Improved Born approximation for $e^+e^- \rightarrow W^+W^-$ in the LEP200 energy region

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Abstract:

We introduce a new representation for the one-loop corrected amplitude to $e^+e^- \rightarrow W^+W^-$, which avoids unitarity cancellations between different invariant functions. Using this representation we demonstrate that the full one-loop amplitude can be accurately described through a Born-like structure, i.e. improved Born approximations are possible. Explicit simple improved Born approximations are given for the total and the differential cross section. For LEP200 energies these reproduce the exact one-loop results within $\sim 0.5\%$ (relative to the Born level) for the total cross section including hard photon bremsstrahlung and within $\sim 1\%$ for the differential cross section in the soft photon approximation.

1 Introduction

The observation of the reaction $e^+e^- \rightarrow W^+W^-$ at LEP200 will provide important precision tests of the Glashow–Salam–Weinberg model. Measurements of the corresponding cross section not only allow an accurate determination of the W-mass, but also yield the first direct experimental results about the self interaction of the electroweak gauge bosons. The expected experimental error at LEP200 is about 1%. Consequently the full $\mathcal{O}(\alpha)$ and the leading $\mathcal{O}(\alpha^2)$ radiative corrections have to be taken into account in the theoretical prediction. Independent calculations for the one-loop corrections, including soft photon bremsstrahlung, have been given in Refs. [1], [2] and [3]. The latter two calculations agree within 0.3%. However, these results consist of long and complicated analytical expressions which lead to involved numerical evaluations. Especially an application as input in Monte Carlo routines seems not to be appropriate. Consequently simple approximative formulae for the radiative corrections are desirable. Such formulae are not only useful for fast numerical programs but also may improve the theoretical understanding of the origin of the dominant corrections. A first attempt to sort out the relevant contributions has been made in Ref. [4], using the electromagnetic and weak multipole moments of the W-boson as guideline. Pursuing a different approach we confirm and extend these results. In addition we present explicit simple approximations.

Starting from the expressions given in Ref. [2] we introduce a new decomposition of the invariant matrix element. This is constructed in such a way that unitarity cancellations between the different contributions are absent. Using this representation we show the strong dominance of those form factors which occur already in the Born amplitude. This demonstrates that improved Born approximations can be constructed. We then set up such approximations for the total and differential cross sections including the leading effects arising from the fermion and Higgs sector, the leading logarithms due to collinear bremsstrahlung, and the Coulomb singularity of the produced W-bosons. In the case of the differential cross section further contributions have to be taken into account to obtain satisfactory approximations in the LEP200 energy region. The improved Born approximations are compared numerically to the exact results.

The paper is organized as follows:

In Section 2 we introduce a new representation for the exact one-loop amplitude to $e^+e^- \rightarrow W^+W^-$. The improved Born approximation for the total cross section is presented and discussed in Section 3, the one for the differential cross section in Section 4. Section 5 gives a short summary. The appendices complete the approximation formulae by providing explicit expressions and the input parameters used for the numerical evaluations.

2 Representation of the invariant matrix element for $e^+e^- \rightarrow W^+W^-$ avoiding unitarity cancellations

In [2, 3] the one-loop correction $\delta\mathcal{M}^\sigma$ to the helicity amplitudes $\mathcal{M}^\sigma$ was calculated systematically by expanding $\delta\mathcal{M}^\sigma$ into a set $\mathcal{M}_i^\sigma$ ($i = 1, \dots, 7$) of standard matrix elements

$$\mathcal{M}^\sigma(\lambda_+, \lambda_-) = \sum_{i=1}^7 F_i^\sigma \mathcal{M}_i^\sigma(\lambda_+, \lambda_-) . \quad (1)$$

Here $\sigma = \pm \frac{1}{2}$ and $\lambda_+, \lambda_- = 0, \pm 1$ denote the helicities of the electron and of the W-bosons, respectively. We use the conventions of Ref. [2] throughout. However, we define the helicities of the W-bosons in the usual way, i.e. we differ from Ref. [2] by a minus sign in the helicity of the W^- . The matrix elements $\mathcal{M}_7^\sigma$ used there can be expressed as

$$\mathcal{M}_7^\sigma = \mathcal{M}_5^\sigma + \frac{1}{2}(M_W^2 - t)\mathcal{M}_3^\sigma + \frac{s}{2}(\mathcal{M}_4^\sigma - \mathcal{M}_1^\sigma - \mathcal{M}_2^\sigma). \quad (2)$$

Then we are left with the minimal number of 12 linear independent standard matrix elements (assuming CP conservation which holds at the one-loop level in $e^+e^- \rightarrow W^+W^-$). The $\mathcal{M}_i^\sigma$ were chosen according to formal simplicity concerning their structure.

For longitudinally polarized W-bosons each term in (1) grows with the CM energy and individually violates unitarity. This is still the case after $\mathcal{M}_7^\sigma$ is eliminated via (2). The unitarity behaviour of $\mathcal{M}^\sigma$ results from cancellations between these individual terms. Already in lowest order considerable unitarity cancellations between the contributions associated with these matrix elements occur. Since these are present in all orders of the perturbation series we introduce a new minimal basis $\hat{\mathcal{M}}_i^\sigma$ ($i = 1, \dots, 6$) which respects unitarity and avoids large compensations between single terms in the sum

$$\mathcal{M}^\sigma(\lambda_+, \lambda_-) = \sum_{i=1}^7 F_i^\sigma \mathcal{M}_i^\sigma(\lambda_+, \lambda_-) = \sum_{i=1}^6 \hat{F}_i^\sigma \hat{\mathcal{M}}_i^\sigma(\lambda_+, \lambda_-). \quad (3)$$

Three of the new standard matrix elements are obtained by rewriting the Born amplitude

$$\mathcal{M}_{\text{Born}}^\sigma(\lambda_+, \lambda_-) = \frac{1}{2s_w^2} \hat{\mathcal{M}}_1^-(\lambda_+, \lambda_-) \delta_{\sigma-} + \hat{\mathcal{M}}_2^\sigma(\lambda_+, \lambda_-). \quad (4)$$

Since the two terms in (4) involve independent parameters the matrix elements

$$\hat{\mathcal{M}}_1^- = \frac{1}{t} \mathcal{M}_1^- - \frac{2}{s - M_Z^2} (\mathcal{M}_2^- - \mathcal{M}_3^-), \quad \hat{\mathcal{M}}_2^\sigma = \frac{2M_Z^2}{s(s - M_Z^2)} (\mathcal{M}_2^\sigma - \mathcal{M}_3^\sigma) \quad (5)$$

must have a good high energy behaviour as can be checked by direct calculation. The same holds for

$$\hat{\mathcal{M}}_1^+ = \frac{1}{t} \mathcal{M}_1^+ - \frac{2}{s - M_Z^2} (\mathcal{M}_2^+ - \mathcal{M}_3^+). \quad (6)$$

Explicit expressions for $\hat{\mathcal{M}}_{1,2}^\sigma$ are listed in App. B. The determination of the remaining matrix elements becomes non-trivial, because for each electron helicity only three linear combinations of the $\mathcal{M}_i^\sigma$ exist which respect unitarity in the high energy limit for the case of at least one longitudinally polarized W-boson, namely

$$\frac{1}{t} \mathcal{M}_1^\sigma - \frac{2}{s} (\mathcal{M}_2^\sigma - \mathcal{M}_3^\sigma), \quad \mathcal{M}_2^\sigma - \frac{2}{s} \mathcal{M}_5^\sigma, \quad \mathcal{M}_1^\sigma - 2\mathcal{M}_4^\sigma - \frac{4}{t} \mathcal{M}_6^\sigma. \quad (7)$$

Together with (5) and (6), this yields eight linear independent standard matrix elements. The remaining four $\hat{\mathcal{M}}_i^\sigma$ would have to be chosen empirically.

