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SOME ASPECTS OF THE THEORY OF SOLAR OSCILLATIONS

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The study of nonradial oscillations of stars, based on the seminal work by Cowling half a century ago, is increasingly providing information about the interiors of stars. Observations of solar oscillations have led to detailed inferences about the structure and rotation of the solar interior. Nonradial oscillations have also been studied in white dwarfs and in rapidly oscillating Ap stars, and indications of solar-like oscillations have been detected in a few stars. Here I give an overview of some properties of nonradial oscillations, and discuss the diagnostic potential of low-degree modes for probing the core of the Sun and other stars. The observed solar frequencies appear to exclude several of the proposed solutions to the solar neutrino problem that involve modifications to solar models.

1. INTRODUCTION

T. G. Cowling's work on stellar oscillations was of limited extent but of fundamental importance. Cowling (1941) characterized the spectrum of nonradial oscillations of a spherically symmetric star, through a combination of analytical insight and numerical calculations. Furthermore, Cowling & Newing (1949) showed how rotation induces splitting of the frequencies according to the azimuthal order m . These two papers are cornerstones for the application of observations of nonradial oscillations to the study of the structure and rotation of the interiors of the Sun and other stars.

Cowling (1941) introduced what is now called the *Cowling approximation*, i.e. the neglect of the perturbation in the gravitational potential (this approximation had previously been used in an erroneous analysis of nonradial oscillations by Emden 1907). It greatly simplifies the asymptotic analysis of the oscillation equations. Such analyses, in combination with numerical calculations, have been extremely valuable in understanding the properties of solar oscillations. Indeed, this approach of combining analytical insight with numerical results, which has played a major role in the development of helioseismic investigations of the solar interior, may be seen as a direct continuation of the work by Cowling (1941).

No extensive overview of the theory of solar oscillations can be attempted in the present paper. In fact, in addition to general tracts on stellar oscillations (see, for example, Ledoux & Walraven 1958; Uno *et al.* 1989) there exist a number of reviews on the application of solar oscillations to helioseismology (e.g. Deubner & Gough 1984; Libbrecht 1988; Vorontsov & Zharkov 1989; Christensen-Dalsgaard 1990; Gough & Toomre 1991; Gough 1991; Christensen-Dalsgaard & Berthomieu 1991). Here I concentrate on the properties of acoustic modes of low degree. These modes penetrate to near the solar centre and hence enable helioseismic investigation of the structure of the solar core. This is of obvious importance to our understanding of solar evolution. Amongst the observed modes of solar oscillations, the departure from the Cowling approximation is most important for those of low degree. Thus, although frequencies of acoustic modes are in general largely determined by the sound speed in the solar interior, the effect of the perturbation in the gravitational potential in low-degree modes introduces a substantial sensitivity of the frequencies to the density structure of the deep solar interior.

The need for information about the solar core is emphasized by the discrepancy between the predicted flux of solar neutrinos from normal solar models and the observed flux (for an overview, see Bahcall 1989); the observed frequencies place stringent constraints on the structure of the solar core, and hence on modifications of solar models proposed to reduce the computed neutrino flux. It is found that the observed frequencies exclude with high significance several models with reduced neutrino flux, rendering it probable that the solution to the neutrino problem is not of astrophysical nature.

Finally, it should be noted that since low-degree modes may in principle be observable in other solar-like stars, analysis of their frequencies might enable determination of the evolutionary state of such stars.

2. THE COWLING APPROXIMATION

2.1 General properties of solar oscillations

For the purpose of using solar oscillations to study the structure of the solar interior, effects of rotation can be neglected to leading order. Then a mode of oscillation can be characterized by its degree l and radial order n , the frequencies being independent of the third wavenumber (the azimuthal order m). The degree is related to the horizontal wavenumber k_h of the mode, when approximated locally at a distance r from the centre by a plane wave, by

$$k_h^2 = \frac{l(l+1)}{r^2}. \quad (2.1)$$

It is generally the case that the radial order may be defined such that when large $|n|$ gives the number of nodes in the radial component of the displacement; as discussed below, n takes both positive and negative values, the sign being used to distinguish between different types of modes.

Here I only consider adiabatic oscillations and neglect effects of viscosity. Then the linearized oscillation equations reduce to the equation of continuity, the momentum equation and Poisson's equation for the perturbation in the gravitational potential. After separation of the dependence on time and angular coordinates in terms of a harmonic function and a spherical harmonic, respectively, the equations reduce to a fourth-order set of ordinary differential equations; these may be written in terms of the amplitudes $\xi_r(r)$, $p'(r)$ and $\Phi'(r)$ of the radial component of displacement and the Eulerian perturbations in pressure and gravitational potential, as functions of r . The equations are supplemented by boundary conditions. Two of these impose regularity of the solution at the centre; one surface boundary condition is obtained by requiring continuity of the gravitational potential and the gravitational acceleration at the surface, whereas the second essentially ensures that the star is isolated from its surroundings.

As was first pointed out by Cowling (1941), the spectrum of oscillations separates into three classes of modes: the p -modes for which the frequency (at least for simplified models) tends to infinity as $|n|$ tends to infinity, the g -modes, for which the frequency tends to zero as $|n|$ tends to infinity, and an intermediate class of modes, which Cowling labelled the f -modes, and which for simple models have no node in the radial displacement. In the words of Cowling, for the p -modes "the motion, which is chiefly radial, is largely actuated by pressure variations"; for the g -modes "the motion is chiefly horizontal and is due to the action of gravity in attempting to smooth out the density differences on any sphere concentric with the star". Hence the p -modes are predominantly standing acoustic waves, whereas the g -modes are standing gravity waves. The nature of the f -modes is less well-defined at low or moderate degree; at high degree it essentially corresponds to a surface gravity wave. Conventionally, the p -modes are assigned positive values of n , the g -modes negative values and hence the f -mode has $n = 0$; with this definition the frequency is a monotonically increasing function of n , at given l .

The observed solar oscillations have frequencies between about 1 and about 7 mHz, corresponding to periods between 15 and 2 minutes (*e.g.* Libbrecht, Woodard & Kaufman 1990; Duval *et al.* 1991). At low or moderate degree such frequencies correspond to p -modes of relatively high radial order. Therefore, in the following I am mainly concerned with the properties of such modes.

2.2 Justification and problems of the Cowling approximation

To justify the neglect of the perturbation in the gravitational potential Cowling (1941) noted that "[i]f the mass is strongly concentrated near the centre of the star, as is generally the case, the variations in density do not produce great variations in the gravitational potential in the outer regions of the star. These regions have considerable influence on the values of [the frequency]. We therefore ignore $[\Phi']$ in deriving a first approximation to the periods." More generally, we may expect to the Cowling approximation to hold for modes of large order and degree. A rough argument for this can be obtained from Poisson's equation which after separation of variables becomes

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi'}{dr} \right) - \frac{l(l+1)}{r^2} \Phi' = -4\pi G \rho', \quad (2.2)$$

where ρ' is the amplitude of the Eulerian density perturbation. Estimating d^2/dr^2 by

$$\frac{d^2}{dr^2} \sim -\frac{n^2}{r^2};$$

we obtain the following order-of-magnitude estimate of Φ' :

$$\Phi' \sim -\frac{4\pi r^2 \rho'}{n^2 + l(l+1)}, \quad (2.3)$$

from which it follows that Φ' is small for modes of large order or degree. Hence, for such modes it may be justified to neglect the perturbation in the gravitational potential. This eliminates Poisson's equation from the system of equations and hence reduces its order to two, substantially simplifying the asymptotic analysis of the equations. (For Cowling the resulting reduction in the work required in a numerical solution of the equations was evidently of great importance; needless to say, this is no longer the case). Cowling (1941) also derived a perturbation expression for the correction to the frequencies arising from the effects of the gravitational potential.

