THE $U(1)$ GOLDBERGER-TREIMAN RELATION AND THE PROTON “SPIN”
– A RENORMALISATION GROUP ANALYSIS

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Abstract

A detailed explanation of the $U(1)$ Goldberger-Treiman relation is presented, and its renormalisation group properties discussed in detail. The implications for the proton “spin” problem are described, and a possible explanation for the small value of $G_A(0)$ inferred from the EMC experiment is given.

1. Introduction

The Goldberger-Treiman relation[1] is one of the classic results of current algebra and PCAC. It relates the isotriplet axial-vector form factor of the nucleon at zero momentum transfer, $g_A \equiv 2G_A^{(3)}(0)$, to the pion decay constant F_π and pion-nucleon coupling $g_{\pi NN}$, i.e.

$$F_\pi g_{\pi NN} = m_N g_A. \quad (1.1)$$

The form factor $G_A^{(3)}(k^2)$ is defined from the matrix element $\langle N | J_{\mu 5}^3(k) | N \rangle$, and is experimentally measurable in nucleon beta decay. The essence of the Goldberger-Treiman relation is therefore in providing a link between the pseudovector and pseudoscalar channels.

Although pre-dating its development, eq.(1.1) has a firm theoretical foundation in terms of QCD with spontaneously broken chiral symmetry. In the chiral limit, it is an exact identity; for $m_\pi^2 \neq 0$, it is an approximate relation, valid experimentally to a few percent, dependent on a smoothness assumption on the momentum dependence of certain dynamical functions in the range $0 \leq k^2 \leq m_\pi^2$.

It is an obvious and interesting question whether this relation can be extended to the flavour singlet channel, i.e. whether there exists a $U(1)$ Goldberger-Treiman relation for the singlet form factor $G_A(0)$ defined from $\langle N | J_{\mu 5}^0 | N \rangle$. The theoretical interest of such an extension centres of course on the existence of the axial anomaly, and the associated $U(1)$ problems in QCD.

Unlike the isovector case, however, it is not at all clear how the singlet form factor $G_A(0)$ could be extracted from a low energy experiment, without invoking the OZI approximation to relate it to the octet form factors. However, it can be measured indirectly in polarised deep inelastic lepton-nucleon scattering, since the OPE relates $\langle N | J_{\mu 5}(0) | N \rangle$ to the large Q^2 limit of the matrix element $\langle N | J_\lambda^{\text{em}}(q) J_\rho^{\text{em}}(-q) | N \rangle$ appearing in the scattering amplitude.

The recent results from the EMC collaboration[2], confirming and extending earlier data from SLAC[3], have provoked considerable excitement and controversy. The value of $G_A(0)$ which emerges after some theoretical analysis (for a review, and references to the early literature, see ref.[4]) is

$$G_A(0)|_{Q^2 \approx 11 \text{ GeV}^2} = 0.10 \pm 0.24. \quad (1.2)$$

This result was initially and somewhat oversimply interpreted as saying that the quark contribution to the spin of the proton was approximately zero. Subsequently, much effort[5,6,7] has gone into attempting to reconcile eq.(1.2) with the QCD-parton model, incorporating the axial anomaly and arguing that $G_A(0)$ is in fact the sum of quark and gluon contributions to the “spin”.* (For further discussion, see also ref.[8].)

* While for free massless Dirac fermions the matrix element of $J_{\mu 5}$ can indeed be identified with spin, the identification fails for interacting field theories and, in particular, in the presence of the anomaly. It is therefore strictly incorrect to refer to $G_A(0)$ as the proton spin in QCD, although they may be related in some simplified models of the proton.

In fact, the real surprise of eq.(1.2) arises from a comparison with the corresponding result for the octet form factor $G_A^{(8)}(0)$, viz.

$$G_A^{(8)}(0) = 0.20 \pm 0.02. \quad (1.3)$$

In the OZI limit, these should satisfy $G_A(0) = 2\sqrt{3}G_A^{(8)}(0)$. The challenge presented by the EMC data is therefore to understand this violation of the OZI rule[9]. While any QCD-based approach incorporating the axial anomaly would be expected to predict OZI violations in the flavour singlet channel, we would like to understand (i) why the OZI-violating contributions to $G_A(0)$ should be so large, and (ii) why the sign of these contributions is such that they, at least approximately, cancel the leading OZI expectation.

Our approach to this question is to try to use a $U(1)$ Goldberger-Treiman identity to relate these results for the pseudovector channel to the flavour singlet pseudoscalar channel, where OZI violations are known to be large.

The extension of the Goldberger-Treiman relation to the $U(1)$ channel is subtle for the very reason we are interested in it, viz. the existence of the axial anomaly and the associated OZI violations. In particular, because of the anomaly, the singlet current is not conserved and so acquires an anomalous dimension. The form factor $G_A(0)$ is therefore not renormalisation group (RG) invariant, which is why the scale had to be specified in eq.(1.2). In order to be consistent with the RG, the l.h.s. of the proposed Goldberger-Treiman relation must therefore be scale dependent, and so cannot simply correspond to the substitution of the η' for the π in eq.(1.1) with a physical coupling and decay constant. Furthermore, again because of the anomaly, the η' is not a Nambu-Goldstone (NG) boson, so a naive analogue of the PCAC formula (1.1) would not be expected to hold.

We find that in fact a straightforward extension of eq.(1.1) does exist, though not in terms of the physical η' . The required state is the unphysical η_0 , the would-be NG boson in the OZI limit of QCD[10]. In terms of this state, the $U(1)$ Goldberger-Treiman relation (abbreviated hereafter to $U(1)$ GT) is,

$$F_{\eta_0}(0)g_{\eta_0 NN}(0) = 2m_N G_A(0). \quad (1.4)$$

This formula was first proposed by one of us in ref.[10] based on effective Lagrangian arguments, and was subsequently precisely defined and proved in our previous letter[11]. A more phenomenological approach was given earlier by Hatsuda[12], and other attempts to construct a $U(1)$ GT relation have been made in refs.[13,6]. The RG non-invariance in eq.(1.4) resides in the quantity $F_{\eta_0}(0)$, whose square is proportional to the first moment of the QCD topological susceptibility.

The $U(1)$ GT relation may be rewritten in terms of the physical η' . This gives,

$$Fg_{\eta' NN} + \frac{1}{2N_F}F^2m_{\eta'}^2g_{GNN}(0) = 2m_N G_A(0). \quad (1.5)$$

Here, F is a RG invariant decay constant, related to the zero-momentum limit of the propagator for the composite operator $i\bar{q}\gamma_5 q$. In addition, it is process independent, appearing in a closely analogous formula[14] for the decay amplitude for $\eta' \rightarrow \gamma\gamma$. We emphasise that

it is *not* the so-called “ η' decay constant” $F_{\eta'}$ defined analogously to F_π as the coupling of η' to the current $J_{\mu 5}^0$. Such a definition would not be RG invariant. The remaining ingredient in eq.(1.5) is the “glue”-nucleon coupling $g_{GNN}(0)$, defined precisely in the text as a proper vertex describing the coupling of the nucleon to the gluonic operator $F\tilde{F}$. The RG properties of this vertex are necessarily highly non-trivial, since the l.h.s. of eq.(1.5) is required to scale with a simple anomalous dimension despite the fact that both F and $g_{\eta'NN}$ are RG invariant.

The use of such proper vertices, which we define in terms of the Zumino effective action[15], is one of the main technical novelties of our paper. The Zumino effective action is an exact generating functional, defined by making a partial Legendre transform omitting the currents, which is constructed specially so that the associated proper vertices have a particularly direct relationship with the physical couplings.

In this paper, we present the derivation of the above generalisations of the Goldberger-Treiman relation, and give a complete and thorough discussion of their RG properties. Then, in section 9, we return to the question of the proton “spin”, and discuss the implications of our results for an understanding of the EMC result (1.2). We suggest a possible explanation for the observed small value of $G_A(0)$ based on the $U(1)$ GT relation (1.4).

The rest of the paper is organised as follows. Sections 2 and 3 establish the basic formalism and introduce the anomalous chiral Ward identities for Green functions and the Zumino effective action. In section 4, we rederive the original Goldberger-Treiman relation. This is an interesting exercise, since it shows clearly how our formalism involving proper vertices works in a simple case where everything can finally be expressed in terms of well-understood physical couplings. Then, in section 5, we turn to the anomalous flavour singlet channel and derive the various forms of the $U(1)$ GT relation.

In the second half of the paper, we present a careful study of the RG properties of all the quantities appearing in this work. Sections 6 and 7 describe the RGEs for the composite operator Green functions and proper vertices respectively, while in section 8 the RG behaviour of the $U(1)$ GT relations themselves are discussed in detail.

Several more technical results are confined to appendices. Appendix A contains an explicit parametrisation of the composite operator Green functions involved in the derivations. This allows, e.g., a clear statement of the smoothness approximation under which the Goldberger-Treiman formula holds away from the chiral limit. Generalised inversion formulae relating Green functions and the proper vertices derived from the Zumino effective action are derived in appendix B. Appendix C discusses the RG properties of the original Goldberger-Treiman relation. Finally, in appendix D, we explain in detail why the quantity $F_{\eta'}$ defined by the coupling of the η' to $J_{\mu 5}^0$ is not the correct η' decay constant appearing in the $U(1)$ GT relation.

2. Chiral Ward Identities.

The derivation of the $U(1)$ GT equation involves the propagators and proper vertices of the composite operators

$$\begin{aligned} J_{\mu 5} &= \bar{q} \gamma_\mu \gamma_5 q & Q &= \frac{\alpha_s}{8\pi} \text{tr} F F^* \\ \phi_5^m &= i \bar{q} \lambda^m \gamma_5 q & \phi^m &= \bar{q} \lambda^m q, \end{aligned} \quad (2.1)$$

where q are the quark fields, $m = 0, i$ and $\lambda^i (i = 1, \dots, N_F^2 - 1)$ are flavour $SU(N_F)$ generators, $\lambda^0 = (2N_F)^{-\frac{1}{2}} \mathbf{1}$. We also introduce interpolating fields $N, \bar{N}$ for the nucleon and antinucleon.

$J_{\mu 5}$ is the gauge-invariant $U(1)$ axial-vector current, which satisfies an anomalous Ward identity. The corresponding conserved but gauge-variant current will not be needed in our discussion.

The implications of the anomalous chiral symmetry are most easily seen by writing the chiral Ward identity for the generating functional $W[V_5^\mu, S^a]$ of connected Green functions defined by

$$\exp iW[V_5^\mu, S^a] = \langle \exp i(V_5^\mu J_{\mu 5} + S^a \Phi_a) \rangle. \quad (2.2)$$

Here, V_5^μ and S^a are sources for $J_{\mu 5}$ and the set of fields $\Phi_a = (Q, \phi_5^m, \phi^m, N, \bar{N})$ resp. Repeated indices imply also a spacetime integration.

In massless QCD, the anomalous chiral Ward identity

$$\partial^\mu \frac{\delta W}{\delta V_5^\mu} - 2N_F \frac{\delta W}{\delta \Theta} - \delta_5 S^a \frac{\delta W}{\delta S^a} = 0, \quad (2.3)$$

where $\delta_5 S^a$ are the infinitesimal chiral variations, $\delta_5 \Theta = 0$, $\delta_5 S_5^i = -2S^i$, $\delta_5 S_5^0 = -2S^0$, $\delta_5 S^i = 2S_5^i$, $\delta_5 S^0 = 2S_5^0$, $\delta_5 S_N = -i\chi S_N \gamma_5$, and we denote S_Q by Θ .

Eq.(2.3) holds for both bare and renormalised sources (non-renormalisation of the anomaly), although the composite operators themselves are renormalised and mix non-trivially. Eq.(2.1), of course, defines only the bare operators. A complete discussion of renormalisation is given later, beginning in section 6.

The Ward identities for the 2-point Green functions of $J_{\mu 5}$, Q and ϕ_5^i follow straightforwardly by functional differentiation of eq.(2.3). We find, in momentum space,

$$ik^\mu \frac{\delta^2 W}{\delta V_5^\mu \delta V_5^\nu} - 2N_F \frac{\delta^2 W}{\delta \Theta \delta V_5^\nu} = 0, \quad (2.4)$$

$$ik^\mu \frac{\delta^2 W}{\delta V_5^\mu \delta \Theta} - 2N_F \frac{\delta^2 W}{\delta \Theta \delta \Theta} = 0, \quad (2.5)$$

$$ik^\mu \frac{\delta^2 W}{\delta V_5^\mu \delta S_5^m} - 2N_F \frac{\delta^2 W}{\delta \Theta \delta S_5^m} - 2 \frac{\delta W}{\delta S^m} = 0. \quad (2.6)$$

The final term, $\frac{\delta W}{\delta S^m} = \langle \phi^m \rangle \equiv \langle \bar{q} \lambda^m q \rangle$, in eq.(2.6) is zero except for $m = 0$. This is the usual bilinear quark condensate responsible for spontaneous chiral symmetry breaking in QCD.

In order to discuss the conventional Goldberger-Treiman relation, we also need the chiral Ward identities for the non-anomalous isovector axial current $J_{\mu 5}^i = \bar{q}\tau^i\gamma_\mu\gamma_5 q$, ($i = 1, \dots, 3$). Introducing sources $V_5^{i\mu}$ and $V^{i\mu}$ for the axial and vector currents respectively, the Ward identity is

$$\partial^\mu \frac{\delta W}{\delta V_5^{i\mu}} - \delta_5^i V_5^{j\mu} \frac{\delta W}{\delta V_5^{j\mu}} - \delta_5^i V^{j\mu} \frac{\delta W}{\delta V^{j\mu}} - \delta_5^i S^a \frac{\delta W}{\delta S^a} = 0, \quad (2.7)$$

where $\delta_5^i V_5^{j\mu} = \epsilon^{ijk} V^{k\mu}$, $\delta_5^i V^{j\mu} = \epsilon^{ijk} V_5^{k\mu}$, $\delta_5^i S_5^j = -2\delta^{ij} S^0$, $\delta_5^i S_5^0 = -\frac{1}{2} S^i$, $\delta_5^i S^j = 2\delta^{ij} S_5^0$, $\delta_5^i S^0 = \frac{1}{2} S_5^i$ and $\delta_5^i S_N = -i S_N \gamma_5 \tau^i$ ($i, j, k = 1, \dots, 3$).

3. The Zumino Effective Action

The $U(1)$ GT relation[10,11] is precisely formulated in terms of the 1PI vertices describing the coupling of the fields Q and ϕ_5^m to the nucleon. For reasons that will become apparent later, the appropriate proper vertices are those generated by an effective action $\Gamma[V_5^\mu, \Phi_a]$, defined from $W[V_5^\mu, S^a]$ by a modification of the standard Legendre transform first introduced by Zumino[15].

The special feature of the Zumino effective action is that the Legendre transform is made only with respect to the fields and their sources, but *not* with respect to the current. That is,

$$\Gamma[V_5^\mu, \Phi_a] = W[V_5^\mu, S^a] - S^a \Phi_a, \quad (3.1)$$

where, as usual, the “classical” fields Φ_a are defined by

$$\Phi_a \equiv \Phi_a(V_5^\mu, S^a) = \frac{\delta W}{\delta S^a}, \quad (3.2)$$

and a spacetime integration is again implied in eq.(3.1). The following relations are a consequence of eqs.(3.1) and (3.2):

$$\frac{\delta \Gamma}{\delta \Phi_a} = -S^a \quad \frac{\delta \Gamma}{\delta V_5^\mu} = \frac{\delta W}{\delta V_5^\mu}. \quad (3.3)$$

The proper vertices defined from $\Gamma[V_5^\mu, \Phi_a]$ by functional differentiation w.r.t. the fields Φ_a are therefore 1PI w.r.t. the fields only, *not* w.r.t. the current $J_{\mu 5}$. As will become clear in what follows, this property makes them particularly well-suited to a careful formulation of the Goldberger-Treiman and $U(1)$ GT relations.

