RENORMALISATION GROUP ASPECTS OF $\eta' \rightarrow \gamma\gamma$.**G.M. Shore and G. Veneziano**CERN – *Geneva***Abstract**

The theoretical issues involved in the calculation of the decay amplitude $\eta' \rightarrow \gamma\gamma$ are discussed, with particular emphasis on the renormalisation group (RG) aspects. In contradiction to the existing literature, it is shown that the straightforward generalisation of the well-known PCAC formula for $\pi^0 \rightarrow \gamma\gamma$ to $\eta' \rightarrow \gamma\gamma$ is inconsistent with the RG, and therefore incorrect in principle. The decay amplitude for $\eta' \rightarrow \gamma\gamma$ is found to involve a new decay constant F , defined in terms of the 2-point Green function of the composite operator $i\bar{q}\gamma_5 q$, which is both RG invariant and process-independent, and a proper vertex, describing the coupling of the “glue” component of the η' to photons, which is not constrained by the anomalous Ward identities for spontaneously broken chiral symmetry. The phenomenological implications of this result are discussed.

1. Introduction

The radiative decays of the neutral pseudoscalar mesons are of special theoretical interest because of their intimate relationship with the axial anomaly. In particular, the essential scale-independence of the anomaly allows us to make contact between certain low-momentum processes which can be described in the framework of PCAC, and the elementary degrees of freedom, which are manifest at high energies in an asymptotically free theory such as QCD.

The successful prediction (for a review of the early literature, see, e.g., ref.[1]) of the $\pi^0 \rightarrow \gamma\gamma$ decay amplitude in terms of F_π and the QED axial anomaly coefficient, viz.

$$F_\pi g_{\pi\gamma\gamma} = \frac{1}{3\pi} N_C \alpha_{\text{em}}, \quad (1.1)$$

in the chiral limit (see eq.(3.9)), is a test of the basic ideas of spontaneously broken chiral symmetry in QCD and the existence of the QED anomaly in the isovector axial current $J_{\mu 5}^3$, and provides direct experimental evidence for $N_C = 3$.

It is natural to try to extend these ideas[2,3] to the decay $\eta' \rightarrow \gamma\gamma$ of the flavour singlet pseudoscalar meson (we take $N_F = 3$ for the purposes of this discussion). A substantial literature, both theoretical and experimental, has been devoted to this process over many years (for reviews, see refs.[4] and [5]). The naive PCAC formula analogous to that for the π^0 is assumed to hold, with F_π replaced by an “ η' decay constant” defined by $\langle 0 | J_{\mu 5}^0 | \eta' \rangle = i k_\mu F_{\eta'}$. In certain references, “nonet symmetry” is further invoked to equate this $F_{\eta'}$ to F_π .

However, it is clear that these simple generalisations cannot be correct. The flavour singlet current is not conserved even in the chiral limit because of the QCD axial anomaly, in addition to the $O(\alpha_{\text{em}})$ QED anomaly. Since the current is not conserved, the quantity $F_{\eta'}$ defined above is not renormalisation group (RG) invariant, and so the naive generalisation of the $\pi^0 \rightarrow \gamma\gamma$ formula cannot hold as a matter of principle. Another immediate implication is that the η' is not a simple Nambu-Goldstone (NG) boson in QCD, so that it should not be surprising that the naive PCAC formula is not correct.

Of course, in the large N_C limit, where the anomaly is a sub-leading effect, these problems do not arise, and it is legitimate to regard the η' as a NG boson, with $F_{\eta'} = \sqrt{2N_F} F_\pi$ (with the normalisations used here), and derive the corresponding decay amplitude for $\eta' \rightarrow \gamma\gamma$ [6].

In this paper, we derive the exact formula for the flavour singlet decay $\eta' \rightarrow \gamma\gamma$ in the chiral limit of QCD, taking proper account of the QCD axial anomaly and with particular emphasis on the RG aspects of the problem.

We show that a direct analogue of eq.(1.1) holds, *not* for the physical η' , but for an unphysical state $|\eta_0\rangle$ which would be the NG boson in the OZI limit of QCD, where the anomaly is absent. This “OZI NG boson” was first introduced in ref.[7] (see also refs.[8,9]) in the context of the $U(1)$ Goldberger-Treiman relation.

Writing this result in terms of the η' , an extra term appears, essentially a precisely-defined “glue”-photon coupling, describing the extent to which the physical η' of QCD is not a simple NG boson. Furthermore, the decay constant, which we denote by F , is a

RG invariant quantity defined from the 2-point Green function of the composite operator $i\bar{q}\gamma_5 q$, and *not* from the coupling of the η' to $J_{\mu 5}^0$. As we emphasise in section 6, where we make contact with our previous work on the $U(1)$ Goldberger-Treiman relation, it is this F which is the RG invariant, process-independent generalisation to the flavour singlet channel of F_π .

On the formal side, our analysis makes important use of proper vertices defined from the Zumino effective action. This is an exact generating functional, specially chosen so that the proper vertices have a particularly direct identification with the physical couplings. This is achieved by making a partial Legendre transform w.r.t. a subset of the relevant composite operators only. The resulting proper vertices are truncated composite operator Green functions with the appropriate one-particle poles removed.

Our aim here is to give a careful and complete discussion of the theoretical issues involved in the calculation of $\pi^0 \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$. The RG aspects are discussed in considerable detail. In view of this, the paper is necessarily rather technical and is perhaps best approached in stages. The more phenomenologically minded reader is advised to start immediately with section 6, where we summarise our results and discuss their immediate phenomenological implications, including a critical reassessment of the inferences drawn from the experimental data in the existing literature. The derivation of our formulae for $\pi^0 \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$ is given in sections 3 and 4, which are reasonably self-contained. For the more theoretically inclined reader, section 2 contains a complete discussion of the renormalisation of the relevant composite operators, the chiral Ward identities and the precise definition of the Zumino effective action, and the full RGEs. The RG properties of the various quantities involved in the $\eta' \rightarrow \gamma\gamma$ amplitude are derived in section 5. Here again, these sections may be read in two stages. For completeness, we have included throughout this paper the terms in the RG equations resulting from the extra counterterms needed to renormalise Green functions with more than one composite operator. In fact, these play little role in the final results, and may be neglected on a first reading. The same is not true, however, of the simple anomalous dimensions for these operators, which play a necessary and crucial role in understanding the physics of the flavour singlet channel.

2. Composite Operators, Ward Identities and RGEs

The derivation presented here of the decay amplitudes for $\pi^0 \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$ involves the Green functions and proper vertices of the set of composite operators $J_{\mu 5}^m$, J_μ^m , ϕ_5^m , ϕ^m , Q and Q_{em} . We therefore begin by defining these operators and discussing their renormalisation, generalising the discussion of ref.[9] to include electromagnetism.

The bare operators are defined in terms of the quarks (q) and gluon and electromagnetic field strengths ($G_{\mu\nu}$ and $F_{\mu\nu}$ resp.) as follows:

$$\begin{aligned} J_{\mu 5}^m &= \bar{q}\gamma_\mu\gamma_5\lambda^m q & J_{\mu B}^m &= \bar{q}\gamma_\mu\lambda^m q \\ \phi_5^m &= i\bar{q}\gamma_5\lambda^m q & \phi_B^m &= \bar{q}\lambda^m q \end{aligned}$$

$$Q_B = \frac{\alpha_s}{8\pi} \text{tr} G^{\mu\nu} G_{\mu\nu}^* \quad Q_{\text{em}B} = c e_B^2 F^{\mu\nu} F_{\mu\nu}^*, \quad (2.1)$$

with $m = 0, i$ where λ^i ($i = 1, \dots, N_F^2 - 1$) are flavour $SU(N_F)$ generators, and $\lambda^0 = 1$. The numerical coefficient $c = \frac{1}{32\pi^2} N_C \sum_{f=1, \dots, N_F} q_f^2$, where q_f are the quark charges. The generating functional $W[V_{\mu 5 R}^m, V_{\mu R}^m, S_R^a, \mathcal{S}_{\mu R}^A]$ of connected Green functions of these operators is defined by

$$\exp iW[V_{\mu 5 R}^m, V_{\mu R}^m, S_R^a, \mathcal{S}_{\mu R}^A] = \langle \exp i(V_5^{m\mu} J_{\mu 5}^m + V^{m\mu} J_\mu^m + S^a \Phi_a + \mathcal{S}^{A\mu} A_\mu)_B \rangle \quad (2.2)$$

where $V_{\mu 5}^m$ and V_μ^m are sources for $J_{\mu 5}^m$ and J_μ^m resp., and S^a are sources for the set of fields $\Phi_a = (Q, Q_{\text{em}}, \phi_5^m, \phi^m)$. The sources $S^Q, S^{Q_{\text{em}}}$ are denoted below by Θ and Θ_{em} . $\mathcal{S}_\mu^A$ is the source for the elementary electromagnetic field A_μ . A spacetime integration is implied by repeated indices.

