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Gauged BRST Symmetry and Covariant Gravitational Anomalies

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Abstract

We derive generalized descent equations which contain both the consistent and covariant gravitational anomalies. The derivation makes use of the natural field-reparametrizations which occur in the process of gauging the (anti-) BRST-symmetry and it leads to an expression for the covariant anomaly which satisfies strong covariance conditions.

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1 Introduction

In this paper, we present an algebraic approach to the determination of covariant anomalies and Schwinger terms of gravitational type. The method has recently been introduced in the simpler case of Yang-Mills theories [1]. It is based on the usual algebraic approach to consistent anomalies [2] and unlike other formalisms [3], it does not require the introduction of various homotopy operators. The basic feature of our formalism is the use of the so-called tilde-fields [4] which are the natural variables of the gauged (anti-) BRST symmetry [5]. They allow the derivation of generalized descent equations which contain consistent and covariant anomalous terms and which express the consistency [6] (resp. covariance [3]) conditions that these terms satisfy. A crucial ingredient of our construction is that the global BRST algebra preserves its form under the field-redefinitions dictated by the gauging of the anti-BRST symmetry.

The gravitational case has already been considered briefly along these lines in reference [1]. But, contrary to the Yang-Mills situation, the crucial criterion we just mentioned was not fulfilled there and thus a systematic extension from YM to gravitational theories could not be given. Here, we overcome this drawback by considering a different scheme for gauging (localizing) the anti-BRST symmetry. The non-uniqueness of this procedure is due to the fact that there exists some freedom in the choice of the variations of the local BRST parameter and gauge field. In the present work, both of these fields are assumed to transform canonically under infinitesimal diffeomorphisms, i.e. with the Lie derivative. Thanks to this choice (which has not been considered previously in the literature), the crucial criterion cited above is satisfied.

The general arena we will work in is a (pseudo-)Riemannian manifold of even dimension d and parametrized by real local coordinates x^μ . For simplicity, we will restrict our attention to local considerations and not deal with global issues. A proper treatment of the latter on non-parallelizable manifolds requires the introduction of a background connection which allows the lifting of vector fields from the base manifold to the bundle of orthonormal frames [7].

Our paper is organized as follows. First, we give a derivation of descent equations for consistent gravitational anomalies which is more concise than any other derivation we are aware of¹ [7][9][10][11]. Then, the tilde-variables which are needed in the sequel are derived and combined with the previous results to obtain the generalized descent equations.

2 Consistent gravitational anomalies

The basic variables are the vielbein forms $e^a = e_\mu^a dx^\mu$, the metric $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$ and the connection 1-form $\Gamma_\mu^\lambda = \Gamma_{\mu\nu}^\lambda dx^\nu$ which is associated with the Levi-Civita connection $\Gamma_{\mu\nu}^\lambda$. Let L_ξ denote the Lie derivative along ξ , the ghost vector field of diffeomorphisms, and let

$$v_\mu^\lambda \equiv \partial_\mu \xi^\lambda, \quad (Dv)^\lambda_\mu \equiv Dv_\mu^\lambda \equiv dv_\mu^\lambda + \Gamma_\mu^\lambda{}_\rho v_\rho^\rho + v_\rho^\lambda \Gamma_\mu^\rho{}_\rho.$$

¹The underlying differential algebra is implicit in [8].

The action of L_ξ on forms is given by $L_\xi = i_\xi d + (-1)^{\deg \xi} di_\xi$ where i_ξ denotes the contraction along ξ and $\deg \xi$ the ghost number of ξ .

The (nilpotent) BRST-operation s is defined on the basic variables by

$$\begin{aligned} s e^a &= L_\xi e^a \\ s g_{\mu\nu} &= (L_\xi g)_{\mu\nu} \\ s \Gamma &= L_\xi \Gamma - Dv \\ s \xi &= \frac{1}{2} L_\xi \xi = \frac{1}{2} [\xi, \xi] = \xi^\mu (\partial_\mu \xi^\nu) \partial_\nu, \end{aligned} \quad (1)$$

where $[\xi, \xi]$ denotes the graded Lie bracket.