The arguments given above may make plausible the validity of the Cowling approximation. Also, it is found numerically that in the limit of high order or degree frequencies computed in the Cowling approximation approaches those obtained from the full set of equations, both for polytropic models (Robe 1968) and for realistic solar models (e.g. Christensen-Dalsgaard 1984a; Kosovichev 1988; Vorontsov 1989; Christensen-Dalsgaard & Berthomieu 1991). Nonetheless, some care is warranted. By reducing the order of the system, two of the four independent solutions of the original set of equations, disregarding the boundary conditions, are eliminated. From asymptotic analysis of the full equations it is found that the equations can be decoupled, both in the limit of high and low frequencies, into two second-order equations. One of these corresponds to the Cowling approximation; as pointed out by Cowling (1941) in both limits it is of the Sturm-Liouville type, with solutions that oscillate rapidly as functions of r . The second equation has solutions which are slowly varying functions of r . The complete solution to the boundary value problem is then obtained as a linear combination of these individual solutions, the weight given to each depending on the boundary conditions. Validity of the Cowling approximation requires that the slowly varying part of the solution makes a negligible contribution.

In a few special cases this has been found not to be the case. Goossens & Smeysers (1974) considered g -modes of a composite polytropic model, consisting of two stably stratified regions separated by an unstably stratified region. They found examples of eigenfunctions for high-order modes where the slowly varying part of the solution dominated in parts of the models, although only in regions where the amplitude of the eigenfunction was small. Thus, although this type of behaviour complicates the assignment of the radial order of the modes according to the number of nodes in the radial displacement, it probably had little effect on the computed frequencies.

A related problem concerns the f -mode with $l = 1$. In the Cowling approximation there is a mode with $l = 1$ having no nodes in the radial displacement, intermediate in frequency between the p - and the g -modes; this must be identified with the f -mode. In the full case there is also a solution to the equations with no nodes in the radial displacement, but this has zero frequency, and corresponds to an infinitely slow uniform translation of the model (see also Christensen-Dalsgaard 1976); this is the limit of the slowly varying part of the

solution as the frequency tends to zero. The connection between these two cases can be investigated by making a continuous transition from the Cowling approximation to the full set of equations; this can be accomplished formally by introducing a factor λ on the right-hand side of equation (2.2), such that $\lambda = 0$ corresponds to the Cowling approximation and $\lambda = 1$ to the full equations (Christensen-Dalsgaard 1978). When λ is increased from 0 to 1, the transition from the dipolar f -mode in the Cowling approximation to the zero-frequency "mode" in the full case takes place through a sequence of avoided crossings with the g -modes, where the frequencies approach very closely without actually crossing; in the branch of the frequencies tending towards the zero-frequency mode the slowly varying part of the solution dominates. A similar transition occurs for solutions of the full equations, between the f -mode with $l = 2$ and the zero-frequency mode with $l = 1$, as l is decreased continuously from 2 to 1^(*) (Aizenman, Smeysers & Weigert 1977; Christensen-Dalsgaard 1978).

The classification of the modes, i.e., the assignment of radial orders to them, is simplified in the Cowling approximation. Eckart (1960) proposed a general, and mathematically precise, scheme for the classification of waves in a stratified medium. It was applied by Scuflaire (1974) and Osaki (1975) to the definition of the radial order n for a nonradially pulsating star, on the basis of the behaviour with varying r of the point $(\xi(r), p(r))$ in the so-called phase diagram. It may be shown (Gabriel & Scuflaire 1979; Christensen-Dalsgaard 1980) that in the Cowling approximation this definition has the desirable property of being invariant under a continuous modification of the equilibrium model or a continuous change of l . The Eckart scheme is not generally applicable, however, when solutions of the full equations are considered. The results of Goossens & Smeysers (1974) which were discussed above provide an example of the problems that may arise. Of more practical concern is the fact that in typical models of the present Sun application of this classification procedure to the lowest-order dipolar p -modes, computed from the full equations, results in a misidentification of the modes with $n = 1$ and 2; for more evolved models the problem extends to modes of higher order. Guenther (1991) considered an evolution sequence of $1M_{\odot}$ models extending to the base of the giant branch. In the final model he found it difficult to determine the order of a number of dipolar modes, of mixed g - and p -mode character; from his plots of the eigenfunctions it is evident that the origin of the problem is the presence in the solution of a significant component of the slowly varying solution, shifting the radial displacement sufficiently to eliminate several nodes. Problems were also encountered by Lee (1985) for dipolar modes in evolved models of δ Scuti stars. Lee proposed an alternative classification scheme based on the behaviour of $(\delta\Phi(r), p'(r))$, $\delta\Phi$ being the Lagrangian perturbation in the gravitational potential; this scheme was successful

(*) Note that after separation of variables, l is just a parameter of the resulting system of ordinary differential equations; hence it is mathematically permissible to let l take on any real value.

for the models he considered, even where the original Eckart classification failed. However, I have found that for models of the present Sun problems remain. Thus there is still a need for a reliable classification scheme, valid for a wide range of models. On the other hand, for those modes that have been observed on the Sun neglect or inclusion of $\mathcal{G}$ does not change the overall structure of the spectrum, or the labelling of the modes according to the Eckart scheme.

2.3 Asymptotic properties of the frequencies

For the purpose of interpreting computed or observed solar oscillations, the most interesting feature is evidently the behaviour of the frequencies and their relation to solar structure. A great deal of insight can be obtained from asymptotic analysis of the oscillations. A rigorous derivation of the asymptotic expressions of the eigenfrequencies requires careful analysis of the oscillation equations, often carried out in the Cowling approximation (see, for example, Gough 1986a; Smeyers & Tassoul 1987; Unno *et al.* 1989); the effects of the perturbation in the gravitational potential can be included subsequently by means of a perturbation analysis. However, it is possible to understand the asymptotic behaviour of p -modes, including the dominant departure from the Cowling approximation, in terms of the propagation of sound waves (see also Gough 1991). Since the wavelength of high-order modes is small compared with the typical scale over which the equilibrium structure changes, the modes can be approximated locally by plane sound waves in a homogeneous medium. Including the effects of self-gravity these have the dispersion relation

$$\omega^2 = k^2 c^2 - 4\pi G\rho \quad (2.4)$$

(Jeans 1929; see, for example, Spitzer 1978 [section 13.2] or Kippenhahn & Weigert 1990 [section 26.1]), where $k^2 \equiv k_r^2 + k_h^2$ is the square of the total wavenumber, k_r and k_h being the magnitudes of the radial and horizontal components of the wave vector. For a wave corresponding to a mode of oscillation, k_h is given by equation (2.1); however, a more careful analysis of the asymptotic behaviour of low-degree modes shows that $(l+1)$ should be replaced by $L^2 \equiv (l+1/2)^2$. From equation (2.4) we therefore obtain

$$k_r^2 = \frac{\omega^2}{c^2} - \frac{L^2}{r^2} + \frac{4\pi G\rho}{c^2}. \quad (2.5)$$

Close to the surface, c is small and hence k_r is large. Here the wave propagates almost vertically. With increasing depth, c increases and k_r decreases, until the *turning point* $r = r_t$ is reached where $k_r = 0$ and the wave propagates horizontally. This is a point of total internal reflection; below r_t , $k_r^2 < 0$, and the wave decays exponentially. The reflection of the wave at the surface is not immediately contained in this simple description; however, it is at least plausible that it results from the steep density gradient. A mode of oscillation is formed through interference between such waves propagating between the surface and r_t .

