GEOMETRY AND QUANTUM SYMMETRIES OF SUPERSTRING VACUA

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ABSTRACT

We report on some recent progress which has been made in the understanding of general features of superstring compactifications. The following aspects are discussed: special geometry of the moduli space of (2,2) vacua, target-space duality symmetries, differential equations for the periods of holomorphic three-forms, and higher loop (genus) string corrections to the tree-level effective action through target-space modular anomaly cancellations.

*Invited Lecture given at the
International School of Subnuclear Physics
29th Course: "Physics at the Highest Energy and Luminosity"
Erice, Italy - July 1991*

CERN-TH.6213/91
August 1991

I. Introduction

The topics covered in this lecture deal with several aspects of superstring theories compactified on (2,2) vacua^[1]. N=2 internal superconformal field theories^[2] are the quantum (stringy) version of Calabi-Yau (C-Y) manifolds^[3], giving N=1 space-time supersymmetry in D=4 dimensions^[4]. In particular we will give in section II some general formulae for the geometry of the moduli space of Calabi-Yau manifolds which encode all relevant information for the effective action. In section III the relation of special geometry with the Kähler class and complex structure deformations, the mirror symmetry hypothesis and target-space duality will be considered. In section IV we will show how, at least in certain cases, equations known as Picard-Fuchs equation, allow the determination of the periods of holomorphic three-forms (for C-Y threefolds). For any given C-Y manifold this is equivalent to the dynamical determination of the four-dimensional effective action. In section V target-space modular anomalies will be discussed and their relation, because of supersymmetry, with higher loop corrections to gauge couplings.

II. Special geometry and effective actions

In the present section we will summarize some general formulae which encode all relevant informations for the low-energy dynamics of light particles. By this we mean all effective interactions which may occur in an effective Lagrangian containing up to two-derivative terms.

The striking fact is that the couplings are simple enough to be discussed on a general footing for strings compactified on Calabi-Yau manifolds whose world-sheet interpretation is a c=9, (2,2) superconformal field theory with a quantization condition for the U(1) charge^[2].

Much progress has recently been achieved in the study of this particular class of string vacua in three different but related areas, N=2 Landau-Ginzburg models^[5], topological N=2 SCFT's^{[6]–[12]} and the C-Y geometry due to its connection with N=2 space-time supersymmetry^{[13]–[20]}.

Moreover the study of C-Y compactifications for large volume, compared to the string scale which relies on standard Kaluza-Klein compactification of D=10, N=1 supergravity, has recently been extended to include string corrections through the notion of mirror symmetry^[21], which is a property of N=2 superconformal field theories under the exchange of the chiral rings of primary fields. In the target-space interpretation this symmetry allows one to define the notion of C-Y quantum geometry relating pairs of C-Y manifolds with isomorphic even and odd cohomology classes i.e.:

$$\sum_{p+q=3} H^{(p,q)}(C) = \sum_{p=0}^3 H^{(p,p)}(C')$$

(and the same for $C \rightarrow C'$).

This means that Kähler class (complex structure) deformations in C have the opposite meaning in C'.

The relevance of the mirror symmetries comes about because the moduli space for complex structure deformations is purely classical and does not receive α' corrections^{[22],[23]}. Therefore the α' corrected moduli space for the Kähler class deformations in (C) must coincide with the classical moduli space for complex structure

deformations in (C') and viceversa. It is well known from string theory that quantum corrections in α' come from world-sheet instanton effects^{[23],[24]}. In the context of algebraic geometry instantons are related to rational curves, so the transition from a classical to a quantum manifold is achieved by properly taking into account all rational curves of any degree in the Kähler class moduli geometry^[25].

As a consequence of this mirror duality for (2,2) vacua it is sufficient to discuss the moduli space for the complex structure. In this case, as we will show with some illustrative examples in section IV, it is possible essentially to resolve the problem by some methods of algebraic geometry^{[26]–[30]}. In fact the periods of the holomorphic 3-form satisfy certain linear differential equations, known as Picard-Fuchs equations, whose solutions encode all relevant quantities of the special geometry of the moduli space and hence give all the couplings in the effective action.

From the string point of view these results are equivalent in solving the superconformal field theory, i.e. in giving all perturbed correlation functions in coupling constant space^{[31],[32]}. The equivalence of these methods has recently been shown in some examples^{[26]–[32]}.

Let us briefly recall the basic elements of special geometry, as derived from N=2 supergravity in D=4 dimensions^{[34],[35],[20]}. In the next section we will recall its connection to C-Y manifolds and deformation theory. The coordinate free description of special Kähler manifolds was given in Refs. [17],[20]. In primis a special Kähler manifold is a restricted (or Hodge) manifold^{[36],[37],[38]}, i.e. the cohomology class of the Kähler form:

$$J = iG_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} \quad , \quad G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \quad \text{II} - 1$$

coincides with the first Chern class of a line bundle $\mathcal{L} : [J] = [c_1]$. J is the exterior derivative of the U(1) connection A of $\mathcal{L}$:

$$J = dA \quad , \quad A = -\frac{i}{2} (K_{,i} dz^i - K_{,\bar{i}} d\bar{z}^{\bar{i}}) \quad \text{II} - 2$$

which under a Kähler transformation:

$$K \rightarrow K - \Lambda - \bar{\Lambda} \quad \text{II} - 3$$

transforms as:

$$A \rightarrow A - d(\text{Im}\Lambda) \quad \text{II} - 4$$

The U(1) covariant differential of a field $\Phi(z, \bar{z})$ of weight p is^[20]:

$$\nabla \Phi = (d + ipA)\Phi \quad \text{II} - 5$$

or in components:

$$\nabla_i \Phi = \left(\partial_i + \frac{1}{2} p K_{,i} \right) \Phi \quad , \quad \nabla_{\bar{i}} \Phi = \left(\partial_{\bar{i}} - \frac{1}{2} p K_{,\bar{i}} \right) \Phi \quad \text{II} - 6$$

A covariantly holomorphic field of weight p satisfies:

$$\nabla_{\bar{i}} \Phi = 0 \quad \text{II} - 7$$

By a change of trivialization, the real $U(1)$ bundle can be reduced to a holomorphic bundle. Indeed by setting:

$$\tilde{\Phi} = e^{-pK/2} \Phi \quad \text{II} - 8$$

we have:

$$\nabla_{\bar{i}} \tilde{\Phi} = (\partial_i + pK_{,i}) \tilde{\Phi} \quad , \quad \nabla_{\bar{i}} \tilde{\Phi} = \partial_{\bar{i}} \tilde{\Phi} \quad \text{II} - 9$$

In particular the $U(1)$ transformation on $\tilde{\Phi}$:

$$\Phi \rightarrow e^{p/2(\Lambda - \bar{\Lambda})} \Phi \quad \text{II} - 10$$

induces a holomorphic transformation on $\tilde{\Phi}$:

$$\tilde{\Phi} \rightarrow e^{p\Lambda} \tilde{\Phi} \quad \text{II} - 11$$

A restricted Kähler manifold is special if and only if there exists a completely symmetric holomorphic three-index section:

$$\partial_{\bar{m}} W_{ijK} = 0 \quad \text{of weight } p = 2 \quad \text{II} - 12$$

such that^{[34],[35],[36],[19]}:

$$R_{i\bar{j}\ell\bar{m}} = +G_{i\bar{j}}G_{\ell\bar{m}} + G_{i\bar{m}}G_{\ell\bar{j}} - W_{i\ell p}\bar{W}_{j\bar{m}\bar{p}}G^{p\bar{p}}e^{2K} \quad \text{II} - 13$$

From the Bianchi identities it follows that the (covariantly holomorphic) tensor $C_{i\ell p} = e^K W_{i\ell p}$ satisfies^{[34],[19]}:

$$C_{i\ell p} = \nabla_i \nabla_{\ell} \nabla_p S \quad \text{II} - 14$$

when S is a scalar of weight $p = 2$. Compatibility of eq. II-13 with $N=2$ space-time supersymmetry allows one to construct the metric and W in terms of some prepotential function F . In Refs. [17], [20] it was shown that on a special n -dimensional Kähler manifold one can always introduce $(n+1)$ holomorphic sections X^I :

$$\partial_{\bar{i}} X^I(z) = 0 \quad I = 0 \dots n \quad \text{II} - 15$$

and a function $F(X)$, holomorphic and homogeneous of degree two in the X^I 's:

$$X^I \frac{\partial F}{\partial X^I} = 2F \quad \text{II} - 16$$

i.e., F is a holomorphic section in $\mathcal{L} \times \mathcal{L}$. Since X^I are holomorphic sections of $\mathcal{L}$, a covariant derivative is defined as:

$$\nabla_i X^I = (\partial_i + K_{,i}) X^I \quad \text{II} - 17$$

All relevant quantities can now be expressed through the holomorphic section $F(X)$. Let us define:

