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EXPLORING THE LIPATOV EQUATION

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Abstract

Properties of the Lipatov equation for the small x behaviour in deep inelastic scattering are explored. In particular, the effect of imposing upper and lower cutoffs on the transverse momentum integral in the equation is analysed. The lower cutoff, needed to avoid integration over momenta where the gluon propagators are nonperturbative, does not change the output power. Imposing both a lower and an upper cutoff, so as to take account of kinematic limitations at subasymptotic values of Q^2 , has an important effect even at HERA energies. This will make it difficult to separate the perturbative pomeron from the nonperturbative one seen already in soft hadronic collisions.

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1. Introduction

For three decades, we have been accustomed to the idea that many high-energy strong interaction scattering processes are dominated by pomeron exchange. In recent years, it has become apparent^[1] that the phenomenology of the pomeron can be described rather simply: to a good approximation, at accessible energies, it is a Regge pole, with trajectory having slope close to 0.25 GeV^{-2} and intercept close to 1.08. This pomeron is relevant for soft hadronic processes, such as the total cross section.

The pomeron that has been revealed by experiments so far cannot be generated by QCD perturbation theory: it is a long-distance effect and therefore nonperturbative. In recent years theorists have turned their attention to another pomeron, the perturbative pomeron. This is generated by the Lipatov equation^[2], which has its origins in perturbative QCD. The hope is that the perturbative pomeron will appear in data where semihard interactions are important. Hard enough, that is, for perturbation theory to be applicable, but yet with soft characteristics so that pomeron exchange is relevant. Typical of such processes are deep inelastic scattering when $x \ll 1$, and heavy flavor production when $\Lambda \ll M \ll \sqrt{s}$. At presently accessible energies, the soft pomeron is modified relatively little by unitarity corrections. But the Lipatov pomeron has its intercept considerably above unity, and so it can readily cause a violation of the Froissart bound, unless modified by unitarity corrections. For the Lipatov pomeron, these are certainly of crucial importance in soft processes, but in hard scattering unitarity corrections need not be dominant.^[3]

In the case of soft scattering: e.g., elastic hadron-hadron scattering at low t , or inelastic lepton scattering at low Q^2 , no trace of the Lipatov pomeron has been found in data^[1]. It is only in hard scattering that we can expect the Lipatov pomeron to be directly accessible, for example, in deep inelastic lepton scattering at small z . Perturbative QCD predicts that the structure functions rise at small z in a manner quite unlike that given by the soft pomeron. Typical fits to parton distributions^[4] have a power law substantially steeper than the $1/z^{1.08}$ that corresponds to the soft pomeron. Much of the steep small z rise in the presently accessible kinematic range is associated with the leading order evolution of the parton distributions to higher Q^2 , the physics of which overlaps with but does not include the whole of the Lipatov equation.

The properties of the equation turn out to be quite subtle. For these reasons, it seems worthwhile exploring them in simple terms.

One complication in treating the Lipatov equation is that it involves integrations over regions of momentum where perturbative calculations are certainly inappropriate. So some modification to the equation is necessary. Collins and Ellis^[5] have shown how to modify the equation to subtract out the non-perturbative region, in just the same fashion as in done in conventional hard-scattering calculations. Their formalism can be readily included in the conventional hard scattering formalism, so that a single calculation suffices for both the conventional hard scattering regime and the small z hard pomeron regime.

In this paper we will follow the more usual approach of applying an ad hoc infrared cutoff on the transverse momentum integrals. Since the standard analytic solution of the Lipatov equation depends on its conformal invariance (which is true only in the absence of cutoffs), it is not manifest which of the properties of the solution are preserved in the presence of a cutoff. In any event, the imposition of an infrared cutoff is a model, for we certainly do not know how to calculate nonperturbative effects correctly. Moreover, the Lipatov equation, as currently formulated, only includes a leading logarithmic approximation to perturbation theory, so merely to replace the gluon propagator by some more complete propagator that comes from a confining solution to QCD would be incorrect. The standard method is to assume that the bulk of the nonperturbative effects can be included in the cutoff perturbative Lipatov equation, by an appropriate choice of the driving term T_0 . This is supposed to be essentially the nonperturbative structure function corresponding to the old parton model. We know of no good comprehensible proof of these assertions.