From this description of the p -modes an asymptotic dispersion relation for their frequencies can be derived. To obtain a standing wave, the change in radial phase between the lower turning point and the surface must be an integral multiple of π , apart from contributions describing the phase change at the turning points. This condition can be written

$$\omega \int_{r_t}^R \left(1 - \frac{L^2 c^2}{\omega^2 r^2} + \frac{4\pi G\rho}{\omega^2} \right)^{1/2} \frac{dr}{c} \approx \pi(n + \alpha). \quad (2.6)$$

Here α contains the phase change at the inner turning point and at the surface; a more careful analysis shows that α is in general a function of ω but not of l . Furthermore, it is found that $\alpha(\omega)$ depends on, and hence may be used to probe, the structure of the outermost parts of the star (*e.g.* Christensen-Dalsgaard & Pérez Hernández 1988, 1991a, 1991b; Brodsky & Vorontsov 1989; Vorontsov, Baturin & Pamyatnykh 1990; Baturin, Kononovich & Mironova 1991; Marchenkov & Vorontsov 1991).

The last term in the bracket in equation (2.6) arises from the effect of the perturbation in the gravitational potential. Since this term is generally small, we can expand the bracket, obtaining

$$\frac{\pi(n + \alpha)}{\omega} \approx \int_{r_t}^R \left(1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{1/2} \frac{dr}{c} + \frac{2\pi G}{\omega^2} \int_{r_t}^R \rho \left(1 - \frac{L^2 c^2}{\omega^2 r^2} \right)^{-1/2} \frac{dr}{c}, \quad (2.7)$$

where r_t is defined by the vanishing of the bracket in the first term on the right hand side, i.e. by

$$\frac{c(r_t)}{r_t} = \frac{\omega}{L}. \quad (2.8)$$

As r_t is a function of ω/L , equation (2.7) may be written as

$$\frac{\pi(n + \alpha)}{\omega} = F\left(\frac{\omega}{L}\right) + \frac{1}{\omega^2} F_\Phi\left(\frac{\omega}{L}\right), \quad (2.9)$$

where the functions $F(\omega)$ and $F_\Phi(\omega)$ are defined by equation (2.7). Equations (2.7) and (2.9), without the last terms on the right hand sides, have been commonly used in the analysis of the asymptotic properties of solar p -modes (see, for example, Christensen-Dalsgaard *et al.* 1985; Christensen-Dalsgaard, Gough & Thompson 1988, 1989, 1991; Shibahashi & Seki 1988; Kosovichev 1988; Vorontsov 1989; Vorontsov & Shibahashi 1990). That a relation of this form is satisfied was first found by Duvall (1982) for observed frequencies. Vorontsov (1989) derived the correction term arising from the perturbation in the gravitational potential from a detailed asymptotic analysis, using Cowling's (1941) expression for the frequency correction; a similar result, but valid only for low-degree modes, was obtained by Tassoul (1990). It should be pointed out also that a more careful analysis shows the need to include additional correction terms in equation (2.9), arising partly from the effects of buoyancy, and of the same functional form as the term in F_Φ (*e.g.* Gough 1986b; Vorontsov 1989, 1990). Higher-order corrections arising from the perturbation in the gravitational potential have been derived by Dziembowski & Gough (in preparation; see Gough 1991).

From equation (2.7) we can derive an approximate expression for the difference $\delta\omega^{(\Phi)} = \omega^{(F)} - \omega^{(C)}$ between the frequency $\omega^{(F)}$ obtained taking the perturbation in the gravitational potential into account, and the frequency $\omega^{(C)}$ obtained in the Cowling approximation. By expanding equation (2.7) in $\delta\omega^{(\Phi)}$, including only the linear term, and neglecting a small term arising from the frequency-dependence of α , one obtains

$$\delta\omega^{(\Phi)} \approx -\frac{1}{\omega} \frac{2\pi G \int_{r_1}^R \rho \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} dr}{\int_{r_1}^R \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{-1/2} \frac{dr}{c}}. \quad (2.10)$$

It is of interest to compare this asymptotic expression with differences between frequencies computed in the Cowling approximation and with the full equations (see also Vorontsov 1989). It follows from equation (2.10) that asymptotically $\omega \delta\omega^{(\Phi)}$ is a function of ω/L alone. Consequently, in Figure 1 I illustrate $\nu \delta\nu^{(\Phi)}$, normalized to $\nu = 3000 \mu\text{Hz}$, as a function of ν/L ; here $\nu = \omega/2\pi$ is the cyclic frequency. It is evident from the figure that except for $l = 0$, equation (2.10) provides a reasonable approximation to the computed differences. Also, it should be noted that for modes penetrating to the core of the Sun, the perturbation of the gravitational potential has a substantial effect on the frequencies, with decreasing ν/L and hence increasing τ , the effect decreases rapidly due to the decrease of ρ with increasing r .

3. FREQUENCY SEPARATIONS BETWEEN LOW-DEGREE MODES

In the following I concentrate on the properties of low-degree modes which according to the analysis in section 2.3 are those that penetrate most closely to the solar centre. An approximate expression for the frequencies of such modes can be obtained in the Cowling approximation from asymptotic analysis of the oscillation equations (Tassoul 1980, 1990), or by expanding equation (2.7), neglecting the last term (Gough 1986a); one finds that the cyclic frequency ν_{nl} of a mode of degree l and radial order n satisfies

$$\nu_{nl} \approx \left(n + \frac{l}{2} + \alpha\right) \Delta\nu - (AL^2 - \delta) \frac{\Delta\nu^2}{\nu_{nl}}. \quad (3.1)$$

Here

$$\Delta\nu = \left(2 \int_0^R \frac{dr}{c}\right)^{-1}, \quad (3.2)$$

and

$$A = \frac{1}{4\pi^2 \Delta\nu} \left[\frac{c(R)}{R} - \int_0^R \frac{dc}{dr} \frac{dr}{r} \right]; \quad (3.3)$$

δ is quantity which is independent of l and is related predominantly to the structure of the outer parts of the model. A detailed discussion of equation (3.1) was given by Gough & Novotny (1990). Gabriel (1989) analyzed further corrections to the relation, including the effects of departures from the Cowling approximation, and compared it with computed frequencies of solar oscillation.