Since $\Gamma[V_5^\mu, \Phi_a]$ as defined here is only a partial Legendre transform, several familiar identities relating derivatives of Γ and W are no longer valid. For example, while it remains true that in the subspace spanned by the fields alone, the matrix of 2-point proper vertices Γ^{ab} (where $\Gamma^{ab} \equiv \frac{\delta^2 \Gamma}{\delta \Phi_a \delta \Phi_b}$) is the inverse propagator matrix W_{ab} , the corresponding

result including the current does not hold. In appendix B, we collect some useful general properties of such partial Legendre transforms.

The chiral Ward identity satisfied by $\Gamma[V_5^\mu, \Phi_a]$ follows directly from the definition (3.1) and the Ward identity (2.3) for W . (For reasons explained in section 7, this definition of the effective action is made w.r.t. the renormalised source. Relations derived from it, such as the Ward identity below, therefore refer to the renormalised 1PI vertices.) We find,

$$\partial^\mu \frac{\delta\Gamma}{\delta V_5^\mu} - 2N_F Q - \delta_5 \Phi_a \frac{\delta\Gamma}{\delta \Phi_a} = 0, \quad (3.4)$$

where $\delta_5 Q = 0$, $\delta_5 \phi_5^i = -2\phi^i$, $\delta_5 \phi_5^0 = -2\phi^0$, $\delta_5 \phi^i = 2\phi_5^i$, $\delta_5 \phi^0 = 2\phi_5^0$ and $\delta N = i\chi\gamma_5 N$. This identity is the basis of our derivation of the $U(1)$ GT relation.

Before deriving the $U(1)$ GT relation itself, however, we review, in section 4, the original Goldberger-Treiman relation in this formalism. For this, we need the corresponding effective action and chiral Ward identity for the non-anomalous isotriplet axial-vector current $J_{\mu 5}^i$. Introducing sources as in eq.(2.7), the Zumino transform now gives an effective action $\Gamma[V_5^{i\mu}, V^{i\mu}, \Phi_a]$ satisfying

$$\partial_\mu \frac{\delta\Gamma}{\delta V_5^{i\mu}} - \delta_5^i V_5^{j\mu} \frac{\delta\Gamma}{\delta V_5^{j\mu}} - \delta_5^i V^{j\mu} \frac{\delta\Gamma}{\delta V^{j\mu}} - \delta_5^i \Phi_a \frac{\delta\Gamma}{\delta \Phi_a} = 0, \quad (3.5)$$

where $\delta_5^i \phi_5^j = -\frac{1}{2}\delta^{ij}\phi^0$, $\delta_5^i \phi_5^0 = -2\phi^i$, $\delta_5^i \phi^j = \frac{1}{2}\delta^{ij}\phi_5^0$, $\delta_5^i \phi^0 = 2\phi_5^i$ and $\delta_5^i N = i\tau^i\gamma_5 N$ ($i, j = 1, \dots, 3$).

4. The Goldberger-Treiman Relation

The Goldberger-Treiman relation[1] is an identity relating the form factor $G_A^{(i)}(k^2)$, at zero momentum transfer, defined in terms of the matrix element of the isovector axial current between on-shell nucleon states,

$$\langle N | J_{\mu 5}^i | N \rangle = G_A^{(i)}(k^2) \bar{u} \tau^i \gamma_\mu \gamma_5 u + G_P^{(i)}(k^2) k_\mu \bar{u} \tau^i \gamma_5 u, \quad (4.1)$$

to the pion-nucleon coupling $g_{\pi NN}$ and pion decay constant F_π . Here, u denotes the isodoublet nucleon wave function and τ^i are $SU(2)$ generators.

In order to derive this relation using our formalism, the first step is to express the matrix element in eq.(4.1) in terms of the Zumino effective action. The matrix element is as usual given by the amputated Green function with wave functions replacing the external propagators, i.e.

$$\langle N | J_{\mu 5}^i | N \rangle = \bar{u} \frac{\delta^3 W}{\delta V_5^{i\mu} \delta S_N \delta S_{\bar{N}}} \left(\frac{\delta^2 W}{\delta S_N \delta S_{\bar{N}}} \right)^{-1} \left(\frac{\delta^2 W}{\delta S_N \delta S_{\bar{N}}} \right)^{-1} u. \quad (4.2)$$

Specialising the identity (B.10) derived in appendix B, we can re-express the r.h.s. of eq.(4.2) as follows,

$$\langle N | J_{\mu 5}^i | N \rangle = \bar{u} \left(\frac{\delta^3 \Gamma}{\delta V_5^{i\mu} \delta \bar{N} \delta N} + \frac{\delta^2 W}{\delta V_5^{i\mu} \delta S_5^j} \frac{\delta^3 \Gamma}{\delta \phi_5^j \delta \bar{N} \delta N} \right) u. \quad (4.3)$$

At zero momentum transfer, and for on-shell nucleon states, we therefore have

$$\begin{aligned} & \left(2m_N G_A^{(i)}(0) + (k^2 G_P^{(i)}(k^2)) \right) \Big|_{k=0} \bar{u} \tau^i \gamma_5 u \\ &= -i \bar{u} \left(\partial^\mu \frac{\delta^3 \Gamma}{\delta V_5^{i\mu} \delta \bar{N} \delta N} + \partial^\mu \frac{\delta^2 W}{\delta V_5^{i\mu} \delta S_5^j} \frac{\delta^3 \Gamma}{\delta \phi_5^j \delta \bar{N} \delta N} \right) \Big|_{k=0} u. \end{aligned} \quad (4.4)$$

From the chiral Ward identity (3.5) (conservation of the isovector axial-vector current), both sides of eq.(4.4) vanish. This is possible because of the $1/k^2$ pole in the pseudoscalar form factor $G_P^{(i)}(k^2)$, which is due to the pion contribution to the matrix element. In eq.(4.4), this pole appears explicitly on the r.h.s. in the propagator* $\frac{\delta^2 W}{\delta V_5^{i\mu} \delta S_5^j} \equiv i \langle 0 | T^* J_{\mu 5}^i \phi_5^j | 0 \rangle$.

At this point, there are two routes to the Goldberger-Treiman relation, and it is instructive to consider both. First, from the chiral Ward identities for the 2-point Green functions (see section 2 and appendix A) we have

$$ik^\mu \frac{\delta^2 W}{\delta V_5^{i\mu} \delta S_5^j} - \frac{1}{2} \langle \phi \rangle \delta^{ij} = 0, \quad (4.5)$$

and so the propagator is entirely fixed by the Ward identity to be $-\langle \phi \rangle i k_\mu / 2k^2$. At zero momentum, therefore, the contribution to the pseudoscalar form factor comes from the second term in eq.(4.4), so we find

$$(k^2 G_P^{(i)}(k^2)) \Big|_{k=0} \bar{u} \tau^i \gamma_5 u = -i \bar{u} \frac{1}{2} \langle \phi \rangle \frac{\delta^3 \Gamma}{\delta \phi_5^i \delta \bar{N} \delta N} \Big|_{k=0} u. \quad (4.6)$$

Since the current conservation Ward identity implies $2m_N G_A^{(i)}(0) + (k^2 G_P^{(i)}(k^2)) \Big|_{k=0} = 0$, we then have

$$2m_N G_A^{(i)}(0) \bar{u} \tau^i \gamma_5 u = i \bar{u} \frac{1}{2} \langle \phi \rangle \frac{\delta^3 \Gamma}{\delta \phi_5^i \delta \bar{N} \delta N} \Big|_{k=0} u. \quad (4.7)$$

An alternative derivation of eq.(4.7) follows if we notice that from Lorentz covariance only the first term on the r.h.s. of eq.(4.4) can contribute to the form factor $G_A^{(i)}(k^2)$. Furthermore, the $1/k^2$ pole which contributes to $G_P^{(i)}(k^2)$ is absent in the proper vertices derived from Γ , since it is defined to be 1PI w.r.t. the pseudoscalar field. So, at zero

* For brevity, we refer throughout to the second derivatives of W as “propagators”. Precisely, $\frac{\delta^2 W}{\delta S^a \delta S^b} = i \langle 0 | T^* \phi_a \phi_b | 0 \rangle$, where T^* denotes the covariant time-ordered product.

momentum, we can further conclude that the first term on the r.h.s. in eq.(4.4) contributes *only* to $G_A^{(i)}$. That is,

$$2m_N G_A^{(i)}(0) \bar{u} \tau^i \gamma_5 u = -i \bar{u} \partial^\mu \frac{\delta^3 \Gamma}{\delta V_5^{i\mu} \delta \bar{N} \delta N} \Big|_{k=0} u. \quad (4.8)$$

Eq.(4.7) is then recovered by use of the chiral Ward identity for the 3-point vertices derived from Γ . Differentiating eq.(3.5) w.r.t. the nucleon fields $\bar{N}$, N , we find immediately,

$$\partial^\mu \frac{\delta^3 \Gamma}{\delta V_5^{i\mu} \delta \bar{N} \delta N} + \frac{1}{2} \langle \phi \rangle \frac{\delta^3 \Gamma}{\delta \phi_5^i \delta \bar{N} \delta N} = 0. \quad (4.9)$$

The first derivation above most closely resembles the standard proof of the Goldberger-Treiman relation. The second, besides being rather more direct, is the one which will generalise to give the $U(1)$ GT relation for the anomalous $U(1)$ axial current.

To complete our proof, we need to relate the r.h.s. of eq.(4.7) to properties of the *physical* pion. The pion is identified as the intermediate state giving rise to the pseudoscalar pole in $1/k^2$ in the propagator matrix for the relevant composite operators, in this case $J_{\mu 5}^i$ and ϕ_5^i . The pion decay constant is defined by $\langle 0 | J_{\mu 5}^i | \pi^j \rangle = i k_\mu F_\pi \delta^{ij}$. It then follows directly from the Ward identity (4.5),* by inserting the pion intermediate state in the propagator, that at zero momentum, with the pion on-shell,

$$\langle 0 | \phi_5^i | \pi^j \rangle = \frac{\langle \phi \rangle}{2} \frac{1}{F_\pi} \delta^{ij}, \quad (4.10)$$

The pion-nucleon coupling, defined by $\langle N \pi^i | N \rangle = g_{\pi NN} \bar{u} \tau^i \gamma_5 u$, is therefore expressed in terms of Γ as follows,

$$\bar{u} \frac{\delta^3 \Gamma}{\delta \phi_5^i \delta \bar{N} \delta N} \Big|_{k=0} u = -i \frac{2}{\langle \phi \rangle} F_\pi g_{\pi NN} \bar{u} \tau^i \gamma_5 u, \quad (4.11)$$

* An alternative approach, which generalises in an interesting way to the flavour singlet case (see section 5), is as follows. Define the interpolating π^i field as the canonically normalised field $\pi^i = b(k^2) \phi_5^i$ such that $\frac{\delta^2 \Gamma}{\delta \pi^i \delta \pi^j} = k^2 \delta^{ij}$. The pion decay constant is defined by

$$\frac{\delta^2 \Gamma}{\delta V_5^{i\mu} \delta \pi^j} = \langle 0 | J_{\mu 5}^i | \pi^j \rangle = i k_\mu F_\pi \delta^{ij}.$$

From the Ward identity,

$$i k^\mu \frac{\delta^2 \Gamma}{\delta V_5^{i\mu} \delta \phi_5^j} + \frac{1}{2} \langle \phi \rangle \frac{\delta^2 \Gamma}{\delta \phi_5^i \delta \phi_5^j} = 0,$$

we then see immediately that $b(0) = \frac{2}{\langle \phi \rangle} F_\pi$. and eqs.(4.10),(4.11) follow.

We can also read this result off directly from the propagator matrices in appendix A. From eq.(A.3), or eq.(A.8) in the case $m_\pi \neq 0$, we see that the interpolating π^i field is defined with $b(k^2) = \frac{2}{\langle \phi \rangle} B^{-\frac{1}{2}}(k^2)$, the relation with F_π following from eqs.(A.5) or (A.11).

where we emphasise that the identification requires that the proper vertices are defined to be 1PI w.r.t. the pseudoscalar and nucleon fields only, and not w.r.t. the current $J_{\mu 5}^i$.

Finally, therefore, we can re-express the identity (4.7) in the traditional form of the Goldberger-Treiman relation,

$$G_A^{(i)}(0) = \frac{1}{2m_N} F_\pi g_{\pi NN}. \quad (4.12)$$

The above relation is exact in QCD in the chiral limit. It is interesting, both for realism and because of the analogy with the $U(1)$ GT relation, to consider what happens when a soft chiral symmetry breaking mass for the quarks is added. For simplicity, we assume no flavour breaking. In that case, the chiral Ward identities for the generating functional W and Zumino effective action Γ are modified, eqs.(2.7) and (3.5) acquiring extra terms $-2m \frac{\delta W}{\delta S_5^i}$ and $-2m \phi_5^i$ respectively.

The second derivation above now goes through exactly as before to the stage of eq.(4.7). Notice in particular that the mass term in the Ward identity for Γ does not contribute at all to eq.(4.9). The only modification to the proof of the Goldberger-Treiman relation comes in the identification of the r.h.s. of eq.(4.7) with the physical quantities. Full details are given in appendix A. The coupling of ϕ_5^i to the on-shell ($k^2 = m_\pi^2$) pion is

$$\langle 0 | \phi_5^i | \pi^j \rangle = \frac{1}{2} \langle \phi \rangle B^{\frac{1}{2}}(m_\pi^2) \delta^{ij}. \quad (4.13)$$

Here, $B(k^2)$ is a function characterising the propagator $\langle 0 | T^* \phi_5^i \phi_5^j | 0 \rangle$. On the other hand, the pion decay constant is given by

$$F_\pi = B^{\frac{1}{2}}(m_\pi^2) B^{-1}(0). \quad (4.14)$$

The standard approximation of PCAC is now to assume that the pole-free function $B(k^2)$ is a slowly-varying function in the momentum range $0 \leq k^2 \leq m_\pi^2$. Approximating $B(0) \simeq B(m_\pi^2)$, and with a similar approximation for the proper vertex, where we take $\Gamma_{\phi_5 \bar{N} N}(k^2 = 0) \simeq \Gamma_{\phi_5 \bar{N} N}(k^2 = m_\pi^2)$, we recover the Goldberger-Treiman relation.

The point we wish to stress here is the following. The relation (4.7) between $G_A^{(i)}(0)$ and $\Gamma_{\phi_5 \bar{N} N}$ is *exact*, even for soft chiral symmetry breaking and a non-zero pion mass. The approximation involved in the Goldberger-Treiman relation for $m_\pi^2 \neq 0$ comes *entirely* from a smoothness assumption on the momentum dependence of the dynamical functions controlling F_π and $g_{\pi NN}$.

5. The $U(1)$ GT Relation

At this point, we have introduced all the techniques necessary to derive the generalisation of the Goldberger-Treiman relation for the anomalous axial $U(1)$ current. This relation was first presented in ref.[10], and was subsequently more rigorously defined and proved in ref.[11].

