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**ON THE CANONICAL STRUCTURE OF HIGHER-DERIVATIVE FIELD
THEORIES: THE GRAVITATIONAL WZW-MODEL[†]**

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Abstract

A general expression for the symplectic structure of a higher-derivative Lagrangian field theory is given. General Relativity and the gravitational WZW-model are considered in this framework. In the second case we work out explicitly the Poisson bracket for both chiral solutions giving rise, in two different ways, to the classical exchange algebra of the $SL_q(2)$ group.

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1.-Introduction

The canonical formalism of Lagrangian theories has usually been considered to require a non-covariant description of phase space. In fact, the classical phase space is often defined as the space of initial data, i.e., the initial co-ordinates q^i and momenta p_i . The Poisson bracket of arbitrary functions f and g ,

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}, \quad (1.1)$$

can be equivalently defined by introducing the symplectic form (see (1.3))

$$\omega = \sum_i dq^i \wedge dp_i. \quad (1.2)$$

More generally, given a symplectic two-form ω (i.e., a closed and non-degenerated two-form) on phase space, the Poisson bracket of any two functions f and g is given by [1]

$$\{f, g\} = \omega^{ij} \partial_i f \partial_j g, \quad (1.3)$$

where ω^{ij} is the inverse matrix of the non-degenerated form ω_{ij} . Unlike formula (1.1), the expression (1.3) exhibits the relevant fact that the canonical formalism is not necessarily tied to a non-covariant parametrization of the phase space. Nevertheless, both expressions (1.1) and (1.3) are related by the Darboux theorem which states that if ω is a symplectic form, one can introduce, at least locally, co-ordinates q^i and p_j verifying (1.1).

The discussion above leads to the idea of looking at phase space as the space of solutions of the equations of motion. The covariance of this definition can also be accomplished by a covariant definition of the symplectic form ω . The symplectic structure on the phase space of a field theory can also be defined in a covariant way [2-6] as

$$\omega = \int_{\Sigma} \omega^{\mu} d\sigma_{\mu}, \quad (1.4)$$

where Σ is a initial value hypersurface, and ω^{μ} , the symplectic current, is a conserved and closed two-form ($\partial_{\mu} \omega^{\mu} = 0$; $\delta \omega^{\mu} = 0$, where δ stands for the exterior derivative on phase space). δ is nilpotent and obeys the Leibniz rule.

For a field theory with a Lagrangian density $\mathcal{L}$ depending on a field Ψ^{α} and its first-order derivative $\partial_{\mu} \Psi^{\alpha}$, the symplectic current is given by [6]

$$\omega^{\mu} = \delta \Psi^{\alpha} \wedge \delta \left(\frac{\partial \mathcal{L}}{\partial \Psi^{\alpha}_{,\mu}} \right), \quad (1.5)$$

where $\delta \Psi^{\alpha}(x)$ is a one-form on the covariant phase space (we shall use, from now on, the notation $\partial_{\mu} \Psi^{\alpha} \equiv \Psi^{\alpha}_{,\mu}$ and the exterior product will be omitted). Obviously, ω^{μ} is a closed two-form and, for fields obeying the equations of motion, ω^{μ} satisfies the conservation law $\partial_{\mu} \omega^{\mu} = 0$, thus ensuring that the definition (1.4) is independent of the choice of Σ . Picking Σ to be the standard hypersurface $t = 0$, (1.4) becomes

$$\omega = \int_{t=0} \delta \Psi^\alpha \delta \frac{\partial \mathcal{L}}{\partial \dot{\Psi}^\alpha} \equiv \int_{t=0} \delta \Psi^\alpha \delta \pi_\alpha, \quad (1.6)$$

which coincides again with the usual expression of the Poisson bracket in field theory. We have to stress that the two-form (1.4) is indeed a presymplectic form. In general, ω can be degenerated and this means that we have a constrained system. The true symplectic form and the physical phase space are then obtained by taking the quotient by the vector fields in the kernel of ω . These vector fields are generators of gauge transformations.

A covariant formulation of ordinary first-order derivative field theories was initiated, inspired by the geometrical approach to the calculus of variations[7], in Refs.[8,9]. The covariant approach sketched above is in fact closely related to the oldest works[8,9]. On the other hand, a canonical formalism of higher-derivative mechanics for non-degenerated Lagrangians was worked out by Ostrogradski[10].

In this paper, and on the basis of the covariant description, we present a procedure for obtaining the symplectic current of field theories which can be applied to higher-order derivative and even non-local theories. In the ordinary first-order case we recover the standard expression (1.5). For any finite higher-order field theory we provide the general expression of the symplectic current. Our procedure can be seen then as a covariantization of the standard finite higher-order canonical formalism but it is also valid for non-local and degenerated Lagrangians. To illustrate how the new approach works we discuss two non-trivial examples: General Relativity and the gravitational WZW-model.

Recently, the covariant, canonical (first-order) formalism has also been considered in connection with WZW-theories and quantum group symmetries [11].