From equation (3.1) it follows that to leading asymptotic order the frequencies of the modes are uniformly spaced in n , with a spacing that is the inverse of the sound travel time across a solar diameter; also, modes with the same value of $n + l/2$ coincide in frequency. The second term in equation (3.1) lifts this accidental degeneracy, giving rise to a frequency separation

$$\delta\nu_{nl} \equiv \nu_{nl} - \nu_{n-1, l+2} \approx -(4l+6) \frac{\Delta\nu}{4\pi^2 \nu_{nl}} \int_0^R \frac{dc}{dr} \frac{dr}{r}, \quad (3.4)$$

where I neglected the small term in the surface sound speed $c(R)$. Equation (3.4) shows that the l -dependence of $\delta\nu_{nl}$ is contained in the factor $4l+6$. Hence in the following I shall often consider the scaled separation

$$d_{nl} \equiv \frac{3}{2l+3} \delta\nu_{nl}, \quad (3.5)$$

thus effectively reducing the separation to $l = 0$. It follows from equation (3.4) that asymptotically d_{nl} may be written as

$$d_{nl} \approx \frac{1}{\nu_{nl}} D_{as}(R); \quad (3.6)$$

for convenience I introduced

$$D_{as}(r) = \int_0^r C_{as}(r') dr', \quad (3.7)$$

where

$$C_{as}(r) = -\frac{3\Delta\nu}{2\pi^2 r} \frac{dc}{dr}. \quad (3.8)$$

Equations (3.4) and (3.6) also show that in this approximation $\delta\nu_{nl}$ and d_{nl} decrease with increasing ν as ν^{-1} .

Because of the weighting with r^{-1} in equation (3.8) the core of the model makes a substantial contribution to d_{nl} . To illustrate this, Figure 2 shows $\nu_{ref}^{-1} C_{as}(r)$ and $\nu_{ref}^{-1} D_{as}(r)$, where a reference frequency $\nu_{ref} = 3000 \mu\text{Hz}$ was used. Results are shown for a zero-age main sequence model, a model of the present Sun, and three models at later evolutionary stages, including a model where hydrogen has been nearly exhausted at the centre. This sequence uses the same physics as, and includes, the model of the present Sun listed as Model 6 in Table 2 (cf. section 4.1). Table 1 lists some parameters of the models, together with the values of $\nu_{ref}^{-1} D_{as}(R)$. The change in $C_{as}(r)$ with evolution can be understood by noting that, approximating the equation of state by the ideal gas law, the sound speed is given by

$$c^2 = \frac{\gamma p}{\rho} \approx \frac{\gamma k_B T}{\mu m_u}, \quad (3.9)$$

where p is pressure, $\gamma \approx 5/3$ is the adiabatic exponent, k_B is Boltzmann's constant, m_a is the atomic mass unit and μ is the mean molecular weight. In the ZAMS model, μ is constant, and c is everywhere a decreasing function of r ; hence C_{as} is positive throughout the Sun. In the evolved models hydrogen has been preferentially consumed, and hence the mean molecular weight increased, nearer the centre; this gives rise to a decrease of μ with increasing r in the core of the model. Near the centre this effect is stronger than the decrease of T , leading to an increase in c with r and hence to a region where $C_{as}(r)$ is negative. This gives a negative contribution to d_{nl} , which increases with the age of the model; hence it causes a decrease of the asymptotic value of d_{nl} with increasing age, as seen in Figure 2b and Table 1. In the last model the presence of a small central region almost devoid of hydrogen leads to the gradient of μ , and hence $C_{as}(r)$, becoming relatively small very close to the centre.

Figure 2b also shows that d_{nl} is not solely determined by the structure of the nuclear-burning core; as was emphasized by Gabriel (1989) the rest of the star makes a significant contribution. Nevertheless, the asymptotic results suggest that d_{nl} should be sensitive to the properties of the core. This is confirmed by the numerical results presented below.

It is clear that the asymptotic analysis must be regarded with some caution. The derivation of equation (3.4) assumed the Cowling approximation; yet, Figure 1 shows that the frequency correction $\delta\nu(\Phi)$ depends strongly on degree at given frequency, and this evidently affects d_{nl} . Furthermore, given the behaviour of $C_{as}(r)$ illustrated in Figure 2a for evolved models, the validity of the asymptotic analysis is questionable; asymptotic theory requires that the equilibrium structure changes on a scale that is large compared with the wavelength of the oscillations, and this is certainly not satisfied near the centre of the evolved models. Figure 3 illustrates the behaviour of d_{nl} as obtained from computed frequencies for some of the models considered in Figure 2. Panel a) is for the ZAMS model, the frequencies being computed in the Cowling approximation. As might have been anticipated from the smooth behaviour of C_{as} in Figure 2, in this case the agreement between the asymptotic and the numerical values of d_{nl} is reasonable, although some differences remain; also, it should be noted that unlike the asymptotic result, the computed d_{nl} shows some dependence on l . The remaining panels are based on frequencies computed with the full equations. Comparison of panel b), which is also for the ZAMS model, with panel a) shows the effect of removing the Cowling approximation; since this leads to a decrease in the frequency which is larger at lower degree (cf. equation (2.10) and Figure 1) d_{nl} is decreased, substantially increasing the discrepancy between the asymptotic and the numerical results. The remaining panels show that the numerically computed d_{nl} also decrease with increasing age, although the effect is less strong than for the asymptotic values. This discrepancy is probably caused by increasing departures from the asymptotic behaviour induced by the rapid variation of $C_{as}(r)$ in the core; this is also evident in the increasing spread with l in d_{nl} . In the model of the present Sun it appears that the errors introduced by the Cowling approximation and by the failure of the

asymptotic description largely cancel, leading to a fortuitous near agreement between the numerical values of d_{nl} for $l = 0$ and 1 and the asymptotic value, near the peak at $\nu = 3000 \mu\text{Hz}$ in the observed spectrum of solar oscillations.

The leading-order effect of the perturbation in the gravitational potential could be included by introducing in equation (3.1) the correction term obtained in equation (2.10). However, Figure 1 shows that this term substantially overestimates the correction for $l = 0$ and hence would lead to erroneous results for the separation d_{nl} ; here, a higher-order asymptotic theory for the effect of Φ' is required. For higher values of l , equation (2.10) would give corrections to the asymptotic d_{nl} which are at least indicative, although possibly not of sufficient accuracy to permit a meaningful comparison with observations. In any case, it is not clear that an asymptotic description can be found which would deal adequately with the rapid variation in C_{as} near the centre.

For comparisons with observations, it is convenient to parameterize the computed d_{nl} in terms of fits with a small number of parameters to the asymptotic form. Commonly used has been a polynomial fit in both n and l to the frequencies of low-degree modes, resulting in the determination of a measure $\Delta\nu_0$ of the average frequency separation between adjacent modes for $l = 0$, and a quantity D_0 defined such that $6D_0$ measures an average $\langle d_{nl} \rangle$ over n and l of d_{nl} (for details, see for example Scherrer *et al.* 1983; Christensen-Dalsgaard 1988). More recently, Elsworth *et al.* (1990) proposed a more detailed description of the behaviour of d_{nl} , in terms of a linear fit of the form

$$d_{nl} = \bar{d}_l + s_l(n - n_1), \quad (3.10)$$

where n_1 is a suitable reference value of the order n . It might be noted that had the asymptotic relation (3.6) been satisfied, it would follow from equation (3.1) taken to leading order that

$$s_l \approx \frac{\bar{d}_l}{n + 1/2 + 1/4 + \alpha}. \quad (3.11)$$

Values of $\bar{d}_l$ and s_l , obtained by fitting the computed values of d_{nl} to equation (3.10), are listed in Table 1. As was also apparent from Figure 3, $\bar{d}_l$ decreases with increasing l . This has led to the suggestion that the frequency separations between low-degree modes, which in principle can be detected in stars other than the Sun, might be used to determine the evolutionary state of solar-like stars (*e.g.* Christensen-Dalsgaard 1984b, 1988; Ulrich 1986).