The chiral Ward identities of interest to us here are those for the flavour-singlet axial-vector current $J_{\mu 5}^0$, and for the flavour non-singlet currents $J_{\mu 5}^r$ where r denotes a generator in the Cartan sub-algebra. These satisfy the following anomalous Ward identities in the chiral limit:

$$\partial_\mu \frac{\delta W}{\delta V_{\mu 5}^0} - 2N_F \frac{\delta W}{\delta \Theta} - \frac{\delta W}{\delta \Theta_{\text{em}}} - \delta_5^0 V_{\mu 5}^m \frac{\delta W}{\delta V_{\mu 5}^m} - \delta_5^0 V_\mu^m \frac{\delta W}{\delta V_\mu^m} - \delta_5^0 S^a \frac{\delta W}{\delta S^a} = 0, \quad (2.3)$$

and

$$\partial_\mu \frac{\delta W}{\delta V_{\mu 5}^m} - a^{(r)} \frac{\delta W}{\delta \Theta_{\text{em}}} - \delta_5^r V_{\mu 5}^m \frac{\delta W}{\delta V_{\mu 5}^m} - \delta_5^r V_\mu^m \frac{\delta W}{\delta V_\mu^m} - \delta_5^r S^a \frac{\delta W}{\delta S^a} = 0, \quad (2.4)$$

where the variations of the sources are $\delta_5^0 V_{\mu 5}^m = 0$, $\delta_5^0 V_\mu^m = 0$, $\delta_5^0 S_5^i = -2S^i$, $\delta_5^0 S_5^0 = -2S^0$, $\delta_5^0 S^i = 2S_5^i$, $\delta_5^0 S^0 = 2S_5^0$, $\delta_5^0 \Theta = 0$, $\delta_5^0 \Theta_{\text{em}} = 0$, and for the non-singlet, $\delta_5^r V_{\mu 5}^m = f^{rij} V_\mu^j$, $\delta_5^r V_{\mu 5}^0 = 0$, $\delta_5^r V_\mu^i = f^{rij} V_{\mu 5}^j$, $\delta_5^r V_\mu^0 = 0$, $\delta_5^r S_5^i = -2\delta^{ri} S^0$, $\delta_5^r S_5^0 = -\frac{1}{2} S^r$, $\delta_5^r S^i = 2\delta^{ri} S_5^0$, $\delta_5^r S^0 = \frac{1}{2} S_5^r$, $\delta_5^r \Theta = 0$, $\delta_5^r \Theta_{\text{em}} = 0$, which follow immediately from the corresponding results for the fields themselves. $a^{(r)}$ is a coefficient giving the relative strength of the electromagnetic anomalies in the non-singlet and singlet currents (e.g. $a^{(3)} = 1/4$).

In eqs.(2.3), (2.4), we have not indicated whether the sources are renormalised or bare. This is because, as a special property of the chiral Ward identities, they hold in the same form for both bare and renormalised quantities.

The renormalisation of the composite operators occurring in the axial anomaly equations in pure QCD has been clearly explained in ref.[10] (see also ref.[11]), and here we shall simply extend these results to the case of QCD with electromagnetism.

In pure QCD, the flavour non-singlet axial currents, being conserved, are not renormalised, while the singlet current is multiplicatively renormalised. Following arguments in ref.[10], we find that with electromagnetism, the flavour non-singlet current mixes with the singlet as follows:

$$J_{\mu 5 R}^0 = (1 + z + z_{\text{em}}) J_{\mu 5 B}^0, \quad (2.5)$$

$$J_{\mu 5 R}^r = J_{\mu 5 B}^r + a^{(r)} z_{\text{em}} J_{\mu 5 B}^0, \quad (2.6)$$

where the new renormalisation constant z_{em} is of $O(\epsilon_R^4)$. For $\epsilon_R = 0$, eqs.(2.5), (2.6) reduce to those of refs.[9,10,11] with $1 + z = Z$.

We now assume that the axial anomaly equations take the same form when written in terms of either the bare or renormalised operators (“non-renormalisation of the anomaly”). The justification for this result is given for pure QCD in ref.[10], and it is clear from this work that the same result holds in the more general case with electromagnetism. It is the expression in the language of composite operator renormalisation of the Adler-Bardeen theorem. This assumption fixes the renormalisation of the operators Q and Q_{em} , and we find

$$Q_R = Q_B + \frac{1}{2N_F} z \partial^\mu J_{\mu 5}^0{}_B, \quad (2.7)$$

$$Q_{\text{em}R} = Q_{\text{em}B} + z_{\text{em}} \partial^\mu J_{\mu 5}^0{}_B. \quad (2.8)$$

Notice in particular that Q_{em} is renormalised and mixes non-trivially with $J_{\mu 5}^0$, although only at $O(\epsilon_R^4)$, a factor of ϵ_R^2 being already included in the definition of $Q_{\text{em}R}$.

In the following sections, we compute the decays $\pi^0 \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$ to lowest order, i.e. to $O(\epsilon_R^2)$. It will be clear from the above discussion that to this order the composite operator renormalisation may be approximated by simply omitting z_{em} altogether, so that $J_{\mu 5}^r$ is not renormalised and $Q_{\text{em}R} = Q_{\text{em}B}$, the elementary field renormalisation of $A_{\mu R}$ being cancelled by that of the coupling ϵ_R .

We may then regard $Q_{\text{em}R}$ simply as $Q_{\text{em}}(A_{\mu R}) = c\epsilon_R^2 F_R^{\mu\nu} F_{\mu\nu R}^*$, i.e. as a product of renormalised elementary fields rather than as an independent composite operator. The source Θ_{em} is dispensed with, and the Legendre transform defining the effective action Γ is made w.r.t. the elementary field $A_{\mu R}$ only.

To lowest order, therefore, the basic RGEs (i.e. those obtained without subsequent use of the chiral Ward identities) for W , Γ , and the Green functions and proper vertices derived from them, are the same as in our previous paper[9]. However, in reviewing these results below, we shall for completeness also indicate the extension of the RGEs to all orders.

