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ONE-JET $O(\alpha_s^3)$ CROSS-SECTION: MASS DEPENDENCE

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Abstract

In this paper, we give a method to compute the partonic jet cross-section analytically to order α_s^3 , with a full dependence on the jet-cone size. The simplified case, where only the subprocess $gg \rightarrow ggg$ contributes, is considered. The jet is defined in such a way that a mass can be associated with it. We meet again with previous calculations in the limit where the jet-cone size goes to zero, and present some results on the mass dependence.

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1. Introduction

The precise study of the jet cross-section in hadronic collisions is worth while being undertaken for two reasons. First, because it provides a decisive test of parton-parton dynamics within the Standard Model and secondly because it may give evidence of new physics. Recently, two complete calculations of the $O(\alpha_s^3)$ corrections have been performed using different jet algorithms [1] [2]. In ref. [1], a jet with four-momentum $P(E, \vec{P})$ is defined in the polar angle-azimuthal angle plane as a set of any number of hadrons ($\vec{p}_i$) contained inside the cone of semi-aperture δ with $\vec{P} = \sum_i \vec{p}_i$; in ref. [2], on the other hand, a jet with components (E_T, η, ϕ) is defined in the pseudorapidity-azimuthal angle plane as a set of hadrons (E_{Ti}, η_i, ϕ_i) that verify $(\eta - \eta_i)^2 + (\phi - \phi_i)^2 < R^2$ and with $E_T = \sum_i E_{Ti}$. These two calculations describe quite well the UA2 and CDF data on the one-jet inclusive production and have led to the same conclusion, namely that the one-loop corrections improve the QCD predicting power sizeable introducing a dependence on the jet-cone size, which is missing at the Born level, thus reducing the theoretical uncertainty. Note that these results have been obtained thanks to the complete calculation of the $O(\alpha_s^3)$ matrix elements for all parton subprocesses done by Ellis and Sexton [3].

There are two contributions in the jet cross-section of ref. [1]: one corresponds to an analytical computation for a cone of semi-aperture $\delta \ll 1$; it has been done first by Furman [4] for quark-quark (of different flavours) scattering and extended to other subprocesses [5]. The other one corresponds to a numerical contribution, which takes care of the finite cone size. Surprisingly it has been shown that the approximation $\delta \ll 1$ (called Furman's approximation) is valid for a large range of δ (up to $\delta \simeq 0.7-0.8$). In addition, it gives an approximate partonic cross-section, which can be evaluated in much less time that it takes to do the full calculation. The aim of this paper is to present an analytical computation of the jet cross-section which is not limited to a half-opening angle $\delta \ll 1$. Its interest is twofold; it is much faster to do an evaluation with an analytical formula than by a numerical integration and it enables us to see more clearly what the dominant contributions to the cross-section or the terms to be re-summed are. We present results for the pure glue subprocess ($g + g \rightarrow g + g + g$) mainly at the partonic level, because our

goal more than to give phenomenological results (already obtained in ref. [1] and [2]), is to discuss the status of the small-cone approximation. The plan of the paper is as follows. In Section 2 we set our notations and describe the jet definition we have used. The explicit calculation is given in Sections 3 and 4. In Section 5 the consequences of our computation are discussed, and some results obtained at the hadronic level are given. We conclude in Section 6.

2. Notation and Formalism

We work in the partonic centre of mass, because the kinematics is simpler there, but the method we shall describe can be used in another frame (hadronic centre of mass, for instance), however with considerably more labour. It is sufficient for our purpose to stay in the partonic centre of mass.

We consider a conical jet of half-angle δ , with no limitation on δ except $0 < \delta < \pi/2$. The jet makes an angle θ with the beam axis, and we assume that $\delta < \theta < \pi - \delta$, this means that the initial direction cannot be contained within the cone. In order to simplify the computation, we change the definition of the jet as given by Furman [4]. For us, a jet is a set of any number of hadrons satisfying the two conditions:

- a) $P_{\text{jet}} = \sum_i P_i$ (we sum the four-momenta instead of the three-momenta);
- b) all $\vec{P}_i$ are contained within the cone of semi-aperture δ .

Note that we can now assign a mass to the jet:

$$M^2 \equiv E_{\text{jet}}^2 - \vec{P}_{\text{jet}}^2 = \left(\sum_i E_i \right)^2 - \left(\sum_i \vec{P}_i \right)^2; \quad (1)$$

M^2 will be maximum when all the hadrons are on the wall of the cone with the same energy E (we neglect all the hadron mass effects), i.e.:

$$M_{\text{max}}^2 = (NE)^2 - (NE \cos \delta)^2 = N^2 E^2 (1 - \cos^2 \delta), \quad (2)$$

where N is the number of hadrons inside the cone; in addition, since $NE \equiv E_{\text{jet}}$, $M_{\text{max}}^2 = E_{\text{jet}}^2 (1 - \cos^2 \delta)$. When we take the limit $\delta \ll 1$, $M^2 \simeq O(\delta^2)$ and we retrieve Furman's definition.

Following Furman's method, we divide the phase space into six regions. If $P_A + P_B \rightarrow P_1 + P_2 + P_3$ is the generic reaction, these regions are:

- 1) P_1 inside the cone, P_2 and P_3 outside,
- 2) P_1 and P_2 inside the cone,
- 3) P_2 inside the cone, P_1 and P_3 outside,
- 4) P_1 and P_3 inside the cone,
- 5) P_3 inside the cone, P_2 and P_1 outside,
- 6) P_2 and P_3 inside the cone.

Then we cut out those regions where there is one parton inside the cone (1, 3, 5) each into three subregions. More precisely, take for instance region 1; it is subdivided into:

- region a_1 : P_1 is the jet; P_2, P_3 are anywhere or one of the two is non-existent,
- region b_1 : P_1 is the jet; P_2 inside the cone,
- region c_1 : P_1 is the jet, P_3 is inside the cone.

Obviously, we have:

$$d\sigma(\text{region } t) = d\sigma(\text{region } a_1) - d\sigma(\text{region } b_1) - d\sigma(\text{region } c_1).$$

It is convenient to do this because $d\sigma(\text{region } a_1)$ is the one-parton (P_1) inclusive cross-section already computed in ref. [5].

Let us now define the kinematical variables we have used. We choose the partonic variables $w = -u/(s + t - M^2)$, $v = 1 + (t - M^2)/s$, where $s = (P_A + P_B)^2$, $t = (P_A - P_{\text{jet}})^2$, $u = (P_B - P_{\text{jet}})^2$, and $M^2 = P_{\text{jet}}^2$. The fact that $w = 1$ implies that $s + t + u = M^2$.

This choice has been dictated by the fact that our goal is to retrieve Furman's result when $\delta \ll 1$. In order to regularize infra-red and collinear divergences, we work in $n(= 4 - 2\epsilon)$ dimensions. So that any confusion due to the introduction of many variables can be avoided, let us describe our notation for the cross-section. The quantity $d\hat{\sigma}/dvdw dM^2$ is the partonic cross-section to order α_s^3 for $gg \rightarrow \text{jet}$, including all the jet-cone size dependence. This cross-section can be cut into two parts:

$$\frac{d\hat{\sigma}}{dvdw dM^2} = \frac{d\hat{\sigma}_F}{dvdw dM^2} + \frac{d\Delta\hat{\sigma}}{dvdw dM^2},$$

where $d\hat{\sigma}_F$ is the $O(\alpha_s^2)$ partonic cross-section computed in Furman's approximation and $d\Delta\hat{\sigma}$ is a correction that takes care of the finite size of the cone. In addition these cross-sections $d\hat{\sigma}, d\hat{\sigma}_F$, and $d\Delta\hat{\sigma}$ will be further divided into two contributions, corresponding

to the region with one parton inside the cone ($d\hat{\sigma}^{(1)}, d\hat{\sigma}_F^{(1)}, d\Delta\hat{\sigma}^{(1)}$) and to that with two partons inside the cone ($d\hat{\sigma}^{(2)}, d\hat{\sigma}_F^{(2)}, d\Delta\hat{\sigma}^{(2)}$).