We will also investigate the effect of incorporating an ultraviolet cutoff in the Lipatov equation. The simple solution of the equation, as we have said, relies on conformal invariance, so it appears to rely on the possibility of both infinitely large and infinitely small momenta. But physically there is a

kinematic cutoff at large momenta. Asymptotically, at large s , the ultraviolet cutoff goes away, so the cutoff will mimic the effects of nonasymptotic corrections.

To be able to investigate analytically the effects of cutoffs, we will replace the Lipatov kernel by a simpler formula that has the essential analytic features. We find that the imposition of an infrared cutoff modifies the exact form of the solution, and the nature of the j plane singularity, but does not affect the position of the j plane singularity. Imposing both an ultraviolet cutoff and an infrared cutoff changes the position and nature of the singularity, but leaves it above unity, as we will see.

When the ultraviolet cutoff is taken to infinity, the power does indeed approach the value obtained by solving the equation without a cutoff, but the approach to asymptopia is rather slow. Thus in practical applications at accessible energies, one should not expect the solution to the simple Lipatov equation to be a good approximation to the actual result. This supports results found numerically by Collins and Ellis^[5] with their modified Lipatov equation.

2. Lipatov equation

In appropriate physical gauges, the Lipatov equation represents a sum of ladder diagrams. Figure 1 shows the case of deep-inelastic scattering. At small z , the diagram with n gluon rungs is

$$F_2^{(n)}(x, Q^2) = \int_x^1 \frac{dx'}{x'} \int \frac{dk_T^2}{k_T^2} T_n(x', k_T^2) C(z/x', k_T, Q^2), \quad (1)$$

where the function C corresponds to the upper quark loop in figure 1 and

$$T_n(z, k_T^2) = \int_z^1 \frac{dx'}{x'} \int dk_T'^2 K(k_T^2, k_T'^2) T_{n-1}(x', k_T'^2), \quad (2)$$

with

$$\int du K(z, u) t(u) = \frac{3\alpha_S}{\pi} z \int \frac{du}{u} \left[\frac{t(u) - t(z)}{|u - z|} + \frac{t(z)}{\sqrt{z^2 + 4u^2}} \right]. \quad (3)$$

In (1), (2) and (3) the kinematics restricts the integrated squared transverse momentum k_T^2 or $k_T'^2 = u$ to be less than $Q^2(x/z - 1)$. However, in (1) the properties of the upper quark loop function C effectively reduce this upper^[6] limit to Q^2 because C becomes small when $z/x' \ll 1$. In the leading logarithm approximation, the values of the x' variables for neighboring rungs differ by a large factor, so in (2) and (3) to first approximation one can set the upper limit to infinity. (Corrections to this can be absorbed in the higher order corrections to the kernel.) To begin with, we shall follow this procedure, and set the upper limit of the $k_T'^2 = u$ integration in (2) and (3) to infinity. Strictly, in (3) the coupling α_S should run, but this is difficult to implement in an analytic calculation, and so we shall adopt the usual course of fixing it. After we have imposed an infrared cutoff on the transverse momentum integrations, this would not seem to be a bad approximation.

Equation (2) is a recurrence relation. One has to input a driving term T_0 ; the equation then gives $T_1, T_2, \dots$. The sum T of the T_n 's satisfies

$$T(z, k_T^2) = T_0(z, k_T^2) + \int_z^1 \frac{dx'}{x'} \int dk_T'^2 K(k_T^2, k_T'^2) T(x', k_T'^2), \quad (4)$$

if the sum converges.