$$F_{\Lambda\Sigma\dots} = \frac{\partial}{\partial X^{\Lambda}} \frac{\partial}{\partial X^{\Sigma}} \dots F \quad , \quad N_{\Lambda\Sigma} = F_{\Lambda\Sigma} + \bar{F}_{\bar{\Lambda}\bar{\Sigma}} \quad \text{II} - 18$$

then:

$$G_{i\bar{j}} = \partial_i X^\Lambda \partial_{\bar{j}} \bar{X}^\Sigma \partial_\Lambda \bar{\partial}_\Sigma K = -f_i^\Lambda f_{\bar{j}}^\Sigma N_{\Lambda\Sigma} \quad \text{II} - 19$$

when:

$$K = -\ln(N_{\Lambda\Sigma} X^\Lambda \bar{X}^\Sigma) = -\ln(X^\Lambda \bar{F}_\Lambda + \bar{X}^\Lambda F_\Lambda) \quad \text{II} - 20$$

$$f_i^\Lambda = \nabla_i L^\Lambda = e^{K/2} \left(\delta_\Sigma^\Lambda - \frac{X^\Lambda (N \bar{X})_\Sigma}{X N \bar{X}} \right) \partial_i X^\Sigma \quad \text{II} - 21$$

with $L^\Lambda = e^{K/2} X^\Lambda$:

$$S = -\frac{1}{4} N_{\Lambda\Sigma} L^\Lambda L^\Sigma = -\frac{1}{4} e^K N_{\Lambda\Sigma} X^\Lambda X^\Sigma \quad \text{II} - 22$$

$$W_{i\ell m} = e^{-K} \nabla_i \nabla_\ell \nabla_m S = e^{-K} f_i^\Lambda f_\ell^\Sigma f_m^\Gamma F_{\Lambda\Sigma\Gamma} = \partial_i X^\Lambda \partial_\ell X^\Sigma \partial_m X^\Gamma F_{\Lambda\Sigma\Gamma} \quad \text{II} - 23$$

In Ref. [17] it was shown that $\left(X^\Lambda, \frac{i\partial F}{\partial X^\Sigma} \right)$ can be viewed as sections of a flat holomorphic $\text{Sp}(2n+2)$ -bundle whose existence is in fact equivalent to eq. II-13.

Connection of the previous formulae with the effective actions comes from the fact that heterotic or type II superstrings compactified on C-Y threefolds give D=4 superstrings with N=1 or N=2 space-time supersymmetry respectively^{[13],[14]}. Because of a special mapping^[14] between heterotic and type II superstrings compactified on a same (2,2) vacuum, the couplings in the effective actions are actually all given by formulae from the special geometry^{[17]-[19]}.

Let us first consider briefly type II superstrings. In type II compactifications on C-Y manifolds the moduli scalar corresponding to Kähler-class deformations ((1,1) moduli) are in N=2 vector multiplets in type IIA theories and in hypermultiplets in type IIB theories. The reverse is true for (2,1) moduli corresponding to complex structure deformations. Also because of mirror symmetry the type IIA theory on C turns out to be identical to the type IIB theory in C'. These arguments can be used to show that the moduli geometry is the direct product of two special geometries, one for (1,1) moduli and the other for (2,1) moduli^{[14],[22],[36]}. Because of the mirror hypothesis and absence of stringy corrections one can then confine the discussion to (2,1) moduli in type IIA and IIB theories.

The Kaluza-Klein analysis was performed in Ref. [18]. In type IIA theories the N=2 partners of the (2,1) moduli are $2(h_{(1,2)} + 1)$ Ramond-Ramond scalars. Together with the dilaton and antisymmetric tensor they form a special quaternionic manifold of dimension $4(h_{(2,1)} + 1)$ studied in Ref. [19] which also gives all the coupling in type II effective actions.

In type IIB theories the (2,1) moduli have $h_{(2,1)}$ vector superpartners, which have a non-trivial coupling to the graviphoton. Again all the couplings are fixed by the special geometry^{[17],[20]}. For example the two-derivative couplings of the $(h_{(2,1)} + 1)$ R-R vectors with field strengths $F_{\mu\nu}^{\Lambda\Sigma}(\mu, \nu = 1 \dots 4)(\Lambda, \Sigma = 0 \dots h_{2,1})$ are given by:

$$-\sqrt{-g}(4\text{Re}\mathcal{N}_{\Lambda\Sigma}(z, \bar{z})F_{\mu\nu}^\Lambda F^{\Sigma\mu\nu} - 2i\text{Im}\mathcal{N}_{\Lambda\Sigma}(z, \bar{z})F_{\mu\nu}^\Lambda F_{\rho\alpha}^\Sigma \epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}) \quad \text{II} - 24$$

where:

$$\mathcal{N}_{\Lambda\Sigma} = -\bar{F}_{\Lambda\Sigma} + \frac{1}{(\text{XNX})} (\text{NX})_{\Lambda} (\text{NX})_{\Sigma} \quad \text{II} - 25$$

with $F_{\Lambda\Sigma}, N_{\Lambda\Sigma}$ defined by eq. II-18.

Let us now consider N=1 heterotic superstrings^[39]. In this case (1,1) and (2,1) moduli are related to $h_{(1,1)}$ (27) E_6 families and $h_{(2,1)}$ ($\bar{27}$) E_6 antifamilies. Denoting by $\Phi_{27}^i, \Phi_{\bar{27}}^{\alpha}$ ($i = 1 \dots h_{(1,1)}, \alpha = 1 \dots h_{(2,1)}$) the complex scalars of the E_6 charged chiral multiplets, all the effective couplings are given by the moduli metrics (given by eq. II-19), by the superpotential:

$$\begin{aligned} W(\phi, \psi, \Phi_{27}, \Phi_{\bar{27}}) &= W_{i\ell m}^1(\phi) \Phi_{27}^i \Phi_{27}^{\ell} \Phi_{27}^m + \\ &+ W_{\alpha\beta\gamma}^2(\phi) \Phi_{\bar{27}}^{\alpha} \Phi_{\bar{27}}^{\beta} \Phi_{\bar{27}}^{\gamma} \end{aligned} \quad \text{II} - 26$$

and the family metrics^[36]:

$$G_{i\bar{j}}^{27} = e^{-(K_1 - K_2)/3} G_{i\bar{j}}^1 \quad \text{II} - 27$$

$$G_{\alpha\bar{\beta}}^{\bar{27}} = e^{-(K_2 - K_1)/3} G_{\alpha\bar{\beta}}^2 \quad \text{II} - 28$$

where $K_1(\phi, \bar{\phi}), K_2(\psi, \bar{\psi})$ are the Kähler potentials for the two special geometries of (1,1) and (2,1) moduli and $G^1 = \partial\bar{\partial}K^1, G^2 = \partial\bar{\partial}K^2$ are the related metrics given by eqs. II-19, II-20.

Here ϕ, ψ denote the coordinates for (1,1) and (2,1) moduli respectively.

In section V we will show that also some higher-loop corrections to the heterotic effective action can be expressed in terms of quantities related to the special geometry of the moduli space.

III. Deformation of the complex structure and target-space duality symmetry

The relation of the special geometry with algebraic geometry arises with the interpretation of the Kähler metric $G_{i\bar{j}}$ for (2,1) moduli as the corresponding metric for the moduli space for the complex structure deformations, i.e. the Weil-Petersson metric^{[40],[41],[16]}:

$$g_{i\bar{j}}^{\text{WP}} = i \int_C \phi_i \wedge \phi_j^* \quad \text{III} - 1$$

when ϕ_i is a (moduli-dependent) basis for the $H^{(2,1)}$ Dolbeault cohomology:

$$\phi_i = \phi_{i,a\bar{b}\bar{c}} dx^a \wedge dx^b \wedge d\bar{x}^c \quad \text{III} - 2$$

(x are complex coordinates on C).

The precise relation is:

$$G_{i\bar{j}} = + \frac{1}{i \int \Omega \wedge \bar{\Omega}} g_{i\bar{j}}^{\text{WP}} = -\partial_i \partial_{\bar{j}} \log i \int \Omega \wedge \bar{\Omega}, i \int \Omega \wedge \bar{\Omega} = X^A \bar{F}_A + \bar{X}^A F_A \quad \text{III} - 3$$

where $\Omega(\psi)$ is the deformed holomorphic three-form and by definition:

$$\int \Omega \wedge \phi_i = \int \Omega \wedge \bar{\phi}_i = 0 \quad \text{III} - 4$$

From the fact that ϕ is a “holomorphic” deformation of the (3,0)-form Ω , it follows that:

$$\phi_i = \nabla_i \Omega = \frac{\partial \Omega}{\partial \psi_i} - \frac{1}{(\Omega, \bar{\Omega})} \left(\frac{\partial \Omega}{\partial \psi_i}, \bar{\Omega} \right) \Omega \quad \text{III} - 5$$

The holomorphic sections $X^A(\psi), F_A(\psi)$ of the special geometry are identified^{[15]–[17],[20]} with the periods of Ω along the $b_3 = 2(h_{2,1} + 1)$ homology cycles A^A, B_A :

$$\Omega(\psi) = X^A(\psi) \alpha_A + i F_A(\psi) \beta^A \quad \text{III} - 6$$

where α_A, β^A is a (fixed) cohomology basis in H^3 dual to the homology cycles:

$$\begin{aligned} \int_M \alpha_A \wedge \beta^B &= - \int_M \beta^B \wedge \alpha_A = \delta_A^B \\ \int_M \alpha_A \wedge \alpha_B &= \int_M \beta^A \wedge \beta^B = 0 \end{aligned} \quad \text{III} - 7$$

From the fact that $\int \Omega \wedge \frac{\partial \Omega}{\partial \psi_i} = 0$, it follows that $F_A = \frac{\partial F}{\partial X^A}$ with $X^A F_A = 2F$ and then eq. II-20 follows from eq. III-3.