The starting point is again the form factor decomposition of the matrix element for the flavour-singlet, axial-vector current $J_{\mu 5}$, viz.

$$\langle N | J_{\mu 5} | N \rangle = G_A(k^2) \bar{u} \gamma_\mu \gamma_5 u + G_P(k^2) k_\mu \bar{u} \gamma_5 u. \quad (5.1)$$

Again, we consider first the ideal case of QCD with massless quarks. For simplicity of notation, we omit the $m = 0$ flavour index on the fields. We consider the Zumino effective action $\Gamma[V_5^\mu, Q, \phi_5, \phi, N, \bar{N}]$ defined in section 3. Expressing the matrix element in terms of the proper vertices derived from Γ , we find, using eq.(B.10),

$$\langle N | J_{\mu 5} | N \rangle = \bar{u} \left(\frac{\delta^3 \Gamma}{\delta V_5^\mu \delta \bar{N} \delta N} + \frac{\delta^2 W}{\delta V_5^\mu \delta \Theta} \frac{\delta^3 \Gamma}{\delta Q \delta \bar{N} \delta N} + \frac{\delta^2 W}{\delta V_5^\mu \delta S_5} \frac{\delta^3 \Gamma}{\delta \phi_5 \delta \bar{N} \delta N} \right). \quad (5.2)$$

Taking the divergence of $J_{\mu 5}$ and going to zero momentum, only the first term on the r.h.s. contributes, since the propagators both vanish. This is because there is, by assumption of the resolution of the $U(1)$ problem, no massless pseudoscalar in the flavour singlet channel. We therefore find

$$2m_N G_A(0) \bar{u} \gamma_5 u = -i \bar{u} \partial^\mu \frac{\delta^3 \Gamma}{\delta V_5^\mu \delta \bar{N} \delta N} \Big|_{k=0} u. \quad (5.3)$$

The next step is to use the following Ward identity, valid for all momenta, derived from eq.(3.4),

$$\partial^\mu \frac{\delta^3 \Gamma}{\delta V_5^\mu \delta \bar{N} \delta N} = -2 \langle \phi \rangle \frac{\delta^3 \Gamma}{\delta \phi_5 \delta \bar{N} \delta N}. \quad (5.4)$$

Notice that the anomalous term proportional to Q in eq.(3.4) does not contribute to this identity. Finally, therefore, we have

$$2m_N G_A(0) \bar{u} \gamma_5 u = i \bar{u} 2 \langle \phi \rangle \frac{\delta^3 \Gamma}{\delta \phi_5 \delta \bar{N} \delta N} \Big|_{k=0} u. \quad (5.5)$$

This is the first expression for the $U(1)$ GT relation. Notice that it takes the same form as the flavour non-singlet formula (4.7), although of course the interpretation of the Zumino effective action is quite different since it now generates proper vertices which are 1PI w.r.t. both ϕ_5 and Q .

Before re-expressing eq.(5.5) in more physical terms, it is interesting to derive a non-forward version of the $U(1)$ GT relation, valid for arbitrary momentum transfer k . To

see this, return to eq.(5.2) and substitute the Ward identities (2.5) and (2.6) for the propagators. Then, using eq.(5.4) for non-zero k , we find

$$ik^\mu \frac{1}{2N_F} \langle N | J_{\mu 5} | N \rangle = \bar{u} \left(\frac{\delta^2 W}{\delta \Theta \delta \Theta} \frac{\delta^3 \Gamma}{\delta Q \delta \bar{N} \delta N} + \frac{\delta^2 W}{\delta \Theta \delta S_5} \frac{\delta^3 \Gamma}{\delta \phi_5 \delta \bar{N} \delta N} \right) u. \quad (5.6)$$

We will use this more general expression for most of the renormalisation group analysis later in the paper.

The $U(1)$ GT relation (5.5) can be written in different ways, each bringing some new physical insight. One interesting form expresses the form factor $G_A(0)$ in terms of the first moment of the topological susceptibility $\chi_t(k^2) = \int dx e^{ik \cdot x} i \langle 0 | T^* Q(x) Q(0) | 0 \rangle$ and the coupling $g_{\eta_0 NN}$ of the nucleon to a state $|\eta_0\rangle$ which we may call the OZI Nambu-Goldstone boson, since it is the massless $U(1)$ NG boson in the OZI limit of QCD, where the anomaly is absent. It is not, of course, a physical state of QCD itself.

We therefore define the η_0 interpolating field as the canonically normalised flavour singlet field (c.f. footnote, p.8)

$$\eta_0 = b(k^2) \phi_5, \quad (5.7)$$

such that

$$\frac{\delta^2 \Gamma}{\delta \eta_0 \delta \eta_0} = k^2. \quad (5.8)$$

Now observe that

$$\frac{\delta^2 \Gamma}{\delta V_5^\mu \delta \eta_0} = \langle 0 | J_{\mu 5} | \eta_0 \rangle = ik_\mu F_{\eta_0}(k^2), \quad (5.9)$$

defining the “decay function” $F_{\eta_0}(k^2)$. It therefore follows from the Ward identity

$$ik^\mu \frac{\delta^2 \Gamma}{\delta V_5^\mu \delta \phi_5} + 2 \langle \phi \rangle \frac{\delta^2 \Gamma}{\delta \phi_5 \delta \phi_5} = 0, \quad (5.10)$$

that

$$\eta_0 = \frac{1}{2 \langle \phi \rangle} F_{\eta_0}(k^2) \phi_5, \quad (5.11)$$

so the $U(1)$ GT relation may be re-expressed as

$$2m_N G_A(0) = F_{\eta_0}(0) g_{\eta_0 NN}(0), \quad (5.12)$$

where the η_0 -nucleon coupling is defined in analogy to eq.(4.11) as

$$\bar{u} \frac{\delta^3 \Gamma}{\delta \eta_0 \delta \bar{N} \delta N} u = -i g_{\eta_0 NN}(k^2) \bar{u} \gamma_5 u. \quad (5.13)$$

As we show below, $F_{\eta_0}(0)$ is proportional to the square root of the first moment of the topological susceptibility. An alternative derivation, using an explicit parametrisation

of the propagator matrix and the matrix of 2-point proper vertices, is given in appendix A. We find

$$2m_N G_A(0) = 2N_F (\chi'_t(0))^{\frac{1}{2}} g_{\eta_0 NN}(0), \quad (5.14)$$

where

$$\chi'_t(0) = \left(\frac{\partial}{\partial k^2} \int dx e^{ik \cdot x} i \langle 0 | T^* Q(x) Q(0) | 0 \rangle \right) \Big|_{k=0}. \quad (5.15)$$

To show the equivalence of these formulae, we first observe that the chiral Ward identities (2.3) and (3.4) for W and Γ imply that at zero momentum,*

$$W_{\Theta\Theta}(0) = 0 \quad W_{\Theta S_5}(0) = -\frac{1}{2N_F} 2\langle \phi \rangle, \quad (5.16)$$

$$\Gamma_{\phi_5 \phi_5}(0) = 0 \quad \Gamma_{Q\phi_5}(0) = 2N_F \frac{1}{2\langle \phi \rangle}, \quad (5.17)$$

assuming the absence of a massless pseudoscalar in the $U(1)$ channel. Since both $W_{\Theta\Theta}$ and $\Gamma_{\phi_5 \phi_5}$ are therefore of $O(k^2)$, the equivalence of eqs.(5.12) and (5.14) is established if we can prove

$$\left(\frac{1}{k^2} W_{\Theta\Theta} \right) \Big|_{k=0} = \frac{1}{(2N_F)^2} 4\langle \phi \rangle^2 \left(\frac{1}{k^2} \Gamma_{\phi_5 \phi_5} \right) \Big|_{k=0}. \quad (5.18)$$

Now, as a special case of the inversion formula (B.4), we have

$$W_{\Theta\Theta} \Gamma_{Q\phi_5} + W_{\Theta S_5} \Gamma_{\phi_5 \phi_5} = 0, \quad (5.19)$$

for all k , and so in particular,

$$\frac{W_{\Theta\Theta}}{\Gamma_{\phi_5 \phi_5}} \Big|_{k=0} = -\frac{W_{\Theta S_5}}{\Gamma_{Q\phi_5}} \Big|_{k=0}. \quad (5.20)$$

Substituting the explicit values determined by the zero-momentum Ward identities completes the proof. The η_0 decay constant is given by

$$F_{\eta_0}(0) = 2N_F \chi'_t(0)^{\frac{1}{2}}. \quad (5.21)$$

Of course, in QCD itself, rather than the OZI limit, the state $|\eta_0\rangle$ is not a physical particle. The physical pseudoscalar in the flavour singlet channel is the η' , with a non-zero mass $m_{\eta'}$, which in the valence approximation is a linear combination of meson ($|\phi_5\rangle$) and glueball ($|Q\rangle$) states.

* Here, and frequently in the rest of the paper, we use an abbreviated notation for functional derivatives, so that

$$W_{\Theta\Theta} \equiv \frac{\delta^2 W}{\delta\Theta\delta\Theta} \quad \Gamma_{\phi_5 \phi_5} \equiv \frac{\delta^2 \Gamma}{\delta\phi_5 \delta\phi_5},$$

etc.

The interpolating field for the physical η' is identified by diagonalisation of the propagator matrix in the pseudoscalar sector. This is achieved by defining

$$\eta' = c(k^2) \phi_5 \quad G = Q - \frac{W_{\Theta S_5}}{W_{S_5 S_5}} \phi_5. \quad (5.22)$$

Following the method above, we then define the interpolating η' field to be the canonically normalised field such that

$$\frac{\delta^2 \Gamma}{\delta \eta' \delta \eta'} = k^2 - m_{\eta'}^2. \quad (5.23)$$

This determines

$$c^2 = -\frac{1}{k^2 - m_{\eta'}^2} \frac{1}{W_{S_5 S_5}}. \quad (5.24)$$

We now re-express the $U(1)$ GT relation in terms of the physical η' -nucleon coupling. As usual, this is identified with the appropriate 3-point proper vertex derived from the Zumino transform through the relation

$$\bar{u} \frac{\delta^3 \Gamma}{\delta \eta' \bar{N} N} u = -i g_{\eta' NN}(k^2) \bar{u} \gamma_5 u, \quad (5.25)$$

the physical coupling being the on-shell value $g_{\eta' NN} = g_{\eta' NN}(m_{\eta'}^2)$. Notice that it is crucial here, and in eq.(5.23), that the orthonormal basis is used, so that the partial derivatives are taken w.r.t. η' at fixed G . A similar equation is taken to define a “glue”-nucleon coupling $g_{GNN}(k^2)$, which however lacks the immediate physical interpretation of $g_{\eta' NN}$. We shall return to this point in section 8.

From eq.(5.6), we find for the non-forward $U(1)$ GT relation

$$ik^\mu \frac{1}{2N_F} \langle N | J_{\mu 5} | N \rangle = \bar{u} \left(W_{\Theta S'_\eta} \Gamma_{\eta' \bar{N} N} + \left(W_{\Theta \Theta} - \frac{W_{\Theta S'_\eta}^2}{W_{S'_\eta S'_\eta}} \right) \Gamma_{G \bar{N} N} \right) u, \quad (5.26)$$

while at zero momentum transfer,

$$2m_N G_A(0) = -2N_F \left(W_{\Theta S'_\eta} g_{\eta' NN}(k^2) - \frac{W_{\Theta S'_\eta}^2}{W_{S'_\eta S'_\eta}} g_{GNN}(k^2) \right) \Big|_{k=0}, \quad (5.27)$$

since $W_{\Theta \Theta}$ vanishes at $k^2 = 0$. The normalisation factor c for the η' field cancels from these identities. (Notice that here we have written the extra gluonic term as $\Gamma_{G \bar{N} N}$ with the derivative at fixed η' . In ref.[11], the corresponding expression was written with $\Gamma_{Q \bar{N} N}$ at fixed ϕ_5 . These are identical.)

We see, therefore, that unlike the conventional Goldberger-Treiman relation of section 4, the analogous relation in the anomalous $U(1)$ channel is not a simple relation between $G_A(0)$ and the η' -nucleon coupling, but contains an extra piece describing the coupling of $|FF^*\rangle$ to the nucleon.

Anticipating the results of section 8, it turns out that the quantity $W_{\Theta S'_\eta}(0)$ is RG invariant. It is therefore reasonable to define the physical constant

$$F = -2N_F W_{\Theta S'_\eta}(0). \quad (5.28)$$

Indeed, as we show in appendix A (see eq.(A.25)), this is a natural generalisation of the pion decay constant to the flavour singlet sector. In fact,

$$F = \frac{1}{m_{\eta'}} 2\langle\phi\rangle \left(\int dx \, i\langle 0|T^* \phi_5(x) \phi_5(0)|0\rangle \right)^{-\frac{1}{2}}. \quad (5.29)$$

We therefore rewrite the $U(1)$ GT relation in its final form,

$$2m_N G_A(0) = F g_{\eta' NN}(0) + \frac{1}{2N_F} F^2 m_{\eta'}^2 g_{GNN}(0). \quad (5.30)$$

If we now make a dynamical assumption on the momentum dependence of the η' -nucleon coupling, viz. that $g_{\eta' NN}(0) \simeq g_{\eta' NN}$, we may rewrite this relation in the following approximate form,

$$2m_N G_A(0) = F g_{\eta' NN} + \frac{1}{2N_F} F^2 m_{\eta'}^2 g_{GNN}(0). \quad (5.31)$$

As will be explained in section 8, there is no similar well-defined approximation relating $g_{GNN}(0)$ to an independently determined physical quantity.

We close this section by mentioning briefly the effect on the $U(1)$ GT formulae of adding quark mass terms to the QCD Lagrangian. Introducing quark masses results in the addition of a term $\sum_m -2m^{(m)}\phi_5^m$ to the chiral Ward identity (3.4) for Γ . Differentiating w.r.t. N and $\bar{N}$, this term does not contribute, and so the basic Ward identity (5.4) is completely unchanged and the $U(1)$ GT formula (5.5) remains valid in its original form. Just as for the pion, the only effect of such terms in the other $U(1)$ GT relations is in the identification of the physical couplings and decay constants with their values at $k^2 = 0$, which requires smoothness assumptions on pole-free dynamical functions as discussed in section 4 and appendix A. We emphasise that with our proper vertex formalism, such assumptions are imposed only for pole-free (and RG invariant) quantities, a point we return to in connection with the proton “spin” problem in section 9.

6. RGEs for Composite Operator Green Functions

We now come to the question of how the various composite operators, Green functions and proper vertices involved in the $U(1)$ GT relation transform under the renormalisation group.

Conventionally, renormalised composite operators $J_{\mu 5}^R$, Q_R , ϕ_{5R} and ϕ_R are defined from the corresponding bare quantities by multiplicative renormalisation, under which the current and anomalous divergence mix. That is, (see, e.g. ref.[16]),

$$\begin{aligned} J_{\mu 5}^R &= Z J_{\mu 5}^B & Q_R &= Q_B - \frac{1}{2N_F}(1-Z)\partial^\mu J_{\mu 5}^B \\ \phi_{5R} &= Z_\phi \phi_{5B} & \phi_R &= Z_\phi \phi_B. \end{aligned} \quad (6.1)$$

In terms of the generating functional $W[V_5^\mu, S^a]$, this is implemented by defining renormalised sources,

$$\begin{aligned} V_{5R}^\mu &= Z^{-1} V_{5B}^\mu - \frac{1}{2N_F} Z^{-1} (1-Z) \partial^\mu \Theta_B & \Theta_R &= \Theta_B \\ S_R^5 &= Z_\phi^{-1} S_B^5 & S_R &= Z_\phi^{-1} S_B. \end{aligned} \quad (6.2)$$

Functional derivatives of W w.r.t. the renormalised sources for a *single* composite operator and any number of elementary fields then produce finite, renormalised Green functions.

However, these renormalisations are not sufficient to make Green functions involving more than one composite operator finite. Here, new divergences arise which are not eliminated by the multiplicative renormalisations (6.1), and further subtractions have to be made.

A formalism which describes this extra renormalisation for Green functions of composite operators in a concise way has recently been presented in ref.[17]. It is clear that to incorporate counterterms for the new divergences, extra terms may be added to the action involving higher powers of the sources. Differentiating to obtain the Green functions, these new terms will only contribute to Green functions with more than one composite operator. The essential proposal of ref.[17] is to regard these extra counterterms as a non-linear renormalisation of the bare source. In this formalism, the anomalous dimension in the renormalisation group equation for W becomes a non-linear function of the sources.