3. Region with one parton inside the cone

Since $g + g \rightarrow g + g + g$ is completely symmetric with respect to all permutations of final four-momentum, $d\sigma(\text{region } 1) = d\sigma(\text{region } 3) = d\sigma(\text{region } 5)$. We also have $d\sigma(\text{region } b_1) = d\sigma(\text{region } c_1)$. Let us take the region c_1 such that P_3 is inside the cone and P_1 is the jet ($P_1 = P_{\text{jet}}$), which means that in this case $M^2 = 0$. The partonic cross-section is given by:

$$\begin{aligned} \frac{d\hat{\sigma}^{(1)}}{d^n P_{\text{jet}}} &= \int d^n P_1 \frac{d\hat{\sigma}^{(1)}}{d^n P_1} \delta^{(n)}(P_{\text{jet}} - P_1) \\ &= \frac{1}{8sV_\epsilon^2} \frac{1}{(2\pi)^{5-4\epsilon}} \int \frac{d^{n-1}P_3}{2E_3} \delta^+(P_{\text{jet}}^2) \delta^+[(P_A + P_B - P_{\text{jet}} - P_3)^2] |M|^2 \\ &= \frac{1}{8sV_\epsilon^2} \frac{1}{(2\pi)^{5-4\epsilon}} \frac{[sv(1-w)]^{-\epsilon} 2^{-2\epsilon} V(n-2)}{4} \delta(M^2) \int_0^1 d\chi \chi^{-\epsilon} (1-\chi)^{-\epsilon} \\ &\quad \times \int_0^1 d\eta \eta^{-\frac{1}{2}-\epsilon} (1-\eta)^{-\frac{1}{2}-\epsilon} |M|^2 \end{aligned} \quad (3)$$

$$(4)$$

where $V(n-2)$ is the volume of angular phase space in $n-2$ dimensions $V(n-2) = 2\pi^{1/2-\epsilon}/\Gamma(1/2-\epsilon)$, $V = 1/[1 + Rv(1-w)]$, $R = (1 - \cos\delta)/(1 + \cos\delta)$, and $V_\epsilon \equiv N^2 - 1$ is the dimensionality of the vector representation of the $SU(N)$ group, while $|M|^2$ is the matrix element of Ellis and Sexton (function D) [3]. With the help of REDUCE, by a cyclic use of the kinematical relation obtained from the conservation laws: $P_A + P_B = P_1 + P_2 + P_3$, we rewrite the matrix element as a sum of basic partial fractions: $1/P_i \cdot P_j, 1/(P_i \cdot P_j \times P_k \cdot P_\ell), (P_i \cdot P_j)^n/P_k \cdot P_\ell$, etc. Then we compute the corresponding phase-space integrals, some of which are collected in Appendix A. Inserting back these basic integrals in the REDUCE program and making simple algebra, we obtain the following result:

$$\begin{aligned} \frac{d\hat{\sigma}^{(1)}}{dvdw dM^2} &= K(\epsilon) \delta(M^2) \left\{ a \left(\frac{1}{\epsilon^2}, \frac{1}{\epsilon}, v, R \right) \delta(1-w) + b \left(\frac{1}{\epsilon}, v, R \right) \right. \\ &\quad \times \left. \frac{1}{(1-w)_+} + c(v) \left[\frac{\log(1-w)}{1-w} \right]_+ + d \left(\frac{1}{\epsilon}, v, w, R \right) \right\}, \end{aligned} \quad (5)$$

where $K(\epsilon)$ is an overall factor common to all cross-sections we have computed. By integrating over all 'azimuthal' angles in n dimensions we have obtained:

$$d^n P_{\text{jet}} = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{sv}{2} (svw(1-v) - M^2 v)^{-\epsilon} dM^2 dv dw.$$

Note that in eq. (5) all the δ dependence is in the term $R \equiv (1 - \cos \delta)/(1 + \cos \delta)$. Let us recall also that eq. (5) is an inelastic cross-section, i.e. we produce a massless jet plus two partons. We can now compare eq. (5) with the same calculation done with Furman's approximation (cf. Section 4 of ref. [4], or ref. [5]); this result has the following form:

$$\begin{aligned} \frac{d\hat{\sigma}_V^{(1)}}{dvdw dM^2} &= K(\epsilon)\delta(M^2) \left\{ a' \left[\frac{1}{\epsilon^2}, v, \log(\delta) \right] \delta(1-w) + b' \left[\frac{1}{\epsilon}, v, \log(\delta) \right] \right. \\ &\quad \times \left. \frac{1}{(1-w)_+} + c'(v) \left[\frac{\log(1-w)}{1-w} \right]_+ + d' \left[\frac{1}{\epsilon}, v, w, \log(\delta) \right] \right\}. \end{aligned} \quad (6)$$

It must be recalled that Furman's computation was done initially in the hadronic centre of mass ($x_A \vec{P}_A + x_B \vec{P}_B = 0$) and the only dependence on δ is via a $\log(\delta^2 E^2/s)$, where E is the jet energy. To go to the partonic centre of mass, we put $x_A = x_B = 1$ and accordingly $E = \sqrt{s}/2(1-v+vw)$, so that $\log(\delta^2 E^2/s)$ becomes $\log(\delta^2/4) + 2\log(1-v+vw)$. The correction term $d\Delta\hat{\sigma}^{(1)}/dvdw dM^2$, which is just the difference: $d\hat{\sigma}^{(1)}/dvdw dM^2 - d\hat{\sigma}_V^{(1)}/dvdw dM^2$, can be written:

$$\begin{aligned} \frac{d\Delta\hat{\sigma}^{(1)}}{dvdw dM^2} &= K(\epsilon)\delta(M^2) \\ &\quad \left\{ \Delta a \left(\frac{1}{\epsilon}, v, \delta \right) \delta(1-w) + \Delta b(v, \delta) \frac{1}{(1-w)_+} + \Delta d(v, w, \delta) \right\} \end{aligned} \quad (7)$$

Several remarks are in order:

The dependence on δ is more complicated; schematically we have:

$$\begin{aligned} \Delta a \left(\frac{1}{\epsilon}, v, \delta \right) &= f_a \left(\frac{1}{\epsilon}, v, R \right) + g_a \left(\frac{1}{\epsilon}, v \right) \left[\log(R) - \log \left(\frac{\delta^2}{4} \right) \right] \\ \Delta b(v, \delta) &= f_b(v, R) + g_b(v) \left[\log(R) - \log \left(\frac{\delta^2}{4} \right) \right] \\ \Delta d(v, w, \delta) &= f_d(v, w, R) + g_d(v, w) \left[\log(R) - \log \left(\frac{\delta^2}{4} \right) \right] \end{aligned}$$

The functions f_a, f_b, f_d go to zero like δ^2 when $\delta \rightarrow 0$ ($R \rightarrow 0$).

As expected, the $1/\epsilon$ divergences associated with initial partons as the $1/\epsilon^2$ singularities disappear in eq. (7) because they do not depend on the cone size: the left-over $1/\epsilon$ terms are associated with final partons.

In eq. (7), there is no term proportional to $[\log(1-w)/(1-w)]_+$, because $c(v) = c'(v)$. When $\delta \ll 1$ ($R \simeq \delta^2/4$), all leading terms in δ cancel, and we have $d\Delta\hat{\sigma}^{(1)} \simeq O(\delta^2)$.

4. Region with two partons inside the cone

Because of the symmetry of $g+g \rightarrow g+g+g$, $d\hat{\sigma}(\text{region } 2) = d\hat{\sigma}(\text{region } 4) = d\hat{\sigma}(\text{region } 6)$. We choose region 2, in which P_1 and P_2 are in the cone. In this case $P_{jet} = P_1 + P_2$ and $M^2 = (P_1 + P_2)^2 \neq 0$, and the computations are more involved. We start with:

$$\begin{aligned} \frac{d\hat{\sigma}^{(2)}}{dv dP_{jet}} &= \int d^n q \frac{d\hat{\sigma}^{(2)}}{d^n q} \delta^n(P_{jet} - q) \quad \text{with} \quad q \equiv P_1 + P_2 \\ &= \frac{1}{8V_c^2 s} \frac{1}{(2\pi)^{5-4\epsilon}} \delta(s+t+u-M^2) \int d^n P_1 d^n P_2 \delta^+(P_1^2) \\ &\quad \times \delta^+(P_2^2) \delta^{(n)}(P_{jet} - P_1 - P_2) |M|^2. \end{aligned} \quad (8)$$

Since we produce a massive jet plus one parton, eq. (8) is an elastic (in the sense that $s+t+u = M^2$) cross-section. Changing the variables we obtain:

$$\frac{d\hat{\sigma}^{(2)}}{dv dw dM^2} = \frac{1}{16V_c^2 s} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{[svw(1-v)-M^2v]^{-\epsilon}}{(2\pi)^{5-4\epsilon}} \delta(1-w) I \otimes |M|^2,$$

where $I \otimes |M|^2$ denotes

$$\int d^n P_1 d^n P_2 \delta^+(P_1^2) \delta^+(P_2^2) \delta^{(n)}(P_{jet} - P_1 - P_2) |M|^2.$$