4. RESULTS ON FREQUENCY SEPARATIONS

Given that the frequency separations between low-degree modes, measured by d_{nl} , are sensitive to the structure of the solar core, it is of obvious interest to compare observed values

with values computed for solar models. Gelly *et al.* (1988) made a precise estimate of the average frequency separation and its variation with frequency by means of a cross-correlation technique; hence they could determine $6D_0$ and a corresponding average $\langle s \rangle$ of the slope s . More recently, Elsworth *et al.* (1990, 1991) obtained accurate measurements of frequencies of low-degree solar five-minute oscillations, which in particular permitted the determination of d_n .

Early comparisons with observations (see, for example, Faulkner, Gough & Vahia 1986; Däppen, Gilliland & Christensen-Dalsgaard 1986) indicated that normal solar models predicted a frequency separation somewhat higher than observed; this seemed to favour models where the structure of the core was modified by the inclusion of energy transport through the motion of so-called WIMPs (Weakly Interacting Massive Particles). Such models were proposed in an attempt to solve the solar neutrino problem: by contributing to the energy transport the WIMPs would reduce the temperature gradient in the solar core, and therefore the central temperature and the neutrino flux. Later work has shown that this apparent difficulty for normal solar models was caused by numerical inaccuracies in the calculation of the solar models and their frequencies; thus there appears to be good overall agreement between the observed separations and those computed for normal solar models (e.g. Cox, Guzik & Kidman 1989; Elsworth *et al.* 1990; Christensen-Dalsgaard & Berthomieu 1991; Guenther 1991).

Here I illustrate the dependence of d_n on the physics of solar models by considering a number of normal models computed with different assumptions about the physics; further detail about the model calculations was given by Christensen-Dalsgaard (1991a). In addition I consider examples of various non-standard models, including models designed to obtain a lower neutrino flux.

4.1 Solar models

All models considered here have been adjusted to have solar luminosity and radius (assumed to be $3.846 \times 10^{33} \text{ erg sec}^{-1}$ and $6.9599 \times 10^9 \text{ cm}$, respectively) at the age of the present Sun, by varying the initial helium abundance Y_0 and the ratio α of mixing length to pressure scale height. Except where otherwise noted the age of the Sun was assumed to be 4.75×10^9 years. Frequencies of the models were computed in the adiabatic approximation. The computational methods were described in some detail by Christensen-Dalsgaard (1982, 1991a) and Christensen-Dalsgaard & Berthomieu (1991). Care has been taken to ensure that the results are essentially unaffected by numerical errors, at the precision with which they are quoted.

The models are summarized in Table 2. Most of those models were also included in Table 1 of Christensen-Dalsgaard (1991a); for convenience the same numbering of these models have been used (with the exception of Model 11 of Christensen-Dalsgaard 1991a, which is here labelled 21). Models 1 – 18 are “standard” solar models, distinguished by the equation of state, opacity and energy generation parameters used in the model computation. Three different formulations were used for the equation of state: EFF refers to the Eggleton, Faulkner & Flannery (1973) prescription, which provides a minimal description of partial ionization, including a rough but thermodynamically consistent treatment of “pressure ionization”; CEFF is an improved version of the EFF equation of state which includes the effects of Coulomb interactions; finally, MHD refers to the Mihalas, Hummer & Däppen formulation (Hummer & Mihalas 1988; Mihalas, Däppen & Hummer 1988; Däppen *et al.* 1988) which includes an extensive treatment of the interactions in the gas. Several different opacity tables were used. CT refers to the Cox & Tabor (1976) tables, with a heavy element abundance of $Z = 0.02$. The remaining tables were all based on the Los Alamos Opacity Library (Huebner *et al.* 1977). G is the Geneva implementation (Lebreton & Maeder 1986), again with $Z = 0.02$. S1 – S5 are tables set up at Saclay with different heavy element abundances and mixtures (Courtraud *et al.* 1990; Turck-Chièze 1990); S1 – S4 were based on the Anders & Grevesse (1989) relative mixture of heavy elements, S1 (with $Z = 0.01934$) and S2 (with $Z = 0.02$) using the photospheric iron abundance, and S3 (with $Z = 0.01917$) and S4 (with $Z = 0.02$) using the meteoritic iron abundance, whereas S5 used the Ross & Aller (1976) mixture of heavy elements, with $Z = 0.0198$. Finally, the energy generation was either computed with parameters from Fowler, Caughlan & Zimmerman (1975; FCZ), or from Parker (1986), supplemented by other, more recent values from Parker (P86).

The range included here is probably representative of the different types of physics that have been used in computations of standard solar models. Therefore, in a loose sense the spread in the computed values gives an indication of the theoretical uncertainty resulting from the uncertainty in the physics. It must be kept in mind, however, that further significant modifications may be required, particularly to the opacity; thus Iglesias & Rogers (1991) found fairly substantial increases in the opacity over the LAOL values which have been used here.

The remaining models in Table 2 illustrate the effects of modifying the standard assumptions. Models 21 and 22 show the result of changing the assumed age; the age of Model 22 was chosen to obtain an average separation $\overline{d_0}$ approximately consistent with the observed value (see below). Models 23 – 28 were computed with reduced core opacity to simulate the effect of WIMPs, as described by Christensen-Dalsgaard (1991b). Here the opacity κ was changed by subtracting from $\log \kappa$, where $\log$ is logarithm to base 10, a function of $\log T$ localized to $\log T \gtrsim 7$ and characterized by the maximum reduction A of $\log \kappa$. Two types of models were considered: “static” models, where the hydrogen abundance X was obtained by scaling by a constant amount the abundance profile, at given mass coordinate, of a reference model (here

Model 7); and full evolutionary models. Finally Models 29 and 30 explore the effects of mixing in the core of the Sun. In Model 29 the hydrogen abundance is uniform (and again adjusted to get solar luminosity). Model 30 is based on the hydrogen profile in a model computed by Schatzman *et al.* (1981) with an assumed "turbulent diffusion" characterized by a so-called turbulent Reynolds number $Re^* = 100$.

Additional information about the models with modified core opacity or composition and the corresponding reference Model 7 is given in Table 3. For models 23 and 24, A was chosen to match the observed $\bar{d}_0$. Models 26 and 28 were computed to have approximately the observed ^{37}Cl neutrino flux. From the values given in Tables 2, 3 and 4 it may be verified that most of the overall properties of the models, including the frequency separations, are approximately linear in the age or the value of A over the ranges considered; also, the logarithm of the ^{37}Cl neutrino flux is roughly linear in A . In the completely mixed model the neutrino flux is also reduced to near the observed value, whereas the diffusive mixing reduces the flux by roughly a factor of two. A similar reduction factor was found by Schatzman *et al.* (1981); however, their value for the unmodified model was only 4.5 SNU, probably as a result of their using the Cox & Stewart (1970) opacities which are somewhat lower than the LAOI opacities used in Model 7.

4.2 Oscillation frequencies

Table 4 shows the values of the fitting coefficients $\bar{d}_l$ and s_l computed for the models in Table 2. The same information is illustrated in Figures 4, in a form first proposed by Elsworth *et al.* (1990).