First, we require the source renormalisations corresponding to eqs.(2.5)-(2.8) for the composite operators. We have,

$$\begin{aligned} V_{\mu 5 R}^0 &= (1 + z + z_{\text{em}})^{-1} \left(V_{\mu 5 B}^0 + \frac{1}{2N_F} z \partial_\mu \Theta_B + z_{\text{em}} (\partial_\mu \Theta_{\text{em}B} - \sum_r a^{(r)} V_{\mu 5 B}^r) \right) \\ V_{\mu 5 R}^r &= V_{\mu 5 B}^r \\ \Theta_R &= \Theta_B \\ \Theta_{\text{em}R} &= \Theta_{\text{em}B}. \end{aligned} \quad (2.9)$$

These renormalisations give rise to an anomalous dimension matrix amongst the sources $V_{\mu 5 R}^0$, $V_{\mu 5 R}^r$, $\partial_m \Theta_R$ and $\partial_m \Theta_{\text{em}R}$, of which the only non-zero entries are $\gamma_{V_5^0 V_5^0}$, $\gamma_{V_5^0 V_5^r}$, $\gamma_{V_5^0 \Theta}$ and $\gamma_{V_5^0 \Theta_{\text{em}}}$. These are not all independent, however, since the chiral Ward identities imply the relations

$$\gamma_{V_5^0 V_5^0} + \gamma_{V_5^0 \Theta} + \gamma_{V_5^0 \Theta_{\text{em}}} = 0, \quad (2.10)$$

$$\gamma_{V_5^0 V_5^r} + a^{(r)} \gamma_{V_5^0 \Theta_{\text{em}}} = 0. \quad (2.11)$$

There are therefore just two independent anomalous dimensions, which we can take to be $\gamma_{V_5^0\Theta} \equiv -\frac{1}{2N_F}\gamma$ and $\gamma_{V_5^0\Theta_{em}} \equiv -\gamma_{em}$. Notice that γ_{em} is $O(\epsilon_R^4)$, while in the $\epsilon_R = 0$ limit, γ is just the usual axial-vector anomalous dimension of refs.[9,10,11].

The remaining fields ϕ_5^m and ϕ^m are multiplicatively renormalised, so that the corresponding source renormalisation is simply

$$S_{\pm R}^m = Z_\phi S_{\pm B}^m, \quad (2.12)$$

where $S_{\pm R}^m = S_R^m \mp iS_{5R}^m$, with anomalous dimension γ_ϕ . The elementary electromagnetic field A_μ is multiplicatively renormalised, and we denote the corresponding anomalous dimension by γ_A .

Finally, we have to take account of the further renormalisations required to remove the new divergences which appear in Green functions involving more than one composite operator. The necessary formalism is explained in refs.[9,11]. For the applications here, we simply require to introduce a source S_B for the unit operator 1, which is non-linearly renormalised. The counterterms are restricted by the chiral Ward identities. The procedure for constructing them is explained in detail in ref.[11], and the extension to include electromagnetism and the operator Q_{em} is straightforward. However, the full list of possible counterterms is rather lengthy, and so we quote here only those which we shall explicitly use in the $O(\epsilon_R^2)$ calculations in the paper. Restricting to second order in the sources, as is sufficient for 2-point Green functions and the proper vertices needed here, we therefore have

$$\begin{aligned} S_B = & S_R + L_5^{(T)}(\partial_\mu B_{\nu R} - \partial_\nu B_{\mu R})^2 + L_5^{(L)}(\partial_\mu B_{\mu R}^\mu)^2 + \tilde{L}_5^{(T)}(\partial_\mu V_{\nu 5 R}^i - \partial_\nu V_{\mu 5 R}^i)^2 \\ & + L^{(T)}(\partial_\mu V_{\nu R}^0 - \partial_\nu V_{\mu R}^0)^2 + \tilde{L}^{(T)}(\partial_\mu V_{\nu R}^i - \partial_\nu V_{\mu R}^i)^2 \\ & + L^{(S)}\partial^\mu S_{+R}^m \partial_\mu S_{-R}^m + O(\epsilon_R^4) + O(S^3), \end{aligned} \quad (2.13)$$

where $B_{\mu R} = V_{\mu 5 R}^0 - \frac{1}{2N_F}\partial_\mu \Theta_R$.

The RGE for the generating functional $W[V_{\mu 5 R}^m, V_{\mu R}^m, \Theta_R, \Theta_{em R}, S_{5 R}^m, S_R^m, \mathcal{S}_{\mu R}^A, S_R]$ is therefore

$$\begin{aligned} \mathcal{D}W = & \gamma(V_{\mu 5 R}^0 - \frac{1}{2N_F}\partial_\mu \Theta_R) \frac{\delta W}{\delta V_{\mu 5 R}^0} + \gamma_{em}(\sum_r a^{(r)}V_{\mu 5 R}^r - \partial_\mu \Theta_{em}) \frac{\delta W}{\delta V_{\mu 5 R}^0} \\ & + \gamma_\phi\left(S_{5 R}^m \frac{\delta W}{\delta S_{5 R}^m} + S_R^m \frac{\delta W}{\delta S_R^m}\right) + \gamma_A \mathcal{S}_{\mu R}^A \frac{\delta W}{\delta \mathcal{S}_{\mu R}^A} \\ & + \left(\beta_5^{(T)}(\partial_\mu B_{\nu R} - \partial_\nu B_{\mu R})^2 + \beta_5^{(L)}(\partial^\mu B_{\mu R}^\mu)^2 + \tilde{\beta}_5^{(T)}(\partial_\mu V_{\nu 5 R}^i - \partial_\nu V_{\mu 5 R}^i)^2\right. \\ & + \beta^{(T)}(\partial_\mu V_{\nu R}^0 - \partial_\nu V_{\mu R}^0)^2 + \tilde{\beta}^{(T)}(\partial_\mu V_{\nu R}^i - \partial_\nu V_{\mu R}^i)^2 \\ & \left. + \beta^{(S)}(\partial^\mu S_{5 R}^m \partial_\mu S_{5 R}^m + \partial^\mu S_R^m \partial_\mu S_R^m) + O(\epsilon_R^4) + O(S^3)\right) \frac{\delta W}{\delta S_R}, \end{aligned} \quad (2.14)$$

where $\mathcal{D} = (\mu\partial/\partial\mu + \beta\partial/\partial g)|_{V,S}$. The RGEs for 2-point Green functions follow immediately by differentiation of eq.(2.14), and the required results, for $\gamma_{em} = 0$, are given in ref.[9].

The remaining ingredient in our approach is the Zumino effective action $\Gamma[V_{\mu 5R}^m, V_{\mu R}^m, Q_R, Q_{\text{em}R}, \phi_{5R}^m, \phi_R^m, A_{\mu R}, 1_R]$ which is the generating functional for proper vertices which are 1PI w.r.t. all the composite and elementary fields except the sources. As will become apparent, this property makes their identification with the physical couplings particularly direct. Γ is defined from W by a partial Legendre transform w.r.t. the fields only, not w.r.t. the currents, i.e.

$$\Gamma[V_{\mu 5R}^m, V_{\mu R}^m, \Phi_{aR}, A_{\mu R}, 1_R] = W[V_{\mu 5R}^m, V_{\mu R}^m, S_R^a, \mathcal{S}_R^A, S_R] - S_R^a \Phi_{aR} - \mathcal{S}_R^{A\mu} A_{\mu R} - S_R 1_R, \quad (2.15)$$

where

$$\Phi_{aR} \equiv \Phi_{aR}(V, S) = \frac{\delta W}{\delta S_R^a}, \quad (2.16)$$

and $S^a = (\Theta, \Theta_{\text{em}}, S_5^m, S^m)$. Notice that the definition is in terms of the renormalised sources. Derivatives of Γ yield finite, renormalised 1PI vertices. We emphasise that here the term “effective action” is used in the above sense as a generating functional of proper vertices. There is no approximation being made, and the following identities are exact results.