As we have said, we suppose that T_0 , the bottom of the ladder in figure 1, is a nonperturbative structure function. It describes only events in which no parton is produced with transverse momentum greater than some value Q_0 , which is usually taken to be a few GeV, or less. A lower cut-off Q_0^2 is imposed on the k_T^2 integration in the Lipatov equation (2); this equation then corrects the nonperturbative input T_0 so as to take account of events in which one or more parton is produced having $k_T^2 > Q_0^2$. $T_0(z, k_T^2)$ is nonperturbative, which means that its small- z behaviour is controlled by the nonperturbative pomeron and its large- k_T^2 behaviour is soft. So we take

$$T_0(z, z) = z^{-\epsilon} (1 - z)^{\epsilon} t_0(z), \quad (5)$$

with $\epsilon \approx 0.08$ and where $t_0(z)$ goes to zero fairly rapidly at large k_T^2 .

3. Solution without cutoff

Let us begin by reviewing the analysis^[2,7] of the Lipatov equation without a cut-off. The eigenfunctions of the kernel $K(z, u)$ are powers $z^{\frac{1}{2} + \omega}$, with eigenvalues

$$K(\omega) = \frac{3\alpha_S}{\pi} [-2\gamma - \psi(\frac{1}{2} + \omega) - \psi(4 - \omega)], \quad (6)$$

where γ is Euler's constant and ψ is the logarithmic derivative of the gamma function. It is important to note that in order for the integral for K acting on an eigenfunction to be finite, ω must lie in the range

$$|\operatorname{Re} \omega| < \frac{1}{2}. \quad (7)$$

Because the eigenfunctions are powers, it is useful to use Mellin-transform techniques and write

$$T_n(z, z) = \int \frac{d\omega}{2\pi i} z^{\frac{1}{2} + \omega} T_n(z, \omega), \quad (8)$$

where, as in (3), we use z to denote k_T^2 . The inverse transform is

$$T_n(z, \omega) = \int_0^\infty dz z^{-\omega - \frac{1}{2}} T_n(z, z). \quad (9)$$

In (8), the integration is along a line $\operatorname{Re} \omega = \omega_R$ in the complex ω plane. The value of ω_R must be chosen such that the z integration in (9) converges. This depends on the behaviour of $T_n(z, z)$ both at large z and also at small z . In particular if, as $z \rightarrow 0$, $T_n(z, z) \sim z^\delta$ then $\omega_R < -\frac{1}{2} + \delta$. Since we also have to take account of (7), we need $\delta > 0$.

It also is useful to take a Mellin transform also with respect to z :

$$T_n(z, \omega) = \int \frac{dN}{2\pi i} z^N T_n(N, \omega). \quad (10)$$

Since $T_n(z, \omega)$ vanishes for $z > 1$, its Mellin transform is certainly defined for $\operatorname{Re} N$ large and negative, so the integration in (10) is chosen to be along a line parallel to the imaginary axis in the complex N -plane, with $\operatorname{Re} N$ sufficiently large and negative. Inserting (10) into the Lipatov equation and dropping nonleading powers of z , we find that

$$T_n(N, \omega) = \left(\frac{-K(\omega)}{N} \right) T_{n-1}(N, \omega) \quad (11)$$

or

$$T_n(N, \omega) = \left(\frac{-K(\omega)}{N} \right)^n T_0(N, \omega). \quad (12)$$

Recall that initially we are concerned with values of ω that satisfy the inequality (7), so that $K(\omega)$ is finite and we can initially take $\operatorname{Re} N$ to be large enough negative for $|N^{-1}K(\omega)|$ to be less than 1. Then the sum over n of (12) converges to

$$T(N, \omega) = \frac{NT_0(N, \omega)}{N + K(\omega)}. \quad (13)$$

Figure 2 shows a plot of $K(\omega)$ against ω for real ω . From it, we see that for N large negative there are two zeros of the denominator of (13) corresponding to real values of ω . Because we needed to have $K(\omega) < -N$ in order to have the sum over n of (12) converge, these zeros lie one to each side of the contour of integration over ω that gives $T(z, z)$, the (double) Mellin transform of (13).