The holomorphic three-index section given by eq. II-23 is given by the following expression:

$$W_{i\ell m} = i \int \Omega \wedge \frac{\partial \Omega}{\partial \psi_i \partial \psi_\ell \partial \psi_m} \quad \text{III} - 8$$

while the matrix N (II-18) and $f_i^\wedge$ (II-2)^[20]:

$$N_{AB} = i \int \frac{\partial \Omega}{\partial X^A} \wedge \frac{\partial \bar{\Omega}}{\partial \bar{X}^B}, \quad f_i^A = e^{K/2} \int \nabla_i \Omega \wedge \beta^A \quad \text{III} - 9$$

If we consider the vector space $H^{(3,0)} + H^{(2,1)} \subset H^3(M)$ which varies holomorphically with ψ , a basis is given by:

$$\frac{\partial \Omega}{\partial X^A} = \alpha_A + i F_{AB} \beta^B \quad \text{III} - 10$$

With respect to this basis the period matrix takes the simple form:

$$\omega = (\mathbb{1}, iF) = (\mathbb{1}, \Omega) \quad \text{III} - 11$$

It then follows that the matrix N is the imaginary part of the period matrix $N = \text{Im} \Omega$. It also follows that^[35]:

$$\begin{aligned} \det(-N) &= \det G e^{-(n+1)K} \\ -\partial_i \partial_{\bar{j}} \lg \det N &= W_{ipq} \bar{W}_{\bar{j}\bar{p}\bar{q}} g^{WP_{p\bar{p}}} g^{WP_{q\bar{q}}} \end{aligned} \quad \text{III} - 12$$

which relates, in special geometry, the determinant of $\text{Im} \Omega$ and the determinant of the moduli metric.

All previous formulae apply both for (2,1) and (1,1) moduli, because of the mirror hypothesis. However the existence of a holomorphic prepotential for (1,1) moduli is also a consequence of the special geometry of N=2 supergravity^[34]. This allows one to identify a classical (large values of the moduli) limit of the holomorphic prepotential, like the one obtained from a C-Y compactification of D=10 supergravity.

In this case the holomorphic prepotential^[14] is a cubic form, related to the intersection matrices^[42]:

$$F_{\text{classical}} = i d_{ABC} \frac{X^A X^B X^C}{X^0} = (\text{id}_{i\ell m} \psi^i \psi^\ell \psi^m) (X^0)^2 \quad \text{III} - 13$$

where:

$$d_{ABC} = \int_M V_A \wedge V_B \wedge V_C \quad \text{III} - 14$$

and V_A is a basis for the $H^{(1,1)}$ cohomology.

Eq. III-13 has been written in the “special coordinates”^{[35],[19]} $\phi_i = X^A/X^0$ ($A = 1 \dots h$). These are actually the coordinates in which N=2 supergravity was first formulated^[34]. In this coordinate basis the Kähler potential and Yukawa couplings become^[15]:

$$K = -\log (2\mathcal{F} + 2\bar{\mathcal{F}} - (\psi_i - \bar{\psi}_i)(\mathcal{F}_i - \bar{\mathcal{F}}_i)) \quad \text{III} - 15$$

$$W_{i\ell m} = \partial_i \partial_\ell \partial_m \mathcal{F} \quad , \quad \mathcal{F} = (X^0)^{-2} F(X) \quad \text{III} - 16$$

The relation of special geometry with Dolbeault cohomology of C-Y manifolds exhibits a symplectic structure $\text{Sp}(b_3; \mathcal{Z})$ corresponding to a change of basis of the homology cycles, as is evident from eq. III-6. From the point of view of N=2, D=4 supergravity, subgroups of $\text{Sp}(b_3; \mathbb{R})$ are related to the so-called duality transformations^{[43],[44],[14]}. They correspond to isometries of special Kähler manifolds. Discrete subgroups Γ of $\text{Sp}(b_3; \mathcal{Z})$ for which:

$$K(\Gamma\psi) = K - \Lambda(\psi) - \bar{\Lambda}(\bar{\psi}) \quad \text{III} - 17$$

correspond to the action of global diffeomorphisms on the homology basis. In string theory Γ is the target-space modular group (target-space duality) and it must be modded out to define the moduli space of the corresponding superconformal field theory^[45].

The identification of points in the moduli space connected by Γ means actually that the moduli space of string theory is not a manifold M but an orbifold M/Γ with fixed points. In the effective action, Γ appears as an exact symmetry^{[46],[47]} and there are arguments that this symmetry survives in (higher genus) string perturbation theory^{[48],[49]}. In section V we will show how target-space modular anomalies are cancelled in string theory and how this mechanism determines the running of gauge couplings induced by dynamical (moduli) scales. To prepare the ground for a discussion on target-space modular anomalies it is important to realize that a duality transformation Γ on the moduli space $\psi \rightarrow \Gamma\psi$ acts with a Kähler phase on the Yukawa couplings and with rescaling and $U(1)$ phases on all the fermions of the theory^{[50]–[53]}.

Since fermions are coupled to gauge and gravitational fields, these (field-dependent) phases can induce mixed σ -model-gauge (or gravitational) anomalies through ordinary (one-loop) triangular graphs in which a σ -model $U(1)$ connection couples to gauge or gravitational gauge fields^{[50]–[53]}. We will give here the transformation properties, under a generic discrete isometry Γ , of all relevant quantities appearing in the effective

Lagrangian for an N=1 heterotic string compactified on a C-Y manifold with gauge group $E_8 \times E_6$.

These transformations, since they act on sections of line bundle, will act accompanied by Kähler phases.

Let us ψ, ϕ be the coordinates on $M_1 \times M_2$ of a generic moduli space of C-Y manifolds. The Kähler potential, under the action of Γ , will transform as:

$$K \rightarrow K - (\Lambda_1 + \bar{\Lambda}_1) - (\Lambda_2 + \bar{\Lambda}_2)$$

where $\Lambda_1(\phi), \Lambda_2(\psi)$ are Kähler phases. Under the same action the Yukawa couplings for 27 and $\bar{27}$ families will change as follows:

$$\begin{aligned} W^1(\Gamma\psi) &\rightarrow e^{2\Lambda_1 - 3\gamma_1\Sigma} W^1(\psi) \\ W^2(\Gamma\psi) &\rightarrow e^{2\Lambda_2 - 3\gamma_2\Sigma} W^2(\psi) \end{aligned} \quad \text{III} - 18$$

and accordingly, the scalar fields of chiral E_6 families:

$$\begin{aligned} \phi_{27} &\rightarrow e^{\gamma_1\Sigma_1 + (\Lambda_2 - \Lambda_1)/3} \phi_{27} \\ \phi_{\bar{27}} &\rightarrow e^{\gamma_2\Sigma_2 + (\Lambda_1 - \Lambda_2)/3} \phi_{\bar{27}} \end{aligned} \quad \text{III} - 19$$

which ensure that the total superpotential W defined by eq. II-26 just cancel the variation of K in the “norm” over the line-bundle:

$$||W||^2 = e^G = WW^* e^K \quad \text{III} - 20$$

The variation of the moduli and family metrics is given by:

$$\begin{aligned} G^1 &\rightarrow G^1 e^{-\gamma_1 + (\Sigma_1 + \bar{\Sigma}_1)} \\ G^2 &\rightarrow G^2 e^{-\gamma_2 + (\Sigma_2 + \bar{\Sigma}_2)} \\ G_{27}^1 &\rightarrow G_{27}^1 e^{-\gamma_1(\Sigma_1 + \bar{\Sigma}_1) + (\Lambda_1 + \bar{\Lambda}_1)/3 - (\Lambda_2 + \bar{\Lambda}_2)/3} \\ G_{\bar{27}}^2 &\rightarrow G_{\bar{27}}^2 e^{-\gamma_2(\Sigma_2 + \bar{\Sigma}_2) - (\Lambda_2 + \bar{\Lambda}_2)/3 - (\Lambda_1 + \bar{\Lambda}_1)/3} \end{aligned} \quad \text{III} - 21$$

Eq. III-21 can easily be derived from eqs. II-13 and II-27, II-28. Therefore the extra (holomorphic) phase $\gamma\Sigma$ (γ is a real normalization constant) is due to the discrete isometry Γ of the modular group which acts on the fermions as a $U(1)$ gauge transformation with gauge connection:

$$\begin{aligned} B_\mu &= i(B_i \partial_\mu z_i - B_{\bar{i}} \partial_\mu \bar{z}_{\bar{i}}) \\ B_i &= \partial_i(\log \det G) \end{aligned} \quad \text{III} - 22$$

The holomorphic phases Σ, Λ are actually related to one another since Σ is given by $\log \det \partial f_i^\Gamma / \partial z_j(z_i \rightarrow f_i^\Gamma(z))$ and charged matter fields are required to be tensor densities under Γ .