In terms of Γ , the chiral Ward identities are

$$\partial_\mu \frac{\delta \Gamma}{\delta V_{\mu 5R}^0} - 2N_F Q_R - Q_{\text{em}R} - \delta_5^0 V_{\mu 5R}^m \frac{\delta \Gamma}{\delta V_{\mu 5R}^m} - \delta_5^0 V_{\mu R}^m \frac{\delta \Gamma}{\delta V_{\mu R}^m} - \delta_5^0 \Phi_R^a \frac{\delta \Gamma}{\delta \Phi_R^a} = 0, \quad (2.17)$$

and

$$\partial_\mu \frac{\delta \Gamma}{\delta V_{\mu 5R}^r} - a^{(r)} Q_{\text{em}R} - \delta_5^r V_{\mu 5R}^m \frac{\delta \Gamma}{\delta V_{\mu 5R}^m} - \delta_5^r V_{\mu R}^m \frac{\delta \Gamma}{\delta V_{\mu R}^m} - \delta_5^r \Phi_R^a \frac{\delta \Gamma}{\delta \Phi_R^a} = 0, \quad (2.18)$$

where the field variations are $\delta_5^0 \phi_5^i = -2\phi^i$, $\delta_5^0 \phi_5^0 = -2\phi^0$, $\delta_5^0 \phi^i = 2\phi_5^i$, $\delta_5^0 \phi^0 = 2\phi_5^0$, $\delta_5^0 Q = 0$, $\delta_5^0 Q_{\text{em}} = 0$ and $\delta_5^r \phi_5^i = -\frac{1}{2}\delta^{ri}\phi^0$, $\delta_5^r \phi_5^0 = -2\phi^r$, $\delta_5^r \phi^i = \frac{1}{2}\delta^{ri}\phi_5^0$, $\delta_5^r \phi^0 = 2\phi_5^r$, $\delta_5^r Q = 0$, $\delta_5^r Q_{\text{em}} = 0$.

The RGE for Γ follows directly from that for W , eq.(2.14). We find,

$$\begin{aligned} \mathcal{D}\Gamma = & \gamma \left(V_{\mu 5R}^0 - \frac{1}{2N_F} \frac{\delta \Gamma}{\delta Q_R} \partial_\mu \right) \frac{\delta \Gamma}{\delta V_{\mu 5R}^0} + \gamma_{\text{em}} \left(\sum_r a^{(r)} V_{\mu 5R}^r - \frac{\delta \Gamma}{\delta Q_{\text{em}R}} \partial_\mu \right) \frac{\delta \Gamma}{\delta V_{\mu 5R}^0} \\ & - \gamma_\phi \left(\phi_{5R}^m \frac{\delta \Gamma}{\delta \phi_{5R}^m} + \phi_R^m \frac{\delta \Gamma}{\delta \phi_R^m} \right) - \gamma_A A_{\mu R} \frac{\delta \Gamma}{\delta A_{\mu R}} \\ & - \left(\beta_5^{(T)} (\partial_\mu B_{\nu R} - \partial_\nu B_{\mu R})^2 + \beta_5^{(L)} (\partial^\mu B_{\mu R})^2 + \tilde{\beta}_5^{(T)} (\partial_\mu V_{\nu 5R}^i - \partial_\nu V_{\mu 5R}^i)^2 \right. \\ & + \beta^{(T)} (\partial_\mu V_{\nu R}^0 - \partial_\nu V_{\mu R}^0)^2 + \tilde{\beta}^{(T)} (\partial_\mu V_{\nu R}^i - \partial_\nu V_{\mu R}^i)^2 \\ & \left. + \beta^{(S)} \left(\partial^\mu \frac{\delta \Gamma}{\delta \phi_{5R}^m} \partial_\mu \frac{\delta \Gamma}{\delta \phi_{5R}^m} + \partial^\mu \frac{\delta \Gamma}{\delta \phi_R^m} \partial_\mu \frac{\delta \Gamma}{\delta \phi_R^m} \right) + O(\epsilon_R^4) + \dots \right), \end{aligned} \quad (2.19)$$

where here $\mathcal{D} = (\mu\partial/\partial\mu + \beta\partial/\partial g)|_{V,\Phi}$, $B_{\mu R} = V_{\mu 5R}^0 + \frac{1}{2N_F} \partial_\mu \frac{\delta \Gamma}{\delta Q_R}$, and the dots denote terms of higher order in V or $\frac{\delta \Gamma}{\delta \Phi}$. The RGEs for the proper vertices again follow straightforwardly by differentiation of eq.(2.19).

3. $\pi^0 \rightarrow \gamma\gamma$ and the Renormalisation Group

We are now in a position to rederive the familiar PCAC formula for $\pi^0 \rightarrow \gamma\gamma$ at $O(e_R^2)$ using chiral Ward identities and the Zumino effective action.

We shall calculate the decay amplitude for QCD with $N_F = 2$ coupled to electromagnetism, initially in the chiral limit. The starting point is therefore the $r = 3$ chiral Ward identity for the Zumino effective action. From eq.(2.18) we have,

$$\begin{aligned} \partial_\mu \frac{\delta\Gamma}{\delta V_{\mu 5R}^3} - a^{(3)} Q_{\text{em}}(A_{\mu R}) - \left(\delta V \frac{\delta\Gamma}{\delta V} \text{terms} \right) \\ + \frac{1}{2} \phi_R^0 \frac{\delta\Gamma}{\delta \phi_{5R}^3} + 2\phi_R^3 \frac{\delta\Gamma}{\delta \phi_{5R}^0} - \frac{1}{2} \phi_{5R}^0 \frac{\delta\Gamma}{\delta \phi_R^3} - 2\phi_{5R}^3 \frac{\delta\Gamma}{\delta \phi_R^0} = 0, \end{aligned} \quad (3.1)$$

where, as explained in the previous section, at this order we may take

$$Q_{\text{em}}(A_{\mu R}) = 4ce_R^2 \epsilon_{\mu\nu\lambda\rho} \partial^\mu A_R^\nu \partial^\lambda A_R^\rho. \quad (3.2)$$

Differentiating w.r.t. the electromagnetic fields and evaluating at the VEVs, we immediately find,

$$ik_\mu \frac{\delta^3\Gamma}{\delta V_{\mu 5R}^3 \delta A_R^\lambda \delta A_R^\rho} + 8a^{(3)} ce_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta + \frac{1}{2} \langle \phi_R^0 \rangle \frac{\delta^3\Gamma}{\delta \phi_{5R}^3 \delta A_R^\lambda \delta A_R^\rho} = 0, \quad (3.3)$$

where p_1, p_2 are the momenta carried by the A_R and $k = -(p_1 + p_2)$, or in an obvious shorthand notation,

$$ik_\mu \Gamma_{V_{\mu 5}^3 A^\lambda A^\rho} + 8a^{(3)} ce_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta + \frac{1}{2} \langle \phi \rangle \Gamma_{\phi_5^3 A^\lambda A^\rho} = 0. \quad (3.4)$$

In the pion on-shell limit $k^2 = 0$, the first term vanishes since there can be no pole at $k^2 = 0$ in the 1PI vertex $\Gamma_{V_{\mu 5}^3 A^\lambda A^\rho}$. We therefore find the key result,

$$\frac{1}{2} \langle \phi \rangle \Gamma_{\phi_5^3 A^\lambda A^\rho}(k^2 = 0) = -8a^{(3)} ce_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta. \quad (3.5)$$

To relate this formula to the physical pion, we need the relation,

$$\langle 0 | \phi_{5R}^3 | \pi^0 \rangle = \frac{1}{2} \langle \phi \rangle F_\pi^{-1} + O(e_R^2), \quad (3.6)$$

where the pion decay constant is defined as usual by $\langle 0 | J_{\mu 5R}^3 | \pi^0 \rangle = ik_\mu F_\pi$. This follows directly from an application of Goldstone's theorem to the Ward identity

$$ik_\mu W_{V_{\mu 5}^3 S_5^3} - a^{(3)} W_{\Theta_{\text{em}} S_5^3} - \frac{1}{2} \langle \phi \rangle = 0, \quad (3.7)$$

derived from eq.(2.4). The decay amplitude for $\pi^0 \rightarrow \gamma\gamma$, which we can write as $\langle \pi^0 | \gamma\gamma \rangle = -ig_{\pi\gamma\gamma} \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta \epsilon^\lambda(p_1) \epsilon^\rho(p_2)$ is expressed in terms of Γ to $O(\epsilon_R^2)$ as

$$\Gamma_{\phi_5^3 A^\lambda A^\rho}(0) = -\frac{1}{\langle \phi \rangle} F_\pi g_{\pi\gamma\gamma} \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta. \quad (3.8)$$

Notice that this identification requires that the proper vertex is 1PI w.r.t. the pseudoscalar (and electromagnetic) fields only, and not w.r.t. $J_{\mu 5 R}^3$. Such a definition would have introduced a further term into eq.(3.8) because of the non-vanishing π^0 - $J_{\mu 5 R}^3$ propagator[9].