Instead we prefer a classification according to the helicities of the produced W-bosons. It turns out that the $\hat{\mathcal{M}}_i^\sigma$ ($i = 3, \dots, 6$) can be constructed such that any polarization of the W-bosons gets contributions from only one of them. With an appropriate normalization we obtain

$$\hat{\mathcal{M}}_3^\sigma = \eta_{++} + \eta_{--}, \quad \hat{\mathcal{M}}_4^\sigma = \eta_{+-} + \eta_{-+}, \quad \hat{\mathcal{M}}_5^\sigma = \eta_{+0} + \eta_{-0} + \eta_{0+} + \eta_{0-}, \quad \hat{\mathcal{M}}_6^\sigma = \eta_{00}, \quad (8)$$

where $\eta_{+0} = \delta_{\lambda_+, +1} \delta_{\lambda_-, 0}$ etc. Thus unitarity cancellations between the $\hat{\mathcal{M}}_i^\sigma$ ($i = 3, \dots, 6$) are completely excluded. One may convince oneself that also cancellations between these and the Born-like amplitudes and between the two Born-like amplitudes are excluded in this formulation. Consequently all unitarity cancellations must occur inside the invariant functions $\hat{F}_i$. The matrix elements $\hat{\mathcal{M}}_i^\sigma$ ($i = 3, \dots, 6$) are given by

$$\begin{aligned}
\hat{\mathcal{M}}_3^\sigma &= (ut - M_W^4)^{-\frac{1}{2}} \left[\mathcal{M}_2^\sigma + \frac{2(2M_W^2 - s)}{s(s - 4M_W^2)} \mathcal{M}_5^\sigma \right], \\
\hat{\mathcal{M}}_4^\sigma &= (ut - M_W^4)^{-\frac{3}{2}} \left[s(t + M_W^2)(\mathcal{M}_4^\sigma - \mathcal{M}_1^\sigma) - (t^2 - M_W^4)\mathcal{M}_3^\sigma + 2(s - 4M_W^2)\mathcal{M}_6^\sigma \right. \\
&\quad \left. - (M_W^2 s + st + ut - M_W^4)\mathcal{M}_2^\sigma + 2M_W^2 \frac{(s + 2t)(s - 4M_W^2) + 2(t + M_W^2)^2}{s(s - 4M_W^2)} \mathcal{M}_5^\sigma \right], \\
\hat{\mathcal{M}}_5^\sigma &= (ut - M_W^4)^{-1} \sqrt{2s} M_W \left[\mathcal{M}_4^\sigma - \mathcal{M}_1^\sigma - \mathcal{M}_2^\sigma + \frac{M_W^2 - t}{s} \mathcal{M}_3^\sigma + \frac{2(u - t)}{s(4M_W^2 - s)} \mathcal{M}_5^\sigma \right], \\
\hat{\mathcal{M}}_6^\sigma &= (ut - M_W^4)^{-\frac{1}{2}} \frac{4M_W^2}{s(s - 4M_W^2)} \mathcal{M}_5^\sigma. \tag{9}
\end{aligned}$$

The corresponding invariant functions $\hat{F}_i^\sigma$ are obtained as follows

$$\begin{aligned}
\hat{F}_1^\sigma &= t(F_1^\sigma + F_4^\sigma), \\
\hat{F}_2^\sigma &= -\frac{s(s - M_Z^2)}{2M_Z^2} F_3^\sigma + \frac{ts}{M_Z^2} (F_1^\sigma + F_4^\sigma) + \frac{(t - M_W^2)(M_Z^2 - s)}{2M_Z^2} F_4^\sigma, \\
\hat{F}_3^\sigma &= \sqrt{ut - M_W^4} \left[F_2^\sigma + F_3^\sigma + \frac{M_W^2 - u}{s} F_4^\sigma + \frac{ut - M_W^4}{2(s - 4M_W^2)} F_6^\sigma \right], \\
\hat{F}_4^\sigma &= \frac{(ut - M_W^4)^{\frac{3}{2}}}{2(s - 4M_W^2)} F_6^\sigma, \\
\hat{F}_5^\sigma &= \frac{ut - M_W^4}{\sqrt{2s} M_W} \left[F_4^\sigma - \frac{s(M_W^2 + t)}{2(s - 4M_W^2)} F_6^\sigma + \frac{s}{2} F_7^\sigma \right], \\
\hat{F}_6^\sigma &= \sqrt{ut - M_W^4} \left[\frac{s - 2M_W^2}{2M_W^2} (F_2^\sigma + F_3^\sigma) + \frac{2M_W^4 - st - 2M_W^2 t - sM_W^2}{2sM_W^2} F_4^\sigma \right. \\
&\quad \left. + \frac{s(s - 4M_W^2)}{4M_W^2} F_5^\sigma + \frac{s(M_W^2 + t)^2}{4M_W^2(s - 4M_W^2)} F_6^\sigma - \frac{s(M_W^2 + t)}{2M_W^2} F_7^\sigma \right]. \tag{10}
\end{aligned}$$

The relation between the $\hat{F}_i^\sigma$ and the $\tilde{F}_i^\sigma$ corresponding to a representation $\mathcal{M}^\sigma(\lambda_+, \lambda_-) = \sum_{i=1}^6 \tilde{F}_i^\sigma \mathcal{M}_i^\sigma(\lambda_+, \lambda_-)$, where $\mathcal{M}_7^\sigma$ has been eliminated according to (2), is given by (10) with the terms containing F_7^σ omitted and F_i^σ replaced by $\tilde{F}_i^\sigma$ for $i = 1, \dots, 6$.