The transformation laws of the fermions are now determined by the bilinear fermionic terms in the effective action which are not chiral invariant (bilinear with an even number of γ matrices) namely:

$$e^{K/2} (W^1 \chi_{27} \chi_{27} \Phi_{27} + W^2 \chi_{\overline{27}} \chi_{\overline{27}} \Phi_{\overline{27}}) \quad \text{III} - 23$$

$$G_{27}^2 \chi_{27} \lambda \Phi_{27} \quad , \quad G_{\overline{27}}^2 \chi_{\overline{27}} \lambda \Phi_{\overline{27}} \quad \text{III} - 24$$

$$e^{K/2} \phi_\mu \sigma^{\mu\nu} \psi_\nu \quad , \quad \chi^i G_{i\bar{j}} \partial_\mu \psi^{\bar{j}} \gamma^\nu \gamma^\mu \psi_\nu \quad \text{III} - 25$$

$$\lambda \sigma^{\mu\nu} F_{\mu\nu} \chi_s \quad \text{III} - 26$$

Eqs. III-13 and III-14, i.e. the Yukawa couplings, fix the phases of the changed E_6 fermions and of the gauginos. Eqs. III-25, III-26 fix the phases of the gravitino, moduli fermions and dilatino. The result is:

$$\begin{aligned} \chi_{27} \rightarrow \chi_{27} \, e^{\frac{\gamma_1}{2} (\Sigma_1 - \overline{\Sigma}_1) - \frac{5}{12} (\Lambda_1 - \overline{\Lambda}_1) - \frac{1}{12} (\Lambda_2 - \overline{\Lambda}_2)} \\ e^{\frac{\gamma_1}{2} (\Sigma_1 + \overline{\Sigma}_1) - \frac{1}{6} (\Lambda_1 + \overline{\Lambda}_1) + \frac{1}{6} (\Lambda_2 + \overline{\Lambda}_2)} \end{aligned} \quad \text{III} - 27$$

(the same for $\chi_{\overline{27}}$ with $1 \leftrightarrow 2$).

$$\begin{aligned} \lambda \rightarrow \lambda \, e^{\frac{1}{4} (\Lambda_1 - \overline{\Lambda}_1) + \frac{1}{4} (\Lambda_2 - \overline{\Lambda}_2)} \\ \psi_\mu \rightarrow \psi_\mu \, e^{\frac{1}{4} (\Lambda_1 - \overline{\Lambda}_1) + \frac{1}{4} (\Lambda_2 - \overline{\Lambda}_2)} \\ \chi_s \rightarrow \chi_s \, e^{-\frac{1}{4} (\Lambda_1 - \overline{\Lambda}_1) - \frac{1}{4} (\Lambda_2 - \overline{\Lambda}_2)} \end{aligned} \quad \text{III} - 28$$

$$\chi^i \rightarrow \chi^i \, e^{\gamma_1 \Sigma_1} \, e^{-\frac{1}{4} (\Lambda_1 - \overline{\Lambda}_1) - \frac{1}{4} (\Lambda_2 - \overline{\Lambda}_2)} \quad \text{III} - 29$$

Note that $\lambda, \psi_\mu \chi_s$ undergo a pure $U(1)$ Kähler phase while the matter and moduli fermions acquire extra terms due to the coordinate charge. Also the pure Kähler phase of charged fermions is not standard because the charged chiral scalars transform as holomorphic sections under a Kähler transformation on $M_1 \times M_2$.

Target-space duality transformations can be discussed, in great detail, in Z_n orbifold models which are limiting cases of C-Y manifolds^{[50]–[53]}. In that case the moduli spaces of orbifolds^[54] correspond to the so-called untwisted moduli and the duality groups are explicitly known^{[55]–[57]}. We will not report here this analysis, which has been discussed in great detail in recent papers, but rather try to extract from this analysis general conclusions.

The topic we would like to discuss in the last part of this section is the concept of automorphic functions for general C-Y compactification by a generalization of automorphic functions for the modular group $SL(2, \mathbb{Z})$ and their connection to special geometry. In this respect the key ingredient is that, for untwisted moduli of (2,2) orbifolds, the moduli space has the structure M/Γ where M is a symmetric space compatible with special geometry and Γ is a discrete subgroup of M determined by the string quantum symmetries^{[54]–[57]}. The symmetric spaces M are actually all given by particular truncation of the homogeneous space:

$$S = \frac{SO(6, 6)}{SO(6) \times SO(6)} \quad \text{III} - 30$$

which is related to the Narain-Sarmadi-Witten^[58] classification of the moduli space of toroidal compactifications. Let us call M_1 and M_2 the manifolds for untwisted (1,1) and (2,1) moduli such that $M_1 \times M_2$ is contained in the toroidal six-dimensional moduli space S . Then all Z_n orbifolds correspond to particular choices of the following special symmetric spaces^{[14],[54]–[57]}:

$$M_1 = \left(\frac{SU(1, 1)}{U(1)} \right)^3, \quad \frac{SU(1, 1)}{U(1)} \times \frac{SU(2, 2)}{SU(2) \times SU(2) \times U(1)},$$

$$\frac{SU(3, 3)}{SU(3) \times SU(3) \times S(1)} \quad \text{III} - 31$$

$$M_2 = 1 \text{ or } \frac{SU(1, 1)}{U(1)}$$

The M_1 spaces correspond to cubic prepotentials of the form:

$$\mathcal{F}^i(\phi_a) = i d_{abc} \phi^a \phi^b \phi^c \quad \text{III} - 32$$

while the M_2 space, if not empty, corresponds to a quadratic prepotential:

$$\mathcal{F}^2(\psi) = c + \psi^2 \quad \text{III} - 33$$

The previous equations are written in a special coordinate system as defined by eqs. III-15, III-16. The corresponding Yukawa couplings are therefore given by:

$$W_{abc}^1 = i d_{abc}, \quad W^2 = 0 \quad \text{III} - 34$$

All (2,2) orbifolds have a complex modulus which corresponds to the breathing mode^[54]. This corresponds to a submanifold of M^1 given by the prepotential^[14]:

$$\mathcal{F}(\phi) = i \lambda \phi^3 \quad \text{III} - 35$$

where λ is a normalization factor. The modular group Γ associated to ϕ is $SL(2, \mathbb{Z})$ ^[46]. As discussed in Ref. [46] a superpotential term for ϕ , which may arise from

non-perturbative stringy effects, must be an automorphic function of Γ . Then the only choice for $W(\phi)$ which is finite and non-zero inside the fundamental domain is^[59]:

$$W(\phi) = \frac{1}{\eta(\phi)^6} = \left[\prod_{m,n} (m + in \phi)^{-3} \right]_{\text{reg.}} \quad \text{III} - 36$$

where $\eta(\phi)$ is the Dedekind function. By interpreting W as a holomorphic section on $M^{[56]}$, its norm is:

$$||W(\phi)|| = |W(\phi)| e^{K/2} = \frac{1}{|\eta(\phi)|^6 |\phi + \bar{\phi}|^{3/2}} \quad \text{III} - 37$$

Eq. III-37 defines a modular invariant function, which, in the context of orbifold compactifications, may be interpreted as the free energy of the (fermionic excitations of) the orbifold lattice, as a function of $\phi = \phi^2 + ib$:

$$\mathcal{F}_{\text{reg.}} = -\log ||W(\phi)||^2 \quad \text{III} - 38$$

More generally, this connection leads to the suggestive interpretation of a holomorphic superpotential as the (regularized) determinant of the holomorphic mass matrix $\mathcal{M}(\phi)$ of the (infinitely many) fermionic excitations of the lattice^[56]:

$$\log \det \mathcal{M}(\phi)_{\text{reg.}} = \log W(\phi) \quad \text{III} - 39$$

A more interesting case is that of three moduli with $M = (\text{SU}(1,1)/\text{U}(1))^3$. This is actually a subspace of all spaces in III-31.