In Sec. 2 we present the new approach and derive the general expression of the symplectic current for finite higher-order field theories. As a first test of the scheme developed in Sec. 2, we analyze, in Sec. 3, the case of General Relativity. The Lagrangian is then considered either as a first-order Lagrangian, following the Palatini variational principle, or as a second-order one depending on the metric and its first and second derivatives. Irrespective of the starting point, the approach proposed in Sec. 2 leads to the same symplectic current and it is also consistent with the symplectic current proposed in Ref. 3 for General Relativity. In Sec. 4 we consider a purely higher-order Lagrangian associated with the induced two-dimensional gravity (i.e., the gravitational WZW model)[12]. Following our general approach we compute explicitly the symplectic structure on the covariant phase space and hence the Poisson brackets for the chiral group solutions. We show that the Poisson bracket relations are of r -matrix nature and this can be seen as the analogue to the $SL(2)$ exchange algebra found in the conformal gauge (i.e., Liouville theory). In addition to its own interest, and due to the consistence of the result, this last example can serve as a non-trivial test of the approach of Sec. 2.

2.-The symplectic current for higher-order field theories.

In this section we attempt to derive a generalization of the symplectic current (1.5) for arbitrary Lagrangians. To begin with, let us consider a first-order Lagrangian $\mathcal{L} = \mathcal{L}(\Psi^\alpha, \partial_\mu \Psi^\alpha \equiv \Psi_\mu^\alpha)$ and an arbitrary variation of the field $\Psi^\alpha(x)$,

$$\Psi^\alpha(x) \rightarrow \Psi^\alpha(x) + \delta \Psi^\alpha(x).$$

Under this variation

$$\delta \mathcal{L} = \partial_\mu j^\mu + (E-L)_\alpha \delta \Psi^\alpha, \quad (2.1)$$

where

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha} \delta \Psi^\alpha, \quad (2.2)$$

and $(E-L)_\alpha = 0$ are the Euler-Lagrange equations. If $\Psi^\alpha(x)$ belongs to phase space, i.e., $\Psi^\alpha(x)$ satisfies the equation of motions ($(E-L)_\alpha = 0$), then

$$\delta \mathcal{L} = \partial_\mu j^\mu. \quad (2.3)$$

Following now the interpretation of $\delta \Psi^\alpha(x)$ as a one-form on phase space, the current (2.2) can be seen as a one-form as well, and the formula (2.3) is consistent with the alternative interpretation of δ as the exterior-derivative operator on phase space. Moreover, comparing (2.2) with (1.5) suggests regarding j^μ in (2.2-3) as some sort of "potential current" for the symplectic current, i.e.,

$$\omega^\mu \equiv -\delta j^\mu = \quad (2.4a)$$

$$-\delta\left(\frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha} \delta \Psi^\alpha\right) = \delta \Psi^\alpha \delta \frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha}. \quad (2.4b)$$

The fundamental properties of the symplectic current (2.4),

$$a) \delta \omega^\mu = 0 \quad (2.5a)$$

$$b) \partial_\mu \omega^\mu = 0, \quad (2.5b)$$

can be now checked in a simple and straightforward way. a) is a consequence of the fact that $\delta^2 = 0$, and from (2.4a) and (2.3)

$$\partial_\mu \omega^\mu = -\delta(\partial_\mu j^\mu) = -\delta(\delta \mathcal{L}) = -\delta^2 \mathcal{L} = 0. \quad (2.6)$$

It is important to stress that a) and b) are essentially consequences of formulas (2.1) and (2.4a), the validity of which is kept for Lagrangians depending on arbitrary higher-order derivatives and even for non-local theories. Let us consider, for instance, a second-order Lagrangian $\mathcal{L} = \mathcal{L}(\Psi^\alpha, \partial_\mu \Psi^\alpha \equiv \Psi_\mu^\alpha, \partial_\mu \partial_\nu \Psi^\alpha \equiv \Psi_{\mu\nu}^\alpha)$. The Euler-Lagrange equations now read

$$\frac{d^2}{dx^\mu dx^\nu} \frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha} - \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha} + \frac{\partial \mathcal{L}}{\partial \Psi^\alpha} = 0, \quad (2.7)$$

and the current j^μ in (2.2) is given by

$$j^\mu = \left[\frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha} - \frac{d}{dx^\nu} \left(\frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha} \right) \right] \delta \Psi^\alpha + \frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha} \delta \Psi_\nu^\alpha. \quad (2.8)$$

The canonical expression for the symplectic current in a second-order field theory is then ($\omega^\mu = -\delta j^\mu$):

$$\omega^\mu = \delta \Psi^\alpha \delta \left[\frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha} - \frac{d}{dx^\nu} \left(\frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha} \right) \right] + \delta \Psi_\nu^\alpha \delta \frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha}. \quad (2.9)$$

For a third-order Lagrangian a similar computation leads to the following expression

$$\begin{aligned} \omega^\mu = \delta \Psi^\alpha \delta \left[\frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha} - \frac{d}{dx^\nu} \left(\frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha} - \frac{d}{dx^\sigma} \frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu\sigma}^\alpha} \right) \right] + \delta \Psi_\nu^\alpha \delta \left[\frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu}^\alpha} - \frac{d}{dx^\sigma} \frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu\sigma}^\alpha} \right] + \\ \delta \Psi_{\nu\sigma}^\alpha \delta \left[\frac{\partial \mathcal{L}}{\partial \Psi_{\mu\nu\sigma}^\alpha} \right]. \end{aligned} \quad (2.10)$$

