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ON EFFECTIVE THEORIES OF TOPOLOGICAL STRINGS

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Dedicated to Daniele Amati on the occasion of his 60th birthday

Abstract

We study the construction of effective target space theories of topological string theories. The example of the CP1 topological sigma model is analysed in detail. There we find that the effective target space theory is itself topological. We obtain the relations between the world sheet couplings of general CP1-like models and the effective target space theory couplings. Any model of this type can be obtained as an effective theory for some exact string theory. Dilaton-type couplings emerge in the full Lagrangian realization of the moduli space of topological theories with n observables. En route, we encounter a non-perturbative duality, an equivalence of theories with different world sheets and discuss the relation between the cosmological constant in these finite theories and the zero-point function.

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1. INTRODUCTION

In order to calculate S-matrix elements in a 26 (10) dimensional theory of gravity reflecting extended objects as the elementary particles, we have instructed ourselves to use a black box featuring very special two-dimensional field theories. It consists of a summation on all two-dimensional topologies of conformal field theories (CFT) with a Virasoro central charge (in the matter sector) of 26 (15). In order to form a tie with point particle theory, we also attempt to find a 26 (10) dimensional effective particle theory which describes small fluctuations around a fixed background and reproduces the same S-matrix elements as string theory. The nature of the particles, symmetries and interactions of the theory is a subject of continuous research.

In this note we obtain the effective action, for a very simple system, accounting for all the physical correlation functions which are calculated exactly to all orders in world-sheet topology. We will discuss some features of this exact result.

The second issue we address concerns relations between the nature of the world-sheet theory and that of the target space theory. The black box has been originally viewed as some special class of two-dimensional gravity. In order to study lower-dimensional gravity such as four-dimensional gravity, an additional attitude has developed. One considers CFT with $c = 26$ (15) realized by some product of a string moving in a flat four-dimensional background [with $c = 4$ (6) in the matter sector] and another CFT carrying the remaining $c = 22$ (9) central charge. Later, another sector of the $c = 26$ (15) manifold was focused on a CFT describing a string moving in a low-dimensional manifold background containing a linear dilaton consummating on its own the full value of c . The latter type of theories were also regarded as two-dimensional gravitational theories with one less matter field. In any case, any way one viewed these theories, be the conformal mode coupled or uncoupled, these are theories of two-dimensional gravity. The world sheet and the target space theory have in common the symmetry of general coordinate invariance. Assume for a moment that a similar relation would have held for gauge theories. Namely one could have obtained gauge theories in higher dimensions as a result of some calculation in a

have obtained gauge theories in higher dimensions as a result of some calculation in a low-dimensional gauge theory. Given the more detailed knowledge of the possible phases of gauge theories, one would have enquired into the relations between the phase of the input gauge theory and that of the output one. The understanding of the phase structure of general coordinate invariant systems is at a much earlier stage. Nevertheless one has already learned that such systems can manifest themselves also in what has been termed a topological manner [1, 2].

Topological matter and gravity field theories (TFT) were found to exist also in two and four dimensions. Such type of theories may be used as inputs on the world sheet and may emerge as part of the output in target space. As these two-dimensional theories have $c = 0$ they may be appended in bunches to $c = 26$ (15) input theories. One needs to study the relations between TFT and suitable non-TFT on the world sheet, and between TFT and non-TFT in target space (TS).

As a first step in that direction we study the effective target space theories generated by a very simple TFT on the world sheet, the CP1 topological sigma model. Our conclusion in this case is that TFT on the world sheet leads to TFT in the TS. We use these results to study an issue which is of a more philosophical nature. As one uses as an input on the world sheet two-dimensional gravitational theories, one may wonder if they themselves should have not been initiated by a similar process. This was articulated by Green in his call for a "world sheet for the world sheet" [3]. In the case at hand we show that any given TFT, of type studied in this note, can be viewed itself as an effective TFT for some string theory. We find that these theories reproduce themselves only in the classical limit (genus zero contributions).

In Section 2 topological sigma models are reviewed, together with the general structure of topological field theories, as expressed by the operator product coefficients and the "metric" on the observable space. The dimensionality of moduli space of such theories and its Lagrangian realization are analysed. In Section 3 the particular case of the CP1 sigma model is studied in detail as a concrete example of the notions of Section 2. Following Ref. [1], all the correlation functions are explicitly calculated. In Section 4, the concept of target space effective

theory is explicitly identified as a similar topological theory. The relation between the couplings of the CP1 class world-sheet theories and the couplings of the effective theory is found. We obtain both perturbative and non-perturbative symmetries (duality) in the moduli space. We discuss the connection between the zero-point function and the cosmological constant in TFT. We conclude by a discussion of the generality of the relations between world-sheet TFT and target-space TFT, as well as the generality of other results.