Together with a general discussion of the non-linear source renormalisation (NLSR) formalism, ref.[17] contains the application to the operators occurring in the axial anomaly equation as a special case. We summarise the main results here. The generating functional is written as

$$\begin{aligned} &\exp iW[\mathcal{S}_R, V_{5R}^\mu, \Theta_R, S_{5R}, S_R, S_R^0] \\ &= \int \mathcal{D}\Phi \exp i \int dx \left(\mathcal{L}(\Phi) + \mathcal{S}_B \Phi + V_{5B}^\mu J_{\mu 5} + \Theta_B Q + S_B^5 \phi_5 + S_B \phi + S_B^0 1 \right), \end{aligned} \quad (6.3)$$

where Φ denotes the elementary (quark, gluon) fields and $\mathcal{S}_B$ their sources. Notice that it is crucial to include the unit operator 1 in the basis of composite operators, together

with its source S_B^0 . The new counterterms are required to be dimension 4 combinations of the sources, and their derivatives, with a single composite (including 1). They are also constrained to satisfy the anomalous Ward identity written in terms of the renormalised sources, viz.

$$\partial^\mu \frac{\delta W}{\delta V_{5R}^\mu} - 2N_F \frac{\delta W}{\delta \Theta_R} + 2S_R \frac{\delta W}{\delta S_{5R}} - 2S_{5R} \frac{\delta W}{\delta S_R} = 0. \quad (6.4)$$

(In fact, by the non-renormalisation of the anomaly, eq.(6.4) takes the same form in terms of both renormalised and bare sources.) This equation is equivalent to demanding invariance of $W[V_{5R}^\mu, S_R^a, S_R^0]$ under the local symmetry $V_{5R}^\mu \rightarrow V_{5R}^\mu + \partial^\mu \alpha$, $\Theta_R \rightarrow \Theta_R + 2N_F \alpha$, $S_R^\pm \rightarrow S_R^\pm e^{\pm 2i\alpha}$ (where $S_R^\pm = S_R \mp iS_{5R}$). It is easy to see on dimensional grounds that the only new counterterms possible are those multiplying the operator 1. The dimension 4 invariant counterterms are therefore the following: $(\partial^\mu A_R^\nu - \partial^\nu A_R^\mu)(\partial_\mu A_{\nu R} - \partial_\nu A_{\mu R})$, $(\partial_\mu A_R^\mu)^2$, $(\partial^\mu - 2iV_{5R}^\mu)S_R^+$, $(\partial_\mu + 2iV_{5R}^\mu)S_R^-$, etc., where A_R^μ is the invariant combination $(V_{5R}^\mu - \frac{1}{2N_F} \partial^\mu \Theta_R)$. Since we shall only be concerned in what follows with the RGEs for 2-point functions, it is sufficient to consider only the counterterms up to second order in the sources. To this order, and expressing the results in the language of NLSR, we find the following expansions of the bare sources,

$$\begin{aligned} V_{5B}^\mu &= ZV_{5R}^\mu + \frac{1}{2N_F}(1-Z)\partial^\mu \Theta_R \\ \Theta_B &= \Theta_R \\ S_B^\pm &= Z_\phi S_R^\pm \\ S_B^0 &= S_R^0 + L^{(T)}(\partial^\mu A_R^\nu - \partial^\nu A_R^\mu)(\partial_\mu A_{\nu R} - \partial_\nu A_{\mu R}) + L^{(L)}(\partial_\mu A_R^\mu)^2 + L^{(S)}\partial^\mu S_R^+ \partial_\mu S_R^- \end{aligned} \quad (6.5)$$

where $L^{(T)}$, $L^{(L)}$ and $L^{(S)}$ are divergent functions of the QCD coupling constant. Substituted into the generating functional (6.3), these expansions are sufficient to render all the 2-point functions of the composite operators finite and consistent with the renormalised Ward identities.

The RGE for the generating functional in the NLSR formalism is

$$\mathcal{D}W + \gamma^{V_5^\mu}(V_{5R}^\mu, S_R^a, S_R^0) \frac{\delta W}{\delta V_{5R}^\mu} + \gamma^{S^a}(V_{5R}^\mu, S_R^a, S_R^0) \frac{\delta W}{\delta S_R^a} + \gamma^{S^0}(V_{5R}^\mu, S_R^a, S_R^0) \frac{\delta W}{\delta S_R^0} = 0, \quad (6.6)$$

where $\mathcal{D} = (\mu \partial / \partial \mu + \beta \partial / \partial g)|_{\text{fixed } V_{5R}^\mu, S_R^a, S_R^0}$, and as usual $S_R^a = \{\Theta_R, S_R^+, S_R^-\}$ and a space-time integration is assumed in the sum over repeated indices. The anomalous dimensions here are non-linear functions of the sources, defined by

$$\gamma^{S^a}(V_{5R}^\mu, S_R^a, S_R^0) = \mu \frac{d}{d\mu} S_R^a, \quad (6.7)$$

etc.

Evaluating the anomalous dimensions using the explicit forms (6.5) for the source renormalisations, we find the RGE

$$\begin{aligned} \mathcal{D}W &= \gamma A_R^\mu \frac{\delta W}{\delta V_{5R}^\mu} + \gamma_\phi \left(S_{5R} \frac{\delta W}{\delta S_{5R}} + S_R \frac{\delta W}{\delta S_R} \right) \\ &+ \left(\beta^{(T)} (\partial^\mu A_R^\nu - \partial^\nu A_R^\mu) (\partial_\mu A_{\nu R} - \partial_\nu A_{\mu R}) + \beta^{(L)} (\partial_\mu A_R^\mu)^2 \right. \\ &\quad \left. + \beta^{(S)} (\partial^\mu S_{5R} \partial_\mu S_{5R} + \partial^\mu S_R \partial_\mu S_R) + \dots \right) \frac{\delta W}{\delta S_R^0}, \quad (6.8) \end{aligned}$$

the dots denoting terms of higher order in the sources. Here, γ and γ_ϕ are the conventional anomalous dimensions corresponding to Z and Z_ϕ , while $\beta^{(T)}$, $\beta^{(L)}$ and $\beta^{(S)}$ are beta functions defined in the standard way from $L^{(T)}$, $L^{(L)}$ and $L^{(S)}$. The particular values do not concern us here. For further details, see ref.[17].

The RGEs for the renormalised 2-point functions of $J_{\mu 5}^R$, Q_R and ϕ_{5R} now follow in the usual way by differentiation of eq.(6.8) w.r.t. the renormalised sources. They can immediately be simplified using the renormalised chiral Ward identities (2.4) – (2.6), and we find

$$\mathcal{D} \frac{\delta^2 W}{\delta V_{5R}^\mu \delta V_{5R}^\nu} = 2\gamma \frac{\delta^2 W}{\delta V_{5R}^\mu \delta V_{5R}^\nu} - 2\beta^{(T)} (k^2 g_{\mu\nu} - k_\mu k_\nu) - 2\beta^{(L)} k_\mu k_\nu, \quad (6.9)$$

$$\mathcal{D} \frac{\delta^2 W}{\delta V_{5R}^\mu \delta \Theta_R} = 2\gamma \frac{\delta^2 W}{\delta V_{5R}^\mu \delta \Theta_R} + \frac{1}{2N_F} 2i\beta^{(L)} k^2 k_\mu k_\nu, \quad (6.10)$$

$$\mathcal{D} \frac{\delta^2 W}{\delta \Theta_R \delta \Theta_R} = 2\gamma \frac{\delta^2 W}{\delta \Theta_R \delta \Theta_R} - \frac{1}{(2N_F)^2} 2\beta^{(L)} k^4, \quad (6.11)$$

$$\mathcal{D} \frac{\delta^2 W}{\delta V_{5R}^\mu \delta S_{5R}} = (\gamma + \gamma_\phi) \frac{\delta^2 W}{\delta V_{5R}^\mu \delta S_{5R}}, \quad (6.12)$$

$$\mathcal{D} \frac{\delta^2 W}{\delta \Theta_R \delta S_{5R}} = (\gamma + \gamma_\phi) \frac{\delta^2 W}{\delta \Theta_R \delta S_{5R}} + \gamma \frac{1}{2N_F} 2\langle\phi\rangle, \quad (6.13)$$

$$\mathcal{D} \frac{\delta^2 W}{\delta S_{5R} \delta S_{5R}} = 2\gamma_\phi \frac{\delta^2 W}{\delta S_{5R} \delta S_{5R}} - 2\beta^{(S)} k^2, \quad (6.14)$$

where we have used $\frac{\delta W}{\delta S_R} = \langle\phi\rangle$ and $\frac{\delta W}{\delta S_R^0} = 1$.

Notice that even without the new terms resulting from the extra divergences in the 2-point functions, not all the RGEs for the propagators are homogeneous. In particular, the RGE for $W_{\Theta S_5}$, which will be particularly important in the analysis of the $U(1)$ GT relation, has a non-homogeneous term proportional to the condensate $\langle\phi\rangle$.

7. RGE for the Zumino effective action

In this section, we discuss the RGE for the Zumino effective action, defined in eq. (3.1) as a partial Legendre transform in which the current itself does not enter. This brings extra complications due to the fact that under renormalisation, $J_{\mu 5}$, which is not transformed, mixes with Q , which is. In addition, we have to take account of the extra renormalisations which appear in proper vertices, as well as Green functions, involving several composite operators.

The Zumino effective action is defined precisely in terms of the renormalised sources as follows,

$$\Gamma[V_{5R}^\mu, Q_R, \phi_{5R}, \phi_R, 1, N, \bar{N}] = W[V_{5R}^\mu, \Theta_R, S_R^5, S_R, S_R^0, S_N, S_{\bar{N}}] - \Theta_R Q_R - S_{5R} \phi_{5R} - S_R \phi_R - S_R^0 1 - \bar{N} S_{\bar{N}} - S_N N. \quad (7.1)$$

The interpolating nucleon fields, of course, are not renormalised and play an entirely passive role in this discussion.

We now discuss the RGE for Γ . Define here $\mathcal{D} = (\mu\partial/\partial\mu + \beta\partial/\partial g)|_{V_{5R}^\mu, Q_R, \phi_{5R}, \phi_R}$, as appropriate for the RG operator acting on the Zumino functional. Taking the total derivative $\mu d/d\mu$, and with the definition (6.7) of the anomalous dimension functions, we find

$$\mathcal{D}\Gamma = -\gamma^{V_5^\mu} \frac{\delta\Gamma}{\delta V_{5R}^\mu} - \gamma^\Theta Q_R - \gamma^{S_5} \phi_{5R} - \gamma^S \phi_R - \gamma^{S^0} 1. \quad (7.2)$$

Substituting the explicit expressions for the anomalous dimensions from section 6 then gives the required RGE, and re-expressing the sources as first derivatives of Γ gives finally

$$\begin{aligned} \mathcal{D}\Gamma = & \gamma \left(V_{5R}^\mu - \frac{1}{2N_F} \frac{\delta\Gamma}{\delta Q_R} \partial^\mu \right) \frac{\delta\Gamma}{\delta V_{5R}^\mu} - \gamma_\phi \left(\phi_{5R} \frac{\delta\Gamma}{\delta \phi_{5R}} + \phi_R \frac{\delta\Gamma}{\delta \phi_R} \right) \\ & - \left(\beta^{(T)} (\partial^\mu A_R^\nu - \partial^\nu A_R^\mu) (\partial_\mu A_{\nu R} - \partial_\nu A_{\mu R}) + \beta^{(L)} (\partial_\mu A_R^\mu)^2 \right. \\ & \left. + \beta^{(S)} \left(\partial_\mu \frac{\delta\Gamma}{\delta \phi_{5R}} \partial^\mu \frac{\delta\Gamma}{\delta \phi_{5R}} + \partial_\mu \frac{\delta\Gamma}{\delta \phi_R} \partial^\mu \frac{\delta\Gamma}{\delta \phi_R} \right) \dots \right), \end{aligned} \quad (7.3)$$

where here we interpret $A_R^\mu = V_{5R}^\mu + \frac{1}{2N_F} \partial^\mu \frac{\delta\Gamma}{\delta Q_R}$.

This is the familiar type of expression for the RGE for an effective action, with two distinctions. The first, of course, is the presence of the new terms proportional to $\beta^{(T)}$, $\beta^{(L)}$ and $\beta^{(S)}$ of second order in $\frac{\delta\Gamma}{\delta Q_R}$, $\frac{\delta\Gamma}{\delta \phi_{5R}}$, etc. The second is the unusual non-linear $\frac{\delta\Gamma}{\delta Q_R} \frac{\delta\Gamma}{\delta V_{5R}^\mu}$ term proportional to the anomalous dimension γ for the axial current. This arises because Γ has been defined as a partial Legendre transform w.r.t. only one of two fields which mix under renormalisation.

This RGE is the basic ingredient in the investigation of the renormalisation group behaviour of the $U(1)$ GT. All the required relations follow from it, the corresponding RGE for W , and the chiral Ward identities.

The RGEs for the 3-point proper vertices follow directly from differentiating eq.(7.3). With the notation $\Gamma_{V\bar{N}N} \equiv \partial^\mu \Gamma_{V_5^\mu \bar{N}N} \equiv \partial^\mu \frac{\delta^3 \Gamma}{\delta V_5^\mu \bar{N}N}$, we easily find (evaluating with the sources set to zero and the fields to their VEVs)

$$\begin{aligned} \left(\mathcal{D} + \gamma_\phi \langle \phi \rangle \frac{\delta}{\delta \phi_R} \right) \Gamma_{V\bar{N}N} = & -\gamma \left(-\Gamma_{V\bar{N}N} + \frac{1}{2N_F} (\Gamma_{VQ} \Gamma_{V\bar{N}N} + \Gamma_{VV} \Gamma_{Q\bar{N}N}) \right) \\ & - 2\beta^{(L)} k^4 \frac{1}{2N_F} \left(-\Gamma_{Q\bar{N}N} + \frac{1}{2N_F} \Gamma_{VQ} \Gamma_{Q\bar{N}N} \right) - 2\beta^{(S)} k^4 \Gamma_{V\phi_5} \Gamma_{\phi_5 \bar{N}N} \end{aligned} \quad (7.4)$$

$$\begin{aligned} \left(\mathcal{D} + \gamma_\phi \langle \phi \rangle \frac{\delta}{\delta \phi_R} \right) \Gamma_{Q\bar{N}N} = & -\gamma \frac{1}{2N_F} \left(\Gamma_{QQ} \Gamma_{V\bar{N}N} + \Gamma_{QV} \Gamma_{Q\bar{N}N} \right) \\ & - 2\beta^{(L)} k^4 \frac{1}{(2N_F)^2} \Gamma_{QQ} \Gamma_{Q\bar{N}N} - 2\beta^{(S)} k^2 \Gamma_{Q\phi_5} \Gamma_{\phi_5 \bar{N}N} \end{aligned} \quad (7.5)$$

$$\begin{aligned} \left(\mathcal{D} + \gamma_\phi \langle \phi \rangle \frac{\delta}{\delta \phi_R} \right) \Gamma_{\phi_5 \bar{N}N} = & -\gamma_\phi \Gamma_{\phi_5 \bar{N}N} - \gamma \frac{1}{2N_F} \left(\Gamma_{\phi_5 Q} \Gamma_{V\bar{N}N} + \Gamma_{\phi_5 V} \Gamma_{Q\bar{N}N} \right) \\ & - 2\beta^{(L)} k^4 \frac{1}{(2N_F)^2} \Gamma_{\phi_5 Q} \Gamma_{Q\bar{N}N} - 2\beta^{(S)} k^2 \Gamma_{\phi_5 \phi_5} \Gamma_{\phi_5 \bar{N}N} . \end{aligned} \quad (7.6)$$

Now, from eq.(3.4), we have the following chiral Ward identities for the 2-point proper vertices,

$$\Gamma_{VV} + 2\langle \phi \rangle \Gamma_{\phi_5 V} = 0 \quad (7.7)$$

$$\Gamma_{VQ} - 2N_F + 2\langle \phi \rangle \Gamma_{\phi_5 Q} = 0 \quad (7.8)$$

$$\Gamma_{V\phi_5} + 2\langle \phi \rangle \Gamma_{\phi_5 \phi_5} = 0. \quad (7.9)$$

Substituting into eqs.(7.4) and (7.6), we immediately verify the consistency of these RGEs with the crucial identity

$$\Gamma_{V\bar{N}N} = -2\langle \phi \rangle \Gamma_{\phi_5 \bar{N}N}, \quad (7.10)$$

used in section 5 to derive the $U(1)$ GT relation.