In general, for an arbitrary r -order Lagrangian $\mathcal{L} = \mathcal{L}(\Psi^\alpha, \Psi_{\mu_1}^\alpha, \dots, \Psi_{\mu_1 \dots \mu_r}^\alpha)$, the symplectic current potential is given by

$$j^\mu = \sum_{s=1}^r \left[\frac{\partial \mathcal{L}}{\partial \Psi_{\mu_1 \dots \mu_{s-1} \mu}^\alpha} + \lambda_\alpha^{\mu_1 \dots \mu_{s-1} \mu} \right] \delta \Psi_{\mu_1 \dots \mu_{s-1}}^\alpha \quad (2.11a)$$

and

$$\lambda_\alpha^{\mu_1 \dots \mu_{s-1} \mu} = -\frac{d}{dx^\mu} \left[\frac{\partial \mathcal{L}}{\partial \Psi_{\mu_1 \dots \mu_{s-1} \mu}^\alpha} + \lambda_\alpha^{\mu_1 \dots \mu_{s-1} \mu} \right]. \quad (2.11b)$$

For non-local theories, the symplectic current should be obtained in the same way as for finite higher-order theories, but we cannot provide the general expression since it depends on the particular form of the Lagrangian.

3.-Derivation of the symplectic current for General Relativity.

The Hilbert-Einstein Lagrangian $\mathcal{L} = \sqrt{g} R$ can be seen as a first-order Lagrangian in the metric and connection variables or as a second-order Lagrangian with respect to the metric. In the first case the Palatini variational principle leads to both Einstein equations, $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$, and to the metricity condition $\nabla g = 0$. In the second case one is led directly to the Einstein equations. In this section we shall derive the symplectic current in both situations following the approach proposed in Sections 1 and 2. Irrespective of the variational principle used we shall obtain an unambiguous result which can also be regarded as a test of the self-consistence of the procedure exposed in Sec. 2. Although this example can be considered on the border between the first- and second-order cases, due to the fact that the second derivative term is a divergence, it is not straightforward to see that we should obtain a unique and covariant result.

Following the general scheme of Sec. 2, and according to the Palatini formulation, the expression of the symplectic current potential is formally given by

$$j^\alpha = \frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} \delta g_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial \Gamma_{\mu\nu,\alpha}^\rho} \delta \Gamma_{\mu\nu}^\rho, \quad (3.1)$$

where $\mathcal{L}$ is understood to be a function of the form $\mathcal{L} = \mathcal{L}(g_{\mu\nu}, \Gamma_{\mu\nu}^\sigma; \partial_\lambda g_{\mu\nu}, \partial_\lambda \Gamma_{\mu\nu}^\sigma)$, i.e., $\mathcal{L}$ depends on g , Γ , as independent co-ordinates, and their corresponding derivatives. We have

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \Gamma_{\mu\nu,\alpha}^\rho} = \sqrt{g}(g^{\mu\nu}\delta_\rho^\alpha - g^{\mu\alpha}\delta_\rho^\nu) \quad (3.2)$$

so that

$$j^\alpha = \sqrt{g}(g^{\mu\nu}\delta \Gamma_{\mu\nu}^\alpha - g^{\mu\alpha}\delta \Gamma_{\mu\nu}^\nu). \quad (3.3)$$

Furthermore, applying the scheme developed in Sec. 2 to the second-order Lagrangian $\mathcal{L} = \sqrt{g}R$ as a function of $g_{\mu\nu}$ and its derivatives $g_{\mu\nu,\sigma}$ and $g_{\mu\nu,\sigma\rho}$ the symplectic current potential is written as follows:

$$j^\alpha = \left(\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} - \partial_\beta \frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha\beta}} \right) \delta g_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha\lambda}} \delta g_{\mu\nu,\lambda}. \quad (3.4)$$

Now we have:

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} = \sqrt{g}(2g^{\lambda\alpha}g^{\nu\rho}\Gamma_{\lambda\rho}^\mu + g^{\mu\nu}g^{\lambda\rho}\Gamma_{\lambda\rho}^\alpha - 2g^{\lambda\rho}g^{\nu\alpha}\Gamma_{\lambda\rho}^\mu - g^{\mu\lambda}g^{\nu\rho}\Gamma_{\lambda\rho}^\alpha), \quad (3.5a)$$

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha\lambda}} = \sqrt{g}(g^{\mu\lambda}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\lambda}), \quad (3.5b)$$

$$\left(\frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} - \partial_\beta \frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha\beta}} \right) = \sqrt{g}(g^{\lambda\alpha}g^{\nu\rho}\Gamma_{\lambda\rho}^\mu - g^{\lambda\rho}g^{\nu\alpha}\Gamma_{\lambda\rho}^\mu), \quad (3.6)$$

and then

$$j^\alpha = \sqrt{g}(g^{\lambda\alpha}g^{\nu\rho}\Gamma_{\lambda\rho}^\mu - g^{\lambda\rho}g^{\nu\alpha}\Gamma_{\lambda\rho}^\mu)\delta g_{\mu\nu} + \sqrt{g}(g^{\mu\lambda}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\lambda})\delta g_{\mu\nu,\lambda}. \quad (3.7)$$

It is straightforward to verify that (3.3) and (3.7) are indeed the same current.