At lowest order only the three invariant functions $\hat{F}_1^-$, $\hat{F}_2^+$ and $\hat{F}_2^-$ contribute. At one-loop order these contain the IR- and UV-sensitive corrections which are known to be potentially large. In order to investigate the relevance of the other invariant functions, we compare the cross section including the complete one-loop correction $\sigma_{\text{full}}^{\text{tot}, \sigma}$ with the approximations $\sigma_{\text{appr}}^{\text{tot}, \sigma}$ obtained from the part of the helicity matrix elements corresponding to the Born structure

$$\mathcal{M}_{\text{appr}}^\sigma = \hat{F}_1^\sigma \hat{\mathcal{M}}_1^\sigma \delta_{\sigma-} + \hat{F}_2^\sigma \hat{\mathcal{M}}_2^\sigma \tag{11}$$

$\sqrt{s}/\text{GeV}$	165	175	200	300	500	1000
$\sigma_{\text{Born}}^{\text{tot},+}/10^{-2}\text{ pb}$	11.4937	41.0100	69.7983	36.7946	10.0482	2.0039
$\delta\sigma_{\text{full}}^{\text{tot},+}/10^{-2}\text{ pb}$	-0.4234	-2.3535	-4.8458	-2.0328	-0.9868	-0.3603
$\delta\sigma_{\text{app2}}^{\text{tot},+}/10^{-2}\text{ pb}$	-0.4245	-2.3598	-4.8882	-2.3256	-0.9563	-0.3556
$\delta\sigma_{\text{appr}}^{\text{tot},+}/10^{-2}\text{ pb}$	-0.4642	-2.6440	-5.5359	-2.3845	-0.9589	-0.3554
$\sigma_{\text{Born}}^{\text{tot},-}/\text{ pb}$	43.0003	63.0206	69.6903	48.6006	25.8554	9.6649
$\delta\sigma_{\text{full}}^{\text{tot},-}/\text{ pb}$	-5.4163	-9.3108	-11.1690	-8.4566	-4.8281	-2.0632
$\delta\sigma_{\text{appr}}^{\text{tot},-}/\text{ pb}$	-5.4098	-9.2921	-11.1579	-8.4853	-4.8361	-2.0612
$\sigma_{\text{Born}}^{\text{tot}}/\text{ pb}$	10.7788	15.8577	17.5971	12.2421	6.4890	2.4212
$\delta\sigma_{\text{full}}^{\text{tot}}/\text{ pb}$	-1.3551	-2.3336	-2.8044	-2.1192	-1.2095	-0.5167
$\delta\sigma_{\text{app2}}^{\text{tot}}/\text{ pb}$	-1.3535	-2.3289	-2.8017	-2.1271	-1.2114	-0.5162
$\delta\sigma_{\text{appr}}^{\text{tot}}/\text{ pb}$	-1.3536	-2.3296	-2.8033	-2.1273	-1.2114	-0.5162

Table 1: Comparison of the full one-loop corrections and the approximations.

and $\sigma_{\text{app2}}^{\text{tot},\sigma}$ obtained from

$$\mathcal{M}_{\text{app2}}^{\sigma} = \sum_{i=1,2} \hat{F}_i^{\sigma} \hat{\mathcal{M}}_i^{\sigma} . \quad (12)$$

The numerical results are obtained from the analytical expressions given in [2]. To remove the IR divergencies of the virtual corrections we added the soft photon bremsstrahlung cross section with the cut-off parameter $\Delta E_{\gamma} = 0.05E$. In Table 1 we give the total cross section in lowest order together with the full one-loop corrections $\delta\sigma_{\text{full}}^{\text{tot}} = \sigma_{\text{full}}^{\text{tot}} - \sigma_{\text{Born}}^{\text{tot}}$ and the corrections $\delta\sigma_{\text{appr}}^{\text{tot}}$, $\delta\sigma_{\text{app2}}^{\text{tot}}$ as obtained by our approximations for right-handed, for left-handed and for unpolarized electrons. In Figure 1 we show the relative difference of $\sigma_{\text{appr}}^{\text{tot}}$ and $\sigma_{\text{full}}^{\text{tot}}$ normalized to the lowest order cross section, i.e. the quantity

$$\Delta_{\text{appr}}^{\text{tot}} = \frac{\sigma_{\text{appr}}^{\text{tot}} - \sigma_{\text{full}}^{\text{tot}}}{\sigma_{\text{Born}}^{\text{tot}}} \quad (13)$$

for unpolarized and left-handed electrons. It is less than 0.07%. Figure 2 shows $\Delta_{\text{appr}}^{\text{tot}}$ and the analogously defined quantity $\Delta_{\text{app2}}^{\text{tot}}$ for right-handed electrons. Here the difference reaches up to 2% at the $t\bar{t}$ threshold. Note that the inclusion of $\hat{F}_1^{+}$ considerably improves the approximation for right-handed electrons in the LEP200 energy region. In particular it is necessary to reproduce the correct behaviour at the ZZ threshold.

The quality of the approximations $\mathcal{M}_{\text{appr}}^{\sigma}$ and $\mathcal{M}_{\text{app2}}^{\sigma}$ for the differential cross section is illustrated in Figures 3 and 4. As can be seen from Figure 3 for left-handed and unpolarized electrons the corresponding relative difference $\Delta_{\text{appr}}^{\text{diff}}$ is negligible at a scattering angle $\theta = 10^{\circ}$ and still below 0.4% at $\theta = 90^{\circ}$. The larger deviation for $\theta = 150^{\circ}$ is probably not of practical relevance since the cross section is very small for scattering in the backward direction. At LEP200 energies the difference $\Delta_{\text{app2}}^{\text{diff}}$ for unpolarized

Figure 1: Relative difference $\Delta_{\text{appr}}^{\text{tot}}$ of the approximated and the full one-loop corrected total cross section for left-handed (—) and unpolarized (---) electrons in per cent.

Figure 2: Relative difference $\Delta_{\text{app2}}^{\text{tot}}$ (—) and $\Delta_{\text{appr}}^{\text{tot}}$ (---) of the approximated and the full one-loop corrected total cross section for right-handed electrons in per cent.

Figure 3: Relative difference $\Delta^{\text{diff}}_{\text{appr}}$ of the approximated and the full one-loop corrected differential cross section for left-handed (—) and unpolarized (---) electrons in per cent.

Figure 4: Relative difference $\Delta^{\text{diff}}_{\text{appr2}}$ (—) and $\Delta^{\text{diff}}_{\text{appr}}$ (---) of the approximated and the full one-loop corrected differential cross section for right-handed electrons in per cent.

electrons coincides practically with $\Delta_{\text{app2}}^{\text{diff},-} = \Delta_{\text{appr}}^{\text{diff},-}$ for left-handed electrons. In the unpolarized case the absolute error for $\frac{d\sigma}{d\Omega}$ always remains below 1 fb and even below 0.2 fb for $500 \text{ GeV} < \sqrt{s} < 1 \text{ TeV}$. The relative agreement for right-handed electrons is worse. Nevertheless the absolute deviation of at most 1 fb at $\sqrt{s} = 2m_t$ is acceptable. Away from this threshold, i.e. for $|\sqrt{s} - 2m_t| > 60 \text{ GeV}$ it is less than 0.1 fb. Figure 4 confirms the relevance of $\hat{F}_1^+$ near the ZZ threshold. For high energies it can be safely neglected.

The fact that σ_{appr} respectively σ_{app2} is almost equal to the full one-loop cross section $\delta\sigma_{\text{full}}$ is of great importance for the existence of improved Born approximations. It shows that approximations can be restricted to Born-like invariant functions $\hat{F}_1^\sigma$, $\hat{F}_2^\sigma$ without introducing remarkable errors.