It is convenient to redefine the differential s by $s_\xi \equiv s - L_\xi$ [8] which implies $s_\xi \xi = -\frac{1}{2} [\xi, \xi]$. By virtue of this operation, the BRST algebra for Γ and v takes the same form as the one for the connection and ghost field in YM-theories:

$$\begin{aligned} s_\xi \Gamma &= -Dv \\ s_\xi v &= -v^2, \quad (s_\xi)^2 = 0. \end{aligned} \quad (2)$$

These transformation laws are equivalent to the 'horizontality condition' for the curvature 2-form $R = d\Gamma + \Gamma\Gamma$:

$$\hat{R} = R, \quad (3)$$

where

$$\hat{R} \equiv d\hat{\Gamma} + \hat{\Gamma}\hat{\Gamma}, \quad d \equiv d + s_\xi, \quad \hat{\Gamma} \equiv \Gamma + v. \quad (4)$$

(Note that $\hat{d}^2 = 0$ by virtue of $L_\xi d + dL_\xi = 0$.)

As usual [2], the descent equations are derived by combining the horizontality condition (3) with the transgression formula involving the Chern-Simons polynomial T_{2n-1} ,

$$dT_{2n-1}(\Gamma) = \text{Tr } R^n, \quad (5)$$

i.e. from the relation

$$\hat{d}T_{2n-1}(\hat{\Gamma}) = \text{Tr } \hat{R}^n. \quad (6)$$

If one expands the $(2n-1)$ -form $T_{2n-1}(\hat{\Gamma})$ with respect to the ghost degree,

$$T_{2n-1}(\hat{\Gamma}) = \sum_{k=0}^{2n-1} T_{2n-1-k}^k(\Gamma, v), \quad (7)$$

where the upper index represents the ghost number, then eq.(5) is equivalent to

$$\begin{aligned} dT_{2n-1}^0 &= \text{Tr } R^n \\ s_\xi T_{2n-k-1}^k + dT_{2n-k-2}^{k+1} &= 0 \quad \text{for } 0 \leq k \leq 2n-2 \\ s_\xi T_{2n-1}^{2n-1} &= 0. \end{aligned} \quad (8)$$

For instance, for $n = 2$ (which is relevant for $d = 2$ space-time dimensions), one finds the explicit expressions

$$\begin{aligned} T_3^0 &= \text{Tr}(\Gamma d\Gamma + \frac{2}{3}\Gamma^3), \quad T_2^1 = \text{Tr}(v d\Gamma), \quad T_1^2 = \text{Tr}(-v^2\Gamma), \quad T_0^3 = \text{Tr}(-\frac{1}{3}v^3). \end{aligned}$$

Now, consider the descent equation with $k = 1$ in $d = 2n - 2$ space-time dimensions. If one integrates over space-time (which is assumed to be boundaryless), then it follows that

$$s \int T_{2n-2}^1(\Gamma, \xi) = 0, \quad (9)$$

because $L_\xi T_{2n-2}^1 = i_\xi dT_{2n-2}^1 - di_\xi T_{2n-2}^1$ and because the $(2n - 1)$ -form dT_{2n-2}^1 vanishes identically in $2n - 2$ dimensions. The last equation is the Wess-Zumino consistency condition [6] and identifies the cocycle $\int T_{2n-2}^1$ as a candidate for the consistent anomaly [2].

Since the previous results coincide with those of YM-theories [2][1] modulo the substitution $A \rightarrow \Gamma$, $c \rightarrow v$, $s \rightarrow s_\xi$, they represent a good basis for the algebraic study of covariant anomalies along the lines of YM-theories [1]. The necessary ingredients for this generalization will be developed in the next section and then applied in section 4.