As in Section 3, we expand $|M|^2$ in partial fractions and then compute the basic integrals (see Appendix B). We obtain:

$$\begin{aligned} \frac{d\hat{\sigma}^{(2)}}{dv dw dM^2} &= K(\epsilon) \delta(1-w) \left\{ A \left(\frac{1}{\epsilon^2}, v, R \right) \delta(M^2) \right. \\ &\quad \left. + B(v, R) \frac{1}{(M^2)_B} + C(v) \left[\frac{\log(M^2/B)}{M^2} \right]_B + D(v, M^2, R) \right\}, \end{aligned} \quad (9)$$

where the 'B distributions' are defined as:

$$\int_0^B dM^2 f(M^2) \frac{1}{(M^2)_B} \equiv \int_0^B dM^2 [f(M^2) - f(0)] \frac{1}{M^2}$$

and

$$\int_0^B dM^2 f(M^2) \left[\frac{\log(M^2/B)}{M^2} \right]_B \equiv \int_0^B dM^2 \frac{[f(M^2) - f(0)]}{M^2} \log \left(\frac{M^2}{B} \right);$$

B is the upper limit of the M^2 integration range; in the partonic centre of mass, $B = Rs$. Note that there are singularities only for the term in $\delta(M^2)$ as we could expect: if they

appeared elsewhere, there would be no means of cancelling them. This is a non-trivial consistency check in the calculation since intermediate steps involve these singularities which, in the end, get cancelled. Furman's result has the following form:

$$\frac{d\hat{\sigma}_F^{(2)}}{dvdw dM^2} = K(\epsilon)\delta(1-w)\delta(M^2)A'\left[\frac{1}{\epsilon^2}, \frac{1}{\epsilon}, v, \log(\delta)\right]. \quad (10)$$

The corrective term has the following form:

$$\begin{aligned} \frac{d\Delta\hat{\sigma}^{(2)}}{dvdw dM^2} &\equiv \frac{d\hat{\sigma}^{(2)}}{dvdw dM^2} - \frac{d\hat{\sigma}_F^{(2)}}{dvdw dM^2} = K(\epsilon)\delta(1-w)\left\{\delta(M^2)\Delta A\left(\frac{1}{\epsilon}, v, \delta\right)\right. \\ &\quad \left.+ B(v, R)\frac{1}{(M^2)_B} + C(v)\left[\frac{\log(M^2/B)}{M^2}\right]_B + D(v, M^2, R)\right\} \end{aligned} \quad (11)$$

The remaining $1/\epsilon$ singularities in ΔA are associated with final partons.

$$\Delta A\left(\frac{1}{\epsilon}, v, \delta\right) = h_A\left(\frac{1}{\epsilon}, v, R\right) + k_A\left(\frac{1}{\epsilon}, v\right)\left[\log(R) - \log\left(\frac{\delta^2}{4}\right)\right] + \alpha_A(v),$$

where $h_A(1/\epsilon, v, R) \simeq \delta^2$ when $\delta \rightarrow 0$. So the $1/\epsilon$ terms are of order δ^2 when $\delta \ll 1$, as in the previous case. Since $M^2 \leq Rs$, to take the limit $\delta \ll 1$, we have first to integrate eq. (11) over M^2 :

$$\int_0^B dM^2 \frac{d\Delta\hat{\sigma}^{(2)}}{dvdw dM^2} = K(\epsilon)\delta(1-w)\left\{\Delta A\left(\frac{1}{\epsilon}, v, \delta\right) + F(v, R)\right\} \quad (12)$$

with

$$\begin{aligned} F(v, R) &= \int_0^B dM^2 \left[B(v, R)\frac{1}{(M^2)_B} + C(v)\left(\frac{\log(M^2/B)}{M^2}\right) + D(v, M^2, R) \right] \\ &= \int_0^B dM^2 D(v, M^2, R) \end{aligned}$$

Naively, one could expect that, when $\delta \ll 1$, $F(v, R) = O(\delta^2)$, since in this limit $B \simeq s\delta^2/4$, but it is not so (see Appendix C). In fact, when $\delta \ll 1$, $F(v, R) = -\alpha_A(v) + O(\delta^2)$, which exactly cancels constant term in ΔA so that

$$\int_0^B dM^2 \frac{d\Delta\hat{\sigma}^{(2)}}{dvdw dM^2} \simeq O(\delta^2)$$

We can note that, from a purely mathematical point of view, the calculation in Furman's approximation [cf. ref. [4], eq. (4.9)] is not simple, since we have to deal with a Taylor expansion of distributions; it is thus interesting to perform the complete calculation to check it.

5. Discussion

5.1 Partonic level

Now, we can add the two contributions $d\Delta\sigma^{(1)}$ and $d\Delta\sigma^{(2)}$. This sum is free of singularities, and it can be written:

$$\begin{aligned} \frac{d\Delta\hat{\sigma}}{dvdw dM^2} &= K(0)\left\{\delta(1-w)\delta(M^2)\alpha(v, \delta) + \delta(M^2)\right. \\ &\quad \times \left[\Delta b(v, \delta)\frac{1}{(1-w)_+} + \Delta d(v, w, \delta)\right] + \delta(1-w) \\ &\quad \times \left\{B(v, R)\frac{1}{(M^2)_B} + C(v)\left[\frac{\log(M^2/B)}{M^2}\right]_B + D(v, M^2, R)\right\} \end{aligned} \quad (13)$$

with

$$\alpha(v, \delta) = \Delta A\left(\frac{1}{\epsilon}, v, \delta\right) + \Delta a\left(\frac{1}{\epsilon}, v, \delta\right) \quad \text{and} \quad K(0) = \frac{\alpha_s^2(\mu^2)}{8V_s^2},$$

where $\alpha_s(\mu^2)$ is the two-loop running coupling constant evaluated at the renormalization scale μ^2 .

The partonic cross-section to order $O(\alpha_s^3)$ is:

$$\frac{d\hat{\sigma}}{dvdw dM^2} = \frac{d\hat{\sigma}_F}{dvdw dM^2} + \frac{d\Delta\hat{\sigma}}{dvdw dM^2}.$$

At this level, $d\hat{\sigma}$ contains $1/\epsilon$ terms (associated with initial partons) that come from $d\hat{\sigma}_F$. The method to handle these left-over singularities is well known. These collinear singularities are absorbed into the bare distribution functions $F_g^H(x)$ to yield the evolved structure function $F_g^H(x, Q^2)$. Since we consider only one subprocess ($gg \rightarrow ggg$), we can separate explicitly the finite part and the divergent one:

$$\begin{aligned} \frac{d\hat{\sigma}_F}{dvdw dM^2} &= K_F(s, v, w, \mu^2, Q^2, \delta)\delta(M^2) + \frac{1}{v}H_{gg}(w)\left(\frac{d\sigma^0}{dv}\right)_{gg \rightarrow gg}(ws, v) \\ &\quad \delta(M^2) + \frac{1}{1-vw}H_{gg}\left(\frac{1-v}{1-vw}\right) \\ &\quad \times \left(\frac{d\sigma^0}{dv}\right)_{gg \rightarrow gg}\left(\frac{1-v}{1-vw}s, vw\right)\delta(M^2) \end{aligned} \quad (14)$$

where

$$H_{gg}(y) = P_{gg}(y)\left(-\frac{1}{\epsilon} - \log(4\pi) + \gamma_E\right) + P_{gg}(y)\log\left(\frac{Q^2}{\mu^2}\right) + f_{gg}(y).$$

The function $P_{gg}(y)$ and $f_{gg}(y)$ are the Altarelli-Parisi kernels and finite $O(\alpha_s)$ corrections to structure functions, respectively (for more details and for different choices of f_{gg} ,

see ref. [5]); $(d\sigma^0/dv)_{gg \rightarrow gg}(s, v)$ is the Born cross-section for the $gg \rightarrow gg$ subprocess; $K_F(s, v, w, \mu^2, Q^2, \delta)$ is free of singularities.

