quad (\text{A.27})$$

Hence the final form of the finite temperature photon propagator for QED₃ to leading order in the $1/N$ -expansion is given by (A.6) as,

$$\Delta_{\mu\nu}(p_0, P, \beta) = \frac{A_{\mu\nu}}{p^2 + \Pi_3^m} + \frac{B_{\mu\nu}}{p^2 + \Pi_1^m + \Pi_2^m} \quad (\text{A.28})$$

B The dependence of observables on v_S

The purpose of this appendix is to demonstrate that observable quantities are approximately independent of the spin-wave velocity, v_S , and hence that the results derived in Section 4 for the special case $v_S = v_F$ are in fact quite general. Units are chosen so that $v_F = \hbar = 1$ and $p^2 = p_0^2 + |\mathbf{p}|^2$. For any three-vector b_μ , it is convenient to define rescaled vectors $\bar{b}_\mu = (b_0/v_S, \mathbf{b})$ and $\hat{b}_\mu = (b_0, \mathbf{b}/v_S)$. The rescaled tensor $\hat{c}_{\mu\nu}$ is defined as

$$\hat{c}_{00} = c_{00} \quad \hat{c}_{i0} = c_{i0}/v_S \quad \hat{c}_{ij} = c_{ij}/v_S^2 \quad (\text{B.1})$$

In this notation, the kinetic term (4.2) for the statistical gauge field is

$$\mathcal{L}_{kin} = -\frac{1}{4g^2} (\bar{\partial}_\mu a_\nu - \bar{\partial}_\nu a_\mu)^2 \quad (\text{B.2})$$

The corresponding bare propagator is,

$$\Delta_{\mu\nu}^{(0)} = \frac{v_S^2}{p_0^2 + v_S^2 |\mathbf{p}|^2} \left(\delta_{\mu\nu} - \frac{v_S^2 \bar{p}_\mu \bar{p}_\nu}{p_0^2 + v_S^2 |\mathbf{p}|^2} \right) \quad (\text{B.3})$$

To leading order in $1/N$ (or, equivalently, to one loop) the statistical photon propagator can be written as

$$\Delta_{\mu\nu}^{-1} = \Delta_{\mu\nu}^{(0)-1} + \hat{\Pi}_{\mu\nu}(p) \quad (\text{B.4})$$

where $\Pi_{\mu\nu}(p)$ is the vacuum polarisation tensor in the case $v_S = 1$,

$$\Pi_{\mu\nu}(p) = \Pi(p) \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad (\text{B.5})$$

with $\Pi(p)$ given by (4.23) in the general case $\Delta > 0$. The dependence of the self-consistent gap equation on v_S is contained in the kernel,

$$K(p_0, \mathbf{p}, v_S) = \text{Tr}(\hat{\gamma}^\mu \Delta_{\mu\nu} \hat{\gamma}^\nu) \quad (\text{B.6})$$

The analysis of the Schwinger-Dyson equation given in Section 2.5 indicates that the magnitude of the gap is controlled by the low-momentum behaviour of the kernel [21]. For $p \ll g^2$,

$$K(p_0, \mathbf{p}, v_S) \simeq \text{Tr}(\hat{\gamma}_\mu (\hat{\Pi})_{\mu\nu}^{-1} \hat{\gamma}^\nu) \quad (\text{B.7})$$

It is easily checked that the dependence of this expression on v_S cancels exactly. Hence, to a good approximation, the gap is independent of the spin-wave velocity. Evaluating the mixed Chern-Simons term at low momenta the effective action is given by,

$$\mathcal{L}_{\text{matter}} = -\frac{1}{12\pi\Delta} f_{\mu\nu}(\hat{a}) f^{\mu\nu}(\hat{a}) - \frac{e}{2\pi} \varepsilon^{\mu\nu\rho} \tilde{A}_\mu f_{\nu\rho}(\hat{a}) \quad (\text{B.8})$$

where $f_{\mu\nu}(\hat{a}) = \partial_\mu \hat{a}_\nu - \partial_\nu \hat{a}_\mu$. This is identical to the matter action in the special case $v_S = v_F$ as given by (4.29), with the replacement of a_μ by $\hat{a}_\mu$. As the London action is derived by performing the functional integral over a_μ all dependence on v_S can be eliminated by the change of variables $\hat{a}_\mu \rightarrow a_\mu$. Hence the final form of the action (4.36) remains exactly unchanged. Consequently, observable quantities such as λ depend on v_S only through the gap. As pointed out above, the dependence of the gap itself on the spin-wave velocity is only weak. We expect these considerations to remain valid at finite temperature.