Now it is very illustrative to obtain again the symplectic current for General Relativity just from the basic idea of our approach and without using the expressions (2.8-11) (it is in this way that one should proceed when dealing with non-local theories). According to Sec. 2 the crucial point for reaching the symplectic current associated with an arbitrary Lagrangian consists of rewriting the variation (or the exterior derivative) of the Lagrangian as a divergence on phase space. We now attempt to implement this approach for the Hilbert-Einstein Lagrangian

$$\delta \mathcal{L} = \delta(\sqrt{g}g^{\mu\nu}R_{\mu\nu}) = \quad (3.8a)$$

$$\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta(\sqrt{g}g^{\mu\nu}). \quad (3.8b)$$

As is well known, the second term in (3.8b) is proportional to Einstein equations ($\delta\sqrt{g} = -1/2\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}$),

$$R_{\mu\nu} \delta(\sqrt{g} g^{\mu\nu}) = \sqrt{g} (R_{\mu\nu} - 1/2 g_{\mu\nu} R) \delta g^{\mu\nu}, \quad (3.9)$$

so that, in phase space, $\delta\mathcal{L}$ becomes

$$\delta\mathcal{L} = \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu}. \quad (3.10)$$

The r.h.s. of (3.10) can be easily converted into a divergence:

$$\sqrt{g} g^{\mu\nu} \delta R_{\mu\nu} = \sqrt{g} \nabla_\alpha (g^{\mu\nu} \delta \Gamma_{\mu\nu}^\alpha - g^{\mu\alpha} \delta \Gamma_{\mu\nu}^\rho) = \partial_\alpha j^\alpha \quad (3.11)$$

where

$$j^\alpha = \sqrt{g} (g^{\mu\nu} \delta \Gamma_{\mu\nu}^\alpha - g^{\mu\alpha} \delta \Gamma_{\mu\nu}^\rho). \quad (3.12)$$

This is, indeed, a key issue for obtaining the Einstein equations from a variational principle since the divergence (3.11) does not contribute to the variation of the action functional. However, it is just the current (3.12) that captures the canonical formalism (symplectic structure) of General Relativity. The symplectic current $\omega^\alpha = -\delta j^\alpha$ is then given by the following expression ($\delta\sqrt{g} = -1/2 g_{\mu\nu} \delta g^{\mu\nu} = 1/2 \sqrt{g} \delta \ln g$):

$$\omega^\alpha = \sqrt{g} \left\{ \delta \Gamma_{\mu\nu}^\alpha (\delta g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} \delta \ln g) - \delta \Gamma_{\mu\nu}^\nu (\delta g^{\mu\alpha} + \frac{1}{2} g^{\mu\alpha} \delta \ln g) \right\}. \quad (3.13)$$

Obviously, from the arguments of Sec. 2, (3.13) verifies $\delta\omega^\alpha = 0$ and $\partial_\alpha \omega^\alpha = 0$, and then the symplectic form is

$$\omega = \int_\Sigma \omega^\mu d\sigma_\mu. \quad (3.14)$$

The expressions above are consistent with the symplectic current of General Relativity proposed in Ref. 3 (see also [4-6]). We have to remark that we have derived the symplectic structure of General Relativity in a systematic and straightforward way of general applicability (valid for arbitrary higher-order Lagrangian field theories). The consistency of this result gives us confidence in the correctness of our approach.

4.- Symplectic current for the gravitational WZW-model, Poisson brackets and the classical exchange algebra.

Two-dimensional gravity in the light-cone gauge ($ds^2 = dx^+ dx^- + h (dx^+)^2$) can be described, in terms of the field H ($\partial_+ H = h \partial_- H$) by the following action [12] ($x^+ = t + x$, $x^- = t - x$; $\kappa = -d/12\pi$)

$$S = -2\kappa \int dx^+ dx^- \left(\frac{\partial_-^2 H \partial_+ \partial_- H}{(\partial_- H)^2} - \frac{(\partial_-^2 H)^2 \partial_+ H}{(\partial_- H)^3} \right). \quad (4.1)$$

It is clear that (4.1) corresponds to a higher-order theory. Our aim in this section is to carry out a detailed analysis of the canonical structure of this theory following the approach sketched in Sec. 2. We will compute explicitly the Poisson bracket for the chiral solutions showing that they are of r -matrix nature.

Now it is convenient to perform a change of variable in (4.1) by introducing a field F defined as $F(x^+, H(x^+, x^-)) = x^-$. One can then rewrite (4.1) as

$$S = 2 \kappa \int d^2x \frac{F_+}{F_-} \left(\frac{F_{--}}{F_-} - 2 \frac{F_-^2}{F_-^2} \right) \quad (4.2)$$

where, from now on, we shall use the notation $F_+ \equiv \partial_+ F$, $F_- \equiv \partial_- F$. Moreover, we shall consider the theory defined on the cylinder $0 \leq x \leq 2\pi$, $-\infty \leq t \leq +\infty$. The convenience of using expression (4.2) can be justified by noting that it can be considered as the gravitational analogue of the WZW model. In fact, the action (4.2) is related to the action functional associated with the coadjoint orbits of the Virasoro group [13] ($F(x^+, x^-)$, for a x^+ fixed, plays the role of a diffeomorphism) in the same way as the WZW models are related to the Kac-Moody groups. It can also be seen as the field model constructed from the Virasoro group by means of a group quantization scheme [14].