3 Improved Born approximation for the total cross section including hard photon bremsstrahlung

For the total cross section of the reaction $e^+e^- \rightarrow W^+W^-, W^+W^-\gamma$ (integrated over the full photon phase space) we propose the following approximation

$$\sigma_{\text{impr}}^{\text{tot},\sigma}(s) = \sigma_{\text{uni}}^{\text{tot},\sigma}(s) + \delta\sigma_{\text{log}}^{\text{tot},\sigma}(s) + \delta\sigma_{\text{coll}}^{\text{tot},\sigma}(s) + \delta\sigma_{\text{coul}}^{\text{tot},\sigma}(s). \quad (14)$$

Here we give only the formulae for unpolarized W-bosons; however, these can be easily extended to the polarized case. The different terms in (14) are motivated as follows¹:

- a) $\sigma_{\text{uni}}^{\text{tot},\sigma}(s)$ refers to the Born level modified by universal radiative corrections arising from the fermion sector. As we use the running electromagnetic coupling constant $\alpha(s)$ and the Fermi constant G_μ in the way described in Ref. [6], the leading logarithms of the light fermions are generated correctly to all orders in α and the m_t^2 -dependence at least up to $\mathcal{O}(\alpha^2)$

$$\begin{aligned} \sigma_{\text{uni}}^{\text{tot},+}(s) &= \left(\frac{\alpha(s)}{\alpha} \right)^2 \sigma_{\text{Born}}^{\text{tot},+}(s), \\ \sigma_{\text{uni}}^{\text{tot},-}(s) &= \left(\frac{G_\mu M_W^2}{\sqrt{2}\pi\alpha} \right)^2 \sigma_{11}^{\text{tot}}(s) + \frac{G_\mu M_W^2 \alpha(s)}{\sqrt{2}\pi\alpha^2} \sigma_{12}^{\text{tot}}(s) + \left(\frac{\alpha(s)}{\alpha} \right)^2 \sigma_{22}^{\text{tot}}(s), \end{aligned} \quad (15)$$

where

$$\alpha(s) = \frac{\alpha}{1 - \Delta\alpha(s)}, \quad \Delta\alpha(s) = -\text{Re}\{\hat{\Pi}^\gamma(s)\} = \frac{\alpha}{\pi} \left[\sum_{f \neq t} Q_f^2 \ln\left(\frac{s}{m_f^2}\right) + \mathcal{O}(1) \right]. \quad (16)$$

Explicit expressions for σ_{11}^{tot} , σ_{12}^{tot} , σ_{22}^{tot} and $\sigma_{\text{Born}}^{\text{tot},\sigma}$ are given in App. B.

- b) $\delta\sigma_{\text{log}}^{\text{tot},\sigma}(s)$ contains large logarithms which might arise from a heavy top-quark or Higgs boson. These $\ln m_t$ and $\ln M_H$ terms have been evaluated for $s \ll m_t^2, M_H^2$ with the result

$$\sigma_{\text{log}}^{\text{tot},+}(s) = 2\hat{F}_{2,\text{log}}^+ \sigma_{\text{Born}}^{\text{tot},+}(s), \quad \sigma_{\text{log}}^{\text{tot},-}(s) = \hat{F}_{2,\text{log}}^- \left(\frac{1}{2s_w^2} \sigma_{12}^{\text{tot}} + 2\sigma_{22}^{\text{tot}} \right),$$

¹A detailed derivation can be found in Ref. [5].

$$\begin{aligned}\hat{F}_{2,\log}^+ &= \frac{\alpha}{24\pi s_w^2 c_w^2} \frac{s}{M_Z^2} \left\{ \frac{1}{4} \ln\left(\frac{M_H^2}{M_W^2}\right) f_{M_H}(s) - \ln\left(\frac{m_t^2}{M_W^2}\right) f_{m_t}(s) \right\}, \\ \hat{F}_{2,\log}^- &= \frac{\alpha}{48\pi s_w^4 c_w^2} \frac{s}{M_Z^2} \left\{ \frac{1}{4} \ln\left(\frac{M_H^2}{M_W^2}\right) f_{M_H}(s) + (3c_w^2 - s_w^2) \ln\left(\frac{m_t^2}{M_W^2}\right) f_{m_t}(s) \right\}.\end{aligned}\quad (17)$$

Because of the factor s/M_Z^2 , these terms seem to violate unitarity. They disappear, however, in the unitarity limit $s \gg m_t^2, M_H^2$. So a kind of weight functions $f_M(s)$ is needed to concentrate the contributions on their ranges of validity. Of course there is no sensitive dependence on the explicit form of $f_M(s)$. We have chosen

$$f_M(s) = \left[1 + \exp\left\{ \frac{\sqrt{s} - M - \frac{1}{2}M_W}{\frac{1}{2}M_W} \right\} \right]^{-1}. \quad (18)$$

As pointed out in Ref. [7] these terms may give rise to large radiative corrections for $M^2 \gg s \gg M_W^2$, where M is any heavy Higgs or fermion mass (delayed unitarity cancellations).

- c) The leading bremsstrahlung effects emerge from the leading logarithms associated with photons collinear to the incoming electron or positron. We have calculated analytically the $\mathcal{O}(\alpha)$ leading logarithms of the form $\alpha \ln(Q^2/m_e^2)$ applying a structure function method as given in Ref. [8]. The explicit expressions of $\delta\sigma_{\text{coll}}^{\text{tot},\sigma}(s)$ are listed in App. C.
- d) The long-ranging Coulomb interaction of the W-bosons gives rise to important corrections if these are produced with small velocity

$$\beta = \sqrt{1 - \frac{4M_W^2}{s}} \ll 1.$$

To restrict the β^{-1} term describing this Coulomb singularity to the threshold region we multiply the function $g(\beta) = (1 - \beta^2)^2$ to obtain

$$\delta\sigma_{\text{coul}}^{\text{tot},\sigma}(s) = \frac{\alpha\pi}{2\beta} g(\beta) \sigma_{\text{Born}}^{\text{tot},\sigma}(s). \quad (19)$$

The approximation (14) contains higher-order contributions through $\alpha(s)$ and G_μ . To allow a meaningful comparison these have to be included in the exact one-loop result in the same way. Thus we split off the leading fermionic $\mathcal{O}(\alpha)$ -terms from $\delta\mathcal{M}^\sigma$ such that again the procedure of Ref. [6] can be used to generate the higher orders in α

$$\begin{aligned}|\mathcal{M}_{\text{Born}}^+ + \delta\mathcal{M}^+|^2 &\rightarrow \left[\frac{1}{(1 - \Delta\alpha(M_W^2))^2} - 2\Delta\alpha(M_W^2) \right] |\mathcal{M}_{\text{Born}}^+|^2 + 2\text{Re}\{\delta\mathcal{M}^+\} \mathcal{M}_{\text{Born}}^+, \\ |\mathcal{M}_{\text{Born}}^- + \delta\mathcal{M}^-|^2 &\rightarrow \frac{1}{(1 - \Delta\alpha(M_W^2))^2} \left| \frac{1}{2s_w^2 + 2c_w^2 \Delta\bar{\rho}} \hat{\mathcal{M}}_1^- + \hat{\mathcal{M}}_2^- \right|^2 + 2\text{Re}\{\delta\mathcal{M}^-\} \mathcal{M}_{\text{Born}}^- \\ &\quad - 2\Delta\alpha(M_W^2) |\mathcal{M}_{\text{Born}}^-|^2 + \frac{c_w^2}{s_w^4} \Delta\rho \hat{\mathcal{M}}_1^- \mathcal{M}_{\text{Born}}^-, \end{aligned}\quad (20)$$