References

- [1] J. G. Bednorz and A. K. Müller, *Z. Phys.* **B64** (1986) 189.
- [2] R. B. Laughlin, *Science* **242** (1988) 525.
- [3] A. L. Fetter, C. B. Hanna and R. B. Laughlin, *Phys. Rev.* **B39** (1989) 9679.
Y. H. Chen, B. I. Halperin, F. Wilczek and E. Witten, *Int. J. Mod. Phys.* **B3** (1989) 1001.
- [4] X. G. Wen and A. Zee, *Phys. Rev. Lett.* **62** (1989) 2873.
- [5] E. Fradkin and M. Stone, *Phys. Rev. B* **38** 7215.
X. G. Wen and A. Zee, *Phys. Rev. Lett.* **61** (1988) 1025.
- [6] J. Lykken, J. Sonnenschein and N. Weiss, *Phys. Rev.* **D42** (1990) 2161.
- [7] N. E. Mavromatos, University of Oxford preprint: OUTP-90-04P.
- [8] T. Banks and J. Lykken, *Nucl. Phys.* **B336** (1990) 550.
- [9] A. Kovner and B. Rosenstein, *Phys. Rev.* **B42** (1990) 4748.
- [10] N. Dorey and N. E. Mavromatos, *Phys. Lett.* **B250** (1990) 107.
- [11] G. W. Semenoff and N. Weiss, *Phys. Lett.* **B250** (1990) 117.
- [12] N. D. Mermin and H. Wagner, *Phys. Rev. Lett.* **22** (1966) 1133.
- [13] R. Shankar, *Phys. Rev. Lett.* **63** (1989) 203 and *Nucl. Phys.* **B330** (1990) 433.
- [14] P. B. Wiegmann, *Physica (Amsterdam)* **153-155C** (1988) 103.
- [15] P. A. Lee, *Phys. Rev. Lett.* **63** (1989) 680.
- [16] N. Dorey and N. E. Mavromatos, *Phys. Rev.* **B44** (1991) 5286.
- [17] S. Deser, R. Jackiw and S. Templeton, *Ann. Phys.* **140** (1982) 372.
- [18] J. Leinaas and J. Myrheim, *Nuovo Cimento* **B37** (1977) 1.
- [19] R. Mackenzie and F. Wilczek, *Int. J. Mod. Phys.* **A3** (1988) 2827.
G. W. Semenoff and P. Sodano, *Nucl. Phys.* **B328** (1989) 753.
- [20] R. D. Pisarski, *Phys. Rev.* **D29** (1984) 2423.
- [21] T. W. Appelquist, M. Bowick, D. Karabali and L. C. R. Wijewardhana, *Phys. Rev.* **D33** (1986) 3704.
T. W. Appelquist, D. Nash and L. C. R. Wijewardhana, *Phys. Rev. Lett.* **60** (1988) 2575.
- [22] A. Kovner and B. Rosenstein, *Mod. Phys. Lett.* **A5** (1990) 2661.
- [23] J. Kosterlitz and D. Thouless, *J. Phys.* **C6** (1973) 1181.
- [24] N. Dorey and N. E. Mavromatos, *Phys. Lett.* **B266** (1991) 163.
- [25] L. Dolan and R. Jackiw, *Phys. Rev.* **D9** (1974) 3320.
- [26] D. J. Gross, R. D. Pisarski and L. G. Yaffe, *Rev. Mod. Phys.* **53** (1981) 43.
- [27] I. S. Gradshteyn and I. M. Ryzhik, 'Table of Integrals, Series and Products' (Academic Press 1965).
- [28] N. Dorey and N. E. Mavromatos, *Phys. Lett.* **B257** (1991) 359.
- [29] V. S. Popov, *Yad. Fiz.* **12** (1970) 429.
- [30] P. I. Fomin, V. P. Gusynin, V. A. Miransky and Yu. A. Sitenko, *Riv. Nuovo. Cim.* **6** (1983) 1.
- [31] M. Abramowitz and I. A. Stegun, 'Handbook of Mathematical Functions' (Dover 1975).
- [32] E. Dagotto, A. Kocic and J. B. Kogut, *Phys. Rev. Lett.* **62** (1989) 1083 and *Nucl. Phys.* **B334** (1990) 279.
- [33] T. Maskawa and H. Nakajima, *Prog. Theor. Phys.* **52** (1974) 1326.
- [34] M. R. Pennington and D. Walsh, University of Durham preprint: DTP-90/62.