3 Gauging of the (anti-)BRST symmetry

The integrated covariant anomaly viewed as a linear functional of $\bar{\xi}$, the antighost vector field of diffeomorphisms, satisfies the covariance condition [3] [1]

$$s\Delta_{cov}[\Gamma, \bar{\xi}] = 0, \quad (10)$$

where the action of s on $\bar{\xi}$ is defined by $s\bar{\xi} = L_\xi \bar{\xi}$. Our algebraic method for constructing solutions of this equation makes use of the so-called tilde-variables of the gauged anti-BRST symmetry [4] [1]. This section is devoted to a derivation of these variables and the determination of their transformation laws under diffeomorphisms.

First, we introduce the global anti-BRST operation $\bar{s}$ which mimics the global BRST-operation (eq.(1)) :

$$\begin{aligned} \bar{s}e^a &= L_\xi e^a \\ \bar{s}\Gamma &= L_\xi \Gamma - \mathcal{D}\bar{v} \quad , \quad \bar{v}_\mu^\lambda \equiv \partial_\mu \bar{\xi}^\lambda \\ \bar{s}\bar{\xi} &= \frac{1}{2} L_\xi \bar{\xi} = \frac{1}{2} [\bar{\xi}, \bar{\xi}] \quad . \end{aligned} \quad (11)$$

The s and $\bar{s}$ actions are related by [12] [8]

$$\bar{s}\xi + s\bar{\xi} = [\xi, \bar{\xi}] \quad (12)$$

which degeneracy can be lifted by assuming that

$$s\bar{\xi} = L_\xi \bar{\xi} = [\xi, \bar{\xi}] \quad , \quad \bar{s}\xi = 0 \quad . \quad (13)$$

The gauged (anti-) BRST algebra results from a combination of the global (anti-) BRST symmetry and the ghost number symmetry [4]. Following this systematic approach, we introduce a ghost field for the localized anti-BRST symmetry, i.e. a real commuting function $\bar{\omega}(x)$, and rescale the basic fields ($\xi, \bar{\xi}$ and e^a) by its exponential according to

$$\bar{\xi} \rightarrow e^{\bar{\omega}} \bar{\xi} \quad , \quad \xi \rightarrow e^{-\bar{\omega}} \xi \quad , \quad e^a \rightarrow e^a \quad . \quad (14)$$

Here, the signs in the exponents are dictated by the ghost number of the corresponding field.

Next, we introduce a gauge field $\bar{\alpha}$ for the localized anti-BRST symmetry $\bar{s}_{loc}$: $\bar{\alpha}$ is an Abelian connection 1-form with ghost number one. There is a certain arbitrariness in choosing the variations of $\bar{\omega}$ and $\bar{\alpha}$. In reference [1], the variable $\bar{\alpha}$ was assumed to be inert under global $\bar{s}$ -transformations and $\bar{\omega}$ was assumed to satisfy $\bar{s}_{loc}\bar{\omega} = 0$. This choice implies that all fields have to be redefined in the tilde-basis associated with $\bar{s}_{loc}$. Then, unlike the situation in Yang-Mills theories [1], the s -operator does not act in the same way on the tilde-fields as it does on the original fields. This in turn implies that the derivation of descent equations for the covariant anomalous terms cannot be done in any obvious way.

In the present work, we make a different choice which is actually more natural from a geometric point of view: we assume that $\bar{\alpha}$ transforms with $\bar{\xi}$ according to

$$\bar{s}\bar{\alpha} = L_{\bar{\xi}}\bar{\alpha} \quad (15)$$

and that $\bar{\omega}$ varies under $\bar{s}_{loc}$ according to

$$\bar{s}_{loc}e^{\bar{\omega}} = e^{\bar{\omega}} L_{\bar{\xi}}e^{\bar{\omega}} \quad . \quad (16)$$

As usual [4] [1], the $\bar{s}$ -variations of the basic fields, eqs.(11),(13), are postulated to preserve their form under the rescaling (14):

$$\begin{aligned} \bar{s}_{loc}(e^{\bar{\omega}}\bar{\xi}^\lambda) &= (e^{\bar{\omega}}\bar{\xi}^\mu)\partial_\mu(e^{\bar{\omega}}\bar{\xi}^\lambda) \\ \bar{s}_{loc}(e^{-\bar{\omega}}\bar{\xi}^\lambda) &= 0 \\ \bar{s}_{loc}e^a &= L_{(e^{\bar{\omega}}\bar{\xi})}e^a \quad . \end{aligned} \quad (17)$$