The chiral Ward identities (7.7)–(7.10) also allow us to simplify the RGEs (7.5), (7.6) for the 3-point vertices which enter directly in the non-forward $U(1)$ GT relation. We find, writing everything in terms of the pseudoscalar vertices,

$$\begin{aligned} \left(\mathcal{D} + \gamma_\phi \langle \phi \rangle \frac{\delta}{\delta \phi_R} \right) \Gamma_{Q\bar{N}N} = & -\gamma \Gamma_{Q\bar{N}N} + \gamma \frac{2\langle \phi \rangle}{2N_F} \left(\Gamma_{Q\phi_5} \Gamma_{Q\bar{N}N} + \Gamma_{QQ} \Gamma_{\phi_5 \bar{N}N} \right) \\ & - 2\beta^{(L)} k^4 \frac{1}{(2N_F)^2} \Gamma_{QQ} \Gamma_{Q\bar{N}N} - 2\beta^{(S)} k^2 \Gamma_{Q\phi_5} \Gamma_{\phi_5 \bar{N}N} \end{aligned} \quad (7.11)$$

$$\begin{aligned} \left(\mathcal{D} + \gamma_\phi \langle \phi \rangle \frac{\delta}{\delta \phi_R} \right) \Gamma_{\phi_5 \bar{N}N} = & -\gamma_\phi \Gamma_{\phi_5 \bar{N}N} + \gamma \frac{2\langle \phi \rangle}{2N_F} \left(\Gamma_{\phi_5 \phi_5} \Gamma_{Q\bar{N}N} + \Gamma_{\phi_5 Q} \Gamma_{\phi_5 \bar{N}N} \right) \\ & - 2\beta^{(L)} k^4 \frac{1}{(2N_F)^2} \Gamma_{\phi_5 Q} \Gamma_{Q\bar{N}N} - 2\beta^{(S)} k^2 \Gamma_{\phi_5 \phi_5} \Gamma_{\phi_5 \bar{N}N} . \end{aligned} \quad (7.12)$$

8. Renormalisation Group Properties of the $U(1)$ GT Relation

We have derived RGEs for all the quantities entering into the $U(1)$ GT relation, and so are now in a position to verify explicitly its RG consistency.

We do this first for the non-forward generalisation of the $U(1)$ GT relation. Recall from section 5 that this states,

$$ik^\mu \langle N | J_{\mu 5}^R | N \rangle = 2N_F \bar{u} \left(W_{\Theta\Theta} \Gamma_{Q\bar{N}N} + W_{\Theta S_5} \Gamma_{\phi_5 \bar{N}N} \right) u. \quad (8.1)$$

Clearly, the l.h.s. satisfies a homogeneous RGE with anomalous dimension γ . We have to verify this for the r.h.s.

First notice that while the RGEs in section 7 involve the combination $(\mu\partial/\partial\mu + \beta\partial/\partial g)|_{V_{5R}^\mu, \Phi_R^a} + \gamma_\phi \langle \phi \rangle \delta/\delta\phi_R$, in section 6 the RGEs for the propagators are written with $(\mu\partial/\partial\mu + \beta\partial/\partial g)|_{V_{5R}^\mu, S_R^a, S_R^0}$. In fact, at the physical limit of zero sources and fields set to their VEVs, these are identical. We simply use the notation $\mathcal{D}$ below for this operator.

We therefore find, from the RGEs (6.11) and (6.13) for $W_{\Theta\Theta}$ and $W_{\Theta S_5}$, and the RGEs (7.11) and (7.12) for $\Gamma_{Q\bar{N}N}$ and $\Gamma_{\phi_5 \bar{N}N}$,

$$\begin{aligned} \mathcal{D}(W_{\Theta\Theta} \Gamma_{Q\bar{N}N} + W_{\Theta S_5} \Gamma_{\phi_5 \bar{N}N}) &= \gamma \left(2W_{\Theta\Theta} - W_{\Theta\Theta} + \frac{2\langle \phi \rangle}{2N_F} (W_{\Theta\Theta} \Gamma_{Q\phi_5} + W_{\Theta S_5} \Gamma_{\phi_5 \phi_5}) \right) \Gamma_{Q\bar{N}N} \\ &\quad + \gamma \left(W_{\Theta S_5} + \frac{2\langle \phi \rangle}{2N_F} (1 + W_{\Theta S_5} \Gamma_{\phi_5 Q} + W_{\Theta\Theta} \Gamma_{QQ}) \right) \Gamma_{\phi_5 \bar{N}N} \\ &\quad + \gamma_\phi (W_{\Theta S_5} - W_{\Theta\Theta}) \Gamma_{\phi_5 \bar{N}N} \\ &\quad - 2\beta^{(L)} k^4 \frac{1}{(2N_F)^2} (1 + W_{\Theta\Theta} \Gamma_{QQ} + W_{\Theta S_5} \Gamma_{\phi_5 Q}) \Gamma_{Q\bar{N}N} \\ &\quad - 2\beta^{(S)} k^2 (W_{\Theta\Theta} \Gamma_{Q\phi_5} + W_{\Theta S_5} \Gamma_{\phi_5 \phi_5}) \Gamma_{\phi_5 \bar{N}N} \\ &= \gamma (W_{\Theta\Theta} \Gamma_{Q\bar{N}N} + W_{\Theta S_5} \Gamma_{\phi_5 \bar{N}N}), \end{aligned} \quad (8.2)$$

as required. In the second step, we have used the inversion formula (B.4) for propagators and 2-point functions in the pseudoscalar sector.

This is our key result, establishing the consistency of the non-forward $U(1)$ GT relation with the renormalisation group. Although, of course, it is just a special case of the above, it is amusing to check this result directly in the zero-momentum limit, where the actual $U(1)$ GT relation (5.5) holds. At first sight, it is far from obvious that the r.h.s., involving $2\langle \phi \rangle \Gamma_{\phi_5 \bar{N}N}|_{k=0}$, can satisfy the required homogeneous RGE with anomalous dimension γ . That this is so follows from eq.(7.12), which, together with the zero-momentum Ward identities $\Gamma_{\phi_5 \phi_5} = 0$ and $\Gamma_{\phi_5 Q} = 2N_F/2\langle \phi \rangle$ (see eq.(5.17)), implies

$$\mathcal{D} \phi_R \Gamma_{\phi_5 \bar{N}N}|_{k=0} = \gamma \phi_R \Gamma_{\phi_5 \bar{N}N}|_{k=0}, \quad (8.3)$$

evaluating at the VEVs. Remarkably enough, while the γ_ϕ dependence cancels as expected, a dependence on γ is induced due to the nature of the 1PI vertex $\Gamma_{\phi_5 \bar{N}N}$ defined with the Zumino functional.

The RG consistency of the $U(1)$ GT relation expressed in terms of the canonically normalised field η_0 is readily established. From the RGE

$$\mathcal{D}\Gamma_{\phi_5\phi_5} = 2\gamma \frac{2\langle\phi\rangle}{2N_F} \Gamma_{Q\phi_5}\Gamma_{\phi_5\phi_5} - 2\gamma_\phi\Gamma_{\phi_5\phi_5} - 2\beta^{(S)}k^2\Gamma_{\phi_5\phi_5}\Gamma_{\phi_5\phi_5}, \quad (8.4)$$

which follows from eq.(7.3) and the Ward identity (7.9), together with $\mathcal{D}\Gamma_{\eta_0\eta_0} = 0$ from the defining normalisation condition (5.8), we deduce the behaviour of the normalisation factor $b(k^2)$, viz.

$$\mathcal{D}b = \left(\gamma \frac{2\langle\phi\rangle}{2N_F} \Gamma_{Q\phi_5} - \gamma_\phi\right)b - \beta^{(S)}k^4b^3. \quad (8.5)$$

Equivalently, we have

$$\mathcal{D}F_{\eta_0}(k^2) = \left(\gamma \frac{2\langle\phi\rangle}{2N_F} \Gamma_{Q\phi_5}\right)F_{\eta_0}(k^2) - \beta^{(S)}k^4 \frac{1}{4\langle\phi\rangle^2} F_{\eta_0}^3(k^2). \quad (8.6)$$

This simplifies at zero momentum, and, from the identities (5.17), we see

$$\mathcal{D}F_{\eta_0}(0) = \gamma F_{\eta_0}(0). \quad (8.7)$$

This is consistent with the identification $F_{\eta_0}(0) = 2N_F(\chi'_t(0))^{\frac{1}{2}}$, since the RGE for the propagator $W_{\Theta\Theta}$ is

$$\mathcal{D}W_{\Theta\Theta} = 2\gamma W_{\Theta\Theta} - \frac{1}{(2N_F)^2}2\beta^{(L)}k^4, \quad (8.8)$$

and the extra term proportional to $\beta^{(L)}$ does not contribute to the first moment $\chi'_t(0)$. The 3-point vertex $\Gamma_{\eta_0\bar{N}N}(0)$ at zero momentum is RG invariant, as is required for the r.h.s. of the $U(1)$ GT relations (5.12) or (5.14) to be identified with $2m_N G_A(0)$. To see this, we show first from eq.(8.5) and the RGE (7.12) for $\Gamma_{\phi_5\bar{N}N}$ that

$$\begin{aligned} \mathcal{D}\Gamma_{\eta_0\bar{N}N} &= \frac{1}{2N_F} \left(\gamma k^2 F_{\eta_0}(k^2) - 2\beta^{(L)}k^4 \frac{1}{2N_F} \Gamma_{Q\eta_0} \right) \Gamma_{Q\bar{N}N} \\ &\quad - \beta^{(S)}k^4 \frac{1}{4\langle\phi\rangle^2} F_{\eta_0}^2(k^2) \Gamma_{\eta_0\bar{N}N}, \end{aligned} \quad (8.9)$$

where we have used the relations $\Gamma_{\phi_5\phi_5} = b^2 k^2$ and $b = \frac{1}{2\langle\phi\rangle} F_{\eta_0}(k^2)$ to simplify the result. At zero-momentum, therefore,

$$\mathcal{D}\Gamma_{\eta_0\bar{N}N}(0) = 0. \quad (8.10)$$

Again notice that, just as in eq.(8.7) for F_{η_0} , the required simplification of the RGE occurs for $k^2 = 0$ only.

Finally, we discuss the RG consistency of the $U(1)$ GT relation expressed in terms of the physical η' . With the definitions $\eta' = c\phi_5$ and $G = Q - \frac{W_{\Theta S_5}}{W_{S_5 S_5}}\phi_5$ of eq.(5.22), together with eq.(5.24), we readily find

$$\mathcal{D}c = -\gamma_\phi c - \beta^{(S)}k^2(k^2 - m_{\eta'}^2)c^3, \quad (8.11)$$

using the RGE for $W_{S_5 S_5}$.

It then follows immediately from the RGE (6.13) for $W_{\Theta S_5}$ that

$$\mathcal{D}W_{\Theta S'_\eta} = \gamma \left(1 + \frac{2\langle\phi\rangle}{2N_F} \frac{1}{W_{\Theta S_5}} \right) W_{\Theta S'_\eta} + \beta^{(S)} k^2 \frac{1}{W_{S_5 S_5}} W_{\Theta S'_\eta}, \quad (8.12)$$

and so at zero momentum

$$\mathcal{D}F = 0, \quad (8.13)$$

recalling the definition $F = -2N_F W_{\Theta S'_\eta}(0)$.

Next, we need the RGE for the 3-point vertex $\Gamma_{\eta' \bar{N} N}$. Writing

$$\Gamma_{\eta' \bar{N} N} = c^{-1} \left(\Gamma_{\phi_5 \bar{N} N} + \frac{W_{\Theta S_5}}{W_{S_5 S_5}} \Gamma_{Q \bar{N} N} \right), \quad (8.14)$$

a straightforward calculation using the RGEs already presented and the inversion formula (B.4) eventually gives

$$\mathcal{D}\Gamma_{\eta' \bar{N} N} = \beta^{(S)} k^2 \frac{1}{W_{S_5 S_5}} \Gamma_{\eta' \bar{N} N}. \quad (8.15)$$

Notice again that while the r.h.s. is non-zero for an arbitrary momentum, it vanishes at $k^2 = m_{\eta'}^2$, and so, as of course required, the physical on-shell η' -nucleon coupling $g_{\eta' \bar{N} N}$ is a RG invariant. The r.h.s. is also zero for $k^2 = 0$, which is essential for the dynamical approximation $g_{\eta' \bar{N} N}(m_{\eta'}^2) \simeq g_{\eta' \bar{N} N}(0)$ used in writing the approximate $U(1)$ GT relation (5.31) to be reasonable.

Remembering that $\Gamma_{G \bar{N} N}|_{\eta'} \equiv \Gamma_{Q \bar{N} N}|_{\phi_5}$, we also have directly from eq.(7.12) that

$$\begin{aligned} \mathcal{D}\Gamma_{G \bar{N} N} &= \left(-\gamma + 2\gamma \frac{2\langle\phi\rangle}{2N_F} \Gamma_{Q \phi_5} - 2\beta^{(L)} k^4 \frac{1}{(2N_F)^2} \Gamma_{QQ} + 2\beta^{(S)} k^2 \Gamma_{Q \phi_5} \frac{W_{\Theta S_5}}{W_{S_5 S_5}} \right) \Gamma_{G \bar{N} N} \\ &+ \left(\gamma \frac{2\langle\phi\rangle}{2N_F} \Gamma_{QQ} c - 2\beta^{(S)} k^2 \Gamma_{Q \phi_5} c \right) \Gamma_{\eta' \bar{N} N}, \end{aligned} \quad (8.16)$$

and so, at zero momentum,

$$\mathcal{D}\Gamma_{G \bar{N} N}(0) = \gamma \left(\Gamma_{G \bar{N} N}(0) - \frac{W_{S'_\eta S'_\eta}}{W_{\Theta S'_\eta}} \Big|_{k=0} \Gamma_{\eta' \bar{N} N}(0) \right). \quad (8.17)$$

The glue-nucleon coupling $g_{G \bar{N} N}(k^2)$ is therefore not RG invariant, but satisfies a non-homogeneous RGE even at $k^2 = 0$. It is therefore not a physical quantity, nor is it sensible to impose any dynamical assumption, such as we have made in section 5 for $g_{\eta' \bar{N} N}(k^2)$, relating its value at different momentum scales.

Finally, putting all this together, we can calculate the RGEs for the individual terms in the expression (5.26) for the non-forward $U(1)$ GT relation. We find,^{*}

$$\mathcal{D}W_{\Theta S'_\eta} \Gamma_{\eta' \bar{N} N} = \left(\gamma \left(1 + \frac{2\langle\phi\rangle}{2N_F} \frac{1}{W_{\Theta S_5}} \right) + 2\beta^{(S)} k^2 \frac{1}{W_{S_5 S_5}} \right) W_{\Theta S'_\eta} \Gamma_{\eta' \bar{N} N}, \quad (8.18)$$

and,

$$\begin{aligned} \mathcal{D} \left(W_{\Theta\Theta} - \frac{W_{\Theta S'_\eta}^2}{W_{S'_\eta S'_\eta}} \right) \Gamma_{G\bar{N}N} &= \gamma \left(W_{\Theta\Theta} - \frac{W_{\Theta S'_\eta}^2}{W_{S'_\eta S'_\eta}} \right) \Gamma_{G\bar{N}N} \\ &\quad - \left(\gamma \frac{2\langle\phi\rangle}{2N_F} \frac{1}{W_{\Theta S_5}} + 2\beta^{(S)} k^2 \frac{1}{W_{S_5 S_5}} \right) W_{\Theta S'_\eta} \Gamma_{\eta' \bar{N} N}. \end{aligned} \quad (8.19)$$

In fact, neither term is individually particularly simple. However, adding to get the combination which is related to the axial current matrix element $\langle N | J_{\mu 5}^R | N \rangle$, we find the simple result

$$\begin{aligned} \mathcal{D} \left(W_{\Theta S'_\eta} \Gamma_{\eta' \bar{N} N} + \left(W_{\Theta\Theta} - \frac{W_{\Theta S'_\eta}^2}{W_{S'_\eta S'_\eta}} \right) \Gamma_{G\bar{N}N} \right) \\ = \gamma \left(W_{\Theta S'_\eta} \Gamma_{\eta' \bar{N} N} + \left(W_{\Theta\Theta} - \frac{W_{\Theta S'_\eta}^2}{W_{S'_\eta S'_\eta}} \right) \Gamma_{G\bar{N}N} \right), \end{aligned} \quad (8.20)$$

as required.