We now define $d\hat{\sigma}_{\text{finite}}/dvdw dM^2$, the finite part of the $O(\alpha_s^3)$ cross-section, as:

$$\begin{aligned} \frac{d\hat{\sigma}_{\text{finite}}}{dvdw dM^2} &= \frac{d\hat{\sigma}}{dvdw dM^2} - \frac{1}{v} H_{gg}(w) \left(\frac{d\sigma^0}{dv} \right)_{gg \rightarrow gg}(ws, v) \delta(M^2) \\ &\quad + \frac{1}{1-vw} H_{gg} \left(\frac{1-v}{1-vw} \right) \left(\frac{d\sigma^0}{dv} \right)_{gg \rightarrow gg} \\ &\quad \times \left(\frac{1-v}{1-vw} s, vw \right) \delta(M^2) \\ &= K_F(s, v, w, \mu^2, Q^2, \delta) \delta(M^2) + \frac{d\Delta\hat{\sigma}}{dvdw dM^2}. \end{aligned} \quad (15)$$

Now, we can discuss the validity of Furman's approximation. The quantity Z is defined by:

$$1 - Z \equiv \frac{\int_{w_{\min}}^s dw \int_0^B dM^2 K_F(s, v, w, \mu^2, Q^2, \delta) \delta(M^2)}{\int_{w_{\min}}^s dw \int_0^B dM^2 [d\Delta\hat{\sigma}/dvdw dM^2 + K_F(s, v, w, \mu^2, Q^2, \delta) \delta(M^2)]} \quad (16)$$

In order to avoid numerical problems, we take $w_{\min} \simeq R$. To fix ideas, we choose: $\sqrt{s} = 630$ GeV, $Q^2 = \mu^2 = s, \theta_{jet} = 90^\circ [v = (s - M^2)/(s(1 + w))]$. In Table I, we have plotted C against δ , and we can see that Furman's approximation is valid up to $\delta \simeq 0.7$ at $\pm 10\%$. This is mainly due to the fact that R behaves like $\delta^2/4$ when $\delta \ll 1$. Let us recall that in the jet cross-section, the uncertainties, coming from a poor knowledge of structure functions and scale variation, amount to around 30% (larger than the errors due to Furman's approximation in the range $0 < \delta \lesssim 0.7$). We have checked that the validity of the approximation is the same if we change the values of s, v, Q^2, μ^2 .

A funny thing can be noticed. If we change the upper bound of the M^2 integration range from B to $M_{\max}^2$, it can take a value for which $Z = 0$. This means that Furman's approximation describes exactly the $O(\alpha_s^3)$ jet cross-section integrated over the jet mass between 0 and $M_{\max}^2$, or in other words that $\int_{w_{\min}}^s dw \int_0^{M_{\max}^2} dM^2 (d\Delta\hat{\sigma}/dvdw dM^2) = 0$. The value $M_{\max}^2$ depends on $\delta, v, s, w_{\min}$ and certainly on the considered subprocess. Since the number of terms in $d\Delta\hat{\sigma}/dvdw dM^2$ is enormous, it is difficult to predict the value of $M_{\max}^2$ analytically.

A final remark can be made. In eq. (13), logarithmic terms such as $\log(M^2/sR)$ can be large if M^2 is much smaller than s . In this case a new scale is present in the problem,

Table I

δ	Z
1×10^{-8}	$1 \cdot 10^{-10}$
0.1	0.0018
0.2	0.0093
0.3	0.025
0.4	0.049
0.5	0.081
0.6	0.121
0.7	0.167
0.8	0.217
0.9	0.272
1	0.332

and terms of order $1/M^2 \alpha_s^3(M^2) \log^m(M^2/Rs) (m \leq 2n - 1)$ must be summed. In order to obtain more reliable predictions, this re-summation can be done in the framework of the double-logarithmic approximation (DLA: $m = 2n - 1$) [6]. The problem is similar to the transverse-momentum distribution of vector bosons in hadron-hadron collisions where the terms to be summed are large and of the type $1/q_T^2 \alpha_s^3(q_T^2) \log^m(Q^2/q_T^2)$ (Q is the vector-boson mass and q_T its transverse momentum) [7]. Unfortunately, in our case having M^2 small implies that the jet is very thin, in which case we must take the hadronization effects into account.

5.2 Hadronic level

Except in the e^+e^- case, the partonic centre of mass (c.m.) does not coincide with the experimental frame; we thus have to apply a Lorentz boost to the cone, which then becomes distorted. Indeed, the definition of the jet is Lorentz-invariant since we have obtained it by summing four-momenta; the definition of the cone is not, since we used azimuthal and polar angles to obtain it. Let β , the parameter of the Lorentz transform to go from the partonic c.m. to hadronic c.m.; β is related to the fraction of momentum x_A, x_B carried by the two initial partons by $|\beta| = |x_A - x_B|/(x_A + x_B)$. In order to do some calculations in the hadronic c.m., we use the following trick. Suppose that in the hadronic c.m., we have a cone of semi-aperture δ^* centred on 90° . For a given Lorentz transformation (a set x_A, x_B), we work, in the partonic c.m., with an effective circular cone of semi-aperture δ such that: $\tan^2(\delta) = (1 - \beta^2) \tan^2(\delta^*)$. It is a rough approximation,

because this relation is only valid for the longitudinal size of the cone, or in other words, we do not take into account the fact that the cone is not circular in the partonic c.m. At the hadronic level, the cross-section is given by:

$$\frac{d\sigma}{d^4P_{\text{jet}}} = \frac{2}{\pi\bar{S}} \int_{v_{\text{min}}}^{v_{\text{max}}} \frac{dv}{1-v} \int_{w_{\text{min}}}^1 \frac{dw}{w} F_g^{H_1}(x_A, Q^2) F_g^{H_2}(x_B, Q^2) \times \left[\frac{1}{v} \left(\frac{d\hat{\sigma}^0}{dv} \right)_{g \rightarrow g} (s, v) \delta(1-w) \delta(M^2) + \frac{1}{v} \frac{d\Delta\hat{\sigma}}{dv} \right] \quad (17)$$

where:

$$x_A = \frac{(1-V)(SVW + M^2) - (1-v)M^2}{(1-V)Sv}, \quad x_B = \frac{1-V}{1-v}, \quad v_{\text{max}} = V, \\ v_{\text{min}} = \frac{(1-V)(SVW + M^2) - M^2}{S(1-V) - M^2}, \quad w_{\text{min}} = \frac{(1-V)(SVW + M^2) - (1-v)M^2}{(1-V)Sv}$$

The hadronic variables V and W are defined below:

$$V = 1 + \frac{T - M^2}{S}, \quad W = \frac{-U}{S + T - M^2}$$

where

$$S = (K_A + K_B)^2, \quad T = (K_A - P_{\text{jet}})^2, \quad U = (K_B - P_{\text{jet}})^2$$

In eq. (17), $d\hat{\sigma}^0$ is the Born cross-section, F_g^H is the distribution function of the gluon inside the hadron H .

We present in Table II the mass distribution $(d\sigma/dM^2 dy)_{y=0}$ because it is a prediction from a next-to-leading calculation. At the Born level, there is no mass dependence. We give $(d\sigma/dM^2 dy)_{y=0}$ as a function of M for $\sqrt{S} = 630$ GeV, $\delta = 0.7$, and $\mu^2 = Q^2 = M^2$, using the set of structure functions of Duke-Owens (set 1) [8]. The cross-section falls rapidly with M . Since, for $M^2 \neq 0$, there are no divergences, we have constructed a program that integrates numerically the matrix elements over the phase space directly in the hadronic c.m. We can thus test the approximation that we made to go from the partonic c.m. to hadronic c.m.; the errors are around 30%-40%. These errors may seem to be important, but two things have to be kept in mind: First, there are no experimental data on this type of distribution, and even if they existed we have calculated only one

Table II

M	$(d\sigma/dM^2 dy)_{y=0}(\text{pb/GeV})$
25	2.05
50	2.38×10^{-3}
75	1.22×10^{-5}

subprocess, so that it is impossible, up to now, to compare these results with experiments. This justifies our rough approximation.

Secondly, we recall that it is possible to compute directly the partonic cross-section in the hadronic c.m., using the same method, although with much more labour because of the more complicated kinematics. But the game may be worth the candle if data exist.

6. Conclusion

We have developed a new method to compute, the one-jet cross-section analytically as far as possible, by giving an explicit mass to the jet. We meet again with old results [4], [5] when the jet-cone size goes to zero. It is an important check of the two calculations. It justifies also the surprising result found in [1], i.e. Furman's approximation is valid over a wide range of δ ($\delta \lesssim 0.7$). We hope that this method will be applied in other reactions involving jets, since the physics can be understood more easily through an analytical expression than through a numerical calculation.