The derivative of $e^{\bar{\omega}}$ on the r.h.s. of the first equation can be eliminated by substituting eq.(16): it immediately follows that

$$\bar{s}_{loc}\bar{\xi}^\lambda = e^{\bar{\omega}}\bar{\xi}^\mu\partial_\mu\bar{\xi}^\lambda = \frac{1}{2}e^{\bar{\omega}}[\bar{\xi}, \bar{\xi}]^\lambda \quad . \quad (18)$$

In view of eq.(15), the local $\bar{s}$ -variation of $\bar{\alpha}$ is postulated to be given by²

$$\begin{aligned} \bar{s}_{loc}\bar{\alpha} &= L_{(e^{\bar{\omega}}\bar{\xi})}\bar{\alpha} + d e^{\bar{\omega}} \\ &= e^{\bar{\omega}} L_{\bar{\xi}}\bar{\alpha} + d e^{\bar{\omega}} (1 - i_{\bar{\xi}}\bar{\alpha}) \quad . \end{aligned} \quad (19)$$

Here, the last term in the first line reflects the fact that $\bar{\alpha}$ represents the gauge field for the symmetry parametrized by $\bar{\omega}$. It is now easy to check that the operation $\bar{s}_{loc}$ defined by eqs.(16),(18),(19) is nilpotent. (One simply uses the standard relations $\{\bar{s}_{loc}, d\} = 0$, $\{\bar{s}_{loc}, L_{\bar{\xi}}\} = L_{(i_{\bar{\xi}}\bar{\alpha})}$ and $[\bar{s}_{loc}, i_{\bar{\xi}}] = i_{(i_{\bar{\xi}}\bar{\alpha})}$.) We remark that the transformation law (19) has previously been considered in reference [13], but the corresponding algebra failed to close, because the parameter $\bar{\omega}$ was assumed to be $\bar{s}_{loc}$ -inert.

²Here and in the sequel, we have omitted the external source field $\bar{t}$, i.e. the connection of the ghost number symmetry [4]: it does not play any rôle in our considerations.

The local $\bar{s}$ -variation of $\bar{\xi}$, eq.(18), does not contain any derivatives of $e^{\bar{\omega}}$ and in the terminology of reference [4], this means that the field $\bar{\xi}$ is *not* redefined in the tilde-basis, i.e.

$$\bar{\xi} = \bar{\xi} . \quad (20)$$

For the local $\bar{s}$ -variation of $e^{\bar{\omega}}$, the last of eqs.(17) implies

$$\bar{s}_{loc} e^{\bar{\omega}} = e^{\bar{\omega}} L_{\bar{\xi}} e^{\bar{\omega}} - (de^{\bar{\omega}}) i_{\bar{\xi}} e^{\bar{\omega}}$$

Expressing the derivative of $e^{\bar{\omega}}$ in terms of $\bar{s}_{loc} \bar{\alpha}$ by virtue of eq.(19) and collecting the resulting terms in a $\bar{s}_{loc}$ -variation on the l.h.s., we obtain

$$\bar{s}_{loc} \left\{ e^{\bar{\omega}} + \Delta \bar{\alpha} (i_{\bar{\xi}} e^{\bar{\omega}}) \right\} = \dots \quad (21)$$

with

$$\Delta \equiv (1 - i_{\bar{\xi}} \bar{\alpha})^{-1} .$$

Here, the dots represent terms which do not involve any derivatives of $e^{\bar{\omega}}$ and therefore, we can read off the tilde-variable associated with $e^{\bar{\omega}}$ from the l.h.s. of this equation :

$$\bar{e}^{\bar{\omega}} = e^{\bar{\omega}} + \Delta \bar{\alpha} (i_{\bar{\xi}} e^{\bar{\omega}}) . \quad (22)$$