Combining these results, we find the familiar PCAC relation determining the $\pi^0 \rightarrow \gamma\gamma$ decay amplitude in terms of the axial anomaly,

$$F_\pi g_{\pi\gamma\gamma} = 16a^{(3)} c \epsilon_R^2, \quad (3.9)$$

valid to $O(\epsilon_R^2)$ in the chiral limit.

It is straightforward to extend this result for explicitly broken chiral symmetry. Introducing a common quark mass m simply includes an extra term $-2m\phi_{5R}^3$ in the Ward identity(3.1). Differentiating w.r.t. A^λ , A^ρ and evaluating at the VEVs, this term does not contribute and the basic identity (3.5) is unchanged. The only modification to the derivation comes in the identification with the physical pion, as explained in detail in ref.[9], so that now eq.(3.9) holds only up to a smoothness approximation on certain pole-free dynamical functions characterising the propagator matrix.

Returning to the chiral limit, we now describe the RG behaviour of the proper vertices involved in this derivation. From eq.(2.19), we easily find at $O(\epsilon_R^2)$,

$$\mathcal{D}\Gamma_{V_{\mu 5}^i A^\lambda A^\rho} = -2\beta^{(S)} k^2 \Gamma_{V_{\mu 5}^i \phi_5^m} \Gamma_{\phi_5^m A^\lambda A^\rho} - 2\gamma_A \Gamma_{V_{\mu 5}^i A^\lambda A^\rho}, \quad (3.10)$$

$$\mathcal{D}\Gamma_{\phi_5^m A^\lambda A^\rho} = -\gamma_\phi \Gamma_{\phi_5^m A^\lambda A^\rho} - 2\beta^{(S)} k^2 \Gamma_{\phi_5^m \phi_5^n} \Gamma_{\phi_5^n A^\lambda A^\rho} - 2\gamma_A \Gamma_{\phi_5^m A^\lambda A^\rho}, \quad (3.11)$$

where here $\mathcal{D} = \mathcal{D}|_{V,\Phi} + \gamma_\phi \langle \phi \rangle \frac{\delta}{\delta \phi_R^0}$ is the appropriate RG operator for fixed sources (see ref.[9]). Simplifying using the identity

$$ik_\mu \Gamma_{V_{\mu 5}^i \phi_5^j} + \frac{1}{2} \langle \phi \rangle \Gamma_{\phi_5^i \phi_5^j} = 0, \quad (3.12)$$

we find

$$(\mathcal{D} + 2\gamma_A) ik_\mu \Gamma_{V_{\mu 5}^3 A^\lambda A^\rho} = \beta^{(S)} k^2 \langle \phi \rangle \Gamma_{\phi_5^3 \phi_5^3} \Gamma_{\phi_5^3 A^\lambda A^\rho}, \quad (3.13)$$

$$(\mathcal{D} + 2\gamma_A) \Gamma_{\phi_5^3 A^\lambda A^\rho} = -\gamma_\phi \Gamma_{\phi_5^3 A^\lambda A^\rho} - 2\beta^{(S)} k^2 \Gamma_{\phi_5^3 \phi_5^3} \Gamma_{\phi_5^3 A^\lambda A^\rho}. \quad (3.14)$$

The RG consistency of the identity (3.4) is then apparent. At $k^2 = 0$, we have simply

$$(\mathcal{D} + 2\gamma_A) \langle \phi \rangle \Gamma_{\phi_5^3 A^\lambda A^\rho}(0) = 0, \quad (3.15)$$

consistent with the identification (3.8) with physical quantities.

4. Decay Amplitude for $\eta' \rightarrow \gamma\gamma$

We now come to the main purpose of this paper, the calculation of the decay amplitude for $\eta' \rightarrow \gamma\gamma$ at $O(e_R^2)$ in the chiral limit. As with $\pi^0 \rightarrow \gamma\gamma$, we start from the appropriate chiral Ward identity for the Zumino effective action, viz. (see eq.(2.17))

$$\partial_\mu \frac{\delta \Gamma}{\delta V_{\mu 5 R}^0} - 2N_F Q_R - Q_{\text{em}}(A_{\mu R}) + \sum_m \left(2\phi_R^m \frac{\delta \Gamma}{\delta \phi_{5 R}^m} - 2\phi_{5 R}^m \frac{\delta \Gamma}{\delta \phi_R^m} \right) = 0, \quad (4.1)$$

where, to this order, we again take $Q_{\text{em}}(A_{\mu R})$ to be a simple product as in eq.(3.2). Differentiating and evaluating at the VEVs, we find

$$ik_\mu \Gamma_{V_{\mu 5}^0 A^\lambda A^\rho} + 8ce_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta + 2\langle\phi\rangle \Gamma_{\phi_5^0 A^\lambda A^\rho} = 0. \quad (4.2)$$

At $k^2 = 0$, therefore, we have simply

$$2\langle\phi\rangle \Gamma_{\phi_5^0 A^\lambda A^\rho}(0) = -8ce_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta. \quad (4.3)$$

We can rewrite this basic relation in different ways. First, we re-express it in terms of the state $|\eta_0\rangle$, the “OZI Nambu-Goldstone boson” of refs.[7,8,9], which would be the $U(1)$ NG boson in the OZI limit of QCD, where the anomaly is absent. It is not a physical state in QCD itself. The η_0 interpolating field is defined to be the canonically normalised field $\eta_0 = b(k^2)\phi_5^0$ such that $\Gamma_{\eta_0\eta_0} = k^2$. (The modifications to these definitions due to electromagnetism are of $O(e_R^2)$ and do not affect the leading order results quoted below.) The η_0 decay function is defined by $\Gamma_{V_{\mu 5}^0 \eta_0} = ik_\mu F_{\eta_0}(k^2)$. As shown in ref.[9], we therefore have

$$\eta_0 = \frac{1}{2\langle\phi\rangle} F_{\eta_0}(k^2) \phi_5^0. \quad (4.4)$$

We can now immediately rewrite eq.(4.3) in terms of the η_0 as follows,

$$F_{\eta_0}(0) g_{\eta_0\gamma\gamma}(0) = 16ce_R^2, \quad (4.5)$$

defining the η_0 -photon coupling in exact analogy to eq.(3.5).

This is the closest $U(1)$ analogue of the standard PCAC relation (3.9) for $\pi^0 \rightarrow \gamma\gamma$. However, the state which is analogous to the pion is *not* the physical η' , which we discuss below, but the unphysical OZI NG state η_0 . Nevertheless, this formulation involving the η_0 is of theoretical interest, and as shown in ref.[9] the decay constant $F_{\eta_0}(0)$ is simply related to the first moment of the topological susceptibility in QCD, $\chi_t'(0)$, a quantity particularly amenable to lattice calculations. Precisely,

$$F_{\eta_0} = 2N_F \left(\frac{\partial}{\partial k^2} \int dx \epsilon^{ik \cdot x} i\langle 0 | T^* Q(x) Q(0) \rangle \right)^{-\frac{1}{2}} \Big|_{k=0}. \quad (4.6)$$

The physical η' is the state responsible for the pole at $k^2 = m_{\eta'}^2$ in the pseudoscalar propagator matrix. Again, for a full discussion, see ref.[9]. Neglecting electromagnetic

corrections, the η' interpolating field is defined by diagonalising the propagator matrix in the pseudoscalar sector,

$$\eta' = c(k^2)\phi_{5R}^0 \quad G = Q_R - \frac{W_{\Theta S_5^0}}{W_{S_5^0 S_5^0}} \phi_{5R}^0, \quad (4.7)$$

and fixing $c = \left(-(k^2 - m_{\eta'}^2)W_{S_5^0 S_5^0}\right)^{-\frac{1}{2}}$ by imposing the canonical normalisation condition $\Gamma_{\eta'\eta'} = k^2 - m_{\eta'}^2$. Notice that we do not make any assumptions about the dynamics of the field G . In particular, its propagator need not contain a physical “glueball” pole.