The space M/Γ with $\Gamma = \text{SL}(2, \mathbb{Z})^3 \times \text{P}(3)$ coincides with the (full) moduli space of the $\mathbb{Z}_3/\mathbb{Z}_3$ orbifold, as discussed in Ref.[55]. In this case the regularized free-energy is:

$$\begin{aligned} \mathcal{F}_{\text{reg.}} &= \sum_{i=1}^3 \sum_{m_i, n_i} \left| \frac{(m_i + n_i \psi_i)^2}{(\phi_i + \bar{\phi}_i)} \right|_{\text{reg.}} \\ &= \sum_{i=1}^3 \log |(\eta(\phi_i))^4 (\phi_i + \bar{\phi}_i)| = \\ &= -\log ||W||^2 \end{aligned} \quad \text{III} - 40$$

where:

$$W^{-1}(\phi_i) = \prod_{i=1}^3 \eta(\phi_i)^2 \quad , \quad K = - \sum_{i=1}^3 \log (\phi_i + \bar{\phi}_i) \quad \text{III} - 41$$

Eqs. III-37 - III-41 suggest the following generalization for the holomorphic eigenvalues of the fermionic mass-matrix on an arbitrary (2,2) internal conformal field theory:

$$M(\phi) = (M_I X^I + i N^I F_I) \quad \text{III} - 42$$

where $X^I(\phi), iF_I(\phi)$ are the periods defined by eq. III-5. Then a modular invariant expression is (formally) obtained as follows^[56]:

$$\begin{aligned} -\log ||W||^2 &= -\log (\det MM^+ e^K) \\ &= \sum_{M_I, N_I} \log \left. \frac{|M_I X^I + iN^I F_I|^2}{\bar{X}^I F_I + X^I \bar{F}_I} \right|_{\text{reg.}} \end{aligned} \quad \text{III} - 43$$

When M_I, N^I transform by the linear action of $\text{Sp}(8; \mathbb{Z})$ induced by Γ , the sum over the integers must be restricted to a “modular orbit” of Γ . In the previous example eqs. III-40 - III-41 are reproduced by making the following choices:

$$F(X) = iX^1 X^2 X^3 / X^0, \quad \phi_i = -iX^i / X^0 \quad \text{III} - 44$$

$$\begin{aligned} M_0 &= m_1 m_2 m_3 & N_0 &= n_1 n_2 n_3 \\ M_1 &= n_1 m_2 m_3 & N_1 &= -m_1 n_2 n_3 \\ M_2 &= n_2 m_1 m_3 & N_2 &= -m_2 n_1 n_3 \\ M_3 &= n_3 m_1 m_2 & N_3 &= -m_3 n_1 n_2 \end{aligned} \quad \text{III} - 45$$

Eq. III-45 specifies the modular orbit of Γ . It is easy to show that the action of the three $\text{SL}(2, \mathbb{Z})$ factors on the three sets of (unrestricted) integers (m_i, n_i) corresponds to the action of Γ , embedded in $\text{Sp}(8; \mathbb{Z})$, on the eight-dimensional integral vectors (M_I, N^I) . It is interesting to stress the fact that existence of the regularized form of eq. III-43, as a norm of a holomorphic section of the line bundle $\mathcal{L}$ is the main assumption of this construction. This is guaranteed provided the objects $M_I X^I + iN^I F_I$ are the eigenvalues of a “good operator” defined over the manifold^[60]. We note that for M^I manifolds which are not simply products of $\frac{SU(1,1)}{U(1)}$ spaces, the holomorphic section W defined through eq. III-43 is no longer given in terms of Dedekind functions and it defines a non-trivial holomorphic modular form over the moduli space^{[56],[57]}.

Let us make some final comments in this section. As far as eq. III-43 is concerned, it should be possible to compute this formula exactly in a pure Kaluza-Klein context by confining oneself to the “lattice” corresponding to deformations of the complex structure. In fact, based on the considerations of the previous section, one would expect this lattice to be purely classical, in contrast with the case of Kähler class deformations. A good example of this phenomenon is given by the two-dimensional torus^[61] in which the duality group Γ is given by the product of two $\text{SL}(2, \mathbb{Z})$ groups but one of them is not connected to winding modes and it is purely “classical”.

Another point that it is worth mentioning is that, as for $\text{SL}(2, \mathbb{Z})$, one expects to find many other holomorphic modular forms for C-Y moduli space M/Γ .

Non-trivial examples of Γ , related to smooth C-Y manifolds, have recently been given^[25] together with procedures^{[25]–[32]} to explicitly construct their related special geometry. In the next section we will show how it is possible to determine the periods $X^I(\phi), F_I(\phi)$ in the case of smooth manifolds, using methods of algebraic geometry.

IV. Picard-Fuchs equations and determination of the periods

The determination of the periods $\omega^a(\psi)$ ($a = 1 \dots b_3$) of the holomorphic 3-form $\Omega(\psi)$ enables us to explore the structure of the moduli space M of complex structure deformations including its global properties, encoded in the (target-space) modular group Γ . Indeed from formulae III-3, III-6, III-8 the metric over M , the three index tensor W , the period matrix Ω are all given in terms of the periods. For instance, the equation for the Kähler potential is given by a bilinear symplectic form over the periods^{[15]–[17],[20]}:

$$e^{-K} = i \int \Omega \wedge \bar{\Omega} = X^A \bar{F}_A + \bar{X}^A F_A = \omega^a \Omega_{ab} \bar{\omega}^b \quad \text{IV} - 1$$

with:

$$\begin{aligned} \omega^a(\psi) &= (X^A(\psi), iF_A(\psi)) \quad a = 1 \dots b_3 \\ \Omega &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \end{aligned} \quad \text{IV} - 2$$

The periods of holomorphic p -forms on certain p -dimensional complex manifolds can be obtained as solutions of linear differential equations, known as Picard-Fuchs equations, which we are going to illustrate.

The periods, at least in the case of manifolds obtained by some defining polynomial in a $\mathbb{CP}^n$ space^{[25]–[27]}:

$$F(x) = 0 \quad \text{IV} - 3$$

can in principle be computed by explicitly integrating the holomorphic p -form $\Omega(\phi)$ along the homology cycles. This was done in Ref. [25] for the mirror manifold of $P_4(5)$. In a more general setting it is possible to obtain^{[26]–[30]} linear differential equations whose independent solutions give exactly the periods $\omega^a(\phi)$.

These equations turn out to connect methods of algebraic geometry with $N=2$ superconformal field theories and their topological version^{[6]–[12]} since the very same equations can be derived (at least in some cases) from the associativity of the fusion rule coefficients of the chiral ring of primary fields in TSCFT's, together with the conservation of the $U(1)$ charge and the relation expressing the three-point function has third-derivative (with respect to the deformation parameters) of the topological free-energy^[10]. The technique of writing differential equations for the periods is particularly simple in the case of one deformation, with parameter ψ , of the defining polynomial^[26]. This applies equally well to a $c=3$ and $c=9$, $N=2$ SCFT namely the (Z_3 orbifoldized) torus^[32] and the C-Y threefold which is the mirror of $P_4(5)$. For the torus ($p = 1, b_1 = 2$) the defining polynomial is a cubic in $\mathbb{CP}^2$ while for the C-Y threefold ($p = 3, b_3 = 4$) the defining polynomial is a quintic in $\mathbb{CP}^4$.

In both cases the periods are given by:

$$\omega^a(\psi) = \int_{A,B} \Omega_p(\psi) \quad \text{IV} - 4$$

when A, B are the homology cycles. It is now possible, by considering the deformed polynomial:

$$F(x, \psi) = F(x) + \psi h(x) \quad |\psi| < 1 \quad \text{IV} - 5$$

to find a linear differential equation for the $(b_p \times b_p)$ matrix C:

$$\frac{dC(\psi)}{d\psi} = C(\psi)B(\psi) \quad \text{IV - 6}$$

which is equivalent to a b_p -order differential equation for the periods $\omega^a(\psi)$. Actually ω^a are the b_p linearly independent solutions of this equation.

The $(b_p \times b_p)$ matrix B is obtained by taking the matrix-elements of an “evolution operator”: $x_0 \frac{dF}{d\psi} = x_0 h(x)$ in a certain basis $X^{(\nu)}$ of polynomials with scalar product defined by:

$$\langle X^{*(\mu)} X^{(\nu)} \rangle = \delta^{\mu\nu} \quad \text{IV - 7}$$

where $X^{*(\mu)}$ is a dual basis (and in the scalar product only constant terms are kept).