- [35] S. Coleman, *Comm. Math. Phys.* **31** (1973) 253.
- [36] D. J. Gross and A. Neveu, *Phys. Rev.* **D10** (1974) 3235.
- [37] E. Witten, *Nucl. Phys.* **B145** (1978) 110.
- [38] S. H. Park, B. Rosenstein and B. J. Warr, SLAC preprint: SLAC-PUB-5349.
- [39] A. C. Davis and A. M. Matheson, *Phys. Rev.* **D40** (1989) 2373.
- [40] A. L. Fetter and J. D. Walecka, '*The Quantum Theory of Many-Particle Systems*' (McGraw-Hill 1971).
- [41] S. H. Park, B. Rosenstein and B. J. Warr, *Phys. Rev.* **D39** (1989) 3088.
- [42] A. Kocic, *Phys. Lett.* **B189** (1987) 449.
- [43] M. Rasolt, T. Edis and Z. Tesanovic, *Phys. Rev. Lett.* **66** (1991) 2927.
- [44] Q. Li et al, *Phys. Rev. Lett.* **64** (1990) 3086.
D. H. Lowndes, D. P. Norton and J. D. Budai, *Phys. Rev. Lett.* **65** (1990) 1160.
- [45] B. Batlogg et al, *Phys. Rev. Lett.* **58** (1987) 2333.
- [46] Z. Schlessinger et al, *Phys. Rev. Lett.* **59** (1987) 1958.
- [47] F. Davydov, *Physics Reports* **190** (1990) 191.
- [48] P. W. Anderson, *Science* **235** (1987) 1196.
- [49] W. F. Brinkman and T. M. Rice, *Phys. Rev.* **B2** (1970) 1324.
- [50] G. Aeppli et al, *Phys. Rev. Lett.* **62** (1989) 2052.
- [51] E. Manousakis, *Rev. Mod. Phys.* **63** (1991) 1.
- [52] J. E. Hirsch and S. Tang, *Phys. Rev. Lett* **62** (1989) 591.
- [53] A. R. Bishop et al, *Z. Phys.* **B76** (1989) 17.
M. S. Hybertsen et al, *Phys. Rev.* **B41** (1990) 11068.
J. Dutka and A. Oleś, *Phys. Rev.* **B42** (1990) 105.
- [54] F. D. M. Haldane, *Phys. Lett.* **A93** (1983) 464 and *Phys. Rev. Lett.* **50**, (1983) 1153.
- [55] I. Affleck, *Nucl. Phys.* **B257** (1985) 397.
- [56] M. V. Berry, *Proc. R. Soc. London* **392** (1984) 45.
- [57] A. M. Polyakov, '*Gauge Fields and Strings*' (Harwood 1987).
S. Deser and A. N. Redlich, *Phys. Rev. Lett.* **61** (1989) 1541.
- [58] I. J. R. Aitchison and N. E. Mavromatos, *Phys. Rev.* **B39** (1989) 6544.
- [59] S. H. Park, B. Rosenstein and B. Warr, *Nucl. Phys.* **B336** (1990) 435.
- [60] R. Blankenbecler, D. Scalapino and R. Sugar, *Phys. Rev.* **D24** (1981) 2278.
- [61] K. Bitar and E. Manousakis, *Phys. Rev.* **B43** (1991) 2615.
- [62] L. B. Ioffe and P. B. Wiegmann, *Phys. Rev. Lett.* **65** (1990) 653.
- [63] I. Affleck and J. B. Marston, *Phys. Rev.* **B37** (1988) 3774 and *Phys. Rev.* **B39** (1989) 11538.
- [64] G. Baskaran and P. W. Anderson, *Phys. Rev.* **B37** (1989) 580.
- [65] S. Elitzur, *Phys. Rev.* **D12** (1975) 3978.
- [66] J. Kogut and L. Susskind, *Phys. Rev.* **D9** (1974) 3501.
- [67] G. Semenoff and L. C. R. Wijewardhana, *Phys. Rev. Lett.* **63** (1989) 2633.
- [68] N. N. Bogoliubov, *Sov. Phys. JETP* **7** (1958) 41.
- [69] A. V. Balatsky, *Phys. Rev. Lett.* **64** (1990) 2078.

- [70] S. Coleman and B. Hill, *Phys. Lett.* **159B** (1985) 184.
- [71] P. W. Higgs, *Phys. Lett.* **12** (1964) 132.
P. W. Anderson, *Phys. Rev.* **130** (1963) 439.
- [72] G. Murthy and S. Sachdev, *Nucl. Phys.* **B344** (1990) 557.
- [73] K. Huang, Y. Koike and J. Polonyi, *Int. J. Mod. Phys* **A6** (1991) 1267.
- [74] A. Kovner, B. Rosenstein and D. Elieser, *Nucl. Phys.* **B350** (1991) 325.
- [75] I. Affleck, J. Harvey, L. Palla and G. Semenoff, *Nucl. Phys.* **B328** (1989) 575.
- [76] I. Affleck, J. Harvey and E. Witten, *Nucl. Phys.* **B206** (1982) 413.
- [77] S. Weinberg, *Prog. Theor. Phys. Suppl.* **86** (1986) 43.
- [78] K. S. Babu, A. Das, P. Panigrahi, *Phys. Rev.* **D36** (1987) 3725.
K. T. Islikawa and A. Matsuyama, *Nucl. Phys.* **B280** [FS 18] (1987) 523.
- [79] A. C. Rose-Innes and E. H. Rhoderick, '*Introduction to Superconductivity*' (Pergamon 1969).
- [80] J. B. Kogut, private communication.
- [81] M. Klein, private communication.