Figure 4a shows results for standard models, and models that deviate only modestly from standard. Also shown are the fitting coefficients for the solar data obtained by Elsworth *et al.* (1990), as well as the average values inferred from the results of Gelly *et al.* (1988). It is evident that the computed coefficients show some dependence on l , in contrast to the prediction of asymptotic theory. Such departure from the asymptotic behaviour was discussed in section 3, and is also evident in Figure 3. Furthermore, the figure shows that within the framework of standard models variations in the physics have comparatively little effect on the frequency separations. A more careful analysis than can be attempted here would be required to understand the causes of the variations that do occur; however, it is clear that as analyzed here the frequency separations are not sensitive diagnostics of uncertainties in the standard physics of solar models. On the other hand, this property means that they are cleaner probes of departures from the standard assumptions. For example, a decrease of the age of the model by 2.5×10^8 years changes $\bar{d}_0$ and $\bar{d}_1$ by considerably more than the scatter amongst the points for given age (see also Gough & Novotny 1990).

The agreement between the models and the observations is generally reasonable good; in particular, for all standard models the slopes s_0 and s_1 are consistent with the observations, within their comparatively large error bars. On the other hand, the standard models show a small, but apparently significant, deviation from the observed $\bar{d}_0$ and $\bar{d}_1$. To investigate the significance of this deviation, I have computed models aimed at reproducing the observed value of $\bar{d}_0$. This can be accomplished by increasing the assumed age of the model to 5.05×10^9 years. This a value might be difficult to reconcile with meteoritic ages and traditional views of the formation of the solar system (Guenther 1989). However, other modifications that give rise to a similar increase in the mean molecular weight in the solar core would have a similar effect. One such possibility might be gravitational settling of helium which causes an increase of the central helium abundance Y_c by about 0.015 (for a detailed overview, see Proffitt & Michaud 1991). This is very close to the difference of 0.014 in Y_c between Models 6 and 22; however, since the change in Y resulting from settling extends over a somewhat larger region of the model than the change caused by increasing the age, the effect on the gradient in μ and hence on d_{nl} may be expected to be smaller (Cox *et al.* (1989) found a very small effect of diffusion on $\delta\nu_{n0}$; however, since they compared models of somewhat different ages, the significance of this result is difficult to judge). Alternatively, one might speculate that helium is not fully ionized in the core of the Sun, thus increasing the mean molecular weight relative to normal models where full ionization is imposed; it may be possible to obtain the observed $\bar{d}_0$ with an appropriate distribution of ionization. As a second example aimed at reproducing the observations, Models 23 and 24 in Figure 4a show that a decrease of the opacity of 8 – 11 per cent, localized to the core, suffices to bring theory and observation into agreement for $l = 0$. (On the other hand, the opacity differences between Models 4 and 6, which are of considerably larger magnitude but extend over the entire radiative interior, have very little effect on $\bar{d}_0$). It is interesting, although barely significant, that for the models where $\bar{d}_0$ agrees with observations a small difference remains for $\bar{d}_1$.

Effects of more substantial reductions of the core opacity, simulating the introduction of WIMPs into the model, and of partial core mixing are illustrated in Figure 4b. As pointed out by Faulkner *et al.* (1986) the overall effects on d_{nl} of these modifications can be understood in terms of the asymptotic expressions: the introduction of WIMPs reduces the central temperature without much effect on the composition profile, and hence accentuates the positive sound-speed gradient in the core of the model, leading to a decrease in d_{nl} ; in contrast, partial mixing reduces the gradient in the mean molecular weight and hence reduces or removes the region where C_{as} is negative, thereby increasing d_{nl} . It is evident from Figure 4b that for the models where the core opacity has been reduced sufficiently to bring the computed neutrino flux into agreement with observations $\bar{d}_1$ is inconsistent with the observed values. Thus, as pointed out by Cox, Guzik & Raby (1990) and Elsworth *et al.* (1990), it seems unlikely that an acceptable solution to the neutrino problem can be found by invoking

WIMPs, or other effects that cause similar reductions in the central temperature of the Sun. Similarly, partial mixing of the nature considered here, although probably insufficient to reduce the neutrino flux to the observed value, results in frequency separations that are greatly in excess of the observed values (see also, for example, Cox & Kidman 1984; Provost 1984; Christensen-Dalsgaard 1986).

5. DISCUSSION

Much of our understanding of the properties of nonradial oscillations is based on asymptotic analyses of the equations of adiabatic oscillations, in the Cowling approximation where the perturbation in the gravitational potential is neglected. Such studies were initiated by Cowling (1941) who elucidated the basic properties of the spectrum of oscillations. From the point of view of observations of solar oscillations, the most important modes are the p -modes, which are essentially standing acoustic waves; asymptotically, their frequencies are determined largely by the variation of the sound speed $c(r)$ with distance r from the centre of the star. For low-degree modes, which penetrate into the dense core of the Sun, the effects of the perturbation to gravity is not negligible, however; a straightforward extension of the analysis shows that for such modes the frequencies also depend directly, in the asymptotic approximation, on the density structure of the star.

While the second-order system of equations resulting in the Cowling approximation is well understood, relatively little is known about the general properties of the fourth-order system describing the oscillations when the perturbation in the gravitational potential is taken into account. A particular problem concerns the classification of the modes of oscillation, i.e., the assignment of their radial order. In the Cowling approximation a scheme exists which has the desirable property of yielding a classification which is invariant under changes to the properties of the underlying equilibrium model, e.g. as a result of stellar evolution. However, application of this scheme, or suggested modifications thereof, to solutions of the full equations leads to mis-identification of some modes with $l = 1$ in models of evolved main-sequence stars, including the Sun. Although this may appear to be a mathematical subtlety, the lack of a proper classification scheme complicates the computation of frequencies of a given model, making it difficult to ensure that all modes within a certain frequency range have been found.

Information about the properties of the solar core can be obtained from analysis of the frequency separation $\delta\nu_{nl} = \nu_{nl} - \nu_{n-1,l+2}$. In taking this difference, the somewhat uncertain influence on the frequencies of the outermost layers of the Sun largely cancels, leaving a relatively clean probe of the radiative interior. The results presented here show that $\delta\nu_{nl}$ is rather insensitive to the choice of physics in calculations of standard solar models. On the other hand, the separation is quite sensitive to changes in the assumed age of the model of

the present Sun, or to modifications aimed at reducing the neutrino flux. As examples of such modifications, I here considered models where the central temperature was reduced by simulating energy transport by WIMPs and models with substantial mixing in the core; these were found to be incompatible with the observed values of $\delta\nu_{nl}$. This clearly strengthens the case for solutions to the solar neutrino problem that do not involve changes to solar structure, such as the so-called MSW effect (Wolfenstein 1978; Mikheyev & Smirnov 1985; see also Bahcall & Bethe 1990). However, it should be mentioned that Gough, Kosovichev & Thompson (in preparation) showed that solar models can be obtained which are consistent with both the observed oscillation frequencies and the neutrino flux, although only by allowing drastic departures from the assumptions of normal solar model calculations.

Simple comparisons, such as those presented here, between observed frequencies or frequency differences and the corresponding computed values for solar models represent only the crudest method of helioseismic investigation of the solar interior. Much more extensive information can be obtained from inverse analyses of the frequencies, whereby properties of the solar interior, such as the sound speed and the density, can be determined as functions of position, overviews of these techniques and their results were given, for example, by Gough (1985) and Gough & Thompson (1991). A detailed summary of the results of such work is outside the scope of the present paper. However, as an example may be mentioned a recent analysis by Däppen *et al.* (1991), who carried out inversion of differences between observed frequencies and those of a reference solar model, to determine the corrections $\delta\rho$ and δu ($u = p/\rho$ being closely related to the sound speed) which should be applied to the model to approximate the Sun. Typical results are illustrated in Figure 5. Here the reference model was Model 13 of Table 2 in Christensen-Dalsgaard *et al.* (1991); this used similar physics to Model 16 of Table 2 of the present paper, except that the opacity was increased somewhat in the vicinity of the base of the convection zone, to increase the convection zone depth to 0.286 R . The figure shows that there are systematic differences between the Sun and the model but that these are generally fairly small. It is probable that the positive values of $\delta u/u$ around $r = 0.5R$ result from errors in the opacities used to compute the reference model. Further work is required to test whether all the discrepancies between the solar data and the model can be accounted for in terms of simple changes to the physics used in the model calculation.