In terms of this variable, the variation (21) takes a simple and compact form :

$$\bar{s}_{loc} \bar{e}^{\bar{\omega}} = e^{\bar{\omega}} \left\{ L_{\bar{\xi}} \bar{e}^{\bar{\omega}} - \frac{1}{2} \Delta \bar{\alpha} (i_{\bar{\xi}} \bar{e}^{\bar{\omega}}) \right\} . \quad (23)$$

Since the metric components $g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}$ do not involve any derivatives of the vielbein, the associated tilde-variables are simply given by

$$\begin{aligned} \bar{g}_{\mu\nu} &= \eta_{ab} \bar{e}^a_{\mu} \bar{e}^b_{\nu} \\ &= g_{\mu\nu} - \Delta \bar{\alpha} (g_{\mu\nu}) \bar{\xi}^{\rho} \end{aligned} \quad (24)$$

and eq.(23) then leads to

$$\bar{s}_{loc} \bar{g}_{\mu\nu} = e^{\bar{\omega}} \left\{ (L_{\bar{\xi}} \bar{g})_{\mu\nu} - \frac{1}{2} \Delta \bar{\alpha} (g_{\mu\nu}) [\bar{\xi}, \bar{\xi}]^{\rho} \right\} . \quad (25)$$

For the components of the inverse metric tensor, these equations imply

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \bar{\alpha}_{\rho} g^{\rho(\mu} \bar{\xi}^{\nu)} \quad (26)$$

and

$$\bar{s}_{loc} \bar{g}^{\mu\nu} = e^{\bar{\omega}} \left\{ (L_{\bar{\xi}} \bar{g})^{\mu\nu} + \frac{1}{2} \Delta \bar{\alpha}_{\rho} [\bar{\xi}, \bar{\xi}]^{\rho(\mu} \bar{g}^{\nu)} \right\} . \quad (27)$$

The connection field depends not only on the metric components, but also on their derivatives, according to

$$\Gamma^{\lambda}_{\mu\nu}(g) = \frac{1}{2} g^{\lambda\rho} \left\{ \partial_{(\mu} g_{\nu)\rho} - \partial_{\rho} g_{\mu\nu} \right\} . \quad (28)$$

Therefore, $\bar{\Gamma}$ is not obtained simply by replacing g by $\bar{g}$ in the previous formula. Using the same procedure as the one which led to the determination of $\bar{e}^{\bar{\omega}}$, one finds that it contains an additional contribution :

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}(\bar{g}) + B^{\lambda}_{\mu\nu} \quad (28)$$

with

$$B^{\lambda}_{\mu\nu} = -\frac{1}{2} \Delta \bar{g}^{\lambda\rho} \left[\bar{\alpha}_{(\mu} \bar{s}_{\bar{0}} \bar{g}_{\nu)\rho} - \bar{\alpha}_{\rho} \bar{s}_{\bar{0}} \bar{g}_{\mu\nu} \right] \quad (29)$$

and $\bar{s}_{\bar{0}}$ defined by $\bar{s}_{loc} \bar{g}_{\mu\nu} = e^{\bar{\omega}} \bar{s}_{\bar{0}} \bar{g}_{\mu\nu}$. This completes our derivation of the tilde-variables.

In conclusion, we look at the transformation laws of all these fields under infinitesimal diffeomorphisms (described by the *global* BRST-operator s). Since the action of s on the variables $e^{\bar{\omega}}$, $\bar{\alpha}$, $\bar{\xi}$ is simply given by the Lie derivative $L_{\bar{\xi}}$, this is also true for $\bar{e}^{\bar{\omega}}$ and $\bar{g}$:

$$s \bar{e}^{\bar{\omega}} = L_{\bar{\xi}} \bar{e}^{\bar{\omega}} , \quad s \bar{g}_{\mu\nu} = (L_{\bar{\xi}} \bar{g})_{\mu\nu} . \quad (30)$$