$\sqrt{s}/\text{GeV}$	165	175	200	300	500	1000
$\sigma_{\text{Born}}^{\text{tot},+}/10^{-2}\text{ pb}$	11.494	41.01	69.80	36.79	10.05	2.00
$\sigma^{\text{tot},+}/10^{-2}\text{ pb}$	9.803 ± 0.006	39.80 ± 0.03	76.15 ± 0.08	47.98 ± 0.08	14.37 ± 0.06	3.03 ± 0.03
$\sigma_{\text{impr}}^{\text{tot},+}/10^{-2}\text{ pb}$	9.623	38.96	74.42	45.94	14.09	3.14
$\Delta_{\text{impr}}^{\text{tot},+}$	-1.8%	-2.1%	-2.3%	-4.3%	-1.9%	4.0%
$\sigma_{\text{Born}}^{\text{tot},-}/\text{ pb}$	43.00	63.02	69.69	48.60	25.86	9.66
$\sigma^{\text{tot},-}/\text{ pb}$	35.59 ± 0.02	58.73 ± 0.05	71.04 ± 0.08	53.67 ± 0.12	29.38 ± 0.09	11.37 ± 0.06
$\sigma_{\text{impr}}^{\text{tot},-}/\text{ pb}$	35.56	58.87	71.51	54.39	30.21	11.77
$\Delta_{\text{impr}}^{\text{tot},-}$	-0.1%	0.2%	0.7%	1.4%	2.8%	3.5%
$\sigma_{\text{Born}}^{\text{tot}}/\text{ pb}$	10.779	15.86	17.60	12.24	6.49	2.42
$\sigma^{\text{tot}}/\text{ pb}$	8.922 ± 0.006	14.78 ± 0.01	17.95 ± 0.02	13.54 ± 0.03	7.38 ± 0.02	2.85 ± 0.01
$\sigma_{\text{impr}}^{\text{tot}}/\text{ pb}$	8.915	14.81	18.06	13.71	7.59	2.95
$\Delta_{\text{impr}}^{\text{tot}}$	-0.1%	0.2%	0.6%	1.3%	2.8%	3.5%

Table 2: Comparison between the completely corrected total cross section and the improved Born approximation.

with

$$\Delta\rho = \frac{3\alpha}{16\pi s_w^2} \frac{m_t^2}{M_W^2}, \quad \Delta\bar{\rho} = \frac{3G_\mu m_t^2}{8\pi^2\sqrt{2}} \left[1 + \frac{G_\mu m_t^2}{8\pi^2\sqrt{2}} (19 - 2\pi^2) \right]. \quad (21)$$

The exact hard bremsstrahlung cross section was computed with a Monte Carlo program [9].

The approximation $\sigma_{\text{impr}}^{\text{tot}}$ is compared with the full one-loop cross section modified by (20) in Table 2. The uncertainty in σ^{tot} arises from the Monte Carlo integration over the photon phase space. The relative error

$$\Delta_{\text{impr}}^{\text{tot}} = \frac{\sigma_{\text{impr}}^{\text{tot}} - \sigma^{\text{tot}}}{\sigma^{\text{tot}}} \quad (22)$$

of $\sim 2\%$ for right-handed electrons is acceptable since the corresponding cross section is suppressed by a factor of $\sim 10^2$. In the case of left-handed or unpolarized electrons we get a deviation of less than 0.7% for LEP200 energies. Beyond the LEP200 region the relative error $\Delta_{\text{impr}}^{\text{tot}}$ increases slightly. Together with the descent of the total cross section this yields an almost constant absolute error of at most 0.2 pb, 0.8 pb, and 0.02 pb for unpolarized, left-handed, and right-handed electrons, respectively. In Figs. 5 and 6 we compare the corrections relative to the lowest order obtained from the complete calculation with those obtained in our approximation for unpolarized and right-handed electrons. The corresponding results for left-handed electrons are nearly identical to the

Figure 5: Relative corrections to the total cross section for the full one-loop result ($\cdots$) and in the improved Born approximation (---) for unpolarized electrons in per cent.

Figure 6: Relative corrections to the total cross section for the full one-loop result ($\cdots$) and in the improved Born approximation (---) for right-handed electrons in per cent.

ones for unpolarized electrons. The distribution of the dots in these figures reflects the uncertainty of the Monte Carlo integration.

The contribution of the $\ln m_t$ and $\ln M_H$ terms is below 0.02% for unpolarized or left-handed and below 0.8% for right-handed electrons if $m_t < 200$ GeV and $M_H < 1$ TeV. The Coulomb term is only relevant in the threshold region.

We note that our approximation is more than a factor 3×10^4 faster in CPU time than the exact calculation.

4 Improved Born approximation for the differential cross section including soft bremsstrahlung

Unfortunately Eq. (14) cannot be extended trivially to describe the differential cross section correctly, too. The reason for this lies in the strong angular dependence of non-leading $\mathcal{O}(\alpha)$ contributions. Up to now there exists no general procedure to approximate the full radiative corrections. So we are forced to make a rough fit of the non-leading terms in the form factors $\hat{F}_{1,2}^\sigma$

$$\begin{aligned} \left(\frac{d\sigma^\sigma}{d\Omega}\right)_{\text{impr}} = \frac{\alpha^2\beta}{4s} \sum_{\lambda_\pm} \left[|\mathcal{M}_{\text{uni}}^\sigma(\lambda_+, \lambda_-)|^2 + \left(\delta_{\text{soft}} + \frac{\alpha\pi}{2\beta}g(\beta)\right) |\mathcal{M}_{\text{Born}}^\sigma(\lambda_+, \lambda_-)|^2 \right. \\ \left. + 2 \sum_{i=1,2} \hat{F}_{i,\text{rem}}^\sigma \hat{\mathcal{M}}_i^\sigma(\lambda_+, \lambda_-) \hat{\mathcal{M}}_{\text{Born}}^\sigma(\lambda_+, \lambda_-) \right]. \end{aligned} \quad (23)$$