In models of solar-like stars the frequency separation $\delta\nu_{nl}$ decreases strongly with increasing age during main-sequence evolution. This suggests that observational determination of this quantity for stars other than the Sun would provide an indication of the evolutionary state of the stars. Unfortunately, detection of such oscillations is greatly complicated by their expected small amplitude. Possible detection of solar-like oscillations in Procyon has been reported (Brown *et al.* 1991), although so far with no definite determination of individual frequencies. However, extensive efforts are being made towards observation of solar-like

oscillations from the ground (see, for example, Schmid *et al.* 1988; Innes *et al.* 1991). Furthermore, the EVRIS experiment on the MARS 94 probe (Baglin 1991) will make photometric observations of oscillation in a limited number of bright stars, and, if selected by the European Space Agency, the PRISMA project (Lemaire *et al.* 1991) will allow seismic investigation of a broad range of stars.

Even given observations of solar-like oscillations, it is evident that the analysis and interpretation of such data is far more difficult than in the solar case. A fundamental limitation is the absence, at least in the foreseeable future, of spatially resolved data; this restricts the detection to modes of low degree, such as those analyzed in section 4. Although the results presented there suggested that the frequency separation $\delta\nu_{nl}$ is relatively insensitive to uncertainties in the overall physics of the stellar models, the lack of precise information about the mass, radius and luminosity of stars other than the Sun limits the information that can be obtained from the observed frequencies (Gough 1987, 1990). Hence accurate measurements of such parameters are required, to supplement the asteroseismic investigations; and theoretical investigations of the diagnostic potential of observed frequencies, given these limitations, are urgently needed. However, against these difficulties must be set the potential for measuring properties of stars over a range of masses, ages and chemical compositions. Indeed, as in other branches of solar-stellar physics, there is complementarity between very detailed information for a single star, *viz.* the Sun, and limited information obtained for a substantial number of stars. In addition to the continually expanding seismic investigations of the solar interior, we may therefore hope to see substantial results of asteroseismology in the coming years. Thus there seems little doubt that the study of nonradial oscillations of the Sun and other stars, based on the work by Cowling, will continue to play an important role in astrophysics.

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Table 1

Age (10^9 years)	X_c	$\nu_{\text{ref}}^{-1} D_{\text{as}}(R)$ (μHz)	$l = 0$		$l = 1$	
			$\bar{d}_0$ (μHz)	s_0 (μHz)	$\bar{d}_1$ (μHz)	s_1 (μHz)
0.000	0.7031	24.80	17.809	-0.612	16.779	-0.526 (a)
0.000	0.7031	24.80	15.591	-0.381	14.940	-0.350
4.750	0.3354	10.24	9.280	-0.303	9.709	-0.255
6.254	0.2019	4.40	7.242	-0.322	7.884	-0.234
7.379	0.1000	0.11	5.828	-0.352	6.451	-0.223
8.696	0.0085	-0.87	4.574	-0.389	4.755	-0.216

Results for evolution sequence of $1M_{\odot}$ models, using the same physics as the calibrated Model 6 of the present Sun in Table 2, and including that model. X_c is the central hydrogen abundance, and $\nu_{\text{ref}}^{-1} D_{\text{as}}(R)$ is the asymptotic value of d_n at the reference frequency $\nu_{\text{ref}} = 3000 \mu\text{Hz}$ (cf. equation (3.6)). The fitting coefficients $\bar{d}_l$ and s_l were obtained from equation (3.10); the fit included modes with n in the range 15–27 and used $n_1 = 21$. The results in the first line, labelled (a), were obtained from frequencies computed in the Cowling approximation; in the remaining cases frequencies computed by solving the full set of equations were used.

Table 2

Model No.	Physics	Y_0	α	T_c (10^6K)	d_b/R	Notes
1	EFF	0.2509	1.6638	14.962	0.2790	
4	EFF	0.2496	1.6425	15.052	0.2767	
5	EFF	0.2848	2.3907	15.883	0.2635	
6	EFF	0.2776	2.5149	15.638	0.2696	
7	EFF	0.2806	2.5167	15.685	0.2712	
8	EFF	0.2709	2.5154	15.493	0.2671	
9	EFF	0.2821	2.4829	15.641	0.2653	
12	EFF	0.2496	1.6401	15.052	0.2769	(a)
13	EFF	0.2849	2.3834	15.683	0.2637	(a)
14	CEFF	0.2404	1.6840	15.009	0.2784	(a)
15	MHD	0.2393	1.6896	15.001	0.2794	(a)
16	MHD	0.2750	2.4602	15.623	0.2666	(a)
17	MHD	0.2706	2.5752	15.624	0.2745	(a)
18	MHD	0.2646	2.5776	15.488	0.2725	(a)
21	EFF	0.2800	2.4779	15.584	0.2667	(b)
22	EFF	0.2747	2.5598	15.703	0.2729	(c)
23	EFF	0.2772	2.5561	15.560	0.2732	(d)
24	EFF	0.2784	2.5426	15.499	0.2733	(e)
25	EFF	0.2698	2.6566	15.012	0.2814	(d)
26	EFF	0.2607	2.7947	14.410	0.2902	(d)
27	EFF	0.2720	2.6220	14.954	0.2790	(e)
28	EFF	0.2650	2.7200	14.313	0.2857	(e)
29	EFF	0.3329	1.8615	14.642	0.1959	(f)
30	EFF	0.2863	2.3546	15.131	0.2567	(g)

Summary of solar model calculations. All models have been calibrated to have solar radius and luminosity. The first column gives an identification number; for convenience common models have the same numbers as in Christensen-Dalsgaard (1991a). The three columns headed *Physics* indicate the equation of state, the opacity and the energy generation parameters, as defined in the text. Y_0 and α are the initial helium abundance and mixing length parameter required to calibrate the model. T_c is the central temperature, and d_b/R is the depth of the convection zone in units of the surface radius.

Notes: (a) obtaining the equation of state from interpolation in tables; (b) assuming the age of the present Sun to be 4.5×10^9 years (rather than the default value of 4.75×10^9 years); (c) assuming the age of the present Sun to be 5.05×10^9 years; (d) static model with reduced core opacity (cf. Table 3); (e) evolved model with reduced core opacity (cf. Table 3); (f) chemically homogeneous model; (g) hydrogen profile obtained from Schatzman *et al.* (1981).