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Appendix A: One parton inside the cone

We do not want to give all the integrals, for this would be boring. Instead, we try to explain the method we used for their computation.

As in section 3, we stay in the region c_1 , i.e. for $P_A + P_B \rightarrow P_1 + P_2 + P_3$, $P_1 \equiv P_{jet}$ and P_3 is inside the cone. In a $n(=4-2\epsilon)$ -dimensional space-time, we orient the vectors $\vec{P}_1, \vec{P}_A, \vec{P}_B, \vec{P}_3$ so that:

$$\begin{aligned} P_1 &= E_1(1, 1, 0, 0, \dots) \\ P_A &= \frac{\sqrt{s}}{2}(1, \cos \alpha, \sin \alpha, 0, \dots) \\ P_B &= \frac{\sqrt{s}}{2}(1, -\cos \alpha, -\sin \alpha, 0, \dots) \\ P_3 &= E_3(1, \cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \dots). \end{aligned}$$

The three-particle phase space may be written as:

$$(PS)_3 = \int \frac{d^n P_1}{(2\pi)^{n-1}} \frac{d^n P_2}{(2\pi)^{n-1}} \frac{d^n P_3}{(2\pi)^{n-1}} \delta^n(P_A + P_B - P_1 - P_2 - P_3) \delta^+(P_1^2) \delta^+(P_2^2) \delta^+(P_3^2) \quad (A1)$$

Defining $\chi = (\sqrt{s} - 2E_3)/2E_1$ and $\eta = (1 - \cos \theta_2)/2$ and using the variables v and w given by $v = 1 + t/s, w = -u/(t+s)$ with $t = (P_A - P_1)^2$ and $u = (P_B - P_1)^2$ (the jet is massless in this case), we obtain:

$$(PS)_3 = \frac{[sv(1-w)]^{-\epsilon} 2^{-2\epsilon}}{(2\pi)^{5-4\epsilon}} \frac{V(n-2)}{8} \int \frac{d^{n-1} \vec{P}_1}{E_1} \int_{\mathcal{V}} d\chi \chi^{-\epsilon} (1-\chi)^{-\epsilon} \times \int_0^1 d\eta \eta^{-\frac{1}{2}-\epsilon} (1-\eta)^{-\frac{1}{2}-\epsilon}, \quad (A2)$$

where

$$\mathcal{V} = \frac{1}{1 + Rv(1-w)} \quad \text{with} \quad R \equiv \frac{1 - \cos \delta}{1 + \cos \delta} \quad \text{and} \quad V(n-2) = \frac{2\pi^{1/2-\epsilon}}{\Gamma(1/2-\epsilon)}.$$

From (A2), it is easy to obtain eq. (4).

Now, it is convenient to rewrite the element $|M|^2$: $|M|^2 = 1/P_2 \cdot P_3 \times H(P_A, P_B, P_1, P_2, P_3) = 2[sv(1-w)]H(P_A, P_B, P_1, P_2, P_3)$.

This enables us to write eq. (4) as:

$$\frac{d\hat{\sigma}^{(1)}}{d^n P_{jet}} = F_A(\epsilon)(1-w)^{-1-2\epsilon} J_A \otimes H(P_A, P_B, P_1, P_2, P_3) \delta(M^2) \quad (A3)$$

where $J_A \otimes$ stands for:

$$(1-w)^\epsilon \int_0^1 d\chi \chi^{-\epsilon} (1-\chi)^{-\epsilon} \int_0^1 d\eta \eta^{-\frac{1}{2}-\epsilon} (1-\eta)^{-\frac{1}{2}-\epsilon}$$

and $F_A(\epsilon)$ is a regular function of ϵ :

$$F_A(\epsilon) = \frac{(sv)^{-\epsilon}}{4s^2 v V_c^2 (2\pi)^{5-4\epsilon}}.$$

We have explicitly exposed in eq. (A3) the soft singularity $(1-w)^{-1-2\epsilon}$, which occurs at $w = 1$, and $J_A \otimes H(P_A, P_B, P_1, P_2, P_3)$ which is a regular function of w . Then we can use the identity, which is valid to order ϵ :

$$(1-w)^{-1-2\epsilon} = -\frac{1}{2\epsilon} \delta(1-w) + \frac{1}{(1-w)_+} - 2\epsilon \left[\frac{\log 1-w}{1-w} \right]_+ \quad (\text{A4})$$

Schematically, the function H can be developed in the following way:

$$\begin{aligned} H(P_A, P_B, P_1, P_2, P_3) = & \sum_{i,j=1,A,B;i \neq j} \left\{ a_i \frac{1}{P_3 \cdot P_i} + a'_i \frac{1}{P_2 \cdot P_i} + b_{ij} \frac{1}{P_3 \cdot P_i \cdot P_j} \right. \\ & + b'_{ij} \frac{1}{P_3 \cdot P_i \cdot P_2 \cdot P_j} + b''_{ij} \frac{1}{P_2 \cdot P_i \cdot P_2 \cdot P_j} \\ & \left. + \sum_{n=1}^3 \left[C_{ij}(n) \frac{(P_3 \cdot P_i)^n}{P_3 \cdot P_j} + C'_{ij}(n) \frac{(P_2 \cdot P_i)^n}{P_2 \cdot P_j} \right] \right\}. \quad (\text{A5}) \end{aligned}$$

Then, we compute the basic integrals. Let us begin with the simplest non-trivial one, where we have redefined the variable $\chi = 1 - (1-b')t$. We then obtain:

$$J_A \otimes \frac{1}{P_3 \cdot P_1} = \frac{1}{s(1-v+vw)} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{1-b'}{1-w} \right)^{-\epsilon} \int_0^1 dt t^{-1-\epsilon} [1 - (1-b')t]^{-\epsilon}.$$

Performing the integration, we get:

$$J_A \otimes \frac{1}{P_3 \cdot P_1} = \frac{1}{s(1-v+vw)} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{1-b'}{1-w} \right)^{-\epsilon} \cdot B(-\epsilon, 1) F(\epsilon, -\epsilon; 1-\epsilon; 1-b') \quad (\text{A6})$$

where F is the ordinary hypergeometric function. Expanding in ϵ we get:

$$J_A \otimes \frac{1}{P_3 \cdot P_1} = \frac{1}{s(1-v+vw)} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{1-b}{1-w} \right)^{-\epsilon} \left(-\frac{1}{\epsilon} \right) [1 - \epsilon^2 \ln_2(1-b')]. \quad (\text{A6})$$

It is useless to expand $\pi^{1-\epsilon}/\Gamma(1-\epsilon)[(1-b')/(1-w)]^{-\epsilon}$, because we will find this factor in front of each integral and will then be able to factorize it out. This integral was

simple because the integrand did not depend on η . More involved is the integral of the type $J_A \otimes 1/P_3 \cdot P_B$, because now $P_3 \cdot P_B = s[2(1-v+vw)][\alpha(\chi) - \beta(\chi)\eta]$ with:

$$\begin{aligned} \alpha(\chi) &= 1-v+\chi[-1+v+v^2w(1-w)]+2v\sqrt{w(1-w)}\sqrt{\chi(1-\chi)} \\ \beta(\chi) &= 4v\sqrt{w(1-v)(1-w)}\sqrt{\chi(1-\chi)}. \end{aligned}$$

Since all the integrals on η are of the same type, it will be helpful to have a generic formula. We define:

$$G(a, b, c, \chi) \equiv \int_0^1 d\eta \eta^{-\frac{1}{2}-\epsilon} (1-\eta)^{-\frac{1}{2}-\eta} [\alpha(\chi) - \beta(\chi)\eta]^{-1}$$

$$\text{with: } \begin{cases} \alpha(\chi) = a\chi + b + 2c\sqrt{\chi(1-\chi)} & \text{and } \frac{\beta(\chi)}{\alpha(\chi)} \leq 1. \\ \beta(\chi) = 4c\sqrt{\chi(1-\chi)} \end{cases}$$