Here

$$\mathcal{M}_{\text{uni}}^\sigma = \frac{G_\mu M_W^2}{\sqrt{2}\pi\alpha} \hat{\mathcal{M}}_1^- \delta_{\sigma-} + \frac{\alpha(M_W^2)}{\alpha} \hat{\mathcal{M}}_2^\sigma \quad (24)$$

includes the leading universal corrections and δ_{soft} describes the leading contributions due to soft bremsstrahlung. It is explicitly given in App. C. The remainder of the form factors $\hat{F}_{i,\text{rem}}^\sigma$ includes the logarithmic terms due to the Higgs and top-quark mass, the ZZ threshold effects, which are only significant for right-handed electrons, and a part obtained from a fitting procedure

$$\hat{F}_{i,\text{rem}}^+ = \hat{F}_{i,\log}^+ + \hat{F}_{i,\text{ZZ}}^+ + \hat{F}_{i,\text{fit}}^+, \quad \hat{F}_{i,\text{rem}}^- = \hat{F}_{i,\log}^- + \hat{F}_{i,\text{fit}}^-, \quad (25)$$

with

$$\begin{aligned} \hat{F}_{1,\log}^\sigma &= 0, & \hat{F}_{2,\log}^\sigma &\text{ from (17) }, \\ \hat{F}_{1,\text{ZZ}}^+ &= \frac{\alpha}{2} \left(\frac{s_w^2}{c_w^2} + 5 \right) \left[\frac{4}{s} (t - M_W^2) + c_w^2 \right] \arctan \sqrt{1 - \frac{s}{4M_Z^2}} \theta(4M_Z^2 - s), \\ \hat{F}_{2,\text{ZZ}}^+ &= \frac{\alpha}{2c_w^2} \left[11 - 24s_w^2 + 16s_w^4 + \frac{2}{s} (t - M_W^2) + \frac{8s_w^2}{s} (t - M_W^2) \right] \\ &\quad \times \arctan \sqrt{1 - \frac{s}{4M_Z^2}} \theta(4M_Z^2 - s), \\ \hat{F}_{1,\text{fit}}^+ &= \hat{C}_1^+ \frac{t}{M_W^2} \left(\frac{4M_Z^2}{s} \right)^{\frac{7}{2}}, & \hat{C}_1^+ &= -0.0054, \\ \hat{F}_{2,\text{fit}}^+ &= \hat{C}_2^+ \left(1 - \frac{4M_W^2}{s} \right), & \hat{C}_2^+ &= +0.017, \end{aligned}$$

Figure 7: Relative corrections to the differential cross section for the full one-loop result (—) and in the improved Born approximation (---) for unpolarized electrons in per cent.

$$\begin{aligned}
\hat{F}_{1,\text{fit}}^- &= \hat{C}_1^- , & \hat{C}_1^- &= -0.016 , \\
\hat{F}_{2,\text{fit}}^- &= \hat{C}_2^- \frac{2}{s} (M_W^2 - t) \left(\frac{4M_Z^2}{s} \right)^2 , & \hat{C}_2^- &= -0.0070 .
\end{aligned} \tag{26}$$

Figures 7 and 8 illustrate the quality of the approximation (23) for the differential cross section in the LEP200 energy region for different scattering angles. The case of left-handed electrons resembles closely the unpolarized one. The relative error

$$\Delta_{\text{impr}}^{\text{diff}} = \frac{d\sigma_{\text{impr}} - d\sigma}{d\sigma_{\text{Born}}} \tag{27}$$

stays mostly below $\lesssim 1\%$ (except for the threshold behaviour of right-handed electrons). The quality is best for scattering in the forward direction ($\theta \rightarrow 0$) and for left-handed electrons, i.e. where the cross section is maximal. In this region our approximation remains valid even for higher energies. We obtain the worst agreement for right-handed electrons near threshold and for left-handed ones in the backward direction ($\theta \rightarrow 180^\circ$), i.e. for regions with a very small cross section.

Table 3 completes our numerical discussion with a comparison of the corresponding total cross sections obtained by numerical integration of the differential cross sections over the full phase space in the soft photon approximation.

Finally we remark that the numerical evaluation of (23) is more than a factor of 300 faster than the evaluation of the full one-loop expressions.

Figure 8: Relative corrections to the differential cross section for the full one-loop result (—) and in the improved Born approximation (---) for right-handed electrons in per cent.

$\sqrt{s}/\text{GeV}$	165	175	200	250	300	500
$\sigma_{\text{Born}}^{\text{tot},+}/10^{-2}\text{ pb}$	11.494	41.010	69.798	55.543	36.795	10.048
$\sigma^{\text{tot},+}/10^{-2}\text{ pb}$	11.241	39.266	65.990	52.304	35.309	9.211
$\sigma_{\text{impr}}^{\text{tot},+}/10^{-2}\text{ pb}$	11.409	39.453	65.738	51.723	34.111	9.224
$\Delta_{\text{impr}}^{\text{tot},+}$	1.5%	0.5%	-0.4%	-1.0%	-3.3%	0.1%
$\sigma_{\text{Born}}^{\text{tot},-}/\text{pb}$	43.000	63.021	69.690	59.087	48.601	25.855
$\sigma^{\text{tot},-}/\text{pb}$	37.938	54.231	59.100	49.556	40.548	21.241
$\sigma_{\text{impr}}^{\text{tot},-}/\text{pb}$	38.065	54.283	59.287	50.104	41.211	21.940
$\Delta_{\text{impr}}^{\text{tot},-}$	0.3%	0.1%	0.3%	0.9%	1.4%	2.7%
$\sigma_{\text{Born}}^{\text{tot}}/\text{pb}$	10.779	15.858	17.597	14.910	12.242	6.489
$\sigma^{\text{tot}}/\text{pb}$	9.513	13.656	14.940	12.520	10.225	5.333
$\sigma_{\text{impr}}^{\text{tot}}/\text{pb}$	9.545	13.669	14.986	12.655	10.388	5.508
$\Delta_{\text{impr}}^{\text{tot}}$	0.3%	0.1%	0.3%	0.9%	1.3%	2.7%

Table 3: Comparison between the approximated and the full one-loop corrected total cross section in the soft photon approximation ($\delta = 0.05$).

5 Summary

We have introduced a decomposition for the helicity matrix elements to $e^+e^- \rightarrow W^+W^-$, which avoids unitarity cancellations between the different invariant functions. Using this representation we show that only three or four of the twelve invariant functions present in general are necessary to describe the one-loop corrected helicity amplitude with sufficient accuracy. The resulting simplified amplitude exhibits the structure of the lowest order contribution. This demonstrates that improved Born approximations are possible for $e^+e^- \rightarrow W^+W^-$.

Explicit simple improved Born approximations are given for the total cross section including the hard photonic corrections integrated over the full phase space and for the differential cross section in the soft photon approximation. The first one allows simple estimates of the total cross section whereas the second one can be used as input in Monte Carlo routines. Both approximations are valid in the LEP200 energy region. The accuracy of the given improved Born approximations amounts to $\sim 0.5\%$ in the case of the total cross section and $\sim 1\%$ in the case of the differential cross section. For energies up to 1 TeV our formulae still reproduce the integrated cross section at the few per cent level; they fail, however, for energies above 250 GeV in the case of the differential cross section. Here more work is needed.

Appendices

A Parameters of the standard model

For the numerical evaluation we use the following set of parameters

$$\begin{aligned} \alpha &= 1/137.0359890, & G_\mu &= 1.166370 \times 10^{-5} \text{ (GeV)}^{-2}, \\ m_e &= 0.5110030 \times 10^{-3} \text{ GeV}, & m_\mu &= 0.10565900 \text{ GeV}, & m_\tau &= 1.78420 \text{ GeV}, \\ m_u &= 0.041 \text{ GeV}, & m_c &= 1.50 \text{ GeV}, & m_t &= 140 \text{ GeV}, \\ m_d &= 0.041 \text{ GeV}, & m_s &= 0.150 \text{ GeV}, & m_b &= 4.50 \text{ GeV}, \\ M_Z &= 91.177 \text{ GeV}, & M_W &= 80.185 \text{ GeV}, & M_H &= 200 \text{ GeV}. \end{aligned}$$

The value of the W-boson mass M_W is calculated from the other parameters using the quantity Δr .