2. TOPOLOGICAL SIGMA MODELS: A REVIEW

In his work [1], Witten has laid out the general framework for calculating correlation functions in two-dimensional topological field theories. It is shown that it is possible to calculate correlation functions without having to resort to an explicit form of the action (or measure). It is assumed however that the target space is a simply connected Kähler manifold. We first review the calculations in topological sigma models (TSM) as performed from "first principles". We then review how the properties of TFT enable us, in practice, to obtain the full information contained in the theory to be obtained.

The starting point is to identify the observables of the topological theory. These are the BRST invariant operators with respect to a certain BRST operator associated with the theory. The general problem is then to calculate the set of correlation functions of these observables.

The rules for calculating the correlation functions are the following. The observables, denoted by O_i , turn out to be in correspondence with the De-Rham cohomology classes of the target space of the TSM. To each cohomology class A_i of the target space T corresponds an observable $O_{A_i}(x)$.

For the purpose of calculating

$$\langle O_{A_1}(x_1) \dots O_{A_n}(x_n) \rangle_g \quad (2.1)$$

each k -dimensional cohomology class A_i is viewed as the Poincaré dual of homology class of degree $d - k$ represented by some closed manifold T_i in the target space T of dimension d . The world-sheet genus is g . It is

then possible to associate with each $O_{A_i}(x)$ a special class of instantons (holomorphic mapping from the world sheet Σ to target space T) that obtain their values at x on the appropriate T_i . The moduli space of instantons belonging to a certain homotopy class is denoted by M and the submanifold of M such that $\phi(x_i) \in T_i$ is denoted by L_i ; $\phi(x)$ is a holomorphic map (instanton).

According to the definitions, only instantons in the respective L_i 's will contribute to (2.1). A non-zero contribution may indeed come only from instantons in the intersection of all the L_i 's; what is more, it is required (and obtained in a field theoretic manner) that the dimension of the intersection be zero. For a non-zero dimension of the intersection, the apparently infinite value of the correlation functions turns out to be zero. For the case of a zero-dimensional intersection, the finite number of intersection-points of L_i is the value of the correlation function.

The contribution of the topologically trivial instantons is very visual; in that case, the L_i can be identified with the T_i 's and as a result

$$\langle O_{A_1}(x_1) \dots O_{A_n}(x_n) \rangle_g = \#(T_1 \cap T_2 \dots \cap T_n); \quad (2.2)$$

while all correlation functions can be calculated in that manner, the procedure may be shortened by the application of various properties of TFT. In fact, it is only necessary to explicitly calculate a set of some low-order (up to three-point function) correlation functions on a single ($g = 0$) world sheet to be able to obtain all n -point correlation functions on any genus g world sheet.

We now review the information needed to characterize the theory and the manner in which it is applied. In a general TFT, let the list of the BRST invariant local operators be $O_a(x)$ ($a = 1, \dots, N$). The required data are:

a) The two-point functions of these operators

$$\langle O_a(x_1) O_b(x_2) \rangle_0 = \eta_{ab}. \quad (2.3)$$

It turns out that the result does not depend on the coordinates x_1, x_2 and that η_{ab} is invertible.

b) The operator product expansion of the observables

$$O_\alpha(x)O_\beta(y) = \sum_\gamma C_{\alpha\beta}^\gamma O_\gamma(y). \quad (2.4)$$

The knowledge of $N, \eta_{\alpha\beta}, C_{\alpha\beta}^\gamma (\alpha, \beta = 1, \dots, N)$ suffices to define the system.

With these data one can reduce recursively the calculation of an n -point function on a world sheet of genus g to that of an $(n-1)$ point function on the same world sheet, and also reduce a calculation of a correlation of an n -point function on a world sheet of genus g to that of a specific $n+1$ point function on a world sheet of genus $g-1$.