The vector space with basis vector $X^{(\mu)}$ is obtained as follows: let us start with a polynomial of degree d in CP^n , then take the sets:

$$\begin{aligned} I &= \{\omega = (\omega_0, \dots, \omega_{n+1}) | \omega_i \in Z_+^{h+2}, d\omega_0 = \omega_1 + \dots + \omega_{n+1}\} \\ A &= \omega \in I, 0 < \omega_i < d, \forall i = 1 \dots n+1 \end{aligned} \quad \text{IV - 8}$$

The basic vectors are given by:

$$X^{(\omega)} = x_0^{\omega_0} x_1^{\omega_1} \dots x_{n+1}^{\omega_{n+1}} \quad \omega \in A \quad \text{IV - 9}$$

Let us now define the following covariant derivatives:

$$D_i = x_i \frac{\partial}{\partial x_i} + x_0 x_i \frac{\partial F}{\partial x_i} \quad i = 1 \dots n+1 \quad \text{IV - 10}$$

Then it can be shown that the quantity:

$$x_0 h(x) X^{(\nu)} \quad \text{IV - 11}$$

can be expanded in terms of $X^{(\omega)} (\omega \in A)$ and covariant derivatives $D_i X^{(\omega)}$ with $\omega \in I$ and $X^{(\omega)}$ having x_0 degree ω_0 less than the polynomial given by eq. IV-11.

Let us generally assume that:

$$x_0 h(x) X^{(\nu)} = B_{\nu\mu}(\psi) X^{(\mu)} + (DX)^{(\nu)}(\psi) \quad \text{IV - 12}$$

Then since:

$$\langle X^{*(\mu)}, (DX)^{(\nu)} \rangle = 0 \quad \text{IV - 13}$$

$B_{\nu\mu}$ is the desired matrix element.

The matrix C is the transpose of the period matrix and eq. IV-6 gives the desired differential equation for $\omega^a(\psi)$.

We would like now to apply these techniques to the two examples in question, the cubic and the quintic, to show the usefulness of the procedure. For the torus we have:

$$F_{\psi=0}(x) = \sum_{i=1}^3 x_i^3, \quad h(x) = -3x_1 x_2 x_3 \quad \text{IV - 14}$$

since $d=3$, $n=2$, $0 < \omega_i < 3$, $\omega_i = (1, 2)$ and we have only two basis polynomials:

$$X^{(1)} = x_0 x_1 x_2 x_3, \quad X^{(2)} = x_0^2 x_1^2 x_2^2 x_3^2 \quad \text{IV - 15}$$

and three derivatives $D_i X^{(\mu)}$.

It follows that:

$$\begin{aligned} x_0 h(x) X^{(1)} &= -3X^{(2)} \\ x_0 h(x) X^{(2)} &= \sum_{i=1}^2 a_i X^{(i)} + D(\dots) \end{aligned} \quad \text{IV - 16}$$

$$\text{with } a_1 = \frac{-\psi}{3(1-\psi^3)}, \quad a_2 = \frac{3\psi^2}{1-\psi^3}.$$

So we get for the B matrix:

$$B_{11} = 0, \quad B_{21} = -3, \quad B_{12} = \frac{-\psi}{3(1-\psi^3)}, \quad B_{22} = \frac{3\psi^2}{1-\psi^3} \quad \text{IV - 17}$$

Eq. IV-6 is solved by:

$$C = \begin{pmatrix} \omega_1 & -1/3\omega_1' \\ \omega_2 & -1/3\omega_2' \end{pmatrix} \quad \text{IV - 18}$$

where $\omega_{1,2}$ are the solutions of the following second-order differential equation:

$$(1 - \psi^3)\omega'' - 3\psi^2\omega' - \psi\omega = 0 \quad \text{IV - 19}$$

If we now define $t = \omega_1/\omega_2$, then the equation for $t(\psi)$ can be re-expressed as an equation for $\psi(t)$ which is related to a ratio of two 3-point functions of the perturbed TSCFT.

This equation was obtained as an equation for the fusion rule coefficients in Ref. [32].

Let us now move to the more interesting case of the C-Y threefold defined by the quintic polynomial:

$$\sum_{i=1}^5 x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0 \quad \text{IV - 20}$$

In this case $h(x) = -5x_1 x_2 x_3 x_4 x_5$ and there are four basis polynomials:

$$\begin{aligned} X^{(1)} &= x_0 x_1 x_2 x_3 x_4 x_5 \\ X^{(2)} &= x_0^2 x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 \\ X^{(3)} &= x_0^3 x_1^3 x_2^3 x_3^3 x_4^3 x_5^3 \\ X^{(4)} &= x_0^4 x_1^4 x_2^4 x_3^4 x_4^4 x_5^4 \end{aligned} \quad \text{IV - 21}$$

As in the previous case we compute:

$$B_{\mu\nu} = \langle X^{*(\mu)}, x_0 h(x) X^{(\nu)} \rangle \quad \text{IV - 22}$$

by expanding $x_0 h(x) X^{(\nu)}$.

We obtain:

$$x_0 h(x) X^{(\nu)} = -5X^{(\nu+1)} \quad , \quad \nu = 1, 2, 3, \quad \text{IV} - 23$$

$$x_0 h(x) X^{(4)} = \sum_{\nu=1}^4 a_\nu X^{(\nu)} + D(\dots) \quad \text{IV} - 24$$

$$\text{with } a_1 = -\frac{1}{125} \frac{\psi}{1 - \psi^5}, a_2 = \frac{3}{5} \frac{\psi^2}{(1 - \psi^5)}, a_3 = \frac{-5\psi^3}{1 - \psi^5}, a_4 = \frac{10\psi^4}{1 - \psi^5}.$$

As before, from eq. IV-6 we get for each row a fourth-order differential equation:

$$(1 - \psi^5)\omega^{\text{IV}} - 10\psi^4\omega^{\text{III}} - 25\psi^3\omega^{\text{II}} - 15\psi^2\omega^{\text{I}} - \psi\omega = 0 \quad \text{IV} - 25$$

Performing the change of variable $z = \psi^5$ we obtain the solutions in terms of generalized hypergeometric functions:

$$z^{\frac{k}{5}} {}_4F_3\left(\frac{k}{5}, \frac{k}{5}, \frac{k}{5}, \frac{k}{5}, \overbrace{\frac{k+1}{5}, \frac{k+2}{5}, \frac{k+3}{5}, \frac{k+4}{5}}; z\right) \quad \text{IV} - 26$$

with $k = 1, 2, 3, 4$ which reproduces the result of Ref. [25].

In the above method there is no real difference between the cubic and the quintic, with the exception that the matrices C, B are 2×2 in one case and 4×4 in the other case. This is so in spite of the fact that as target-space manifolds they are completely different and the second corresponds, from the world-sheet point of view, to an interacting σ -model which is therefore solved exactly by algebraic methods.

It is also clear that the techniques displayed here can in principle be generalized to other spaces and to the case of multidimensional varieties ($h > 1$ for C-Y threefolds). In that case one generalizes the present equation to partial differential equations giving rise to $b_3 = 2(h+1)$ linearly independent solutions. One interesting point is that these equations together with their monodromy will also determine the “quantum” target-space duality group Γ , which is a subgroup of $\text{Sp}(b_3; \mathbb{Z})$ and has a linear action on the periods $\omega^a(\psi_i)$. For the torus this group trivializes to $\text{SL}(2, \mathbb{Z})$ and for the mirror of $P_4(5)$ it was recently discussed in Ref. [25].

In the next section we will see that the duality group Γ plays an important role in discussing the effect of higher orders in the effective action. In particular it relates the moduli dependence of gauge couplings to mixed modular-gauge anomalies because of space-time supersymmetry.

The Picard-Fuchs equations, for the $c=9$ $N=2$ superconformal theories, corresponding to the C-Y threefolds, can be derived^[33] on general grounds, as “holomorphic identities” following from the special geometry of the moduli space or equivalently as $N=2$ supergravity identities.

Indeed, as explained in sect. III, the deformed holomorphic three-form $\Omega(\psi)$ (eq. III-6) and the $(2,1)$ forms ϕ_i (eq. III-5) are entirely equivalent to the two symplectic vectors:

$$V = (X^\Lambda, iF_\Lambda) \quad , \quad U_i = (\nabla_i X^\Lambda, i\nabla_i X^\Sigma F_{\Sigma\Lambda}) \quad \text{IV} - 27$$

according to the formulae given in sect. II. These vectors are the basis elements for the flat $\text{Sp}(2h + 2; \mathbb{R})$ bundle defined by Strominger in Ref. [17]. In particular, from eqs. IV-21, IV-22, V-15 (second reference of) Ref. [20] it follows that:

$$\mathcal{D}_i V = U_i \quad \text{IV} - 28$$

$$\mathcal{D}_i \overline{\mathcal{D}}_{\bar{j}} \overline{V} = G_{i\bar{j}} \overline{V} \quad \text{IV} - 29$$

$$\mathcal{D}_i \mathcal{D}_{\bar{j}} V = C_{i\bar{j}k} G^{k\bar{k}} \overline{U}_{\bar{k}} \quad \text{IV} - 30$$

where $C_{i\bar{j}k}$ are the covariantly holomorphic Yukawa couplings defined by eqs. II-13, II-14. These equations are of course the same as the ones derived from deformation theory of C-Y threefolds^[16].