B Dirac matrix elements and total Born cross section

The explicit expressions for the lowest order helicity matrix elements read

$$\begin{aligned} \hat{\mathcal{M}}_1^\sigma(+1, +1) &= 2\sqrt{ut - M_W^4} \left[\frac{t + M_W^2}{t(s - 4M_W^2)} - \frac{1}{s - M_Z^2} \right], \\ \hat{\mathcal{M}}_2^\sigma(+1, +1) &= 2\sqrt{ut - M_W^4} \frac{M_Z^2}{s(s - M_Z^2)}, \end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{M}}_1^\sigma(\pm 1, \mp 1) &= \frac{\sqrt{ut-M_W^4}}{t(s-4M_W^2)} \left[t - u \mp 2\sigma s \sqrt{1 - \frac{4M_W^2}{s}} \right], \\
\hat{\mathcal{M}}_2^\sigma(\pm 1, \mp 1) &= 0, \\
\hat{\mathcal{M}}_1^\sigma(\pm 1, 0) &= -\frac{\sqrt{2s}}{M_W(s-4M_W^2)} \left[\frac{M_W^4}{t} + t + \frac{s}{2} \mp \sigma \sqrt{1 - \frac{4M_W^2}{s}} \left(\frac{2M_W^4}{t} - 2M_W^2 + s \right) \right] \\
&\quad + \frac{\sqrt{s}}{\sqrt{2}M_W(s-M_Z^2)} \left[t - u \mp 2\sigma s \sqrt{1 - \frac{4M_W^2}{s}} \right], \\
\hat{\mathcal{M}}_2^\sigma(\pm 1, 0) &= -\frac{M_Z^2}{\sqrt{2s}M_W(s-M_Z^2)} \left[t - u \mp 2\sigma s \sqrt{1 - \frac{4M_W^2}{s}} \right], \\
\hat{\mathcal{M}}_1^\sigma(0, 0) &= \sqrt{ut - M_W^4} \left[-\frac{st+4M_W^4}{tM_W^2(s-4M_W^2)} + \frac{s+2M_W^2}{(s-M_Z^2)M_W^2} \right], \\
\hat{\mathcal{M}}_2^\sigma(0, 0) &= -\sqrt{ut - M_W^4} \frac{M_Z^2(s+2M_W^2)}{M_W^2 s(s-M_Z^2)}, \\
\hat{\mathcal{M}}_{1,2}^\sigma(\lambda_+, \lambda_-) &= \hat{\mathcal{M}}_{1,2}^\sigma(-\lambda_-, -\lambda_+). \tag{28}
\end{aligned}$$

From these the lowest order cross section is obtained as follows

$$\begin{aligned}
\sigma_{11}^{\text{tot}}(s) &= \frac{\alpha^2 \beta}{4s} \sum_{\lambda_\pm} \int d\Omega |\hat{\mathcal{M}}_1^-(\lambda_+, \lambda_-)|^2 \\
&= \frac{\alpha^2 \pi \beta}{6sM_W^4} \left[\frac{s(s-4M_W^2)}{(s-M_Z^2)^2} - \frac{2(s-2M_W^2)}{(s-M_Z^2)} \right] (12M_W^4 + 20M_W^2 s + s^2) \\
&\quad + \frac{\alpha^2 \pi \beta}{6sM_W^4} (s^2 + 20M_W^2 s - 48M_W^4) + \frac{8\alpha^2 \pi}{s^2} \ln\left(\frac{1+\beta}{1-\beta}\right) \left[\frac{2M_W^2(2s+M_W^2)}{s-M_Z^2} + s - 2M_W^2 \right], \\
\sigma_{12}^{\text{tot}}(s) &= \frac{\alpha^2 \beta}{4s} \sum_{\lambda_\pm} \int d\Omega (\hat{\mathcal{M}}_1^{-*}(\lambda_+, \lambda_-) \hat{\mathcal{M}}_2^-(\lambda_+, \lambda_-) + \hat{\mathcal{M}}_1^-(\lambda_+, \lambda_-) \hat{\mathcal{M}}_2^{-*}(\lambda_+, \lambda_-)) \\
&= \frac{\alpha^2 \pi \beta M_Z^2}{3M_W^4 s(s-M_Z^2)} \left[1 - \frac{2M_W^2}{s} - \frac{s-4M_W^2}{s-M_Z^2} \right] (12M_W^4 + 20M_W^2 s + s^2) \\
&\quad - \frac{16\alpha^2 \pi M_W^2 M_Z^2}{s^3} \ln\left(\frac{1+\beta}{1-\beta}\right) \left(\frac{2s+M_W^2}{s-M_Z^2} \right), \\
\sigma_{22}^{\text{tot}}(s) &= \frac{\alpha^2 \beta}{4s} \sum_{\lambda_\pm} \int d\Omega |\hat{\mathcal{M}}_2^-(\lambda_+, \lambda_-)|^2 \\
&= \frac{\alpha^2 \pi \beta}{6s^2} \frac{M_Z^4}{M_W^4} \frac{(s-4M_W^2)}{(s-M_Z^2)^2} (12M_W^4 + 20M_W^2 s + s^2), \\
\sigma_{\text{Born}}^{\text{tot},+}(s) &= \sigma_{22}^{\text{tot}}(s), \\
\sigma_{\text{Born}}^{\text{tot},-}(s) &= \frac{\sigma_{11}^{\text{tot}}(s)}{4s_w^4} + \frac{\sigma_{12}^{\text{tot}}(s)}{2s_w^2} + \sigma_{22}^{\text{tot}}(s). \tag{29}
\end{aligned}$$

C Leading bremsstrahlung corrections

The structure function method of Ref. [8] determines the leading contribution of the hard bremsstrahlung only up to an energy scale Q^2 . For Q^2 we choose

$$Q^2 = \begin{cases} E^2 = s/4 & \text{for } \sigma = +\frac{1}{2}, \\ -t_{\min} = t(\theta = 0) = 2E^2(1 - \beta) - M_W^2 & \text{for } \sigma = -\frac{1}{2} \end{cases} \quad (30)$$

following Ref. [9] in the case of left-handed electrons. The contribution of the collinear photons to the cross section can be obtained through an analytic integration. Using the abbreviations

$$L_{m_e} = \ln\left(\frac{Q^2}{m_e^2}\right), \quad \kappa = \sqrt{4c_w^2 - 1}, \quad (31)$$