A straightforward application of formulae (2.11) provides both chiral components of the symplectic current potential:

$$j^+ = 2 \left(\frac{F_{--}}{F_-^2} - 2 \frac{F_-^2}{F_-^2} \right) \delta F \quad (4.3a)$$

$$j^- = 2 \left(\frac{F_{+-}}{F_-^2} \delta F - \left(\frac{F_{+-}}{F_-^2} + 2 \frac{F_+ F_{--}}{F_-^3} \right) \delta F_- + \frac{F_{+-}}{F_-^2} \delta F_{--} \right) \quad (4.3b)$$

It should be noted that it is just the chiral form j^+ which is associated with the Virasoro group [13,14]. The other component is out of reach of the Virasoro group.

The canonical structure of the theory is then given by (choosing the standard hypersurface $t = \text{cte}$)

$$\begin{aligned} \omega &= -\kappa \int_0^{2\pi} dx \delta j^t = -\frac{\kappa}{2} \int_0^{2\pi} dx (\delta j^+ + \delta j^-) \\ &= -\kappa \int_0^{2\pi} dx \left\{ \delta \left(\frac{F_{--}}{F_-^2} - 2 \frac{F_-^2}{F_-^2} \right) \delta F + \delta \left(\frac{F_{+-}}{F_-^2} \right) \delta F - \delta \left(\frac{F_{+-}}{F_-^2} + 2 \frac{F_+ F_{--}}{F_-^3} \right) \delta F_- + \right. \\ &\quad \left. + \delta \left(\frac{F_{+-}}{F_-^2} \right) \delta F_{--} \right\} \end{aligned} \quad (4.4)$$

By partial integration we can transform (4.4) into a more simplified expression

$$\omega = \int_0^{2\pi} dx \left\{ 2\delta F \delta J_{(0)} - \delta F_- \delta \left(\frac{F_{+-}}{F_-^2} \right) - \delta F_- \delta \left(\frac{F_{--}}{F_-^2} \right) \right\} \quad (4.5)$$

where $J_{(0)}$ is a conserved Noether current, $\partial_- J_{(0)} = 0$, given by one of the generators of the $SL(2, \mathbb{R})$ symmetry of the theory [12]:

$$J_{(0)} = \frac{F_{+-}}{F_-^2} - \frac{F_{+-} F_{--}}{F_-^3} \quad (4.6)$$

Since we want to compute the canonical structure on the covariant phase space, and hence the Poisson brackets for the chiral solutions, we have to project down (4.5) to the space of classical solutions. The equation of motion is $\partial_+ \{F; x^-\} = 0$, where $\{F; x^-\}$ is the Schwartzian derivative, whose general solution is:

$$F(t, x) = \frac{A(x^+)f(x^-) + B(x^+)}{C(x^+)f(x^-) + D(x^+)} \quad (4.7)$$

where f is an arbitrary function ($f' > 0$) and A, B, C, D are also arbitrary functions verifying the $SL(2, \mathbb{R})$ condition: $AD - BC = 1$. A, B, C, D and f are not necessarily periodic, but the requirement of periodicity of F , $F(t, x + 2\pi) = F(t, x)$ implies the following monodromy properties

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} (y + 2\pi) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} (y) \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \quad (4.8a)$$

$$f(y - 2\pi) = \frac{af(y) + b}{cf(y) + d} \quad (4.8b)$$

where the monodromy matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a constant $SL(2, \mathbb{R})$ matrix.

Substituting the general solution (4.7) in (4.5) the expression for ω becomes ($Q = B/D$):

$$\begin{aligned} \omega = \kappa \int_0^{2\pi} dy \{ & 4[\delta Q \delta(CD' - DC')] - \frac{1}{D^2} \delta D \delta D'] + \frac{1}{f'^2} \delta f' \delta f' \\ & + \partial_y \left[\frac{1}{(Cf + D)} \left(-\frac{4f}{D} \delta C \delta D - 4\delta f \delta C + \frac{2}{f'} \delta f' \delta (Cf + D) \right) \right] \} \end{aligned} \quad (4.9)$$

where the prime in (4.6) stands for the derivative with respect to the argument. In obtaining the first line of (4.9)—where the left- and right-movers are decoupled—we have employed a partial integration, but the total derivative must be maintained because of the non-periodicity of the chiral functions. It is in just this way that the monodromy group parameters — which are indeed parameters of the covariant phase space — appear explicitly in the canonical structure and, as we shall see, it is a key point for obtaining the proper Poisson brackets. For now on we will restrict ourselves to the case of diagonal monodromy ($b = 0 = c, d = a^{-1}$), although the analysis can be carried out in the general case. In so doing, the resulting expression for ω is

$$\begin{aligned} \omega = \kappa \int_0^{2\pi} dy \{ & 4[\delta Q \delta(CD' - DC')] - \frac{1}{D^2} \delta D \delta D'] + 4 \frac{\kappa}{aD(0)} \delta a \delta D(0) \\ & + \kappa \int_0^{2\pi} dy \frac{1}{f'^2} \delta f' \delta f'' + 2 \frac{\kappa}{f'(0)a} \delta f'(0) \delta a \end{aligned} \quad (4.10)$$