From these transformation laws and the very definition of $\Gamma(\bar{g})$, it immediately follows that (cf. eq.(1))

$$s \Gamma(\bar{g}) = L_{\bar{\xi}} \Gamma(\bar{g}) - \mathcal{D}(\Gamma(\bar{g})) v ,$$

where $\mathcal{D}(\Gamma(\bar{g})) v \equiv dv + [\Gamma(\bar{g}), v]$. A straightforward calculation leads to

$$\begin{aligned} s(\bar{s}_{\bar{0}} \bar{g})_{\mu\nu} &= L_{\bar{\xi}}(\bar{s}_{\bar{0}} \bar{g})_{\mu\nu} \\ s B^{\lambda}_{\mu} &= (L_{\bar{\xi}} B - [B, v])^{\lambda}_{\mu} , \end{aligned}$$

where $B^{\lambda}_{\mu} \equiv B^{\lambda}_{\mu\nu} dx^{\nu}$. Combining the previous results, we conclude that $\bar{\Gamma} = \Gamma(\bar{g}) + B$ changes in the same way under general coordinate transformations as Γ :

$$s \bar{\Gamma} = L_{\bar{\xi}} \bar{\Gamma} - \bar{\mathcal{D}} v \quad (31)$$

with $\bar{\mathcal{D}} v \equiv dv + [\bar{\Gamma}, v]$. This confirms the correctness of the formulae for the tilde-variables and exhibits the general covariance of the whole procedure.

4 Covariant gravitational anomalies

Now, we can forget about the origin of the redefinition (28) and simply use this variable in a global BRST context. Expanding $\bar{\Gamma}$ with respect to $\bar{\alpha}$, one finds

$$\Gamma^{\lambda}_{\mu}(g) = \bar{\Gamma}^{\lambda}_{\mu} - \bar{\alpha} \bar{\Lambda}^{\lambda}_{\mu} + \dots \quad (32)$$

with

$$\bar{\Lambda}^{\lambda}_{\mu}(\bar{\xi}) \equiv \partial_{\mu} \bar{\xi}^{\lambda} + \bar{\xi}^{\nu} \bar{\Gamma}^{\lambda}_{\mu\nu} = \bar{\mathcal{D}}_{\mu} \bar{\xi}^{\lambda} .$$

Substituting eq.(32) into the standard descent equations, eqs.(7), we get

$$s_{\xi} T^k_{2n-k-1}(\bar{\Gamma} - \bar{\alpha} \bar{\Lambda} + \dots, v) + d T^{k+1}_{2n-k-2}(\bar{\Gamma} - \bar{\alpha} \bar{\Lambda} + \dots, v) = 0 . \quad (33)$$

Next, we expand the cocycles $T^{\bar{g}}_p$ with respect to $\bar{\alpha}$ and $d\bar{\alpha}$,

$$T^{\bar{g}}_p(\bar{\Gamma} - \bar{\alpha} \bar{\Lambda} + \dots, v) = T^{\bar{g}}_p(\bar{\Gamma}, v) - \bar{\alpha} \bar{T}^{g-1}_{p-1}(\bar{\Gamma}, v, \bar{\Lambda}) - d\bar{\alpha} \bar{T}^{g-1}_{p-2}(\bar{\Gamma}, v, \bar{\Lambda}) + \dots , \quad (34)$$

where the dots stand for terms containing $\bar{\alpha}_\mu, \partial_\mu \bar{\alpha}_\nu dx^\nu, \dots$ and higher order contributions. Since $\tilde{\Gamma}$ transforms in the same way under s_ξ as Γ and since $s_\xi \bar{\alpha} = 0$, equation (33) has to be satisfied term by term if the T_p^y are expanded according to eq.(34). This yields the following sets of ladder equations :

$$\begin{aligned} \mathcal{O}(1) : \quad & s_\xi T_{2n-k-1}^k(\tilde{\Gamma}, v) + dT_{2n-k-2}^{k+1}(\tilde{\Gamma}, v) = 0 \quad (0 \leq k \leq 2n-1) \\ \mathcal{O}(\bar{\alpha}) : \quad & s_\xi \tilde{T}_{2n-k-2}^{k-1}(\tilde{\Gamma}, v, \Lambda) + d\tilde{T}_{2n-k-3}^k(\tilde{\Gamma}, v, \Lambda) = 0 \quad (0 \leq k \leq 2n-2) \\ \mathcal{O}(d\bar{\alpha}) : \quad & s_\xi \tilde{T}_{2n-k-3}^{k-1}(\tilde{\Gamma}, v, \Lambda) + d\tilde{T}_{2n-k-4}^k(\tilde{\Gamma}, v, \Lambda) = \tilde{T}_{2n-k-3}^k(\tilde{\Gamma}, v, \Lambda) \quad (0 \leq k \leq 2n-3) \end{aligned}$$