Differentiating once more eq. IV-29 one obtains:

$$\mathcal{D}_i \mathcal{D}_{\bar{j}} \overline{U}_{\bar{k}} = 0 \quad \text{IV} - 31$$

By regarding $C_{i\bar{j}k}$ as a square matrix (with $j\bar{k}$ entries and by assuming it is invertible, for fixed i) one finally gets the identity:

$$\mathcal{D}_i \mathcal{D}_{\bar{j}} (C_k^{-1})^{\ell m} \mathcal{D}_k \mathcal{D}_{\bar{\ell}} V = 0 \quad \text{IV} - 32$$

Since V is holomorphic, eq. IV-32 implies an holomorphic differential relation for $V^{[25],[33]}$:

$$\left(\sum_{s=0}^4 a_{i\bar{j}k m, i_1 \dots i_s}^{(s)} \partial_{i_1} \dots \partial_{i_s} \right) V = 0 \quad \text{IV} - 33$$

where the $a^{(s)}$ are holomorphic functions of the moduli, computable from eq. IV-32. Note that if the matrix C vanishes then the differential equation degenerates to a second order relation $\mathcal{D}_i \mathcal{D}_{\bar{j}} V = 0$ with solution $F(X) = a_{\Lambda\Sigma} X^\Lambda X^\Sigma$ ($a_{\Lambda\Sigma}$ is a constant matrix). In eq. IV-33 the holomorphic coefficients are functions of the Yukawa couplings $W_{i\bar{j}k}$ and of the Kähler connections $\Gamma_{\ell m}^i, K_i$. These equations are actually holomorphic identities if we plug for W and K their expression given by eqs. II-20 and II-23 in terms of the holomorphic prepotential defined by eq. II-16. Of course eq. IV-32 are much more general than what comes from the deformation of the Hodge structure of C-Y threefolds since they are just a consequence of N=2 supergravity, special geometry and not all N=2 supergravities come from superstring C-Y compactifications.

The same identities can be interpreted^[33] as differential equations if we imagine to give, as data, the holomorphic coefficients $a^{(s)}(\psi)$ rather than the prepotential $F(X^\Lambda(\psi))$. This is the case in deformation theory of C-Y threefolds where the Picard-Fuchs equations^{[26]–[30]} for the periods (the holomorphic vector V) allow to directly compute the $a^{(s)}$ coefficients and then to find V (and therefore F) through the fourth order (partial) differential equations given by eq. IV-32. These equations are equivalent to the Picard-Fuchs equations in “arbitrary coordinate systems” and an explicit holomorphic expression of the coefficients $a^{(s)}$ can be given in terms of quantities defined in the special geometry.

For example the $a^{(3)}$ coefficient is always related to the Yukawa coupling (we consider for simplicity the one parameter case) through the equation:

$$W' = -\frac{1}{2} a^{(3)} W \quad (\text{where } a^{(4)} \text{ is normalized to } 1) \quad \text{IV} - 34$$

given in Ref. [25]. More complicated equations are found for the other coefficients $a^{(s)}$ ($s = 2, 1, 0$). This approach to Picard-Fuchs systems gives new insight on the relation which occurs between the special geometry of the moduli space and perturbed N=2 topological superconformal field theories. Indeed one can show^[33] that a holomorphically flat connection can be obtained within the framework of special geometry acting on a flat holomorphic bundle (which vanishes in the supergravity special coordinates) in complete analogy with N=2 topological field theories^{[5]–[12]}.

V. Moduli dependence of gauge couplings and target-space duality anomaly cancellation

Recently, loop-corrections to gauge couplings have been computed^{[61]–[67]} in heterotic string compactification (in four dimensions) with an unbroken gauge group G.

The analysis has been mainly confined to (2,2) orbifold vacua with $G = E_8 \times G'$ by a number of authors^{[62],[50]–[53],[63]} with string and field theory calculations, exploiting the relation between renormalization of gauge couplings and anomalies in N=1 supersymmetric gauge theories^{[51],[52],[65]–[67]}.

The peculiar outcome of these investigations is that moduli corrections to gauge couplings are not holomorphic (in the moduli chiral fields) so the question whether these results, which are manifestly target-space duality invariants, compatible with standard supergravity arguments^{[37],[38]}, was raised^[50]. The apparent paradox came from the fact that a gauge invariant coupling to matter has the general form:

$$g_{ab}(\phi, \bar{\phi}) F_{\mu\nu}^a F_{\mu\nu}^b + \theta_{ab}(\phi, \bar{\phi}) F_{\mu\nu}^a F_{\rho\sigma}^b \epsilon^{\mu\nu\rho\sigma} \quad V - 1$$

where, by supersymmetry, the field dependent gauge couplings and θ -angles are required to be harmonic and in fact related to a “complex” holomorphic coupling:

$$f_{ab}(\phi, \bar{\phi}) = f_{ab}(\phi) = g_{ab}(\phi, \bar{\phi}) + \theta_{ab}(\phi, \bar{\phi}) \quad V - 2$$

Consistency with (target-space) modular invariance would seem to require that f is a modular invariant holomorphic function^[46] while explicit calculations^{[62],[63],[50]–[53]} show that f is a modular invariant non-holomorphic function. The apparent paradox finds a solution by noticing that target-space duality relates massless to massive modes and a modular invariant result is obtained when all string states circulate in a loop. On the other hand the contribution of massless states does not satisfy the decoupling and gives rise to infra-red divergences which generate non-local terms^{[50]–[53]} in the effective action S_{eff} .

The overall result is summarized by a single (superfield) formula which is manifestly modular invariant and contains both the local and non-local term^{[70],[51]–[53]}.

$$S_{\Gamma}^{1-\text{loop}} = \text{Re} \int d^2\theta W_A^2 \frac{\bar{D}^2 D^2}{\square} \sum_a C_A^a (H_a + \Lambda_a + \bar{\Lambda}_a) \quad V - 3$$

where H_a is a real (non-chiral) superfield $H_a(\phi, \bar{\phi})$ and $\Lambda_a(\phi)$ is a chiral superfield function of the (chiral) moduli superfields ϕ , i.e.: $\bar{D}\Lambda_a = 0$.

When added to the three-level term:

$$\text{Re} \int S W_A^2 \quad V - 4$$

eqs. V-3 and V-4 define an effective coupling of the form:

$$\frac{1}{g_{A^{\text{eff}}}^2}(\phi, \bar{\phi}) = S + \bar{S} + \sum_s C_A^a (H_a + \Lambda_a + \bar{\Lambda}_a) \quad \text{V-5}$$

(a refer to different types of moduli) and we see that the non-holomorphy of $g_{A^{\text{eff}}}^2(\phi, \bar{\phi})$ is related to the non-local superfield in V-3 coming from the triangular graph when massless particles circulate. The term containing Λ_a is local since $\frac{\bar{D}^2 D^2}{\square} \Lambda_a = \Lambda_a$ and in fact can be interpreted as a local counterterm^{[51]–[53]}:

$$\text{Re} \sum \int d^2\theta W_A^2 C_A^a \Lambda_a(\phi) \quad \text{V-6}$$

needed to cancel the anomaly. In string theory eq. V-6 is just the result of the integration over the heavy string modes for which the decoupling theorem applies. Finally there is an extra source of anomaly cancellation mechanism, i.e. the Green-Schwarz mechanism^{[71]–[73]}. In the S field formation this means that S acquires, through one-loop effects, an anomalous transformation under target-space duality transformations^[51]:

$$S \rightarrow S + \delta(\phi) \quad \text{V-7}$$

Eq. V-7 gives a universal contribution to the cancellation of all anomaly diagrams, equal for all gauge group factors, but model dependent. Actually in certain orbifolds, like the $\mathcal{Z}_3$ and $\mathcal{Z}_7$ orbifolds, all anomalies coincide and the contribution of the massive states $\Lambda_a(\phi)$ is completely absent^{[51]–[53]}. When a G.S. term is present the S-Kähler potential gets modified with a non-trivial mixing to the moduli^{[63],[66]}:

$$-\log(S + \bar{S}) \rightarrow -\log(S + \bar{S} + \Delta(\phi, \bar{\phi})) \quad \text{V-8}$$

where, in order to preserve modular invariance, Δ transforms as:

$$\Delta \rightarrow \Delta - \delta - \bar{\delta} \quad \text{V-9}$$

Δ is the universal term in eq. V-5, which is cancelled by the G.S. mechanism. The non-holomorphic term $\sum_a C_A^a H_a(\phi, \bar{\phi})$ comes from the computation of a mixed σ -model gauge anomaly and depends only on the effective couplings of the massless states and their U(1) charges under modular transformations, as defined by eqs. III-27 – III-29. For smooth C-Y manifolds $A = E_8$ or E_6 and “a” refer to (1,1) or (2,1) moduli so we have only four terms to determine. For $A = E_6$:

$$\begin{aligned} C_A^1 H_1(\psi, \bar{\psi}) = & \left[T(G = E_6) - \frac{5}{3} T(R = 27) h_{1,1} - \frac{1}{3} T(\bar{R} = \bar{27}) h_{1,2} \right] K_1(\psi, \bar{\psi}) \\ & + 2T(R = 27) \log \det G^1(\psi, \bar{\psi}) \end{aligned}$$

V-10

$C_A^2 H_2(\psi, \bar{\psi})$ is the same with $h_{(1,1)} \leftrightarrow h_{(2,2)}$, $2, R \leftrightarrow \bar{R}$.