In calculating an n -point function on a world sheet of genus g , $\langle O_{A_1}(x_1) \dots O_{A_n}(x_n) \rangle_g$, the x_i independence of the result is used to replace $O_\alpha(x)O_\beta(y)$ in (2.1) by $C_{\alpha\beta}^\gamma O_\gamma(y)$ as given by (2.4). The substitution relates any given n -point correlation function on genus g to a linear combination of $(n-1)$ -point functions on the same genus. A repeated application of the substitution for genus $g=0$, for example, combined with the knowledge of $\eta_{\alpha\beta}$ [defined by (2.3)] allows the explicit calculation of any n -point function, on $g=0$, in terms of $C_{\alpha\beta}^\gamma$ and $\eta_{\alpha\beta}$. For example

$$\langle O_\alpha(x_1)O_\beta(x_2)O_\gamma(x_3) \rangle_{g=0} = C_{\alpha\beta}^\delta \langle O_\delta(x_2)O_\gamma(x_3) \rangle_{g=0} = C_{\alpha\beta}^\delta \eta_{\delta\gamma} = C_{\alpha\beta\gamma}. \quad (2.5)$$

For a general g , the n -point functions on g are expressed at this stage in terms of $C_{\alpha\beta}^\gamma$ and one-point functions $\langle O_\alpha(x) \rangle_g$. To conclude the computation one-point functions on g are to be expressed in terms of lower genus correlations.

The method used in Ref. [1] actually relates any n -point function $\langle A \rangle_g$ to a certain $(n+1)$ -point function on genus $g-1$ and, by recursion, finally to $(n+g)$ -point function on $g=0$. As all n -point functions on $g=0$ are expressed in terms of the data, the knowledge of the explicit recursion procedure will completely relate all n -point correlation functions to the data.

To obtain the explicit recursion relation, the amplitude $\langle A \rangle_g$ is factorized along a one-dimensional boundary, such as a circle, of a genus-one manifold, Σ , which does not contain the points of insertion of the operators contained in A (Fig. 1a).

Performing the functional integral on Σ , the factorized genus-one part can be replaced by a physical state and represented by the insertion of an operator, denoted by W in Ref. [1], in the remaining part of the diagram (Fig. 1b).

W is independent of A and g , and can thus be calculated once and for all: it is expressed in terms of the observables O_α as

$$W = \sum_{\sigma=1}^N w^\sigma O_\sigma \quad (2.6)$$

where the w^σ are to be determined. Repeating the factorization one indeed obtains:

$$\langle A \rangle_g = \langle AW^g \rangle_0, \quad (2.7)$$

relating the general n -point function on g to a specific $(n+g)$ -point function on $g=0$. Thus the data $O_\alpha, \eta_{\alpha\beta}, C_{\alpha\beta}^\gamma$, and w^σ define the theory. Moreover, the w^σ themselves are determined by $\eta_{\alpha\beta}$ and $C_{\alpha\beta}^\gamma$. This has been shown in the following manner [1].

The one-point function $\langle O_\alpha \rangle_1$ was considered from the two-point function point of view: on the one hand, from (2.7):

$$\langle O_\alpha \rangle_1 = \langle O_\alpha W \rangle_0 = \left\langle O_\alpha \sum_{\sigma=1}^N w^\sigma O_\sigma \right\rangle_0 = \sum_{\sigma=1}^N w^\sigma \eta_{\alpha\sigma}; \quad (2.8)$$

on the other hand using the factorization properties of TFT (Fig. 2), one has

$$\langle O_\alpha \rangle_1 = \eta^{\beta\gamma} \langle O_\alpha O_\beta O_\gamma \rangle_0 = \eta^{\beta\gamma} C_{\alpha\beta}^\delta \eta_{\delta\gamma} = C_{\alpha\beta}^\delta \eta_\delta^\beta = C_{\alpha\beta}^\beta = \text{Tr}(O_\alpha). \quad (2.9)$$

Thus

$$w_\alpha = C_{\alpha\beta}^\beta = \text{Tr}(O_\alpha). \quad (2.10)$$

This completes the description of the calculation of all correlation functions in terms of the data. Using the definition (2.9) of the trace of an observable, one obtains

$$\langle A \rangle_g = \text{Tr}(AW^{g-1}). \quad (2.11)$$

This includes the $g = 0$ case, if W is invertible:

$$\langle A \rangle_0 = \langle AW^{-1}W \rangle_0 = \langle AW^{-1} \rangle_1 = \text{Tr}(AW^{-1}). \quad (2.12)$$

By summing over g with weight λ^{g-1} , the exact result is

$$\langle A \rangle_{\text{exact}} = \sum_{g=0}^{\infty} \lambda^{g-1} \text{Tr}(AW^{g-1}) = \text{Tr}(A(\lambda W)^{-1}(1 - \lambda W)^{-1}). \quad (2.13)$$