The first set is nothing but the ordinary descent equations with the substitution $\Gamma \rightarrow \tilde{\Gamma}$; the other two sets are new descent equations. For $k=0$, the second set contains a candidate for the covariant anomaly.

Substituting the third set into the second one, we find

$$s_\xi(\tilde{T}_{2n-k-2}^{k-1} - d\tilde{T}_{2n-k-3}^{k-1}) = 0, \quad (35)$$

which, for $k=0$, becomes

$$s_\xi(\tilde{T}_{2n-2}^{-1} - d\tilde{T}_{2n-3}^{-1}) = 0. \quad (36)$$

Thus, the quantity $\mathcal{A}_{cov} \equiv \tilde{T}_{2n-2}^{-1} - d\tilde{T}_{2n-3}^{-1}$ satisfies the covariance condition

$$\begin{aligned} s\mathcal{A}_{cov} &= L_\xi \mathcal{A}_{cov} \\ &= -d\tilde{\xi} \mathcal{A}_{cov}, \end{aligned} \quad (37)$$

where the last equality holds in $d = 2n - 2$ dimensions due to the fact that a $(2n-1)$ -form identically vanishes there. Integrating over space-time, we conclude that $\int \mathcal{A}_{cov}(\tilde{\Gamma}, \tilde{\Lambda})$ satisfies the covariance condition (10) and thus represents the solution we have been looking for.

For concreteness, we give the explicit expression for the non-integrated *covariant anomaly* $\mathcal{A}_{cov}$ which one obtains in two dimensions by substituting the polynomials (8):

$$d = 2 : \quad \mathcal{A}_{cov}(\tilde{\Gamma}, \tilde{\xi}) = 2(\tilde{D}_\mu \tilde{\xi}^\lambda) \tilde{R}_\lambda^\mu.$$

This result (in which R represents the curvature 2-form) has first been found in reference [10] by other methods. The fact that this expression satisfies the strong covariance condition (37) has been noticed before [1] following an explicit calculation, but it did not result from a general formalism.

5 Conclusion

The set of ladder equations following from eq.(33) is not limited to those given above. For n large enough, an expansion up to orders $\bar{\alpha}'$ and $\bar{\alpha}'^{l-1} d\bar{\alpha}$ ($l = 1, 2, \dots$) leads to relations which can be combined into

$$s_\xi(\tilde{T}_{2n-k-l-1}^{k-l} - \frac{1}{l} d\tilde{T}_{2n-k-l-2}^{k-l}) = 0 \quad (l = 1, 2, \dots) \quad (38)$$

These equations contain and generalize the covariance condition (35). For $l=2$ and $k=0$, they provide an explicit expression for the covariant Schwinger term [1]. Further sets of covariance conditions of the same type follow from eq.(33) by combining the equations which occur at order $\bar{\alpha}_\mu$ and $d\bar{\alpha}_\mu$ or at order $(\partial_\mu \bar{\alpha}_\nu) dx^\nu$ and $d(\partial_\mu \bar{\alpha}_\nu) dx^\nu$ and so on.

To summarize, we have given a concise derivation of generalized descent equations which contain both the consistent and covariant gravitational anomalies. The Schwinger terms [14] [11] and the Bardeen-Zumino counterterm [10] which relates the consistent and covariant currents can be extracted from these equations in analogy to YM-theories [1]. The tilde-variables which have been determined in section 3 should be of independent interest for the field theoretical realizations of higher cocycles and for the study of current algebras [5] [4] [13] [14] [15].

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