A further simplification occurs in the zero-momentum limit, which gives the true $U(1)$ GT relation (5.27). There, using the zero-momentum Ward identities $W_{\Theta\Theta} = 0$, $W_{\Theta S_5} = -2\langle\phi\rangle/2N_F$, we find

$$\mathcal{D} \left(W_{\Theta S'_\eta} \Gamma_{\eta' \bar{N} N} \right) \Big|_{k=0} = 0, \quad (8.19)$$

$$\mathcal{D} \left(\frac{W_{\Theta S'_\eta}^2}{W_{S'_\eta S'_\eta}} \Gamma_{G\bar{N}N} \right) \Big|_{k=0} = \gamma \left(\frac{W_{\Theta S'_\eta}^2}{W_{S'_\eta S'_\eta}} \Gamma_{G\bar{N}N} - W_{\Theta S'_\eta} \Gamma_{\eta' \bar{N} N} \right) \Big|_{k=0}. \quad (8.21)$$

In this form, therefore, the $U(1)$ GT matrix element $\langle N | J_{\mu 5}^R | N \rangle$ is decomposed into a piece, proportional to the η' -nucleon coupling at $k = 0$, which is RG invariant, and the remainder, proportional to $g_{G\bar{N}N}(0)$, with the required inhomogeneous RGE to allow the correct overall RG behaviour for the sum.

^{*} It is useful, in deriving eq.(8.19), to note the individual component identities from the matrix inversion formula (B.4). Explicitly, this reads,

$$\begin{pmatrix} \Gamma_{QQ} & \Gamma_{Q\phi_5} \\ \Gamma_{\phi_5 Q} & \Gamma_{\phi_5 \phi_5} \end{pmatrix} = -(W_{\Theta\Theta} W_{S_5 S_5} - W_{\Theta S_5}^2)^{-1} \begin{pmatrix} W_{S_5 S_5} & -W_{\Theta S_5} \\ -W_{S_5 \Theta} & W_{\Theta\Theta} \end{pmatrix}$$

9. Implications for the Proton “Spin”

The QCD-parton model suggests a decomposition of $G_A(0)$ into quark (Δq) and gluon ($\Delta\Gamma$) components of the proton spin[4-7], viz.

$$G_A(0) = \Delta q - N_F \Delta\Gamma. \quad (9.1)$$

This approach requires that the quark content Δq should be RG invariant. As we have just seen, the form (5.30) of the $U(1)$ GT relation satisfies this requirement. Indeed, this is the unique decomposition of $G_A(0)$ into a RG invariant piece and piece depending only on the 1PI “glue”-nucleon vertex Γ_{QNN} . In this respect, therefore, we may identify

$$\Delta q = \frac{1}{2m_N} F g_{\eta' NN} \quad \Delta\Gamma = -\frac{1}{2m_N} \frac{1}{2N_F^2} F^2 m_{\eta'}^2 g_{GNN}, \quad (9.2)$$

as non-perturbative definitions of the “quark” and “gluon” components. However, it is not clear to us whether this is a mere definition, or whether there really exists a deeper connection between the two pictures. It would be interesting to find other hard processes involving the QCD-parton Δq , $\Delta\Gamma$ [7] and see whether the same identification with the proper vertices could be made.

A related issue concerns the quark mass dependence of these results[10,18]. As we have shown in section 5, the simple expression (5.5) of the $U(1)$ GT relation is independent of the quark masses, the only dependence in the final formulae arising from the usual momentum extrapolation from the chiral limit. In fact, the proper vertices $\Gamma_{\phi_5 NN}$ and Γ_{QNN} defined here depend smoothly on the quark masses. This is a further necessary condition for the above identification with quark and gluon components of the “spin”, since these should not exhibit any strong quark mass dependence. This property is not shared by the simple nucleon matrix elements of the pseudoscalar operators ϕ_5^m and Q which, as shown in ref.[10], have a quite singular dependence on the quark masses. The reason for the smooth mass dependence of the proper vertices is simply that, by definition, they are 1PI and therefore free of the propagator poles which give rise to the large, undesired isospin violations in the pseudoscalar matrix elements themselves. Since the effects of quark masses on $G_A(0)$ itself are therefore small and smooth, it seems clear that the resolution of the proton “spin” problem cannot reside in quark mass effects but should be understood already in the chiral limit of QCD.

Independently of QCD-parton ideas, we can ask to what extent the $U(1)$ GT relation provides an explanation of the basic problem of why $G_A(0)$ is so small compared to $2\sqrt{3}G_A^{(8)}(0)$. The first observation is simply that the very existence of a $U(1)$ Goldberger-Treiman formula relating the singlet form factor to quantities defined in the isosinglet pseudoscalar channel is indicative of large OZI violations. However, we would of course like to be much more quantitative.

Assuming $g_{\eta' NN}$ to be experimentally known (in fact the measurement[19] is surprisingly imprecise), the $U(1)$ GT relation has two unknowns, F and $\Gamma_{QNN}(0)$ in the form (5.31), and $F_{\eta'}(0)$ and $\Gamma_{\phi_5 NN}(0)$ in eq.(5.12). In principle, the two decay constants should be calculable in, e.g., lattice QCD, since they have straightforward definitions in terms

of the 2-point functions $\langle \phi_5 \phi_5 \rangle$ and $\langle Q Q \rangle$, although it should be kept in mind that for self-consistency we require to go beyond the OZI (and therefore quenched, see ref.[9]) approximations. Even then, however, the predictivity of the $U(1)$ GT relations depends on our ability to determine the proper vertices Γ_{QNN} or $\Gamma_{\phi_5 NN}$ in non-perturbative QCD or at least in some QCD-inspired dynamical model. In this respect, it is worthwhile to re-express the two components of the proton “spin” in terms of Green functions since, although the decomposition is motivated by properties of the proper vertices, the Green functions are the quantities which would be most directly calculated on the lattice. A straightforward calculation leads to the expressions,

$$2m_N \Delta q \bar{u} \gamma_5 u = \frac{1}{2\langle \phi \rangle} F^2 m_{\eta'}^2 \bar{u} \hat{W}_{S_5 S_{\bar{N}} S_N} u, \quad (9.3)$$

$$2m_N N_F \Delta \Gamma \bar{u} \gamma_5 u = 2N_F \bar{u} \hat{W}_{\Theta S_N S_{\bar{N}}} u + \frac{1}{2\langle \phi \rangle} F^2 m_{\eta'}^2 \bar{u} \hat{W}_{S_5 S_{\bar{N}} S_N} u, \quad (9.4)$$

where $\hat{W}_{S_X S_{\bar{N}} S_N} \equiv W_{S_{\bar{N}} S_N}^{-1} W_{S_X S_N S_{\bar{N}}} W_{S_{\bar{N}} S_N}^{-1}$. Finally, using eq.(5.29) for F , we find the following simple expression for the quark component Δq ,

$$m_N \Delta q \bar{u} \gamma_5 u = -i\langle \phi \rangle \langle 0 | \phi_5(k) \phi_5(-k) | 0 \rangle^{-1} \Big|_{k=0} \langle N | \phi_5 | N \rangle. \quad (9.5)$$

One possible explanation* for the small value of $G_A(0)$ is however suggested by comparing the simple form of the $U(1)$ GT relation in terms of the coupling of the OZI NG boson η_0 to the nucleon,

$$F_{\eta_0}(0) g_{\eta_0 NN}(0) = 2m_N G_A(0), \quad (5.12)$$

with its OZI approximation

$$\sqrt{6} F_\pi \sqrt{2} g_{\eta_8 NN}(0) = 2m_N 2\sqrt{3} G_A^{(8)}(0). \quad (9.7)$$

It seems plausible that the coupling $g_{\eta_0 NN}(0)$ in full QCD should be approximately equal to its value in the OZI limit, viz. $\sqrt{2} g_{\eta_8 NN}(0)$. Recall that $g_{\eta_0 NN}$ is RG invariant even in QCD, so such an approximation is quite consistent with the renormalisation group. Comparing eqs.(5.12) and (9.7), we then see that a possible, and perhaps necessary, explanation of the small value of $G_A(0)$ in QCD (compared to the OZI value $2\sqrt{3} G_A^{(8)}(0)$) is that $F_{\eta_0}(0)$ in QCD is small** compared with its value in the OZI limit, viz. $\sqrt{6} F_\pi$. Notice that $F_{\eta_0}(0)$

* In a sense, this is the opposite of the explanation offered in ref.[20] (see also ref.[21]) in terms of the Skyrme model. In the Skyrme picture, the nucleon is a soliton constructed from the Goldstone fields of the coset space $SU(3)_L \times SU(3)_R / SU(3)$ and has no $U(1)$ component. The coupling $g_{\eta_0 NN}$ therefore vanishes, providing, via the $U(1)$ GT formula (5.12), an explanation of the EMC result.

** This possibility was briefly considered then rejected in ref.[11] because it appeared that a small value of this decay constant would be inconsistent with previous results[22] on $\eta' \rightarrow \gamma\gamma$. However, in the light of our new analysis (see ref.[14] and the discussion below), we now see that in fact a small value of $F_{\eta_0}(0)$ is entirely compatible with $\eta' \rightarrow \gamma\gamma$.

is RG invariant in the OZI limit, whereas in QCD it scales with anomalous dimension γ . Perhaps it is then not surprising that the two values should be quite different. The EMC result would therefore be explained if the first moment of the topological susceptibility $\chi'_t(0)$ in N_F -flavour QCD is anomalously small compared with $(2N_F)^{-1}F_\pi^2$. This is at least a well-defined possibility which could be investigated in lattice QCD.

We can perhaps learn more by comparing the $U(1)$ GT formula written in terms of the η' -nucleon coupling, viz.

$$Fg_{\eta'NN}(0) + \frac{1}{2N_F}F^2m_{\eta'}^2g_{GNN}(0) = 2m_NG_A(0), \quad (5.30)$$

with the corresponding formula for the decay $\eta' \rightarrow \gamma\gamma$. We have recently re-analysed this decay using the methods described in this paper, with the result[14] (to $O(\epsilon_R^2)$),

$$Fg_{\eta'\gamma\gamma}(0) + \frac{1}{2N_F}F^2m_{\eta'}^2g_{G\gamma\gamma}(0) = \frac{4}{3\pi}N_C\alpha_{\text{em}}. \quad (9.8)$$

The similarity with the $U(1)$ GT formula is due to the fact that in both cases the straightforward generalisation of the usual non-anomalous PCAC formula to the flavour singlet sector involves the η_0 . The l.h.s. of eq.(9.8) is equal to $F_{\eta_0}(0)g_{\eta_0\gamma\gamma}(0)$. It follows from this that the decay constant $F_{\eta_0}(0)$ is not constrained by the data on $\eta' \rightarrow \gamma\gamma$, and may, as conjectured above, be small.

There is, however, an important difference between these two formulae. Whereas in eq.(5.30) only the first term is RG invariant, and the scaling behaviour of $G_A(0)$ requires the complicated RG non-invariance of $g_{GNN}(0)$ described in section 8, in eq.(9.8) each term is separately RG invariant[14]. If we approximate RG invariant quantities by their OZI limits, then the second term, proportional to $g_{G\gamma\gamma}(0)$, vanishes* and we find simply

$$\sqrt{6}F_\pi g_{\eta'\gamma\gamma}(0) = \frac{4}{3\pi}N_C\alpha_{\text{em}}. \quad (9.9)$$

In fact, allowing for the fact that here we are using the notation η' for the singlet pseudoscalar, and that to compare with experiment for the true η' we have to take account of singlet-octet mixing, this relation is quite accurately satisfied experimentally (see ref.[23] and the discussion in ref.[14]).

This apparent experimental success of the OZI approximation for $\eta' \rightarrow \gamma\gamma$ suggests (though certainly does not require) a similar interpretation of the $U(1)$ GT relation (5.30).

* The OZI relations quoted here may alternatively be justified in terms of the $1/N_C$ expansion. The large N_C behaviour of the various quantities appearing here is as follows:

$$\begin{array}{llll} F = O(\sqrt{N_C}) & m_{\eta'}^2 = O(1/N_C) & G_A = O(N_C) & \\ g_{\eta'\gamma\gamma} = O(\sqrt{N_C}) & g_{G\gamma\gamma} = O(1) & g_{\eta'NN} = O(\sqrt{N_C}) & g_{GNN} = O(1) \end{array}$$

In this case, however, we apply the OZI approximation only to those quantities which have the same RG behaviour in QCD itself and in the OZI limit. For quantities which are not RG invariant in QCD, but which are RG invariant in the OZI limit, we may expect the OZI approximation to be strongly violated.

Suppose, therefore, that we approximate the first term, $Fg_{\eta' NN}(0)$, in the $U(1)$ GT relation by its OZI value $2\sqrt{3}F_{\pi}g_{\eta_8 NN}(0)$. This term is leading in the OZI limit and is RG invariant. On the r.h.s., $2m_N G_A(0)$ is also leading in the OZI limit, but is RG non-invariant and so cannot be assumed to be well approximated by its OZI value $2m_N 2\sqrt{3}G_A^{(8)}(0)$. Finally, the second term, proportional to $g_{GNN}(0)$, is OZI suppressed. However, since it is also not RG invariant, we do not make the assumption that it is necessarily small.

We would then conclude that (in the interpretation of eq.(9.2)) the quark component of the proton “spin” is not small, and is well approximated by its OZI value, i.e. $\Delta q \simeq \sqrt{3}m_N^{-1}F_{\pi}g_{\eta_8 NN}(0) \simeq 0.68$. The observed small value of $G_A(0)$ would then be due to an approximate cancellation between Δq and the gluon component $\Delta\Gamma$, which must therefore display large OZI violations. This picture agrees qualitatively with the expectations based on the parton model. The ultimate explanation of the cancellation nevertheless still resides in the conjecture that $\chi'_t(0)$ is anomalously small.

In conclusion, although a quantitative prediction appears still to be beyond the reach of current calculational techniques in non-perturbative QCD, the $U(1)$ GT relation and its RG properties provide an important insight into the nature of the large OZI violation in $G_A(0)$, and provides a well-defined framework in which to discuss the resolution of the proton “spin” problem.

Appendix A. Propagator matrices

We show here how the chiral Ward identities may be used to restrict the form of the propagator matrices for the composite operators $J_{\mu 5}$, Q and ϕ_5 , and how the decay constants for the physical pseudoscalar particles π and η' are identified.

(i) Non-anomalous axial current

The most general form for the propagator matrix of the operators $J_{\mu 5}^i$, ϕ_5^i compatible with the Ward identities for the 2-point functions derived from eq.(2.7), viz.

$$ik^\mu \frac{\delta^2 W}{\delta V_5^{i\mu} \delta V_5^{j\nu}} = 0, \quad (A.1)$$

$$ik^\mu \frac{\delta^2 W}{\delta V_5^{i\mu} \delta S_5^j} - \frac{1}{2} \langle \phi \rangle \delta^{ij} = 0, \quad (A.2)$$

is, letting $\mathcal{S} = \{V_5^{i\mu}, S_5^i\}$,

$$\frac{\delta^2 W}{\delta \mathcal{S} \delta \mathcal{S}} = \begin{pmatrix} \frac{1}{k^2} \Delta_{\mu\nu} A(k^2) & -i \frac{k_\mu}{k^2} \langle \phi \rangle \\ i \frac{k_\nu}{k^2} \langle \phi \rangle & -\frac{1}{k^2} \frac{1}{4} \langle \phi \rangle^2 B(k^2) \end{pmatrix} \delta^{ij}, \quad (A.3)$$

where $\Delta_{\mu\nu} = (k^2 g_{\mu\nu} - k_\mu k_\nu)$, and $A(k^2)$, $B(k^2)$ are functions of the momentum which are not determined by the chiral Ward identities. The pseudoscalar pole in $1/k^2$ required by Goldstone's theorem (an immediate consequence of eq.(A.2)) is explicitly displayed. The $\langle 0 | T^* J_{\mu 5}^i J_{\nu 5}^j | 0 \rangle$ propagator is transverse, as required by current conservation (eq.(A.1)).