Let us count the dimension of moduli space of topological theories with N operators. As discussed above, the data required to fix such a theory are the coefficients $C_{\alpha\beta}^\gamma$ and $\eta_{\alpha\beta}$ subject to the conditions of duality:

$$C_{\alpha\beta}^\sigma C_{\sigma\gamma}^\delta = C_{\alpha\gamma}^\sigma C_{\sigma\beta}^\delta \quad (2.14)$$

and, for operators of the bosonic type, the condition of symmetry, is that

$$\langle O_\alpha O_\beta O_\gamma \rangle_0 = C_{\alpha\beta\gamma}^\delta = C_{\alpha\gamma\beta}^\delta \eta_{\delta\gamma} \quad (2.15)$$

is totally symmetric in α, β, γ . However, while $C_{\alpha\beta}^\gamma$ and $\eta_{\alpha\beta}$ fix the TFT completely, a given TFT does not fix them uniquely, since if we just change the names of the operators O_α into O'_α , defined by:

$$O'_\alpha = \sum_\beta S_\alpha^\beta O_\beta \quad (2.16)$$

(S being any non-singular matrix), we will obtain, for the same TFT in the new basis, with different values for the data, $C_{\alpha\beta}^\gamma$ and $\eta_{\alpha\beta}$ appropriately transformed according to (2.16). A TFT corresponds, then, to a set of values for $C_{\alpha\beta}^\gamma$ and $\eta_{\alpha\beta}$ modulo the transformation induced on them by (2.16).

To count the number of parameters, notice, following Ref. [4], that if we define the N matrices $C_\alpha, \alpha = 1 \dots N$, by $(C_\alpha)_\beta^\gamma = C_{\alpha\beta}^\gamma$, then the duality condition (2.14) means that these N matrices are mutually commuting. In the generic case in which each of these N matrices can be diagonalized, they can then be simultaneously diagonalized by some matrix S . Choosing this particular matrix S in the passage to a new basis (2.16) we obtain, in this new basis, diagonal OP coefficients:

$$C_{\alpha\beta}^\gamma = \lambda_{\alpha\beta} \delta_\beta^\gamma = \lambda_{\beta\alpha} \delta_\alpha^\gamma.$$

Since the substitution (2.16) preserves the symmetry of $C_{\alpha\beta}^\gamma$ in its lower indices, $\lambda_{\alpha\beta}$ is different from zero only when $\alpha = \gamma$, i.e. $\lambda_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$. In the new basis we have then $O_\alpha O_\beta = \lambda_\alpha \delta_{\alpha\beta} O_\alpha$. Further normalization of the operators leads to the algebra:

$$O_\alpha O_\beta = \delta_{\alpha\beta} O_\alpha. \quad (2.17)$$

The freedom (2.16) has been used then to completely fix the OP coefficients $C_{\alpha\beta}^\gamma$ to the values (2.17), leaving only the $\eta_{\alpha\beta}$ parameters. These are also constrained by (2.17):

$$\eta_{\alpha\beta} = \langle O_\alpha O_\beta \rangle_0 = \delta_{\alpha\beta} \langle O_\alpha \rangle_0 \equiv \delta_{\alpha\beta} \eta_\alpha. \quad (2.18)$$

The actual number of parameters is now N , describing, in this particular parametrization, the zero-genus, one-point function of each of the N operators in the basis (2.17). There, using (2.9), the operator W of (2.5) is given by:

$$W = \sum_\alpha \frac{O_\alpha}{\eta_\alpha}. \quad (2.19)$$

The genus g correlation functions are:

$$\langle (O_\alpha)^n \rangle_g = \langle O_\alpha \rangle_g = \langle O_\alpha W^g \rangle_0 = (\eta_\alpha)^{1-g}, \quad (2.20)$$

all the other correlation functions being zero.

Suppose that a given TFT, given by some Lagrangian L , corresponds to some point in this N -dimensional moduli space of TFTs. One expects that it will be possible to modify the theory and move to neighbouring points on moduli space by changing the Lagrangian, condensing into it local operators from the content of the theory. In the framework of conformal field theory, this is usually done by condensing truly marginal $(1, 1)$ operators into the Lagrangian. Since the condensed operators have to be integrated over the world sheet, they must be two-form valued operators. The usual way to construct such operators in field theory is to multiply zero-form ordinary field operators by the volume form $\sqrt{g} d^2x$ on the world sheet. For a topological theory, this

procedure introduces an explicit metric dependence which destroys the topological nature of the theory. If, however, the zero-form operator $O_\alpha(x)$ is multiplied by the curvature form of the world sheet, $\sqrt{g(x)}R(x)d^2x$, the resulting modified Lagrangian:

$$L' = L + \frac{1}{8\pi\alpha} \sum_\alpha \mu_\alpha O_\alpha(x) \sqrt{g(x)} R(x) d^2x \quad (2.21)$$