It is then straightforward to re-express the relation (4.3) in terms of the proper vertices $\Gamma_{\eta' A^\lambda A^\rho}$ and $\Gamma_{G A^\lambda A^\rho}$. We find, using eqs.(4.7) and the Ward identity $W_{\Theta S_5^0}(0) = -\frac{2\langle\phi\rangle}{2N_F}$,

$$-2N_F \left(W_{\Theta S_{\eta'}} \Gamma_{\eta' A^\lambda A^\rho} - \frac{W_{\Theta S_{\eta'}}^2}{W_{S_{\eta'} S_{\eta'}}} \Gamma_{G A^\lambda A^\rho} \right) \Big|_{k^2=0} = -8ce_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta. \quad (4.8)$$

Defining the physical constant

$$F = -2N_F W_{\Theta S_{\eta'}}(0), \quad (4.9)$$

as in ref.[9], and with the definitions of $g_{\eta'\gamma\gamma}$ and $g_{G\gamma\gamma}$ analogous to eq.(3.5), we then have

$$F g_{\eta'\gamma\gamma}(0) + \frac{1}{2N_F} F^2 m_{\eta'}^2 g_{G\gamma\gamma}(0) = 16ce_R^2. \quad (4.10)$$

If we also make the dynamical assumption that the physical on-shell η' -photon coupling $g_{\eta'\gamma\gamma} \equiv g_{\eta'\gamma\gamma}(m_{\eta'}^2)$ is approximately equal to the $k^2 = 0$ value $g_{\eta'\gamma\gamma}(0)$, we can rewrite this exact relation in the final, approximate form,

$$F g_{\eta'\gamma\gamma} + \frac{1}{2N_F} F^2 m_{\eta'}^2 g_{G\gamma\gamma}(0) = 16ce_R^2. \quad (4.11)$$

This is our main result. Two points are worth commenting on. First, the physical constant playing the role of an “ η' decay constant” is F , defined in eq.(4.9), and *not* the naive analogue of F_π defined by $\langle 0 | J_{\mu 5 R}^0 | \eta' \rangle = i k_\mu F_{\eta'}$, which is not a RG invariant quantity. An alternative rewriting of eq.(4.9) shows that F is entirely determined by the zero-momentum limit of the propagator $W_{S_5^0 S_5^0}$, i.e.

$$F = \frac{1}{m_{\eta'}'} 2\langle\phi\rangle \left(\int dx \, i \langle 0 | T^* \phi_{5R}^0(x) \phi_{5R}^0(0) | 0 \rangle \right)^{-\frac{1}{2}}, \quad (4.12)$$

and, as demonstrated in ref.[9], this is the natural RG-invariant generalisation of F_π to the flavour singlet channel.

Second, the expression (4.11) differs from a naive PCAC expectation by the appearance of the term proportional to $g_{G\gamma\gamma}(0)$. This reflects the existence of the anomaly in the $U(1)$ channel. It appears because the lightest physical state in the pseudoscalar channel

is not a NG boson, but the more complicated glue-enriched state η' . We will return to a discussion of eq.(4.11) in section 6, after investigating the RG properties of its different elements.

5. RGEs for $\eta' \rightarrow \gamma\gamma$

We now derive the RGEs for the Green functions and proper vertices involved in the various expressions related to the $\eta' \rightarrow \gamma\gamma$ amplitude found in section 4. As already noted, to lowest order in ϵ_R^2 it is sufficient to consider the basic RGEs in the absence of the anomalous dimension γ_{em} . We may therefore take over freely many formulae already derived for pure QCD in ref.[9]. In particular, the RGEs we need for the 2-point functions $W_{\Theta S_5^0}$, $W_{S_5^0 S_5^0}$, etc. are precisely those of ref.[9].

The RGEs for the proper vertices $\Gamma_{Q A^\lambda A^\rho}$ and $\Gamma_{\phi_5^0 A^\lambda A^\rho}$ are found by differentiating eq.(2.19) and simplified using the Ward identity (4.2). We find,

$$\begin{aligned} (\mathcal{D} + 2\gamma_A) \Gamma_{Q A^\lambda A^\rho} = & -\gamma \Gamma_{Q A^\lambda A^\rho} + \gamma \frac{2\langle\phi\rangle}{2N_F} (\Gamma_{Q\phi_5^0} \Gamma_{Q A^\lambda A^\rho} + \Gamma_{QQ} \Gamma_{\phi_5^0 A^\lambda A^\rho}) \\ & + \gamma \frac{1}{2N_F} \Gamma_{QQ} 8ce_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta \\ & - 2\beta_5^{(L)} \frac{1}{(2N_F)^2} k^4 \Gamma_{QQ} \Gamma_{Q A^\lambda A^\rho} - 2\beta^{(S)} k^2 \Gamma_{Q\phi_5^0} \Gamma_{\phi_5^0 A^\lambda A^\rho}, \end{aligned} \quad (5.1)$$

and,

$$\begin{aligned} (\mathcal{D} + 2\gamma_A) \Gamma_{\phi_5^0 A^\lambda A^\rho} = & -\gamma_\phi \Gamma_{\phi_5^0 A^\lambda A^\rho} + \gamma \frac{2\langle\phi\rangle}{2N_F} (\Gamma_{\phi_5^0 \phi_5^0} \Gamma_{Q A^\lambda A^\rho} + \Gamma_{\phi_5^0 Q} \Gamma_{\phi_5^0 A^\lambda A^\rho}) \\ & + \gamma \frac{1}{2N_F} \Gamma_{\phi_5^0 Q} 8ce_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta \\ & - 2\beta_5^{(L)} \frac{1}{(2N_F)^2} k^4 \Gamma_{\phi_5^0 Q} \Gamma_{Q A^\lambda A^\rho} - 2\beta^{(S)} k^2 \Gamma_{\phi_5^0 \phi_5^0} \Gamma_{\phi_5^0 A^\lambda A^\rho}. \end{aligned} \quad (5.2)$$

These expressions simplify remarkably at $k^2 = 0$. Using the identities $\Gamma_{\phi_5^0 \phi_5^0}(0) = 0$, $\Gamma_{\phi_5^0 Q}(0) = \frac{2\langle\phi\rangle}{2N_F}$, together with eq.(4.3), we find

$$(\mathcal{D} + 2\gamma_A) \Gamma_{Q A^\lambda A^\rho}(0) = 0, \quad (5.3)$$

$$(\mathcal{D} + 2\gamma_A) \Gamma_{\phi_5^0 A^\lambda A^\rho}(0) = -\gamma_\phi \Gamma_{\phi_5^0 A^\lambda A^\rho}(0). \quad (5.4)$$

From the latter, we immediately have

$$(\mathcal{D} + 2\gamma_A) \langle\phi\rangle \Gamma_{\phi_5^0 A^\lambda A^\rho}(0) = 0, \quad (5.5)$$

verifying the RG consistency of the basic identity (4.3).