we arrive at

$$\begin{aligned} \delta\sigma_{\text{coll}}^{\text{tot},+}(s) &= \frac{2\alpha}{\pi} L_{m_e} \left[\ln\left(\frac{s\beta^2}{M_W^2}\right) + \frac{3}{4} \right] \sigma_{\text{Born}}^{\text{tot},+}(s) \\ &+ \frac{2\alpha^3}{3s} L_{m_e} \left\{ \beta \left(-\frac{1}{4c_w^4} + 32 + 48c_w^2 + \frac{32M_W^2}{s} \right) - \frac{1}{4c_w^4} \ln\left(\frac{1+\beta}{1-\beta}\right) \left[1 + \frac{M_Z^2}{s} (2 + 14c_w^2) \right] \right. \\ &+ \beta \left(1 - \frac{1}{4c_w^2} \right) \frac{s^2 + M_Z^4}{(s - M_Z^2)^2} \left(\frac{1}{c_w^2} + 20 + 12c_w^2 \right) - \frac{s^2 \kappa}{(s - M_Z^2)^2} \\ &\times \arctan\left(\frac{\beta}{\kappa}\right) \left[\left(\frac{1}{2c_w^4} + \frac{1}{c_w^2} + 54 + 60c_w^2 \right) + \frac{M_Z^2}{s} \left(\frac{7}{c_w^2} - 88 - 84c_w^2 \right) \right. \\ &\left. \left. + \frac{M_Z^4}{s^2} \left(\frac{3}{2c_w^4} + \frac{17}{c_w^2} - 14 + 12c_w^2 \right) - \frac{M_Z^6}{s^3} \left(\frac{1}{c_w^4} + \frac{9}{c_w^2} + 20 + 36c_w^2 \right) \right] \right\}, \quad (32) \end{aligned}$$

$$\begin{aligned} \delta\sigma_{\text{coll}}^{\text{tot},-}(s) &= \frac{2\alpha}{\pi} L_{m_e} \left[\ln\left(\frac{s\beta^2}{M_W^2}\right) + \frac{3}{4} \right] \sigma_{\text{Born}}^{\text{tot},-}(s) \\ &- \frac{2\alpha^3}{s s_w^4} L_{m_e} \left(1 + \frac{2M_W^2}{s} \right) \left[2\text{Li}_2\left(\frac{1-\beta}{2}\right) - \frac{\pi^2}{6} - \frac{1}{2} \ln^2\left(\frac{1+\beta}{1-\beta}\right) + \ln^2\left(\frac{1-\beta}{2}\right) \right] \\ &- \frac{4\alpha^3 c_w^2}{s^4 w} L_{m_e} \frac{(2+c_w^2)(2c_w^2-1)(s^2+M_Z^4)}{s^2(s-M_Z^2)} \left[4 \arctan(\kappa) \arctan\left(\frac{\beta}{\kappa}\right) \right. \\ &\left. + 2\text{Re}\left\{ \text{Li}_2\left(\frac{\kappa^2-\beta+i\kappa(1+\beta)}{4c_w^2}\right) - \text{Li}_2\left(\frac{\kappa^2+\beta+i\kappa(1-\beta)}{4c_w^2}\right) \right\} \right] \\ &+ \frac{4\alpha^3}{s_w^2 c_w^2} L_{m_e} \frac{1}{s-M_Z^2} \left[2\text{Li}_2\left(\frac{2\beta}{1+\beta}\right) + \frac{1}{2} \ln^2\left(\frac{1+\beta}{1-\beta}\right) \right] \\ &\times \left[-\frac{c_w^2}{s_w^2} + \frac{M_W^2}{s} \left(2 - \frac{1}{s_w^2} \right) + \frac{2M_W^4}{s^2} \left(5 - \frac{2}{s_w^2} \right) + \frac{4M_W^6}{s^3} \right] \\ &+ \frac{\alpha^3 \beta}{c_w^8} L_{m_e} \frac{s}{(s-M_Z^2)^2} \left[\frac{M_W^6}{s^3} \left(\frac{100}{3} - \frac{6}{s_w^2} - \frac{146s_w^2}{3} + \frac{64s_w^4}{3} \right) \right. \\ &+ \frac{M_W^4}{s^2} \left(\frac{352}{3} + \frac{51}{4s_w^4} - \frac{331}{6s_w^2} - \frac{355s_w^2}{3} + \frac{122s_w^4}{3} + \frac{8s_w^6}{3} \right) \\ &+ \frac{M_W^2}{s} \left(-\frac{862}{3} - \frac{69}{4s_w^4} + \frac{1087}{12s_w^2} + 534s_w^2 - \frac{1574s_w^4}{3} + \frac{742s_w^6}{3} - \frac{128s_w^8}{3} \right) \\ &\left. + \frac{4117}{12} + \frac{51}{4s_w^4} - \frac{278}{3s_w^2} - \frac{1483s_w^2}{2} + \frac{2824s_w^4}{3} - \frac{2051s_w^6}{3} + \frac{782s_w^8}{3} - 40s_w^{10} \right] \\ &+ \frac{\alpha^3}{c_w^{10}} L_{m_e} \frac{s(2c_w^2-1)\kappa}{(s-M_Z^2)^2} \arctan\left(\frac{\beta}{\kappa}\right) \left[\frac{M_W^6}{s^3} \left(55 - \frac{45}{2s_w^2} - \frac{134s_w^2}{3} + 12s_w^4 \right) \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{M_W^4}{s^2} \left(\frac{68}{3} + \frac{33}{4s_w^4} - \frac{97}{4s_w^2} - \frac{4s_w^2}{3} - \frac{28s_w^4}{3} + 4s_w^6 \right) \\
& + \frac{M_W^2}{s} \left(\frac{2363}{6} + \frac{33}{2s_w^4} - \frac{265}{2s_w^2} - \frac{1159s_w^2}{2} + 453s_w^4 - \frac{538s_w^6}{3} + 28s_w^8 \right) \\
& - \frac{3893}{12} - \frac{33}{4s_w^4} + \frac{343}{4s_w^2} + \frac{2501s_w^2}{4} - 683s_w^4 + \frac{1286s_w^6}{3} - 144s_w^8 + 20s_w^{10} \Big] \\
& + \frac{\alpha^3}{s c_w^6} L_{m_e} \ln \left(\frac{1+\beta}{1-\beta} \right) \left[2 + \frac{5}{8s_w^4} - \frac{47}{24s_w^2} - \frac{2s_w^2}{3} - \frac{12c_w^6}{s_w^2} \frac{M_W^4}{s^2} \right. \\
& \left. + \frac{M_W^2}{s} \left(\frac{370}{3} + \frac{19}{2s_w^4} - \frac{233}{4s_w^2} - \frac{356s_w^2}{3} + 52s_w^4 - 8s_w^6 \right) \right] . \tag{33}
\end{aligned}$$

The factor δ_{soft} contains the exact $\ln \delta$ - and $\ln \beta$ -dependence of the soft bremsstrahlung taken from Ref. [2]; $\delta = \Delta E_\gamma/E$ denotes the soft photon cut-off which is chosen to be 0.05 for all numerical evaluations. The collinear $\ln m_e$ terms follow again from Ref. [8]

$$\delta_{\text{soft}} = \frac{2\alpha}{\pi} \left[\ln \left(\frac{s}{m_e^2} \right) \left(\ln \delta + \frac{3}{4} \right) + \ln \delta \left\{ -2 + \ln \left(\frac{M_W^2 - t}{M_W^2 - u} \right) + \frac{s - 2M_W^2}{s\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) \right\} - \ln \beta \right] . \tag{34}$$

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