Diagonalising the matrix (A.3) gives

$$\begin{pmatrix} \left(\frac{1}{k^2} \Delta_{\mu\nu} A(k^2) + \frac{k_\mu k_\nu}{k^2} B^{-1}(k^2) \right) & 0 \\ 0 & -\frac{1}{k^2} \frac{1}{4} \langle \phi \rangle^2 B(k^2) \end{pmatrix} \delta^{ij}. \quad (A.4)$$

The presence of a physical pion pole requires $B(0) \neq 0$. On the other hand, the absence of a pole in $1/k^2$ in the pseudovector channel clearly requires that $A(0) = B^{-1}(0)$.

The pion decay constant, defined by $\langle 0 | J_{\mu 5}^i | \pi^j \rangle = i k_\mu F_\pi \delta^{ij}$, can be identified given this assumption that the $1/k^2$ pole in the propagator $\langle 0 | T^* J_{\mu 5}^i J_{\nu 5}^j | 0 \rangle$ in eq.(A.3) is entirely due to the pion. We therefore find, using the relation $A(0) = B^{-1}(0)$,

$$F_\pi = B^{-\frac{1}{2}}(0). \quad (A.5)$$

Now consider the inclusion of an explicit chiral symmetry breaking mass for the quarks. For simplicity, we take the masses equal, so there is no flavour breaking. The Ward identity

(2.7) acquires an extra term $-2m \frac{\delta W}{\delta S_5^i}$, and the identities for the propagators become

$$ik^\mu \frac{\delta^2 W}{\delta V_5^{i\mu} \delta V_5^{j\nu}} - 2m \frac{\delta^2 W}{\delta S_5^i \delta V_5^{j\nu}} = 0, \quad (\text{A.6})$$

$$ik^\mu \frac{\delta^2 W}{\delta V_5^{i\mu} \delta S_5^j} - 2m \frac{\delta^2 W}{\delta S_5^i \delta S_5^j} - \frac{1}{2} \langle \phi \rangle \delta^{ij} = 0. \quad (\text{A.7})$$

The propagator matrix now takes the form

$$\frac{\delta^2 W}{\delta \mathcal{S} \delta \mathcal{S}} = \begin{pmatrix} \frac{1}{k^2} \Delta_{\mu\nu} A(k^2) + \frac{k_\mu k_\nu}{k^2} \tilde{A}(k^2) & -i \frac{k_\mu}{k^2} \frac{1}{2} \left(\langle \phi \rangle - \frac{1}{k^2 - m_\pi^2} m \langle \phi \rangle^2 B(k^2) \right) \\ \text{h.c.} & -\frac{1}{k^2 - m_\pi^2} \frac{1}{4} \langle \phi \rangle^2 B(k^2) \end{pmatrix} \delta^{ij}, \quad (\text{A.8})$$

where we have again explicitly displayed the pion pole in the $\langle 0 | T^* \phi_5^i \phi_5^j | 0 \rangle$ propagator. Notice that now the current-current propagator is no longer transverse, and, from eq. (A.6), $\tilde{A}(k^2) = \frac{m}{k^2} \left(\langle \phi \rangle - \frac{1}{k^2 - m_\pi^2} m \langle \phi \rangle^2 B(k^2) \right)$.

Following the method above, we can again identify the decay constant F_π by requiring that the pion couples only to the physical pseudoscalar channel. This implies

$$F_\pi = -m \langle \phi \rangle \frac{1}{m_\pi^2} B^{\frac{1}{2}}(m_\pi^2). \quad (\text{A.9})$$

Since there are no poles at $k^2 = 0$ in the spectrum, we see by inspection of the propagator matrix that $A(0) = B^{-1}(0)(1 + m_\pi^2 B^{-1}(0)B'(0))$, and that

$$B(0) = -\frac{1}{m \langle \phi \rangle} m_\pi^2. \quad (\text{A.10})$$

We therefore find

$$F_\pi = B^{\frac{1}{2}}(m_\pi^2) B^{-1}(0), \quad (\text{A.11})$$

which should be contrasted with eq.(A.5).

If we make the standard smoothness assumption that $B(k^2)$ is a slowly-varying function in the momentum range $0 \leq k^2 \leq m_\pi^2$, it is a reasonable approximation to set $B(m_\pi^2) \simeq B(0)$. We then recover the identification $F_\pi = B^{-\frac{1}{2}}(0)$. In this approximation, therefore, the Goldberger-Treiman relation continues to hold (see section 4). It is also interesting to note that this is the same approximation required for the Dashen formula

$$F_\pi^2 m_\pi^2 = -m \langle \phi \rangle, \quad (\text{A.12})$$

to hold, as can be seen from eqs.(A.9), (A.10). In this context, the approximation is usually formulated as “pole dominance” of the pion contribution to the appropriate 2-point functions.

(ii) Anomalous U(1) axial current

In the flavour singlet sector, the propagator matrix for the operators $J_{\mu 5}$, Q and ϕ_5 is required to satisfy the chiral Ward identities (2.4)–(2.6). A convenient parametrisation satisfying these constraints is, with $\mathcal{S} = \{V_5^\mu, \Theta, S_5\}$,

$$\frac{\delta^2 W}{\delta \mathcal{S} \delta \mathcal{S}} = \begin{pmatrix} \frac{1}{k^2} \Delta_{\mu\nu} A - \frac{k_\mu k_\nu}{k^2} \frac{1}{k^2 - m_{\eta'}^2} D & + i k_\mu \frac{1}{k^2 - m_{\eta'}^2} \frac{1}{2N_F} D & - i \frac{k_\mu}{k^2} 2\langle\phi\rangle \left(1 + \frac{1}{k^2 - m_{\eta'}^2} C\right) \\ -k^2 \frac{1}{k^2 - m_{\eta'}^2} \frac{1}{(2N_F)^2} D & \frac{1}{k^2 - m_{\eta'}^2} \frac{2\langle\phi\rangle}{2N_F} C & \\ h.c. & -\frac{1}{k^2 - m_{\eta'}^2} 4\langle\phi\rangle^2 B & \end{pmatrix} \quad (\text{A.13})$$

where A , B , C and D are pole-free functions of k^2 which are not determined by the Ward identities alone. We have displayed the expected massive pseudoscalar η' pole at $k^2 = m_{\eta'}^2$ explicitly. With this assumption that the $U(1)$ problem is satisfied so that there is no massless pseudoscalar pole, we also require, from inspection of the $W_{V_5^\mu V_5^\nu}$ and $W_{V_5^\mu S_5}$ propagators, that $A(0) = D(0)/m_{\eta'}^2$ and $C(0) = m_{\eta'}^2$. A further relation follows from factorisation of the pole residues, giving

$$(C^2 - BDk^2)|_{k^2=m_{\eta'}^2} = 0. \quad (\text{A.14})$$

We now provide an alternative, explicit derivation of the relation established in section 5 between the η_0 decay function $F_{\eta_0}(0)$ and the first moment of the propagator $\langle 0|T^*Q Q|0\rangle$. First, we have to extract the normalisation factor $b(k^2)$ for the η_0 from the propagator matrix. Inverting in the pseudoscalar sector only, according to eq.(B.4), we find the matrix of 2-point proper vertices in the form, ($\mathcal{F} = \{Q, \phi_5\}$),

$$\frac{\delta^2 \Gamma}{\delta \mathcal{F} \delta \mathcal{F}} = -\frac{(2N_F)^2}{(2\langle\phi\rangle)^2} (k^2 - m_{\eta'}^2) (C^2 - BDk^2)^{-1} \begin{pmatrix} (2\langle\phi\rangle)^2 B & \frac{2\langle\phi\rangle}{2N_F} C \\ \frac{2\langle\phi\rangle}{2N_F} C & k^2 \frac{1}{(2N_F)^2} D \end{pmatrix}, \quad (\text{A.15})$$

Notice that since $(C^2 - BDk^2)$ is of $O(k^2 - m_{\eta'}^2)$, from eq. (A.14), there is no overall zero at $k^2 = m_{\eta'}^2$. The required zero of $\Gamma_{\phi_5 \phi_5}$ at $k^2 = 0$ is, however, manifest, and we can also check using $C(0) = m_{\eta'}^2$ that $\Gamma_{Q\phi_5}(0) = 2N_F/2\langle\phi\rangle$, as required by the zero-momentum Ward identity (5.17).

Since we demand $\frac{\delta^2 \Gamma}{\delta \eta_0 \delta \eta_0} = k^2$ with $\eta_0 = b(k^2)\phi_5$ (eqs.(5.7),(5.8)), the normalisation factor for the η_0 is determined to be

$$b = \frac{1}{2\langle\phi\rangle} \left(-\frac{k^2 - m_{\eta'}^2}{C^2 - BDk^2} \right)^{\frac{1}{2}} D^{\frac{1}{2}}, \quad (\text{A.16})$$

and so, from eq.(5.11), we find the η_0 decay function,

$$F_{\eta_0}(k^2) = \left(-\frac{k^2 - m_{\eta'}^2}{C^2 - BDk^2} \right)^{\frac{1}{2}} D^{\frac{1}{2}}. \quad (\text{A.17})$$

At zero momentum, therefore,

$$F_{\eta_0}(0) = \frac{1}{m_{\eta'}} D^{\frac{1}{2}}(0). \quad (\text{A.18})$$

It is now easy to see the relation with the propagator $\langle 0|T^*Q Q|0\rangle$. From eq.(A.13), we have at zero momentum,

$$\frac{\partial}{\partial k^2} W_{\Theta\Theta}|_{k=0} = \frac{1}{(2N_F)^2} \frac{1}{m_{\eta'}^2} D(0). \quad (\text{A.19})$$

We therefore establish the required identity,

$$F_{\eta_0}(0) = (2N_F) \left(\frac{\partial}{\partial k^2} i \langle 0|T^*Q Q|0\rangle \right)^{\frac{1}{2}} \Big|_{k=0}. \quad (\text{A.20})$$

Finally, we rewrite the propagator matrix in the pseudoscalar sector in terms of the interpolating fields η' and G introduced in section 5. From eq.(5.22) we have,

$$\eta' = c \phi_5 \quad G = Q + \frac{1}{2N_F} \frac{1}{2\langle\phi\rangle} \frac{C}{B} \phi_5, \quad (\text{A.21})$$

and so, with $\mathcal{S} = \{S_G, S_{\eta'}\}$,

$$\frac{\delta^2 W}{\delta \mathcal{S} \delta \mathcal{S}} = \begin{pmatrix} \frac{1}{(2N_F)^2} \frac{1}{k^2 - m_{\eta'}^2} (C^2 - BDk^2) B^{-1} & 0 \\ 0 & -\frac{1}{k^2 - m_{\eta'}^2} 4\langle\phi\rangle^2 B c^2 \end{pmatrix}. \quad (\text{A.22})$$

Notice here that the pole at $k^2 = m_{\eta'}^2$ cancels in the $\langle 0|T^*G G|0\rangle$ propagator by virtue of eq.(A.14). Whether or not there is a physical glueball pole in this propagator depends on the form of the functions B , C and D . Specifying this behaviour requires further dynamical input over and above the constraints of the chiral Ward identities and the assumption we have made throughout of the resolution of the $U(1)$ problem with a physical pseudoscalar η' of non-zero mass.

The normalisation factor for the η' is specified from eq.(5.23) to be such that $\frac{\delta^2 \Gamma}{\delta \eta' \delta \eta'} = k^2 - m_{\eta'}^2$ in the G , η' basis. Clearly, therefore,

$$c = \frac{1}{2\langle\phi\rangle} B^{-\frac{1}{2}}(k^2). \quad (\text{A.23})$$

The propagator $W_{\Theta S_{\eta'}}$ is then expressed as

$$W_{\Theta S_{\eta'}} = \frac{1}{2N_F} \frac{1}{k^2 - m_{\eta'}^2} C B^{-\frac{1}{2}}, \quad (\text{A.24})$$

and so, at zero momentum, we find

$$F = -2N_F W_{\Theta S_{\eta'}}(0) = B^{-\frac{1}{2}}(0), \quad (A.25)$$

which may be compared with the corresponding result (A.5) in the non-anomalous sector.

Appendix B. Partial Legendre transforms

We collect here some relations between Green functions, defined as functional derivatives of the generating functional, and the proper vertices defined from the Zumino action. For generality, we derive these results for the case of a partial Legendre transform where the transform is made only on a subset S^a of the complete set $\{V^r, S^a\}$ of sources. That is, we define the effective action $\Gamma[V^r, S^a]$ by

$$W[V^r, S^a] = \Gamma[V^r, \Phi_a] + S^a \Phi_a. \quad (B.1)$$

(As usual, Lorentz indices are implicit and a spacetime integration is assumed in the sum over repeated indices.)

By definition,

$$\Phi_a = \frac{\delta W}{\delta S^a}, \quad (B.2)$$

and it follows from eq.(B.1) that

$$\frac{\delta \Gamma}{\delta \Phi_a} = -S^a \quad \frac{\delta \Gamma}{\delta V^r} = \frac{\delta W}{\delta V^r}. \quad (B.3)$$

Various relations amongst the second derivatives of W and Γ are easily established. For example, using eqs.(B.2), (B.3), we find

$$\begin{aligned} \delta^a_b &= \frac{\delta}{\delta \Phi_a} \Phi_b = \frac{\delta V^t}{\delta \Phi_a} \frac{\delta^2 W}{\delta V^t \delta S^b} + \frac{\delta S^c}{\delta \Phi_a} \frac{\delta^2 W}{\delta S^c \delta S^b} \\ &= -\frac{\delta^2 \Gamma}{\delta \Phi_a \delta \Phi_c} \frac{\delta^2 W}{\delta S^c \delta S^b}, \end{aligned} \quad (B.4)$$

where in the final step we have used the fact that V^r is independent of Φ_a , i.e. $\delta V^r / \delta \Phi_a = 0$. This is, of course, the analogue of the usual statement that the 2-point proper vertex is the inverse of the propagator. Notice that here it is only valid for quantities restricted to the S^a, Φ_a sector, where the Legendre transform is performed.