It is simple to compute G : we get a hypergeometric function and, after some manipulations on its arguments, we obtain,

$$\begin{aligned} - \text{ for } b/(a+2b) \leq \chi_{\min} \\ G(a, b, c, \chi) &= B\left(\frac{1}{2} - \epsilon, \frac{1}{2} - \epsilon\right) [\chi(a+b)]^{2\epsilon} [(a+2b)\chi - b]^{-1-2\epsilon} \\ &\quad \times F\left[-\epsilon, -2\epsilon; 1-\epsilon; \frac{b(1-\chi)}{(a+b)\chi}\right]; \quad (\text{A7}) \\ - \text{ for } b/(a+2b) \geq \chi_{\max}: \end{aligned}$$

$$\begin{aligned} G(a, b, c, \chi) &= B\left(\frac{1}{2} - \epsilon, \frac{1}{2} - \epsilon\right) [(b(1-\chi))^{2\epsilon} [b - \chi(a+2b)]^{-1-2\epsilon} \\ &\quad \times F\left[-\epsilon, -2\epsilon; 1-\epsilon; \frac{(a+b)\chi}{b(1-\chi)}\right], \quad (\text{A8}) \end{aligned}$$

where $\chi_{\max}$ and $\chi_{\min}$ are the upper and lower bound of the integral over χ ; let us recall that χ is related to E_3 , so it is not possible that there is a pole for a value of $E_3 \neq 0$, i.e. that $b/(a+2b) \notin [\chi_{\min}, \chi_{\max}]$. A similar calculation has been done in ref. [9]. Once we have checked that $\beta(\chi)/\alpha(\chi) \leq 1$ and $b/(a+2b) \leq \chi_{\min} \equiv b'$ in our case, we can integrate over η and we obtain:

$$\begin{aligned} J_A \otimes \frac{1}{P_3 \cdot P_B} &= \frac{(1-v+vw)}{s} \frac{\pi^{1-\epsilon}}{\Gamma(1-2\epsilon)} [(v^2w(1-w))^\epsilon \\ &\quad \times \int_0^1 d\chi \chi^\epsilon (1-\chi)^{-\epsilon} \{ \chi[1-v+v^2w(1-w)] - (1-v) \}^{-1-2\epsilon} \\ &\quad \times F\left[-\epsilon, -2\epsilon; 1-\epsilon; \frac{(1-v)(1-\chi)}{v^2w(1-w)\chi}\right]. \quad (\text{A9}) \end{aligned}$$

Working to order ϵ , we can take $F(-\epsilon, -2\epsilon; 1 - \epsilon; P) \simeq 1 + 2\epsilon^2 \text{li}_2(P) \simeq 1$. In fact we can see that $1/P_3 \cdot P_B$ diverges only when $E_3 \rightarrow 0$ (i.e. when $w \rightarrow 1$), so that the $1/\epsilon$ singularity related to this infra-red divergence will come from $(1 - w)^{-1-2\epsilon}$, and we thus expect that $J_A \otimes 1/P_3 \cdot P_B \simeq \alpha + \epsilon\beta$ (i.e. is regular when $\epsilon \rightarrow 0$). In $J_A \otimes 1/P_3 \cdot P_B$, we can safely drop all powers of ϵ greater than 1.

By making the change of variable $\chi = 1 - (1 - b')t$ and defining $A = [(1 - b')(1 - vw)(1 - v + vw)]/[v^2 w(1 - w)]$, eq. (A9) can be written:

$$J_A \otimes \frac{1}{P_3 \cdot P_B} = \frac{(1 - v + vw)}{s} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{1-b'}{1-w} \right)^{-\epsilon} \frac{(1-b')}{v^2 w(1-w)} \times \int_0^1 dt t^{-\epsilon} [1 - (1-b')t]^\epsilon (1 - At)^{-1-2\epsilon}. \quad (\text{A10})$$

Since $-\cos \alpha < \cos \delta < \cos \alpha$, we know that $A < 1$, i.e. there are no divergences in the integral (as expected); so we can expand the integrand in ϵ and we get:

$$J_A \otimes \frac{1}{P_3 \cdot P_B} = \frac{1}{s(1-vw)} \frac{\pi^{1-\epsilon}}{\Gamma(1-2\epsilon)} \left(\frac{1-b'}{1-w} \right)^{-\epsilon} \times \left\{ -\log(1-A) + \epsilon [\text{li}_2(A) + \log^2(1-A)] \right\}. \quad (\text{A11})$$

From eq. (A11), we easily obtain $J_A \otimes 1/P_3 \cdot P_A$ by changing $v \rightarrow 1 - vw$ and $w \rightarrow (1 - v)/(1 - vw)$. Using the same method, we get without difficulty $J_A \otimes 1/P_2 \cdot P_i$ ($i = A, B, 1$). For integrals with two terms in the denominator $J_A \otimes 1/(K \cdot P_i \times K \cdot P_j)$ ($K = P_3, P_2$ and $i = A, B, 1$), there are two classes. First, if $K \cdot P_i$ and $K \cdot P_j \neq P_3 \cdot P_1$, by using partial fractioning we fall on the same type of integral as for $J_A \otimes 1/P_3 \cdot P_B$, and there are no more complications. Secondly, if $K \cdot P_i$ or $K \cdot P_j = P_3 \cdot P_1$, we can do the integration over η , but we have to be careful for the integral on χ : there, we cannot drop terms of order ϵ^2 . But these terms will be multiplied by $\delta(1 - w)$ and fortunately they cancel when $w = 1$; the integration on χ is thus the same as in the preceding case. For integrals with only one term in the denominator, such as $J_A \otimes (K \cdot P_i)^n / K \cdot P_j$, it is a matter of patience, the method being the same as in the case of $J_A \otimes 1/P_3 \cdot P_B$. After a straightforward integration over η , we are left with two cases:

- either $K \cdot P_j \neq P_3 \cdot P_1$; in this case we can drop all ϵ^2 terms and deal with simple integrals;

- or $K \cdot P_j = P_3 \cdot P_1$; we have a $1/\epsilon$ singularity coming from the integration over χ , but the integration over η is trivial, i.e. it does not generate complicated functions of χ (only polynomes).

Appendix B: Two partons inside the cone

Now, we have P_1 and P_2 inside the cone and $P_1 + P_2 = P_{jet}$. The cone is centred on $\vec{P}_{jet}$.

We orient our axis so that:

$$\begin{aligned} P_{jet} &= (E_{jet}, F_{jet}, 0, 0, \dots), \\ P_1 &= (E_1, E_1 \cos \alpha, E_1 \sin \alpha \vec{A}), \\ P_2 &= (E_2, E_2 \cos \beta, E_2 \sin \beta \vec{B}). \end{aligned}$$

where the unit vectors $\vec{A}$ and $\vec{B}$ vary freely in the $(n-2)$ dimensional subspace perpendicular to $\vec{P}_{jet}$ ($F_{jet} = |\vec{P}_{jet}|$). Starting from eq. (8), we define:

$$\begin{aligned} I &\equiv \int d^n P_1 d^n P_2 \delta^+(P_1^2) \delta^+(P_2^2) \delta^{(n)}(P_{jet} - P_1 - P_2) \\ &= \frac{1}{4} \int E_1^{n-3} dE_1 \int E_2^{n-3} dE_2 \delta(E_{jet} - E_1 - E_2) \int_{cone} d\Omega_{n-1}(\vec{A}) \\ &\quad \times \int_{cone} d\Omega_{n-1}(\vec{B}) \delta^{(n-1)}(\vec{P}_{jet} - \vec{P}_1 - \vec{P}_2). \end{aligned} \quad (B1)$$

The angular integration can be written:

$$\begin{aligned} &\int_0^\delta d\alpha \int_0^\delta d\beta (\sin \alpha \sin \beta)^{n-3} \int d\Omega_{n-1}(\vec{A}) \int d\Omega_{n-1}(\vec{B}) \\ &\times \delta(F_{jet} - E_1 \cos \alpha - E_2 \cos \beta) \delta^{(n-2)}(E_1 \sin \alpha \vec{A} + E_2 \sin \beta \vec{B}). \end{aligned} \quad (B2)$$

We can easily perform the integrals over $\vec{B}$, α , and β with the help of δ distributions, and the integral (B2) becomes:

$$\begin{aligned} &\frac{1}{(E_1 E_2)^{n-3}} \frac{1}{2^{n-4} F_{jet}^{n-3}} \left\{ (E_1 + E_2)^2 - F_{jet}^2 [F_{jet}^2 - (E_1 - E_2)^2]^{(n-4)/2} \right\} \\ &\times \int d\Omega_{n-1}(\vec{A}) \theta(1 - Y_0) \theta(Y_0 - \cos \alpha_0), \end{aligned} \quad (B3)$$

where

$$Y_0 \equiv \frac{F_{jet}^2 + E_1^2 - E_2^2}{2E_1 F_{jet}} \quad \text{and} \quad \alpha_0 = \min \left[\delta, \arcsin \left(\frac{E_2 \sin \delta}{E_1} \right) \right].$$