In terms of the η_0 , we already have from ref.[9] that

$$\mathcal{D}F_{\eta_0} = \gamma \frac{2\langle\phi\rangle}{2N_F} \Gamma_{\phi_5^0 Q} F_{\eta_0} - \beta^{(S)} k^4 \frac{1}{4\langle\phi\rangle^2} F_{\eta_0}^3, \quad (5.6)$$

for the decay function $F_{\eta_0}(k^2)$, and find, from eqs.(4.4) and (5.2),

$$\begin{aligned} (\mathcal{D} + 2\gamma_A) \Gamma_{\eta_0 A^\lambda A^\rho} &= \gamma \frac{1}{2N_F} k^2 F_{\eta_0} \Gamma_{Q A^\lambda A^\rho} + \gamma \frac{1}{2N_F} \Gamma_{\eta_0 Q} 8c e_R^2 \epsilon_{\lambda\rho\alpha\beta} p_1^\alpha p_2^\beta \\ &\quad - 2\beta_5^{(L)} \frac{1}{(2N_F)^2} k^4 \Gamma_{\eta_0 Q} \Gamma_{Q A^\lambda A^\rho} - \beta^{(S)} k^4 \frac{1}{4\langle\phi\rangle^2} F_{\eta_0}^2 \Gamma_{\eta_0 A^\lambda A^\rho}. \end{aligned} \quad (5.7)$$

Again, these expressions simplify at $k^2 = 0$, and we easily verify that

$$\mathcal{D}F_{\eta_0}(0) = \gamma F_{\eta_0}(0), \quad (5.8)$$

and

$$(\mathcal{D} + 2\gamma_A) \Gamma_{\eta_0 A^\lambda A^\rho}(0) = -\gamma \Gamma_{\eta_0 A^\lambda A^\rho}(0), \quad (5.9)$$

showing the RG consistency of eq.(4.5).

Finally, for the η' , we need the following pure QCD RGEs derived in ref.[9]:

$$\mathcal{D}W_{\Theta S_5^0} = (\gamma + \gamma_\phi) W_{\Theta S_5^0} + \gamma \frac{2\langle\phi\rangle}{2N_F}, \quad (5.10)$$

$$\mathcal{D}W_{S_5^0 S_5^0} = 2\gamma_\phi W_{S_5^0 S_5^0} - 2\beta^{(S)} k^2, \quad (5.11)$$

and

$$\mathcal{D}c = -\gamma_\phi c - \beta^{(S)} k^2 (k^2 - m_{\eta'}^2) c^3, \quad (5.12)$$

for the normalisation factor. Combining these with eq.(5.2), we eventually find

$$(\mathcal{D} + 2\gamma_A) \Gamma_{\eta' A^\lambda A^\rho} = \beta^{(S)} k^2 \frac{1}{W_{S_5^0 S_5^0}} \Gamma_{\eta' A^\lambda A^\rho}. \quad (5.13)$$

We see, therefore, that while $g_{\eta'\gamma\gamma}(k^2)$ is not in general RG invariant, it is invariant both on-shell ($k^2 = m_{\eta'}^2$), as required for a physical coupling, and at $k^2 = 0$. This is a necessary condition for the dynamical approximation made in writing eq.(4.11) to be valid. For the remaining factors in eq.(4.11), we have already found in ref.[9] that

$$\mathcal{D}F = 0, \quad (5.14)$$

and since $\Gamma_{GA^\lambda A^\rho}|_{\eta'} \equiv \Gamma_{QA^\lambda A^\rho}|_{\phi_{5R}^0}$, we have

$$(\mathcal{D} + 2\gamma_A) g_{G\gamma\gamma}(0) = 0. \quad (5.15)$$

This completes the demonstration that each factor in the expression (4.11) for the $\eta' \rightarrow \gamma\gamma$ decay amplitude is individually RG invariant.

6. Discussion

In the isotriplet (more generally, flavour non-singlet) channel, there exist two celebrated consequences of PCAC, the Goldberger-Treiman relation,

$$F_\pi g_{\pi NN} = 2m_N G_A^{(3)}(0), \quad (6.1)$$

and the formula for the $\pi^0 \rightarrow \gamma\gamma$ decay amplitude,

$$F_\pi g_{\pi\gamma\gamma} = \frac{N_C}{3\pi} \alpha_{\text{em}}. \quad (3.9)$$

In previous papers[7,8,9], we have generalised the relation (6.1) to the flavour singlet channel, with the result

$$F g_{\eta' NN} + \frac{1}{2N_F} F^2 m_{\eta'}^2 g_{GNN}(0) = 2m_N G_A(0). \quad (6.2)$$

The most important result of this paper is the formula (4.11) for the decay amplitude $\eta' \rightarrow \gamma\gamma$, viz.

$$F g_{\eta'\gamma\gamma} + \frac{1}{2N_F} F^2 m_{\eta'}^2 g_{G\gamma\gamma}(0) = \frac{4N_C}{3\pi} \alpha_{\text{em}}. \quad (4.11)$$

This completes the generalisation of the PCAC results above to the flavour singlet case.

Clearly, eqs.(4.11) and (6.2) are closely analogous. Two points are worth emphasising. The first is the appearance in both formulae of the RG invariant decay constant F . This is the proper generalisation to the $U(1)$ channel of the pion decay constant F_π . We stress again that it is *not* the RG non-invariant quantity $F_{\eta'}$ defined from $\langle 0 | J_{\mu 5 R}^0 | \eta' \rangle$.

The second point is the essential loss of predictivity compared to the flavour non-singlet channel. While the decay constant F is universal, each process involves a new and unknown proper vertex $g_{GXX}(0)$. In particular, the presence of the $g_{G\gamma\gamma}(0)$ term in eq.(4.11) prevents the experimental determination of F from the decay amplitude for $\eta' \rightarrow \gamma\gamma$.

In retrospect, this is not so surprising. The dynamical input to eqs.(4.11) and (6.2) has been essentially limited to the anomalous chiral Ward identities. Unlike the pion, however, which is a genuine NG boson and for which eqs.(3.9),(6.1) are predictive relations, the η' is not a simple NG boson, but may be thought of at the valence level as an admixture of meson and “glue” states. In this case, the Ward identities are correspondingly less restrictive, and the current algebra formulae involve extra vertices $g_{GXX}(0)$. Determining these requires further QCD dynamics, over and above the anomalous chiral Ward identities.

The decay constants $F_{\eta_0}(0)$ and F are straightforwardly defined in terms of the 2-point Green functions $\langle Q Q \rangle$ and $\langle \phi_5 \phi_5 \rangle$ respectively. In principle at least, they should therefore be calculable using, e.g. lattice QCD.

On the other hand, the definition of the proper vertices is more subtle. They may of course be rewritten as 3-point Green functions multiplied by certain inverse composite operator propagators, according to the generalised inversion formulae relating derivatives

of W and Γ (see ref.[9], sect.9 and appx.B, for a complete discussion). However, to some extent this has the effect of undoing the formulae above. The virtue of the proper vertices used here is that they are precisely defined to be the pole-free quantities most directly related to the physical couplings. The price to be paid for this definition is that they are not simply identifiable with certain classes of perturbative Feynman diagrams. It would therefore be interesting to develop techniques to calculate these proper vertices directly, without resorting to a prior calculation of the corresponding Green functions.

Our purpose in this paper has been to give a thorough treatment of the basic theoretical issues involved in the decay $\eta' \rightarrow \gamma\gamma$, rather than to discuss its detailed phenomenology. Nevertheless, we would like to include here some comments on phenomenology and the existing literature.

To apply our results to the actual experimental situation, we need to take $N_F = 3$ and allow for mixing between the octet state η_8 and the singlet η_1 to give the observed mass eigenstates η and η' . (Recall that so far we have been simply referring to the singlet pseudoscalar as the η' . From now on, to make more precise contact with the experimental situation, we use the notation η_1 for the flavour singlet to distinguish it from the observed η' .) The most recent experimental results for $\eta \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$ are given in ref.[12]. They are interpreted by writing formulae for the decay amplitudes in precise analogy to eq.(3.9) for the pion, taking into account an η - η' mixing angle θ_P and denoting the singlet and octet “decay constants” by f_1 and f_8 resp. In ref.[12], the chiral perturbation result[13] incorporating $SU(3)$ breaking is used to set $f_8 = 1.25F_\pi$. The two equations are then solved using the decay amplitude data to give $\theta_P = -19.8 \pm 2.2^\circ$ and $f_1 = (1.02 \pm 0.14)F_\pi$.

As is clear from our work, however, the naive PCAC formula does not apply for the singlet, and this procedure therefore appears to be unjustified. Nevertheless, the experimental result suggests a dynamical pattern, which we now discuss.