The small- z behaviour of $T(z, \omega)$ is given by the leftmost singularity of $T(N, \omega)$ in the N -plane. We locate this singularity by moving the contour of integration over N in this double transform to the right in the complex N -plane. The two zeros of the denominator of (13) move in the ω -plane, and approach the ω -integration contour from opposite sides, until eventually they pinch it. This yields^[8] a singularity in the ω integration. It occurs when $N = -K(0) = -(12\alpha_S/\pi) \log 2$. This is the value of the power in the leading behaviour of F_2 as $z \rightarrow 0$; it corresponds to the exchange of one perturbative pomeron. $T(N, \omega)$ also has the singularities of the nonperturbative input driving term $T_0(N, \omega)$, which results in a term $z^{-\epsilon}$ in F_2 . Which of the two is dominant depends on the value chosen for α_S ; it is usually taken to be about 0.2, so that the perturbative singularity dominates and produces behaviour close to $z^{-\frac{1}{2}}$.

4. Solution with infrared cutoff

The fact that $K(\omega)$ is singular at $\omega = \frac{1}{2}$ is important in this analysis; it prevents one from moving the contour of ω integration to the right and so avoiding the pinch. However, this singularity is associated with the behaviour of the kernel $K(k_T^2, k_T^2)$ at small k_T^2 , which immediately suggests that the imposition of a small- k_T^2 cut-off on the integration in the Lipatov equation (3) might change the $z^{-K(0)}$ behaviour.

Imposing such a cut-off makes the equation difficult to analyse. It is also not clear that one should in fact impose the same cut-off on all the terms in the integration. The term involving $t(u)$ in (3) describes the emission of real partons, while those involving $t(z)$ are virtual-parton corrections. As we have argued, a lower cut-off Q_0^2 is imposed on the real-emission term because the production of small- k_T partons is supposed to be largely accounted for in the driving term T_0 . However, the high- k_T emission still has to be corrected, not only with high- k_T virtual partons, but also small- k_T ones. Hence there is some sense in imposing the cut-off only on the integration over $t(u)$ in (3). This has the advantage that it makes the mathematics much more tractable and, although it may not be completely the correct thing to do, it will demonstrate that imposing even a partial lower cut-off changes the $z^{-\frac{1}{2}}$ behaviour.

To implement such a cut-off, one merely multiplies the integral in the Lipatov equation (2) by a step function $\theta(k_T^2 - Q_0^2)$. (A similar step function has to be included in the driving term T_0 in the integral equation, and the part of T_0 so removed has to be added back into the final solution $T(z, k_T^2)$.) The effect of this step-function can be seen immediately: the contour of integration over ω in the Mellin transform of each T_n must pass to the right of all the singularities. The same applies to the sum T of all the T_n . So the pinch of this contour of integration cannot now occur. This means that the imposition of the cut-off has changed the character of the N -plane singularity. However, we shall see that its position is not changed. The explanation of this is that in the original equation the upper and lower parts of the k_T^2 integration both contribute an N -plane singularity at the same point, so that imposing a low- k_T^2 cut-off modifies the singularity but does not remove it.