Then, introducing $\tau = M^2/s$ and $\chi = (2E_1 + F_{jet} - E_{jet})/(2F_{jet})$ we obtain for eq. (B1) I :

$$I = \frac{(M^2)^{-\epsilon}}{4} \int_0^\delta d\chi \chi^{-\epsilon} (1 - \chi)^{-\epsilon} \int d\Omega_{n-1}(\vec{A}).$$

If we choose P_A and P_B such that

$$P_A = \frac{\sqrt{s}}{2} (1, \cos \theta, \sin \theta, 0, \dots) \quad \text{and} \quad P_B = \frac{\sqrt{s}}{2} (1, -\cos \theta, -\sin \theta, 0, \dots),$$

only the first component of $\vec{A}$ is relevant; let us call it $\cos a$. We define $\eta = (1 - \cos a)/2$, so that I is:

$$I = C(\epsilon) \int_0^\delta d\chi \chi^{-\epsilon} (1 - \chi)^{-\epsilon} \int_0^1 d\eta \eta^{-\frac{1}{2}-\epsilon} (1 - \eta)^{-\frac{1}{2}-\epsilon}, \quad (B4)$$

with

$$\begin{aligned} C(\epsilon) &= \frac{(M^2)^{-\epsilon}}{4} 2^{-2\epsilon} V(n-2) \\ c' &= \frac{R}{R+\tau}, b' = \frac{\tau}{R+\tau}. \end{aligned}$$

Note that τ can vary between 0 and R ($M \in [0, sR]$). As in Appendix A, it is convenient to rewrite $|M|^2$ such that:

$$|M|^2 = \frac{1}{P_1 \cdot P_2} H(P_A, P_B, P_1, P_2, P_3) = \frac{2}{M^2} H(P_A, P_B, P_1, P_2, P_3).$$

Schematically eq. (8) becomes:

$$\frac{d\sigma^{(2)}}{d^n P_{jet}} = F_B(\epsilon) (M^2)^{-1-\epsilon} J_B \otimes H(P_A, P_B, P_1, P_2, P_3), \quad (B5)$$

where $J_B = (M^2)^{-\epsilon}$ and F_B is a regular function of ϵ , such that

$$F_B(\epsilon) = \frac{\delta(1-w)}{4s^2 v V_c^2 (2\pi)^{5-4\epsilon}}.$$

Here also, we have explicitly shown the soft singularity when $M^2 \rightarrow 0$. To order ϵ , we use that:

$$(M^2)^{-1-\epsilon} \equiv -\frac{B^{-n\epsilon}}{n\epsilon} \delta(M^2) + \frac{1}{(M^2)^B} - n\epsilon \left[\frac{\log(M^2)}{M^2} \right]_B, \quad n = 1, 2,$$

B is the upper limit of the M^2 -integration range ($B = Rs$), and the ' B distributions' have been defined in Section 4.

The development of H is the same as in Appendix A. So let us compute:

$$\begin{aligned} J_B \otimes \frac{1}{P_1 \cdot P_A} &= \frac{2^{-2\epsilon} V(n-2)}{4} \int_0^\delta d\chi \chi^{-\epsilon} (1 - \chi)^{-\epsilon} \\ &\quad \times \int_0^1 d\eta \eta^{-\frac{1}{2}-\epsilon} (1 - \eta)^{-\frac{1}{2}-\epsilon} \frac{1}{s} \frac{1}{(\alpha_1 - \beta_1 \eta)}, \end{aligned} \quad (B6)$$

$$\text{with} \begin{cases} \alpha_1 = (1 - \tau - v - v\tau)\chi + \tau v - 2\sqrt{\tau v(1 - v - \tau)}\sqrt{\chi(1 - \chi)} \\ \beta_1 = -4\sqrt{\tau v(1 - v - \tau)}\sqrt{\chi(1 - \chi)} \end{cases}$$

Using formulas (A7) and (A8), the integration over η can be easily done:

$$J_B \otimes \frac{1}{P_1 \cdot P_A} = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{(1-\tau)}{s} (1-\tau-v)^{2\epsilon} \times \int_{b'}^c d\chi \chi' (1-\chi)^{-\epsilon} [(1-v)(1-\tau)\chi - \tau v]^{-1-2\epsilon} \times F \left[-\epsilon, -2\epsilon; 1-\epsilon; \frac{\tau v(1-\chi)}{(1-v-\tau)\chi} \right]. \quad (\text{B7})$$

Note that $\chi_{\min} \leq b'$ because $\cos \theta < \cos \delta$ [and thus $v < (1-\tau)/(1+R)$]. Expanding F around $\epsilon = 0$, we get:

$$F \left[-\epsilon, -2\epsilon; 1-\epsilon; \frac{\tau v(1-\chi)}{(1-v-\tau)\chi} \right] \simeq 1 + 2\epsilon^2 \text{li}_2 \left[\frac{\tau v(1-\chi)}{1-v-\tau\chi} \right].$$

The only way the $O(\epsilon^2)$ term could contribute would be by being multiplied by $\delta(M^2)$; however, in this case, it would become zero, and so

$$F[-\epsilon, -2\epsilon; 1-\epsilon; \tau v(1-\chi)/(1-v-\tau)\chi] \simeq 1.$$

Let us study the integral:

$$L \equiv \int_{b'}^c d\chi \chi' (1-\chi)^{-\epsilon} [(1-v)(1-\tau)\chi - \tau v]^{-1-2\epsilon}.$$

When $\tau = 0$, $L \simeq \int_0^1 d\chi \chi'^{1-\epsilon} (1-\chi)^{-\epsilon}$, which is proportional to $1/\epsilon$. The divergence is for $\chi = 0$, so we can expand $(1-\chi)^{-\epsilon}$:

$$L \simeq \int_{b'}^c d\chi \chi' [(1-v)(1-\tau)\chi - \tau v]^{-1-2\epsilon} - \epsilon \int_{b'}^c d\chi \chi' \log(1-\chi) [(1-v)(1-\tau)\chi - \tau v]^{-1-2\epsilon}.$$

The second integral contributes only at $\tau = 0$, for it can be picked out by $(1/\epsilon)\delta(M^2)$ and it is free of divergences. We drop terms with powers of $\epsilon \geq 2$. Changing the variable $\chi = c' - (c' - b')t$, we obtain:

$$L \simeq (b')^\epsilon (c' - b')^\epsilon [(1-v)(1-\tau)c' - \tau v]^{-1-2\epsilon} (1-A)^{-2\epsilon} \times \int_0^1 du (1 - Nu)^\epsilon (1 - Au)^{-1+\epsilon} + \frac{\epsilon}{1-v} \frac{\pi^2}{6},$$

where

$$A = \frac{(1-v)(1-\tau)(R-\tau)}{[(1-v-\tau)R - \tau^2 v]}, \quad N = 1 - \frac{R(1-v-\tau-vR)}{(1-v-\tau)R - \tau^2 v}.$$

We can see that $A \geq 1$ and $N < 1$ because $v < (1-\tau)/(1+R)$ (i.e. $\cos \theta < \cos \delta$). Then integrating by parts we obtain:

$$L \simeq \frac{1}{(1-v)(1-\tau)} \tau^{-\epsilon} \frac{(R+\tau)^\epsilon}{(1-v-\tau-vR)^{2\epsilon}} \times \left[-\frac{[(1-N)(1-A)]^\epsilon}{\epsilon} + \frac{1}{\epsilon} + \log(1-N) + \epsilon \{\log^2(1-N) + \text{li}_2(N)\} \right]. \quad (\text{B8})$$

The final result is

$$J_B \otimes \frac{1}{P_1 \cdot P_A} = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{s(1-v)} \left\{ -\frac{F_1^\epsilon}{\epsilon} + \frac{\pi^2}{6} + (s)^\epsilon (M^2)^{-\epsilon} H_1^\epsilon \right. \\ \left. \times \left(\frac{1}{\epsilon} + \log(G_1) + \epsilon [\text{li}_2(1-G_1) + \log^2(G_1)] \right) \right\}. \quad (\text{B9})$$

where

$$F_1 = \frac{R(R+\tau)(1-v-\tau)^2}{(R(1-v-\tau) - \tau^2 v)^2}, \quad H_1 = \frac{(R+\tau)(1-v-\tau)^2}{(1-v-\tau-Rv)^2}, \\ G_1 = \frac{R(1-v-\tau-Rv)}{(R(1-v-\tau) - \tau^2 v)}.$$

Note that when multiplying (B9) by $(M^2)^{-1-\epsilon}$ and expanding around $\epsilon = 0$, the $1/\epsilon$ singularity disappears for terms proportional to $1/(M^2)_B$ as it should (see Section 4).