The form ω splits into two chiral pieces ω_R, ω_L , where ω_R is given by the first line in (4.10) and ω_L by the second one. This fact, together with the closeness

of ω_R and ω_L allows, from now on, an independent analysis of the left (right) movers. We should remark that the symplectic form ω , without the monodromy contribution, would have been degenerated and this would have made the definition of the Poisson bracket for the chiral solutions ambiguous, due to the fact that it involves the inverse matrix of ω (1.3).

The Poisson brackets between two arbitrary functions g and $\tilde{g}$ (see (1.3)) can be equivalently defined as [1]

$$\{g, \tilde{g}\} = X_g \cdot \tilde{g} = \delta \tilde{g}(X_g) \quad (4.11)$$

where X_g is the Hamiltonian vector field associated with g through the relation

$$iX_g \omega = - \delta g. \quad (4.12)$$

Let us work out the Hamiltonian vector field X_f corresponding to the chiral solution $f(u)$. X_f satisfies

$$\begin{aligned} iX_f \omega_L &= \kappa \int_0^{2\pi} dz \left[\left(-\frac{X''(z)}{f'^2(z)} \right)' + \left(-\frac{X'(z)}{f'^2(z)} \right)'' \right] \delta f - \\ &- 4\kappa \left(\frac{X''(0)f(0)}{f'^2(0)} - \frac{X'(0)f(0)f'(0)}{f'^3(0)} + \frac{X'(0)}{f'(0)} + \frac{X_a}{a} \right) \frac{da}{a} \\ &= - \delta f \end{aligned} \quad (4.13)$$

where $X(z)$ and X_a are the components of the vector field $X_f \equiv \int dz X(z) \delta / \delta f(z) + X_a \delta / \delta a$. From (4.13) we obtain the following equations:

$$\kappa \left[\left(\frac{X''(z)}{f'^2(z)} \right)' + \left(\frac{X'(z)}{f'^2(z)} \right)'' \right] = \delta(u-z) \quad (4.14a)$$

$$\frac{X''(0)f(0)}{f'^2(0)} - \frac{X'(0)f(0)f'(0)}{f'^3(0)} + \frac{X'(0)}{f'(0)} = - \frac{X_a}{a} \quad (4.14b)$$

The general solution of the differential equation (4.14a) is

$$\begin{aligned} X_f(u) &= \int dz \frac{1}{4\kappa} \{ \epsilon(u-z) [f(z)-f(u)]^2 + \\ &+ L f(z)^2 + M f(z) + N \} \delta / \delta f(z) + X_a \frac{\delta}{\delta a} \end{aligned} \quad (4.15)$$

where L, M, N are arbitrary constants exhibiting the non-triviality of the kernel of ω_L , and the associated ambiguity in the Poisson bracket, when the monodromy contribution is ignored. We will see now how the additional condition coming from the monodromy fixes the coefficients L, M, N . The monodromy condition

$$\delta f(z + 2\pi) = a^2 \delta f(z) + 2f(z) \delta a$$

acting on the Hamiltonian vector field (4.2) implies

$$L = \frac{a + a^{-1}}{a - a^{-1}} \quad (4.16a)$$

$$N = - \frac{a + a^{-1}}{a - a^{-1}} f(u)^2 \quad (4.16b)$$

$$X_a = \frac{af(u)}{2\kappa} \quad (4.16c)$$

Moreover, substituting (4.16c) and (4.15) in (4.14b) we obtain that $M = 0$. Thus the Poisson bracket between two chiral solutions $f(u)$ and $f(v)$ is given by the following expression

$$\{f(u), f(v)\} = \frac{1}{4\kappa} [\varepsilon(u-v)(f(u) - f(v))^2 + \coth \ln a (f(u)^2 - f(v)^2)] \quad (4.17)$$

It should be mentioned that we can recover a monodromy independent Poisson bracket in the limit $a \rightarrow 0$.

The Poisson bracket above is just the chiral sector of those obtained in the r -matrix approach to Liouville theory [15,16]. In fact, the fields $\Psi_{1/2} = 1/\sqrt{f'}$, $\Psi_{1/2} = f/\sqrt{f'}$ satisfy the classical $SL(2, \mathbb{R})$ exchange relations, in tensor notation:

$$\{\Psi(u) \otimes \Psi(v)\} = -\frac{1}{4\kappa} [r^+ \vartheta(u-v) + r^- \vartheta(v-u)] \Psi(u) \otimes \Psi(v) \quad (4.18)$$

with the r -matrices

$$r^+ = L_0 \otimes L_0 - L_{-1} \otimes L_1, \quad (4.19a)$$

$$r^- = -L_0 \otimes L_0 + L_1 \otimes L_{-1} \quad (4.19b)$$

which verify the classical Yang-Baxter equation.