Table 3

No.	Model name	A	Neutrinos (SNU) ³⁷ Cl ⁷¹ Ga
7	-	0.0	8.89 134.3
23	SW0	0.0353	7.91 130.2
24	EW0	0.0486	7.51 128.5
25	SW1	0.2	4.87 117.2
26	SW2	0.4	2.96 107.6
27	EW1	0.2	4.59 115.2
28	EW2	0.4	2.67 105.0
29	MX1	-	2.37 98.6
30	MX2	-	4.68 112.1

Properties of nonstandard models and the corresponding standard reference Model 7. In Models 23 – 28 the opacity κ in the core has been reduced; A is the maximum reduction in $\log \kappa$, $\log$ being the logarithm to base 10. Models 23, 25 and 26 are static models (see text), whereas models 24, 27 and 28 are based on complete evolution sequences. Model 29 is a chemically homogeneous model, whereas the hydrogen profile $X(q)$ in Model 30 ($q = m/M$ being the mass fraction) was obtained by scaling from the hydrogen profile in the model by Schatzman *et al.* (1981) with turbulent diffusive mixing with $Re^* = 100$. The neutrino fluxes for the ³⁷Cl and ⁷¹Ga experiments were calculated using cross sections from Bahcall (1989); the unit is *Solar Neutrino Units* (or SNU), defined to be 10^{-36} reactions per target atom per second.

Table 4

Model No.	$l = 0$		$l = 1$		$l = 2$		$l = 3$	
	$\bar{d}_0$ (μHz)	s_0 (μHz)	$\bar{d}_1$ (μHz)	s_1 (μHz)	$\bar{d}_2$ (μHz)	s_2 (μHz)	$\bar{d}_3$ (μHz)	s_3 (μHz)
1	9.145	-0.309	9.616	-0.263	10.497	-0.295	10.812	-0.272
4	9.233	-0.306	9.688	-0.260	10.572	-0.293	10.878	-0.270
5	9.143	-0.284	9.557	-0.246	10.401	-0.279	10.678	-0.263
6	9.280	-0.303	9.709	-0.255	10.512	-0.277	10.765	-0.259
7	9.250	-0.303	9.681	-0.255	10.493	-0.277	10.750	-0.258
8	9.235	-0.299	9.681	-0.253	10.491	-0.278	10.754	-0.260
9	9.155	-0.299	9.604	-0.253	10.438	-0.278	10.712	-0.260
12	9.232	-0.307	9.685	-0.260	10.571	-0.293	10.878	-0.271
13	9.144	-0.284	9.556	-0.246	10.400	-0.279	10.677	-0.263
14	9.249	-0.309	9.685	-0.262	10.554	-0.292	10.853	-0.270
15	9.303	-0.310	9.724	-0.264	10.580	-0.292	10.868	-0.269
16	9.196	-0.288	9.589	-0.250	10.401	-0.279	10.665	-0.262
17	9.304	-0.308	9.714	-0.259	10.496	-0.277	10.739	-0.257
18	9.249	-0.304	9.677	-0.257	10.467	-0.278	10.723	-0.259
21	9.508	-0.298	9.889	-0.252	10.615	-0.273	10.817	-0.255
22	9.010	-0.309	9.496	-0.258	10.392	-0.282	10.704	-0.263
23	8.971	-0.314	9.501	-0.259	10.384	-0.283	10.705	-0.264
24	8.990	-0.306	9.541	-0.257	10.388	-0.281	10.699	-0.265
25	7.833	-0.360	8.765	-0.275	9.970	-0.306	10.558	-0.288
26	6.635	-0.414	7.956	-0.295	9.550	-0.333	10.422	-0.312
27	8.222	-0.319	9.111	-0.262	10.085	-0.292	10.566	-0.283
28	7.331	-0.344	8.575	-0.271	9.748	-0.309	10.435	-0.303
29	14.180	-0.207	13.601	-0.206	12.843	-0.194	12.019	-0.178
30	12.434	-0.183	12.202	-0.184	11.793	-0.191	11.267	-0.189

Obs. 9.00 ± 0.06 -0.29 ± 0.03 9.36 ± 0.06 -0.28 ± 0.04

Results of linear least-squares fits to frequency separations d_n computed by solving the full oscillation equations for the models listed in Table 2. The fit was of the form given in equation (3.10) and included modes with n in the range 15 – 27, and taking $n_1 = 21$. The bottom row gives the results of a corresponding fit by Elsworth *et al.* (1990) to observed frequency separations.

Figure captions:

Figure 1. Scaled frequency correction $\delta\nu^{(q)}$ resulting from the effect of the perturbation in the gravitational potential, corresponding to a reference frequency $\nu_{ref} = 3000 \mu\text{Hz}$. The lower abscissa is ν/L , where ν is the cyclic frequency and $L = l + 1/2$. The upper abscissa is the asymptotic location of the lower turning point, which is related to ν/L through equation (2.8). Results are shown for Model 6 in Table 2. The heavy continuous curve shows the asymptotic correction given by equation (2.10). The thin curves are based on differences between frequencies computed from the full equations and in the Cowling approximation; a continuous line is used for $l = 0$, short dashes for $l = 1$ and long dashes for $l = 2 - 5$.

Figure 2. Integrands (panel a) and partial integrals (panel b) for the asymptotic frequency spacing d_{nl} at the reference frequency $\nu_{ref} = 3000 \mu\text{Hz}$ (cf. equations (3.6) – (3.8)). Results are shown for the models listed in Table 1, with the following line styles: age 0 (-----); age 4.75×10^9 years (—); age 6.254×10^9 years (— · — · — · —); age 7.379×10^9 years (— · — · — · —); age 8.696×10^9 years (.....).

Figure 3. Scaled frequency separations d_{nl} for some of the models listed in Table 1. The dotted curves show the asymptotic behaviour (cf. equation (3.6)). The remaining curves are based on computed frequencies and use the following line styles: $l = 0$ (—); $l = 1$ (-----); $l = 2$ (— · — · — · —); $l = 3$ (— · — · — · —). Panel a) shows results for frequencies computed in the Cowling approximation for the ZAMS model. The remaining panels are based on solutions to the full equations; panel b) is for the ZAMS model, panel c) for the model of the present Sun, and panel d) for the model of age 6.254×10^9 years.

Figure 4. Computed and observed coefficients $\bar{d}_l$ and s_l in the fit in equation (3.10) to the scaled frequency separation d_{nl} . The symbols show the computed values, which are also given in Table 4. The symbols are identified in the figure with the model number in that table, and with the physics used in the computation. In general the symbol type identifies the opacity table, whereas the line thickness denotes the equation of state. Models 21 and 22 are distinguished by having smaller and larger age, respectively, than the remaining models. Models 23 and 24, and in panel b) Models 25 – 28, were computed with reduced core opacity (cf. Table 3); in particular, Models 26 and 28 have approximately the observed flux of ^{37}Cl neutrinos. Model 30 used a hydrogen profile obtained by Schatzman *et al.* (1981) by assuming partial mixing by turbulent diffusion. For clarity, the results for the static and evolutionary models with reduced core opacity, and the model with partial mixing, have been connected to the corresponding point for the reference model by solid, dotted and dashed lines, respectively, in panel b).

The solid (for $l = 0$) and dashed ($l = 1$) error boxes show the observational results obtained by Elsworth *et al.* (1990), whereas the dotted error box indicates the results of Gelly *et al.* (1988); although the latter were obtained by a somewhat different technique, they correspond approximately to averages of $\bar{d}_0$, $\bar{d}_1$ and s_0 , s_1 , respectively.

Figure 5. Corrections to a reference solar model, obtained by inverting differences between the observed frequencies and the frequencies of the model. Panel a) shows corrections to $u = p/\rho$, where p is pressure and ρ is density, and panel b) shows corrections to ρ . The vertical bars indicate the errors in the results, based on the errors in the observed frequencies, whereas the horizontal bars provide a measure of the resolution in the inversion (from Däppen *et al.* 1991).

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.