These formulae can be simplified by using the equation III-12, i.e. $\log \det N = \log \det G - (h+1)K$.

Then one obtains:

$$\sum_{a=1}^2 C_{A=E_6}^a H_a(\psi, \bar{\psi}) = [T(E_6) + 2T(27)](K_1 + K_2) + \frac{1}{6}T(27)\chi(K_1 - K_2) \quad V-11$$

$$+ 2T(27)(\log \det N^1 + \log \det N^2)$$

and $\chi = 2(h_{1,1} - h_{2,2})$ is the Euler characteristic of C . Also $T(E_8) = 30, T(E_6) = 12, T(27) = T(\bar{27}) = 3$.

For the E_8 anomaly one obtains:

$$T(E_8)(K_1 + K_2) \quad V-12$$

Eqs. V-11 and V-12 show that on smooth C-Y manifolds threshold effects of massive states are always present (Λ_a cannot vanish) since for the difference of effective gauge couplings $\frac{1}{g_{E_6}^2} - \frac{1}{g_{E_8}^2}$ we have a non-trivial non-holomorphic function. Also the anomaly cancellation seems to require at least two different automorphic functions for each type of moduli since in general K and $\log \det N$ have different transformation properties under Γ . Candidate holomorphic functions for threshold corrections were studied in Ref. [56] and discussed in section III.

We would like to add few comments on eq. V-5 as derived from the superfield expression V-3 and formula V-5. The resolution of the puzzle regarding holomorphicity is that the chiral non-local superfield^{[51],[52]}:

$$f_A = \frac{\bar{D}^2 D^2}{\square} H_A \left(H_A = \sum_a C_A^a H_a \right) \quad V-13$$

has a $\theta = 0$ component given by:

$$f_{A_{\theta=0}} = \left(H_{A_{\theta=0}} + \frac{D}{\square} \right) + i \frac{\partial_\mu A^\mu}{\square} \quad V-14$$

For the constant moduli fields $\phi \rightarrow \langle \phi \rangle$ the second and third terms are just absent. As a consequence the non-local chiral superfield f_A at $\theta = 0$ and $\phi = \langle \phi \rangle$ has a local non-holomorphic component $H_a C_A^a H_a(\phi, \bar{\phi})$. Therefore we conclude that non-holomorphic effective gauge couplings are perfectly compatible with the standard supergravity form, which will produce all desired couplings provided the correct expression for f_A (given by eq. V-13) is used.

For those orbifold models in which the target-space modular group is a product of $SL(2, \mathbb{Z})$ factors, the automorphic functions obtained by integrating out the massive string modes (eq. V-5) are logs of products of Dedekind functions, according to the discussion of section III. Also in this case, since the moduli space M is a symmetric space, the Kähler potential K_a and the curvature term $\det \log G_a$ are just proportional, so all holomorphic terms are expressible in terms of the same automorphic function^{[50]–[53]}.

As a final topic of this section we would like to report on higher loop corrections to gauge couplings in string theories which may also be understood from the anomaly cancellation mechanism^[73].

The argument will be made for the E_8 gauge group for which only gauginos contribute to the anomalies. The other gauge groups can be discussed as well but will not be treated here. The idea is to obtain the all-loop β function for a pure $N=1$ super Yang-Mills theory from purely string arguments.

That this must be so, comes from the fact that the all-loop result^{[66]–[68]}:

$$\frac{\beta(g)}{g^3} = \frac{-3T(G)/16\pi^2}{1 - [T(G)/8\pi^2]g^2} \quad V - 15$$

comes from the multiplet structure of the anomaly multiplet of the supercurrent containing the $U(1)$ R-current and the dilatation current^{[65]–[67]}. In string theory all couplings and scales are dynamical variables, so one expects a relation between g_{eff}^2 and g_{bare}^2 as coming from field equations. This can be seen, in string theory, as a relation between the gauge couplings in the linear multiplet and in the chiral multiplet formulation.

Let us consider the anomaly cancellation mechanism for the E_8 gauge group in superfield notations and in the linear multiplet representation. We also consider, for definiteness, the $\mathcal{Z}_3$ and $\mathcal{Z}_7$ orbifolds for which no counterterms other than the Green-Schwarz mechanism are present^{[51],[52]}.

The local effective Lagrangian which cancels the E_8 - $U(1)$ σ -model anomaly is:

$$S_\Gamma^{1-\text{loop}} = \int d^4\theta [\hat{L}^{-1/2} e^{-1/2K} (S_0 \bar{S}_0)^{3/2} + \lambda \hat{L} K] \quad V - 16$$

(here S_0 is a compensating multiplet which sets up the choice of a dynamical scale i.e. gives the normalization of the Einstein term). The combination:

$$\hat{L} = L - \Omega \quad V - 17$$

is a gauge-invariant multiplet containing a real linear multiplet L with the Chern-Simons multiplet Ω . Note that $\hat{L}$ is a modular invariant expression (K is the moduli Kähler potential) so a modular transformation:

$$K \rightarrow K - \Lambda - \bar{\Lambda} \quad V - 18$$

acts on the compensating multiplet S_0 as:

$$S_0 \rightarrow S_0 e^{-1/3\Lambda} \quad V - 19$$

To go to the S field formulation for the dilaton we have to replace $\hat{L}$ by an unconstrained real superfield V and enforce the linear constraint through a Lagrange multiplier, i.e., a chiral field S :

$$\mathcal{L} \rightarrow \mathcal{L}(\hat{L} \rightarrow V) - (S + \bar{S})(V + \Omega) \quad V - 20$$

By defining $U = \frac{1}{2} \left(\frac{S_0 \bar{S}_0}{2V_e^{k/3}} \right)^{3/2}$, the equation of motion for U becomes:

$$U = \frac{S + \bar{S}}{2} - \frac{T(E_8)}{16\pi^2} K \quad V - 21$$

which gives the one-loop effective coupling in the presence of the Green-Schwarz mechanism by identifying:

$$U = \frac{1}{g_{\text{eH}}^2}, \quad \frac{S + \bar{S}}{2} = \frac{1}{g_0^2} \quad \text{V-22}$$

The Green-Schwarz term is not the most general term compatible with the superconformal dimensions and with the anomaly. It can be further completed by replacing:

$$\hat{L}K \rightarrow 3\hat{L}\log e^{K/3}\hat{L} \quad \text{V-23}$$

With this modification eq. V-21 gets replaced by:

$$\frac{1}{g_{\text{eH}}^2} = \frac{1}{g_0^2} \frac{T(E_8)}{8\pi^2} \left(-\log g_{\text{eff}}^2 + c - \frac{3}{2} \log S_0 \bar{S}_0 \right) \quad \text{V-24}$$

where, to get the one-loop result, we have to substitute $S_0 \bar{S}_0 = e^{K/3}$ and neglect the second term on the right-hand side of eq. V-24.

Remembering that $S_0 \bar{S}_0$ sets the unit of measure of $(\text{mass})^2$, eq. V-15 is obtained as follows:

$$\beta(g) = -\frac{1}{g} \frac{dg^2}{d(\log S_0 \bar{S}_0)} \quad \text{V-25}$$

A modular-invariant g_{eff}^2 is obtained by setting $S_0 \bar{S}_0 \propto e^{K/3}$. Then we get:

$$\frac{1}{g_{\text{eff}}^2} = \left(\frac{S + \bar{S}}{2} - \frac{T(E_8)}{16\pi^2} K + c \right) \frac{T(E_8)}{8\pi^2} \log g_{\text{eff}}^2 \quad \text{V-26}$$

and the modified S-Kähler potential becomes:

$$\log \left\{ g_{\text{eff}}^2 \left[1 + \frac{T(E_8)}{16\pi^2} g_{\text{eff}}^2 \right]^{-3} \right\} \quad \text{V-26}$$

where g_{eff}^2 is solved in terms of $S + \bar{S}$ by eq. V-26.

To connect the modular invariant effective gauge coupling V-26 with the running gauge coupling defined in a renormalizable low-energy theory in a given renormalization scheme one should impose on the two couplings appropriate boundary conditions. Note that in eq. V-26 the constant c is undetermined by modular invariant arguments.

Acknowledgements

I would like to thank the following colleagues for the fruitful and stimulating collaboration, on which this review is based:

M. Bodner, C. Cadavid, L. Castellani, A. Ceresole, R. D'Auria, J.P. Derendinger, P. Fré, C. Kounnas, J. Louis, D. Lüst, S. Sabharwal, A. Shapere, A. Strominger, S. Theisen and F. Zwirner.

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