Let us make this explicit. We use the representation

$$\theta(z - Q_0^2) = \int \frac{d\mu}{2\pi i} \frac{z^\mu Q_0^{-2\mu}}{\mu}$$

where the integration is along a line in the complex plane parallel to the μ -axis with $\operatorname{Re} \mu > 0$. Then in place of (11) we have

$$T_n(N, \omega) = \int \frac{d\nu}{2\pi i} \frac{Q_0^{2(\nu-\omega)}}{\omega - \nu} \left(\frac{-K(\nu)}{N} \right) T_{n-1}(N, \nu), \quad (14)$$

where the integration is along a contour parallel to the imaginary ν axis with $|\operatorname{Re} \nu| < \frac{1}{2}$ and $\operatorname{Re} \nu < \operatorname{Re} \omega$. So we have to have $\operatorname{Re} \omega > -\frac{1}{2}$, but it need no longer be less than $\frac{1}{2}$. The singularities in the ω -plane of the integral in (14) are those of $T_{n-1}(N, \omega)$ and of $K(\omega)$, but only those in the left half-plane, because they arise from a pinch of the contour of ν integration between the pole $(\nu - \omega)^{-1}$ and the singularity of either T_{n-1} or K .

If, as is likely for N sufficiently large negative, the sum over n of $T_n(N, \omega)$ converges, it will satisfy the integral equation

$$T(N, \omega) = \int \frac{d\nu}{2\pi i} \frac{Q_0^{2(\nu-\omega)}}{\omega - \nu} \left(\frac{-K(\nu)}{N} \right) T(N, \nu) + T_0(N, \omega). \quad (15)$$

In order to get a feeling for the solution of this equation, consider a simple model where $\theta(k_T^2 - Q_0^2)T_0(z, k_T^2)$ has double Mellin transform

$$T_0(N, \omega) = \frac{AQ_0^{2(\omega_0-\omega)}}{(N+\epsilon)(\omega_0-\omega)}, \quad (16)$$

so that $T_0(z, k_T^2)$ behaves as $z^{-\epsilon}$ for small z and as $(k_T^2)^{\omega_0+\frac{1}{2}}$ for large k_T^2 , where we take $\omega_0 < -\frac{1}{2}$. Also, we use a simplification of $K(\omega)$ which has its essential features in $|\omega| < \frac{1}{2}$:

$$K(\omega) = \frac{K_0}{1-4\omega^2}, \quad (17)$$

with $K_0 = (12\alpha_5/\pi) \log 2$. Then we can solve (15) exactly. The general solution is a linear combination of $T_0(N, \omega)$ and the poles $(\omega \pm \omega_N)^{-1}$, where

$$\omega_N = \frac{1}{2} \sqrt{1 + K_0/N}. \quad (18)$$

However, if the solution is required not only to satisfy the integral equation (15), but also to agree with the iterative equation (14), the pole $(\omega - \omega_N)^{-1}$ must be absent. Hence

$$T(N, \omega) = \frac{1}{1 + N^{-1}K(\omega_0)} \left[T_0(N, \omega) + \frac{T_0(N, -\frac{1}{2})(\frac{1}{2} - \omega_N)}{\omega + \omega_N} Q_0^{-1-2\omega} \right]. \quad (19)$$

This solution is rather different from (13). However, although there is now no pinch in the ω integration when (19) is inverted, the fixed singularity in the N plane, corresponding to the branch point at $\omega_N = 0$, results in the same power of z as before.

5. Ultraviolet cutoff

Consider now the effect of including also an upper cut-off Q_1^2 on the k_T^2 integration in (2). As we have indicated, Q_1 is of order $\sqrt{s}$. The integrand in (14) acquires an additional factor

$$1 - \left(\frac{Q_1}{Q_0} \right)^{2(\nu-\omega)},$$

and we should also include in (16) the extra factor

$$1 - \left(\frac{Q_1}{Q_0} \right)^{2(\omega_0-\omega)}.$$

Then (19) is replaced with

$$T(N, \omega) = \frac{1}{1 + N^{-1}K(\omega_0)} [T_0(N, \omega) + F(\omega, \omega_N) + F(\omega, -\omega_N)], \quad (20)$$

with

$$F(\omega, \omega_N) = \frac{(Q_0^{2(\omega_N-\omega)} - Q_1^{2(\omega_N-\omega)})C(\omega_N)}{\omega - \omega_N},$$