What is being measured as f_1 in this analysis is *not* the universal singlet decay constant F , but a process-dependent combination of F and $g_{G\gamma\gamma}(0)$ extracted from eq.(4.11). The result $f_1 \simeq F_\pi$ therefore determines,

$$F \left(1 + \frac{1}{6} F m_{\eta_1}^2 \frac{g_{G\gamma\gamma}(0)}{g_{\eta_1\gamma\gamma}(0)} \right) \simeq \sqrt{6} F_\pi, \quad (6.3)$$

setting $N_F = 3$ and allowing for our different normalisation in the singlet channel, and defining $m_{\eta_1}^2 = (\cos^2 \theta_P / m_{\eta'}^2 + \sin^2 \theta_P / m_\eta^2)^{-1}$. This is consistent with (but of course does not imply) the OZI approximation*, which predicts

$$F = \sqrt{6} F_\pi \quad g_{\eta_1\gamma\gamma} = 2\sqrt{\frac{2}{3}} g_{\pi\gamma\gamma} \quad g_{G\gamma\gamma} \Lambda_{\text{QCD}}^3 \ll g_{\eta_1\gamma\gamma}. \quad (6.4)$$

* Alternatively, eq.(6.4) can be derived in the $1/N_C$ expansion, where the orders of the various quantities appearing here are:

$$\begin{array}{llll} F = O(\sqrt{N_C}) & m_{\eta'}^2 = O(1/N_C) & G_A = O(N_C) & \\ g_{\eta'\gamma\gamma} = O(\sqrt{N_C}) & g_{G\gamma\gamma} = O(1) & g_{\eta'NN} = O(\sqrt{N_C}) & g_{GNN} = O(1) \end{array}$$

One problem with assuming the validity of the OZI approximation to eq.(4.11), however, is that the same assumption is quite unsuccessful in the case of the $U(1)$ GT relation (6.2). Here, the OZI approximation

$$F = \sqrt{6}F_\pi \quad g_{\eta_1 NN} = \sqrt{2}g_{\eta_8 NN} \quad g_{GNN}\Lambda_{\text{QCD}}^3 \ll g_{\eta_1 NN}, \quad (6.5)$$

is in complete disagreement with the experimental result[14] on the proton “spin”, which implies $G_A(0) \ll 2\sqrt{3}G_A^{(8)}(0)$. There is, however, a crucial difference between eqs.(4.11) and (6.2). While in the former, each term is separately RG invariant, in the $U(1)$ GT relation only the first term is RG invariant, the coupling $g_{GNN}(0)$ satisfying a complicated RGE which permits $G_A(0)$ to scale simply with the anomalous dimension γ . An OZI assumption of the form (6.5) is therefore inevitably in conflict with the renormalisation group.

In general, we may expect quantities which have different RG behaviour in QCD itself and the OZI limit to exhibit strong violations of the OZI approximation. On the other hand, for quantities which do have the same RG behaviour, this objection is removed and it is interesting to see to what extent the OZI approximation may be valid.

For the $U(1)$ GT relation, applying the OZI approximation to the RG invariant quantities only, we would set the term $(2m_N)^{-1}Fg_{\eta' NN}$ (the quark component of the proton “spin” in the interpretation discussed in refs.[8,9]) equal to its OZI value $\sqrt{3}m_N^{-1}F_\pi g_{\eta_8 NN} = 0.68$. The observed small value of $G_A(0) = 0.10 \pm 0.24$ would then be interpreted as due to an approximate cancellation between the two terms on the l.h.s., implying a large OZI violation for the “gluon component” proportional to $g_{GNN}(0)$. As explained in ref.[9], the origin of this cancellation would be traced back to an anomalously small value of the first moment of the QCD topological susceptibility, $\chi'_t(0)$.

Returning to $\eta' \rightarrow \gamma\gamma$, since all the quantities appearing in eq.(4.11) are RG invariant, the validity of the OZI approximation would imply simply,

$$\sqrt{6}F_\pi g_{\eta_1 \gamma\gamma} \simeq \frac{4N_C}{3\pi}\alpha_{\text{em}}, \quad (6.6)$$

for the flavour singlet, neglecting explicit $SU(3)$ breaking.

This is one consistent interpretation of the experimental result (6.3). However, it is also quite possible that eq.(6.3) is satisfied in a different way, with OZI violations in all the terms. In this case though we would have no understanding of why the OZI formula (6.6) is so well satisfied experimentally.

It is therefore interesting to test the validity of the OZI approximation for other processes. Of particular interest are the decays $\eta(\eta') \rightarrow \pi^+\pi^-\gamma$, since, e.g., it has been suggested in refs.[3,4] that by combining the data on these decays with $\eta(\eta') \rightarrow \gamma\gamma$, we could distinguish experimentally between the fractional quark charge assignments of conventional QCD and the integer charge Han-Nambu model. (The difference in the charge assignments in these models results in an extra parameter ξ in the PCAC formulae for $\eta(\eta') \rightarrow \gamma\gamma$, with $\xi = 1$ for QCD and $\xi = 2$ for the Han-Nambu model.)

The decays $\eta(\eta') \rightarrow \pi^+\pi^-\gamma$ are controlled by the box anomaly for one vector and three axial currents, and are evaluated in the same way as $\eta(\eta') \rightarrow \gamma\gamma$. Again, we find that

the naive PCAC formula is reproduced in the singlet channel only for the unphysical OZI NG boson η_0 , the formula for the physical singlet being exactly analogous to eq.(4.11), involving F and a new proper vertex $g_{G\pi\pi\gamma}(0)$.

The naive PCAC parameter count would therefore be f_8, f_1, θ_P and ξ to describe the four decays. In principle, therefore, we would have a clear experimental determination of ξ and could distinguish the fractional and integer charge quark models. In practice, it is not so easy experimentally to extract the anomaly-mediated direct $\eta' \rightarrow \pi^+\pi^-\gamma$ amplitude, since the decay is dominated by the ρ resonance and could proceed via $\eta' \rightarrow \rho\gamma, \rho \rightarrow \pi^+\pi^-$. Nevertheless, even discarding this decay, we would still predict ξ if we accept the chiral perturbation result $f_8 = 1.25F_\pi$ as input.

Our more complete analysis shows, however, that the correct parameter list is $f_8, F, \theta_P, \xi, g_{G\gamma\gamma}(0)$ and $g_{G\pi^+\pi^-\gamma}(0)$, which appears to rule out any prediction. On the other hand, the situation changes if we apply the OZI rule. Since again each term in the formula for the flavour singlet decay $\eta_1 \rightarrow \pi^+\pi^-\gamma$ is RG invariant at $O(e_R^2)$, there is no conflict in applying the OZI approximation, in particular assuming

$$g_{G\pi\pi\gamma}\Lambda_{\text{QCD}}^3 \ll g_{\eta_1\pi\pi\gamma}, \quad (6.7)$$

as in eq.(6.4) for $g_{G\gamma\gamma}$. With this assumption, it should again be possible to determine ξ . It would also be interesting to check that the singlet decay constant deduced with the naive formula from $\eta' \rightarrow \pi^+\pi^-\gamma$ (after subtraction of the ρ contribution) is in agreement with that found from $\eta' \rightarrow \gamma\gamma$, since this would provide additional support for the validity of the OZI rule for η' couplings.

The phenomenological situation therefore remains interesting. Although each new process involving the η' is parametrised by F plus a new unknown proper vertex $g_{GX\dots X}(0)$, the existing data on η' radiative decays and the proton “spin” is still consistent with the OZI approximation, provided it is not applied to quantities which have different RG behaviour in QCD and the OZI limit. It would clearly be interesting, therefore, to extend this analysis and test the validity of the OZI approximation in further processes involving the η' .

Finally, we should mention that the results and renormalisation group analysis described here will be used in a forthcoming paper[15] in which we derive a sum rule for the polarised photon structure function g_1 and show how the axial anomaly amplitude $\langle 0|J_{\mu 5} J_\lambda^{\text{em}} J_\rho^{\text{em}}|0\rangle$ may be measured in deep-inelastic, polarised e^+e^- scattering.

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