Our task now is to determine the Poisson brackets of the right-movers following essentially the same steps as in the previous calculation. The general form of an arbitrary Hamiltonian vector field for the chiral symplectic structure ω_R is

$$X = \int dz [X_Q \frac{\delta}{\delta Q} + X_C \frac{\delta}{\delta C} + X_D \frac{\delta}{\delta D}] + X_a \frac{\delta}{\delta a} \quad (4.20)$$

and the contraction with ω_R gives:

$$\begin{aligned} i_X \omega_R = & \kappa \int dz [(DX_Q + (DX_Q)' \delta C + (-(CX_Q)' - C'X_Q + (\frac{1}{D^2}X_D)' + \frac{1}{D^2}X_D') \delta D + \\ & + (-DX_C - CX_D' + DX_C' + C'X_D) \delta Q] + \\ & + 4\kappa (CX_Q \delta D - DX_Q \delta C - \frac{1}{D^2}X_D \delta D) \Big|_0^{2\pi} + \\ & + 4\kappa (\frac{1}{aD} X_a \delta D - \frac{1}{aD} X_D \delta a)(0) \end{aligned} \quad (4.21)$$

The monodromy independent terms of (4.21) lead to differential equations

for the Hamiltonian vector fields and, in particular, for the ones corresponding to the functions $Q(u)$, $C(u)$, and $D(u)$. The general solution for these Hamiltonian vector fields is

$$\begin{aligned} X_{Q(u)} &= \frac{1}{\chi} \left(\int dz \left[-\frac{D(z)}{D(u)^2} \varepsilon(z-u) \right] \frac{\delta}{\delta C(z)} + X_{\text{Kernel}}^{Q(u)} + X_a^{Q(u)} \frac{\delta}{\delta a} \right) \\ X_{D(u)} &= \frac{1}{\chi} \left(\int dz \left[-\frac{D(z)D(u)}{2} \varepsilon(z-u) \right] \frac{\delta}{\delta D(z)} + X_{\text{Kernel}}^{D(u)} + X_a^{D(u)} \frac{\delta}{\delta a} \right) \\ X_{C(u)} &= \frac{1}{\chi} \left(\int dz \varepsilon(z-u) \left[-\frac{D(u)}{D(z)^2} \frac{\delta}{\delta Q(z)} + (-C(z)D(u) + \frac{1}{2} D(z)C(u)) \frac{\delta}{\delta D(z)} - \right. \right. \\ &\quad \left. \left. - \frac{1}{2} C(z)C(u) \frac{\delta}{\delta C(z)} \right] + X_{\text{Kernel}}^{C(u)} + X_a^{C(u)} \frac{\delta}{\delta a} \right) \end{aligned} \quad (4.22)$$

where the vector fields X_{kernel} belong to the kernel of ω_R (without the monodromy term) and are given by

$$X_{\text{Kernel}} = \int dz \left\{ L \left(\frac{1}{D(z)^2} \frac{\delta}{\delta Q(z)} + C(z) \frac{\delta}{\delta D(z)} \right) + M \left(D(z) \frac{\delta}{\delta D(z)} - C(z) \frac{\delta}{\delta C(z)} \right) + N D(z) \frac{\delta}{\delta C(z)} \right\} \quad (4.23)$$

where L , M , N are again arbitrary constants. The monodromy conditions:

$$\begin{aligned} \delta D(z+2\pi) &= a \delta D(z) + D(z) \delta a \\ \delta C(z+2\pi) &= a^{-1} \delta C(z) - C(z) \frac{\delta a}{a^2} \\ \delta Q(z+2\pi) &= \delta Q(z) \end{aligned} \quad (4.24)$$

together with the remaining equation of (4.21) determine the constants L , M , N and the component $X_a^{(\cdot)}$ for each one of the Hamiltonian vector fields X_Q , X_D , X_C :

$$\begin{aligned} X_{Q(u)}: \quad N &= \frac{a + a^{-1}}{a - a^{-1}} \frac{1}{D(u)^2}, \quad M = 0, \quad L = 0, \quad X_a = 0 \\ X_{D(u)}: \quad N &= \frac{a + a^{-1}}{a - a^{-1}} C(u), \quad M = 0, \quad L = 0, \quad X_a = -a D(u) \\ X_{C(u)}: \quad N &= 0, \quad M = 0, \quad L = -\frac{a + a^{-1}}{a - a^{-1}} D(u), \quad X_a = a C(u) \end{aligned} \quad (4.25)$$

We have now all the ingredients to determine the Poisson brackets for the right-mover solutions. To obtain a transparent result we have to return to the group variables A , B , C and D . If $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the Poisson brackets take the form

$$\{g(u), g(v)\} = \frac{1}{\chi} g(u) \otimes g(v) (C \varepsilon(u-v) + \frac{a + a^{-1}}{a - a^{-1}} T) \quad (4.26)$$

where C is the Casimir operator $C = 4K_{ab} X^a \otimes X^b = (L_+ \otimes L_- + L_- \otimes L_+) + 2L_0 \otimes L_0$. $T = L_+ \otimes L_- - L_- \otimes L_+$. It is easy to see that (4.26) is equivalent to the exchange relations

$$\{g(u), g(v)\} = g(u) \otimes g(v) r^{\pm} \quad (4.27)$$

where the sign $\pm$ corresponds to $u < v$ and $u > v$ and $r^{\pm}$ are the r -matrices

$$r^{\pm} = \pm \frac{1}{2\kappa} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 4(a^{\pm 2} - 1)^{-1} & 0 \\ 0 & 4(a^{\mp 2} - 1)^{-1} & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (4.28)$$

In the limit $a \rightarrow \infty$ the matrices above become monodromy independent and satisfy the classical Yang-Baxter equation.