$$\Delta(\omega_N)C(\omega_N) = \frac{(\frac{1}{2} + \omega_N)Q_0^{1-2\omega_N}T_0(N, \frac{1}{2})}{R^{-2\omega_0} - R^{-1}} - \frac{(\frac{1}{2} - \omega_N)Q_1^{-1-2\omega_N}T_0(N, -\frac{1}{2})}{R^{2\omega_0} - R^{-1}}$$

$$\Delta(\omega_N) = \frac{\frac{1}{2} + \omega_N}{\frac{1}{2} - \omega_N} R^{2\omega_N} - \frac{\frac{1}{2} - \omega_N}{\frac{1}{2} + \omega_N} R^{-2\omega_N}, \quad (21)$$

where $R = Q_1/Q_0$.

Let us note in passing that the inverse Mellin transform of (20) to transverse momentum is quite simple:

$$T(N, k_T^2) = \frac{1}{1 + K(\omega_0)/N} \left\{ \frac{A}{N + \epsilon} z^{\frac{1}{2} + \omega_0} + C(\omega_N) z^{\frac{1}{2} + \omega_N} + C(-\omega_N) z^{\frac{1}{2} - \omega_N} \right\}. \quad (22)$$

for $Q_0 < k_T < Q_1$ and 0 otherwise.

When $R \rightarrow \infty$, (20) becomes identical with (19), but for finite R there are no singularities in the finite ω -plane and there is no square-root branch point, because it cancels between the terms $F(\omega, \omega_N)$ and $F(\omega, -\omega_N)$ of $T(N, \omega)$ in (20). Apart from the singularity at $N = -\epsilon$ in the driving input term $T(N, \omega)$, the singularities in the N plane are the zeroes of $\Delta(\omega_N)$. The exception is the zero of $\Delta(\omega_N)$ at $\omega_N = 0$, which also cancels between the terms $F(\omega, \omega_N)$ and $F(\omega, -\omega_N)$ of $T(N, \omega)$. This leaves an infinite number of zeros at imaginary values of $\omega_N = i\bar{\omega}_N$ satisfying

$$\bar{\omega}_N = \frac{1}{2} \cot(\bar{\omega}_N \log R) \quad \text{or} \quad -\frac{1}{2} \tan(\bar{\omega}_N \log R). \quad (23)$$

$T(N, \omega)$ has simple poles at

$$N = -\frac{K_0}{1 + 4\bar{\omega}_N^2}$$

and for large R the leading one is close to

$$N = -K_0 \left[1 - \left(\frac{\pi}{\log R} \right)^2 \right], \quad (24)$$

while the separation between the leading and next-to-leading one is close to

$$\Delta N = 3K_0 \left(\frac{\pi}{\log R} \right)^2. \quad (25)$$

Ultimately, for very large R , the infinite number of simple poles merge together and, as we have seen, become a square-root branch point at $N = -K_0$. However, for the leading pole to be within a distance of, say, 0.1 from $-K_0$ we need the ratio $R = Q_1/Q_0$ of the upper to the lower cut-off to be at least 10^4 . If Q_0 is 1 GeV or more, this cannot be achieved at HERA. And although ultimately all the poles coalesce, R must be extremely large before even the first two are close to each other. So the conclusion is that the necessary cut-offs on the k_T^2 integration in the Lipatov equation (2) have a very significant moderating effect on the behavior $z^{-0.5}$ predicted for $F_2(z)$ at asymptotic values of Q^2 . When the effects of multiple exchanges are also taken into account, it will be difficult to separate the perturbative pomeron behaviour from the $z^{-0.08}$ that arises from the nonperturbative pomeron.

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Figure captions

1. Ladder diagram for deep inelastic scattering.
2. The function $K(\omega)$ of equation (6) plotted against ω .

Figure 1

Figure 2