Table captions

Table 1: Symmetry properties of QED₃ and τ_3 -QED₃ with and without a parity-conserving mass term.

Table 2: The zero-temperature fermion mass and the critical temperature at $\alpha/\Lambda = 1$ for $n = 1, 5$.

Table 3: The zero-temperature fermion mass and the critical temperature at $\alpha/\Lambda = \infty$ for $n = 1, 5$.

Table 4: The zero-temperature fermion mass and the critical temperature at $n = 1$ for a range of α .

Figure captions

Figure 1: The one-loop vacuum polarisation diagram.

Figure 2: The one-loop contribution to the matrix element $\langle 0 | J_\mu^T | 1, \mathbf{p} \rangle$. A blob indicates an insertion of the current J_μ^T .

Figure 3: The photon propagator to leading order in $1/N$.

Figure 4: The leading order contribution in the $1/N$ expansion for the effective static potential, $V(R)$.

Figure 5: Graph of $V(R)$ for $T/\alpha=0, 0.025$ and 0.25 .

Figure 6: The Schwinger-Dyson equation for the fermion propagator.

Figure 7: Graph of m/Λ against $k_B T/\Lambda$.

Figure 8: Sublattice structure for hole hopping.

Figure 9: The lowest-order contribution to the current-current correlator $\langle 0 | J_\mu(p) J_\nu(-p) | 0 \rangle$. The blob in each fermion loop indicates an insertion of the current J_μ .

Figure 10: The one-loop contribution to the mixed Chern-Simons term.

Figure 11: The photon propagator to leading order in $1/N$.

Tables

$M = 0$	QED ₃	τ_3 -QED ₃	$M > 0$	QED ₃	τ_3 -QED ₃
γ_5, γ_3	Global	-	γ_5, γ_3	Broken	-
τ_3	Global	Local	τ_3	KT	Unbroken
E	Local	Global	E	Unbroken	KT

(1)

n	1	2	3	4	5
$m(T=0)/\Lambda$	0.229	6.50×10^{-2}	2.46×10^{-2}	9.67×10^{-3}	4.23×10^{-3}
$k_B T_c/\Lambda$	4.90×10^{-2}	1.39×10^{-2}	5.09×10^{-3}	2.17×10^{-3}	6.86×10^{-2}
$r = 2m/k_B T_c$	9.35	9.38	9.65	9.36	12.34

(2)

n	1	2	3	4	5
$m(T=0)/\Lambda$	1.15	0.434	0.191	8.61×10^{-2}	3.87×10^{-2}
$k_B T_c/\Lambda$	0.207	8.94×10^{-2}	4.02×10^{-2}	1.80×10^{-2}	8.03×10^{-3}
$r = 2m/k_B T_c$	11.11	9.70	9.49	9.58	9.64

(3)

α/Λ	0.01	0.1	1	10	100	1,000
$m(T=0)/\Lambda$	2.42×10^{-3}	2.84×10^{-2}	0.229	0.803	1.10	1.15
$k_B T_c/\Lambda$	6.22×10^{-4}	6.12×10^{-3}	4.90×10^{-2}	0.159	0.201	0.207
$r = 2m/k_B T_c$	7.69	9.29	9.35	10.10	10.94	11.11
r^*	3.24	3.76	3.74	3.83	3.87	3.89

(4)

Figures

FIGURE 1

FIGURE 2

$$\text{wavy line with dot} = \text{wavy line} + \text{circle with wavy line} + \text{circle with wavy line and circle} + \dots$$

FIGURE 3

FIGURE 4

FIGURE 5

$$(\text{---}\bullet\text{---})^{-1} = (\text{---})^{-1} - \text{---}\overset{\text{---}\bullet\text{---}}{\underset{\text{---}\bullet\text{---}}{\text{---}\bullet\text{---}}}\text{---}$$

FIGURE 6

FIGURE 7

FIGURE 8

FIGURE 9

FIGURE 10

$$(\sim \bullet \sim)^{-1} = (\sim \sim)^{-1} - \sim \bigcirc \sim \sim \bullet \sim \sim \bigcirc \sim$$

FIGURE 11