By changing $v \rightarrow 1-v-\tau$, $J_B \otimes 1/P_1 \cdot P_B$ is easily obtained from eq. (B9). Note also that J_B is symmetric when we exchange P_1 and P_2 [as can be seen in eq. (B1)], so that the number of basic integrals is reduced. For integrals with two terms in the denominator, the integration over η is done by partial fractioning and the integration over χ is of the same kind as those in the calculation of $J_B \otimes 1/P_1 \cdot P_A$. As in Appendix A, the case, $J_B \otimes (K \cdot P_i)^n / K \cdot P_j$ ($i = A, B, 3$ and $K \equiv P_1, P_2$) is long but straightforward to compute.

Just in order to have a better idea of how the limit $\delta \ll 1$ works, we give another integrals:

$$J_B \otimes 1 = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{2} \left(\frac{R-\tau}{R+\tau} + 2\epsilon \right). \quad (\text{B10})$$

Appendix C: Limit $\delta \ll 1$

In this appendix we describe some subtleties about the case of the limit $\delta \ll 1$. Let us first consider the result (A6). When $\delta \ll 1$

$$1 - b' \equiv \frac{Rv(1-w)}{1+Rv(1-w)} \simeq \frac{\delta^2}{4} v(1-w) \quad \text{and} \quad \text{li}_2(1-b') \simeq O(\delta^2),$$

and so

$$(A6) \simeq \frac{1}{s(1-v+vw)} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{\delta^{-2\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{v}{4}\right)^{-\epsilon}. \quad (C1)$$

We then have to compare eq. (A3) with eq. (4.5) of ref. [4], which is:

$$E_{jet} \frac{d\sigma}{d^{n-1}\vec{P}_{jet}} = \frac{R(E_1, E_3) (v(1-w)\sqrt{s}/2)^{-2\epsilon}}{64V_c^2(2\pi)^{5-4\epsilon} s^2(1-v+vw)} \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{\delta^{-2\epsilon}}{\Gamma(1-\epsilon)} (-\epsilon) \quad (C2)$$

[Note that in ref. [4] $\epsilon = n-4$, whilst we take it to be $(4-n)/2$]. In eq. (C2), $R(E_1, E_3) = (P_1 + P_3)^2 |M|^2$. Now we take $H(P_A, P_B, P_1, P_2, P_3) = K(1/P_3 \cdot P_1)$, where K is a constant factor which gives the right dimension to H . So $R(E_1, E_3) = 2K/P_2 \cdot P_3$. With this value of H , eq. (A3) can be written:

$$E_{jet} \frac{d\sigma}{d^{n-1}\vec{P}_{jet}} = \frac{F_A(\epsilon)}{2} (1-w)^{-1-2\epsilon} K J_A \otimes \frac{1}{P_3 \cdot P_1}, \quad (C3)$$

Inserting (C1) into (C3) we get:

$$E_{jet} \frac{d\sigma}{d^{n-1}\vec{P}_{jet}} = \frac{((1-w)v\sqrt{s}/2)^{-2\epsilon}}{64V_c^2(2\pi)^{5-4\epsilon} s^2(1-v+vw)} \frac{2K}{P_2 \cdot P_3} \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{\delta^{-2\epsilon}}{\Gamma(1-\epsilon)} (-\epsilon), \quad (C4)$$

which is exactly eq. (C2). In the case of eq. (A11), when the limit is taken to be $\delta \ll 1$, and because:

$$1 - A = [1 - R(1-v)/(vw)]/[1 + Rv(1-w)] \simeq 1 - \frac{\delta^2}{4} \left[\frac{1-v}{vw} + v(1-w) \right] + O(\delta^4),$$

we find

$$J_A \otimes \frac{1}{P_3 \cdot P_B} \simeq O(\delta^2).$$

The same result was also found in ref. [4].

For two partons inside the cone, the situation is less simple. In order to recover Furman's result, we have to integrate eq. (B5) over M^2 , i.e.

$$\int_0^B dM^2 \frac{d\hat{\sigma}^{(2)}}{d^n \vec{P}_{jet}} = \int_0^B dM^2 F_B(\epsilon) (M^2)^{-1-\epsilon} J_B \otimes H(P_A, P_B, P_1, P_2, P_3). \quad (C5)$$

First we consider the case where $H(P_A, P_B, P_1, P_2, P_3) = K'$, a constant term; here, (C5) becomes:

$$\int_0^B dM^2 \frac{d\hat{\sigma}^{(2)}}{d^n \vec{P}_{jet}} = \frac{F_B(\epsilon)}{2} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} K' \int_0^B dM^2 \left[\frac{-B^{-\epsilon}}{\epsilon} \delta(M^2) + \frac{1}{(M^2)_B} \right] \times \left(\frac{R-\tau}{R+\tau} + 2\epsilon \right). \quad (18)$$

Keeping only leading terms when $\delta \ll 1$, we get:

$$E_{jet} \frac{d\hat{\sigma}^{(2)}}{d^{n-1}\vec{P}_{jet}} = \frac{\delta(1-w)K'}{16s^2v^2(2\pi)^{5-4\epsilon}} \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \left(-\frac{1}{\epsilon} \right) \left(\frac{\delta^2 s}{4} \right)^{-\epsilon} \times [1 + 2\epsilon + 2\epsilon \log(2)]. \quad (C6)$$

It is not difficult, after some manipulations, to match eq. (4.10) of ref. [4] from eq. (C6) [in this case, $K' = R(E_1, E_3)/2$].

Note that, at first sight, terms like $\int_0^B dM^2 [1/(M^2)_B] [(R-\tau)/(R+\tau)]$ seems negligible $[O(\delta^2)]$, since B behaves like $\delta^2 s/4$ when $\delta \ll 1$. But

$$\int_0^B dM^2 \frac{1}{(M^2)_B} \frac{R-\tau}{R+\tau} \equiv \int_0^B \frac{dM^2}{M^2} \left(\frac{-2\tau}{R+\tau} \right) = -2 \int_0^B \frac{dM^2}{sR+M^2}.$$

The integrand diverges when R goes to zero, so it is of the order of $1/\delta^2$ and the integral is not negligible. For each basic integral we have to take this point into account in the limit $\delta \ll 1$.

Finally, let us look at the case where $H(P_A, P_B, P_1, P_2, P_3) = K''/P_1 \cdot P_A$. We have to compute

$$\begin{aligned} \int_0^B dM^2 (M^2)^{-1-\epsilon} J_B \otimes \frac{1}{P_1 \cdot P_A} \\ = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{K''}{s(1-w)} \left\{ \int_0^B dM^2 \left(-\frac{B^{-\epsilon}}{\epsilon} \delta(M^2) + \frac{1}{(M^2)_B} \right) \right. \\ \left. - \epsilon \left[\frac{\log(M^2)}{M^2} \right]_B \right\} \left(-\frac{F_1^\epsilon}{\epsilon} + \frac{\pi^2}{6} \right) \\ + \left(-\frac{B^{-2\epsilon}}{2\epsilon} \delta(M^2) + \frac{1}{(M^2)_B} - 2\epsilon \left[\frac{\log(M^2)}{M^2} \right]_B \right) \\ \times (sH_1)^\epsilon \left(\frac{1}{\epsilon} + \log(G_1) + \epsilon [\text{li}_2(1-G_1) + \log^2(G_1)] \right). \end{aligned} \quad (C7)$$

After expanding around $\epsilon = 0$, we can see that only the integral with $\delta(M^2)$ distributions gives non-negligible term (there are no dangerous terms as in the preceding case). We

obtain:

$$\int_0^B dM^2 (M^2)^{-1-\epsilon} J_B \otimes \frac{1}{P_1 \cdot P_A} \simeq \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{K''}{s(1-v)} \left(\frac{\delta^2 s}{4} \right)^{-\epsilon} \times \left(-\frac{1}{\epsilon^2} \right) \left(-\frac{1}{2} + \epsilon^2 \frac{\pi^2}{6} \right).$$

In this case as well, the result is the same as Furman's [4].