Similarly, we can derive

$$\begin{aligned} 0 &= \frac{\delta}{\delta V^r} \Phi_b = \frac{\delta V^t}{\delta V^r} \frac{\delta^2 W}{\delta V^t \delta S^b} + \frac{\delta S^c}{\delta V^r} \frac{\delta^2 W}{\delta S^c \delta S^b} \\ &= \frac{\delta^2 W}{\delta V^r \delta S^b} - \frac{\delta^2 \Gamma}{\delta V^r \delta \Phi_c} \frac{\delta^2 W}{\delta S^c \delta S^b}. \end{aligned} \quad (B.5)$$

Also,

$$\begin{aligned}\frac{\delta^2 W}{\delta S^a \delta V^s} &= \frac{\delta}{\delta S^a} \frac{\delta \Gamma}{\delta V^s} = \frac{\delta V^t}{\delta S^a} \frac{\delta^2 \Gamma}{\delta V^t \delta V^s} + \frac{\delta \Phi_c}{\delta S^a} \frac{\delta^2 \Gamma}{\delta \Phi_c \delta V^s} \\ &= \frac{\delta^2 W}{\delta S^a \delta S^c} \frac{\delta^2 \Gamma}{\delta \Phi_c \delta V^s},\end{aligned}\tag{B.6}$$

and

$$\begin{aligned}\frac{\delta^2 W}{\delta V^r \delta V^s} &= \frac{\delta}{\delta V^r} \frac{\delta \Gamma}{\delta V^s} = \frac{\delta V^t}{\delta V^r} \frac{\delta^2 \Gamma}{\delta V^t \delta V^s} + \frac{\delta \Phi_c}{\delta V^r} \frac{\delta^2 \Gamma}{\delta \Phi_c \delta V^s} \\ &= \frac{\delta^2 \Gamma}{\delta V^r \delta V^s} + \frac{\delta^2 W}{\delta V^r \delta S^c} \frac{\delta^2 \Gamma}{\delta \Phi_c \delta V^s}.\end{aligned}\tag{B.7}$$

Eqs. (B.4) to (B.7) exhaust the relations amongst the 2-point functions and vertices. Other identities can always be reduced to this set. One interesting expression, which is worth listing separately, is found by combining eqs.(B.5) and (B.7), viz.

$$\frac{\delta^2 W}{\delta V^r \delta V^s} = \frac{\delta^2 \Gamma}{\delta V^r \delta V^s} + \frac{\delta^2 \Gamma}{\delta V^r \delta \Phi_c} \frac{\delta^2 W}{\delta S^c \delta S^d} \frac{\delta^2 \Gamma}{\delta \Phi_d \delta V^s}.\tag{B.8}$$

In the course of deriving the Goldberger-Treiman and $U(1)$ GT relations, we also need an identity relating a 3-point Green function to the 3-point proper vertices derived from Γ . Clearly there are several such relations, which we shall not list here. The particular identity we need is found by differentiating the relation (B.4) w.r.t. the source V^r . We find

$$\begin{aligned}0 &= \frac{\delta}{\delta V^r} \left(\frac{\delta^2 W}{\delta S^a \delta S^c} \frac{\delta^2 \Gamma}{\delta \Phi_c \delta \Phi_b} \right) \\ &= \frac{\delta^3 W}{\delta V^r \delta S^a \delta S^c} \frac{\delta^2 \Gamma}{\delta \Phi_c \delta \Phi_b} + \frac{\delta^2 W}{\delta S^a \delta S^c} \left(\frac{\delta^3 \Gamma}{\delta V^r \delta \Phi_c \delta \Phi_b} + \frac{\delta^2 W}{\delta V^r \delta S^d} \frac{\delta^3 \Gamma}{\delta \Phi_d \delta \Phi_c \delta \Phi_b} \right),\end{aligned}\tag{B.9}$$

and, using the inversion relation (B.4),

$$\frac{\delta^2 \Gamma}{\delta \Phi_a \delta \Phi_d} \frac{\delta^3 W}{\delta V^r \delta S^d \delta S^c} \frac{\delta^2 \Gamma}{\delta \Phi_c \delta \Phi_b} = \frac{\delta^3 \Gamma}{\delta V^r \delta \Phi_a \delta \Phi_b} + \frac{\delta^2 W}{\delta V^r \delta S^d} \frac{\delta^3 \Gamma}{\delta \Phi_d \delta \Phi_a \delta \Phi_b}.\tag{B.10}$$

Specialised to the set of currents and fields occurring in the Zumino transform in the text, this is the key identity used to obtain eqs.(4.3) and (5.2).

Appendix C. RGEs and the conventional Goldberger-Treiman relation

We present here expressions for the RG behaviour of the various quantities entering into the conventional Goldberger-Treiman relation. The analysis follows the methods described in sections 6–8, but is of course very much simpler. In fact, the main results can be read off directly from those in section 8 by suitable truncations, but it is interesting to display them explicitly.

The RGE for $\Gamma[V_{5R}^{i\mu}, V_R^{i\mu}, \phi_{5R}^i, \phi_R^i, \phi_{5R}^0, \phi_R^0]$ is

$$\begin{aligned} \mathcal{D}\Gamma = & -\gamma_\phi \sum_m \left(\phi_{5R}^m \frac{\delta\Gamma}{\delta\phi_{5R}^m} + \phi_R^m \frac{\delta\Gamma}{\delta\phi_R^m} \right) - \sum_i \beta_5^{(T)} (\partial^\mu V_{5R}^{i\nu} - \partial^\nu V_{5R}^{i\mu}) (\partial_\mu V_{\nu 5R}^i - \partial_\nu V_{\mu 5R}^i) \\ & - \sum_i \beta^{(T)} (\partial^\mu V_R^{i\nu} - \partial^\nu V_R^{i\mu}) (\partial_\mu V_{\nu R}^i - \partial_\nu V_{\mu R}^i) \\ & - \sum_m \beta^{(S)} \left(\partial^\mu \frac{\delta\Gamma}{\delta\phi_{5R}^m} \partial_\mu \frac{\delta\Gamma}{\delta\phi_{5R}^m} + \partial^\mu \frac{\delta\Gamma}{\delta\phi_R^m} \partial_\mu \frac{\delta\Gamma}{\delta\phi_R^m} \right) \\ & + \dots \end{aligned} \quad (C.1)$$

where $m = 0, i$ with $i = 1, 2, 3$. The dots denote terms of higher order in the sources or first derivatives of Γ . The second order terms are restricted by the requirement that Γ satisfies the chiral Ward identity (3.5). Notice that since it is the source for a conserved current, there is no anomalous dimension corresponding to $V_{5R}^{i\mu}$.

First, we can check the RG consistency of the essential Ward identity (4.9) relating $\Gamma_{V_5^i \bar{N} N}$ to $\langle \phi \rangle \Gamma_{\phi_5^i \bar{N} N}$. We have, from eq.(C.1),

$$\mathcal{D}\Gamma_{V_5^i \bar{N} N} = -2\beta^{(S)} k^2 \Gamma_{V_5^i \phi_5^j} \Gamma_{\phi_5^j \bar{N} N}, \quad (C.2)$$

and

$$\mathcal{D}\Gamma_{\phi_5^i \bar{N} N} = -\gamma_\phi \Gamma_{\phi_5^i \bar{N} N} - 2\beta^{(S)} k^2 \Gamma_{\phi_5^i \phi_5^j} \Gamma_{\phi_5^j \bar{N} N}. \quad (C.3)$$

Recalling that $\mathcal{D}$ is being used here as the RG operator at fixed sources (i.e. $\mathcal{D}|_{V, \Phi} + \gamma_\phi \langle \phi \rangle \frac{\delta}{\delta\phi_R}$), we easily check

$$\mathcal{D}\Gamma_{V_5^i \bar{N} N} = \mathcal{D} \frac{1}{2} \langle \phi \rangle \Gamma_{\phi_5^i \bar{N} N}, \quad (C.4)$$

using the identity $\Gamma_{V_5^i \phi_5^j} + \frac{1}{2} \langle \phi \rangle \Gamma_{\phi_5^i \phi_5^j} = 0$.

On-shell, the terms proportional to $\beta^{(S)}$ vanish, and we verify that, as required, $G_A(0)$ is RG invariant. To check the RG behaviour of the non-forward version (4.3), we also need the RGE for $W_{V_5^{i\mu} S_5}$, and find from eqs.(C.2), (C.3) that

$$\mathcal{D} \left(\Gamma_{V_5^{i\mu} \bar{N} N} + W_{V_5^{i\mu} S_5} \Gamma_{\phi_5^j \bar{N} N} \right) = 0, \quad (C.5)$$

consistent with the identification with $\langle N | J_{\mu 5}^i | N \rangle$.

Next, we calculate the RGEs for the quantities involving the interpolating π field, defined as $\pi^i = b\phi_{5R}^i$ such that $\Gamma_{\pi^i\pi^j} = k^2\delta^{ij}$. From the RGE

$$\mathcal{D}\Gamma_{\phi_5^i\phi_5^j} = -2\gamma_\phi\Gamma_{\phi_5^i\phi_5^j} - 2\beta^{(S)}k^2\Gamma_{\phi_5^i\phi_5^k}\Gamma_{\phi_5^k\phi_5^j}, \quad (C.6)$$

we find immediately

$$\mathcal{D}b = -\gamma_\phi b - \beta^{(S)}k^4b^3. \quad (C.7)$$

The pion “decay function”, defined by the off-shell matrix element $\langle 0|J_{\mu 5}^i|\pi^j\rangle$, is related to the 2-point vertex $\Gamma_{V_5^{i\mu}\pi^j}$ from the Zumino transform. Differentiating eq.(C.1), we find the RGE

$$\mathcal{D}\Gamma_{V_5^{i\mu}\phi_5^j} = -\gamma_\phi\Gamma_{V_5^{i\mu}\phi_5^j} - 2\beta^{(S)}k^2\Gamma_{V_5^{i\mu}\phi_5^k}\Gamma_{\phi_5^k\phi_5^j}, \quad (C.8)$$

and so for the pion decay function

$$\mathcal{D}F_\pi(k^2) = -\beta^{(S)}k^4\frac{4}{\langle\phi\rangle^2}F_\pi^3(k^2), \quad (C.9)$$

consistent with the identification $b(k^2) = 2\langle\phi\rangle^{-1}F_\pi(k^2)$. Again, notice that because of the extra terms in the RGE resulting from the need to renormalise Green functions of composite operators, $F_\pi(k^2)$ is not RG invariant off-shell. At $k^2 = 0$, however, it is identified with the physical pion decay constant, and we indeed have

$$\mathcal{D}F_\pi = 0. \quad (C.10)$$

Finally, from eqs.(C.3), (C.6) and (C.7), we find the RGE for the 3-point pion-nucleon proper vertex,

$$\mathcal{D}\Gamma_{\pi^i\bar{N}N} = -\beta^{(S)}k^4\frac{4}{\langle\phi\rangle^2}F_\pi^2(k^2)\Gamma_{\pi^i\bar{N}N}. \quad (C.11)$$

Notice that while the r.h.s. is non-vanishing for general k , we have the required property that $\Gamma_{\pi^i\bar{N}N}$ is RG invariant on-shell (at $k^2 = 0$) allowing the identification (4.11) with $g_{\pi NN}$.

We therefore verify explicitly that all the quantities in the physical representation (4.12) of the conventional Goldberger-Treiman relation are RG invariant.

Appendix D. Remarks on the “ η' decay constant” $F_{\eta'}$.

At first sight, it would seem natural to expect the “ η' decay function” $F_{\eta'}(k^2)$, defined by

$$\frac{\delta^2 \Gamma}{\delta V_5^\mu \delta \eta'} = \langle 0 | J_{\mu 5} | \eta' \rangle = i k_\mu F_{\eta'}(k^2), \quad (D.1)$$

to play a role in the $U(1)$ GT relation, and perhaps, modulo some smoothness assumption, for its on-shell value $F_{\eta'} \equiv F_{\eta'}(m_{\eta'}^2)$ to occur in the final $U(1)$ GT formula (5.31). In this appendix, we explain why this is not correct.

First, following the approach in sections 4 and 5 for the π and η_0 , we express the interpolating η' field in terms of ϕ_5 using $F_{\eta'}(k^2)$. From the definitions (D.1) and (5.22), we have

$$F_{\eta'}(k^2) = -\frac{i k_\mu}{k^2} c^{-1} \left(\frac{\delta^2 \Gamma}{\delta V_5^\mu \delta \phi_5} + \frac{W_{\Theta S_5}}{W_{S_5 S_5}} \frac{\delta^2 \Gamma}{\delta V_5^\mu \delta Q} \right). \quad (D.2)$$

This can be simplified using the identity

$$\frac{\delta^2 \Gamma}{\delta V_5^\mu \delta \phi_5} \frac{\delta^2 W}{\delta S_5 \delta S_5} + \frac{\delta^2 \Gamma}{\delta V_5^\mu \delta Q} \frac{\delta^2 W}{\delta \Theta \delta S_5} = \frac{\delta^2 W}{\delta V_5^\mu \delta S_5}, \quad (D.3)$$

which is a special case of eq.(B.5). Noting that in the orthonormal basis for the pseudoscalars, $W_{S_{\eta'} S_{\eta'}} = -(\Gamma_{\eta' \eta'})^{-1} = -(k^2 - m_{\eta'}^2)$, and using the chiral Ward identity (2.6), we find

$$\eta' = \frac{k^2}{k^2 - m_{\eta'}^2} \left(1 + \frac{2N_F}{2\langle \phi \rangle} W_{\Theta S_5} \right)^{-1} \frac{1}{2\langle \phi \rangle} F_{\eta'}(k^2) \phi_5. \quad (D.3)$$

Notice that the normalisation factor is not so explicitly determined as in the case of the pion, because the anomalous Ward identities are less restrictive. As a consistency check, notice that in the OZI limit, where there is no anomaly and $m_{\eta'}^2 = 0$, eq.(D.3) reduces to (5.11) with the identification of η' and η_0 .

To compare $F_{\eta'}(k^2)$ with the quantities in the $U(1)$ GT relation, it is simplest to use the parametrisation of the propagator matrix in appendix A. From eqs.(D.3) and (A.23), we have

$$F_{\eta'}(k^2) = \frac{1}{k^2} (k^2 - m_{\eta'}^2 + C) B^{-\frac{1}{2}}. \quad (D.4)$$

It then follows using results from appendix A that at zero momentum,

$$F_{\eta'}(0) = B^{-\frac{1}{2}}(0)(1 + C'(0)), \quad (D.5)$$

while on-shell,

$$F_{\eta'} \equiv F_{\eta'}(m_{\eta'}^2) = \frac{1}{m_{\eta'}} D^{\frac{1}{2}}(m_{\eta'}^2). \quad (D.6)$$

Comparing with the propagator $W_{\Theta S_{\eta'}}$ (eq.(A.24)), we see that the residue of the pole at $k^2 = m_{\eta'}^2$ is indeed given by $(2N_F)^{-1} m_{\eta'}^2 F_{\eta'}$, as required directly by the Ward identities.

However, because of the extra term in eq.(D.5) compared to (A.25), we cannot identify the physical constant F of the $U(1)$ GT relation with $F_{\eta'}(0)$.

In fact, such an identification is not possible simply because of the different RG properties of the quantities concerned. As shown in section 8, $F \equiv -2N_F W_{\Theta S_{\eta'}}(0)$ is RG invariant and therefore is a physical constant. On the other hand, $F_{\eta'}(k^2)$ is not RG invariant even for special values of the momentum such as $k^2 = 0$ or $k^2 = m_{\eta'}^2$, and so is not physical.

To see this, we use the RGE (8.11) for the normalisation factor c of the η' in terms of ϕ_5 , together with the relation between $F_{\eta'}(k^2)$ and c implicit in eq.(D.3) above, to deduce the RGE for $F_{\eta'}(k^2)$ itself. We find,

$$\mathcal{D}F_{\eta'}(k^2) = \gamma F_{\eta'}(k^2) + \beta^{(S)} k^2 \frac{1}{W_{S_5 S_5}} F_{\eta'}(k^2). \quad (D.7)$$

The second term vanishes both at $k^2 = 0$ and on-shell ($k^2 = m_{\eta'}^2$), so we find both

$$\mathcal{D}F_{\eta'}(0) = \gamma F_{\eta'}(0), \quad (D.8)$$

and

$$\mathcal{D}F_{\eta'} = \gamma F_{\eta'}. \quad (D.9)$$

In particular, therefore, the putative “ η' decay constant” $F_{\eta'}$ is not RG invariant, and consequently is unphysical.

We also emphasise again that it is not sensible to impose a smoothness approximation, relating values at different momenta, to a function which is not RG invariant at least at those momenta. So, eg., it is not appropriate to try to relate $F_{\eta'}$ (eq.(D.6)) to $F_{\eta_0}(0)$ as given in eq.(A.18) by a smoothness assumption on the pole-free but RG non-invariant function $D(k^2)$.

Our conclusion, therefore, is that because of its non-invariant RG behaviour, the quantity $F_{\eta'}$ is not useful in formulations of the $U(1)$ GT relation, despite its natural analogy to F_π .

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