5. - Conclusions and final comments.

We have presented a general scheme, based on the covariant description of phase space, to derive the canonical structure of higher-derivative and non-local field theories. The symplectic structure proposed has emerged with all its relevant properties by reinterpreting the variation of the Lagrangian as the exterior derivative operator on phase space. ω is a well-defined, closed two-form on the covariant phase space since it is the "Noether" charge of a conserved, exact current.

As a first test of our approach, and also as an illuminating example, we have analysed the case of ordinary General Relativity. In Sect. 4 we have considered a purely higher order theory: the gravitational WZW model. Since the classical solutions of this theory are known we have been able to determine explicitly, following our general approach, the canonical structure on the covariant phase space, i.e., the left-moving and right-moving sectors. We have also computed the Poisson brackets, which depend crucially on the proposed symplectic form, for the chiral solutions giving rise to the classical exchange algebra of the $SL_q(2, \mathbb{R})$ group. This result has its own interest in the sense that it can be interpreted as the classical counterpart of the quantum group structure in CFT and this is not surprising, since the quantum group structure also appears in 2d-gravity in the conformal gauge (i.e., Liouville theory). In the context of the covariant, canonical formalism, this result is a non-trivial test of the consistency of the general method of Sec.2.

It is worthwhile to mention again that the derivation of the canonical structure for finite higher-order field theories is based on the general equation (2.1) and, therefore, it also applies for non-local theories. This suggests that the covariant approach could also be useful to define the canonical structure of non-local theories without being forced to define it as the limit $N \rightarrow \infty$ of the N^{th} order derivative truncated theory. This last procedure has the serious drawback that the canonical structure of a higher-order perturbative correction to a lower-order Lagrangian changes the theory dramatically. Additional

"perturbative constraints" should then be imposed to convert the higher-derivative expansion into a true perturbative one of a lower-derivative theory, or to make it convergent to an original non-local theory (see in this respect ref. [17]). We plan to return to the issue of non-locality in a future publication.

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References

- 1.- R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Benjamin, New York (1967)
- 2.- E. Witten, Nucl. Phys. **B276** (1986) 291
- 3.- C. Crnkovic and E. Witten, in *Three Hundred Years of Gravitation*, S.W. Hawking and W. Israel eds., Cambridge, (1987) 676
- 4.- G. Zuckerman, in *Mathematical Aspects of String Theory*, S.T. Yau ed., World Scientific, Singapore, (1987) 259
- 5.- C. Crnkovic, Nucl. Phys. **B288** (1987) 419
- 6.- C. Crnkovic, Class. Quantum Grav. **5** (1988) 1557
- 7.- P. Dedecker, *Calcul des variations, formes differentielles et champs geodesiques*, in *Colloque Internationale de Geometrie Differentielle, Strasbourg, 1953* (Paris, 1954)
- 8.- J. Kijowski, Commun. Math. Phys. **30** (1973) 99
- 9.- J. Kijowski and W. Szczyrba, Commun. Math. Phys. **46** (1976) 183
- 10.- M. Ostrogradski, Mem. Ac. St. Petersburg VI **4** (1850) 385
- 11.- K. Gawedzki, Commun. Math. Phys. **139** (1991) 201. M. Chu, P. Goddard, I. Halliday, D. Olive and A. Schwimmer, Phys. Lett. **B266** (1991) 71. F. Falceto and K. Gawedzki, IHES/P/91/59 (1991)
- 12.- A.M. Polyakov, Mod. Phys. Lett. **A2** (1987) 892
- 13.- A. Alekseev and S. Shatashvili, Nucl. Phys. **B 323** (1989) 719
- 14.- V. Aldaya, J. Navarro-Salas and M. Navarro, Phys. Lett. **B 260** (1991) 311; Imperial College Preprint TP/90-91/30; V. Aldaya and J. Navarro-Salas, Commun. Math. Phys. **126** (1990) 575.
- 15.- J.L. Gervais and A. Neveu, Nucl. Phys. **B224** (1983) 329; Nucl. Phys. **B238** (1984) 125. J.L. Gervais, Commun. Math. Phys. **130** (1990) 257.
- 16.- L. Faddeev and L. Takhtajan: *Liouville model on the lattice*. Lecture Notes in Physics, 246, 166-179. Berlin, Heidelberg, New York; Springer (1986)
- 17.- D.A. Eliezer and R. Woodard, Nucl. Phys. **B325** (1989) 389; J.Z. Simon, Phys. Rev **D41** (1990) 3720