(μ_α being arbitrary coupling constants) retains its topological nature. Indeed, any correlation function of the perturbation $O_\alpha(x)$ is x -independent: its only x dependence comes from $\sqrt{g(x)}R(x)d^2x$, which gives a topological invariant upon integration over the world sheet. Note also that since we are dealing here with a matter topological theory, the metric g and the curvature R in (2.21) are numerical background functions, not dynamical variables, so there is no problem in multiplying $O(x)$ and $R(x)$ in (2.21) at coincident points. This way of modifying the Lagrangian can be termed addition of topological dilaton couplings.

The operator product formula (2.4), $O_\alpha(x)O_\beta(y) = \sum_\gamma C_{\alpha\beta}^\gamma O_\gamma(y)$, means that the correlator of the product of its l.h.s. with any operator equals that of the r.h.s. with the same operator. The modification (2.21) changes the definition of correlator of an operator by multiplying it with various combinations of the operator $\sum_\alpha \mu_\alpha O_\alpha$. Since (2.4) is true in correlators multiplying any operator, it is clear that the new theory corresponding to L' has the same OP coefficients $C_{\alpha\beta}^\gamma$ as the original theory with the Lagrangian L . Hence if in the original theory we define operators according to the gauge (2.17), we will remain in this gauge after turning on the perturbation. The new values η'_α for the η_α parameters are:

$$\eta'_\alpha = \left\langle O_\alpha \exp \left(\frac{1}{8\pi\beta} \sum_\beta \mu_\beta (O_\beta \sqrt{g} R d^2x) \right) \right\rangle_0 = e^{\mu_\alpha} \langle O_\alpha \rangle_0 = \eta_\alpha e^{\mu_\alpha}. \quad (2.22)$$

The N parameters η_α correspond, then, to the N possible coupling constants μ_α . Note, however, that the signs of η_α cannot be changed by the variation (2.21). It can be checked directly that the theory remains indeed topological after the modification (2.21). This amounts to showing that the connection (2.11) or, in the special gauge, (2.20), among correlation functions on different genera are still satisfied. In the

gauge (2.17), consider the genus g correlation, $\langle (O_\alpha)^n \rangle_g$ in the modified system. On the one hand it is given by,

$$\langle (O_\alpha)^n \rangle_g = \langle (O_\alpha)^n \exp \left(\frac{1}{8\pi} u_\alpha \int O_\alpha \sqrt{g} R d^2x \right) \rangle_g = \eta_\alpha^{1-g} e^{\mu_\alpha(1-g)}. \quad (2.23)$$

On the other hand, if the modified theory satisfies (2.11),

$$\langle (O_\alpha)^n \rangle_g = \langle (O_\alpha)^n (W')^g \rangle_0 = (\eta'_\alpha)^{1-g};$$

which by (2.22) agrees with (2.23). Note that this consistency depends crucially on the presence of the factor $1-g$ in the exponential in (2.23), which results from coupling the perturbation (2.21) to the curvature form.

In the case that the topological theory is defined through a BRST projection of a richer, non-topological theory, as is the case for topological sigma models and twisted $N=2$ models, a different way to modify the Lagrangian and wander along moduli space is suggested in Refs. [1, 4]. In the presence of the BRST operator Q , to each Q -invariant zero-form operator O_α of the topological theory there corresponds a hierarchy of three operators $O_\alpha^i, i=0,1,2$, where O_α^0 is an i -form valued operator, $O_\alpha^0 = O_\alpha$, and $[Q, O_\alpha^{i+1}] = dO_\alpha^i$. One can use the two-form valued operators O_α^2 to modify the Lagrangian, getting

$$L' = L + \sum_\alpha t_\alpha O_\alpha^2. \quad (2.24)$$

While the modification (2.21) preserves the form of the $C_{\alpha\beta}^\gamma$ coefficients and changes the value of $\eta_{\alpha\beta}$, hence preserving the gauge (2.17), the variation (2.24) was shown in Ref. [4] to preserve the metric $\eta_{\alpha\beta}$ and change $C_{\alpha\beta}^\gamma$. After the modification (2.24), a further transformation of the kind (2.16) is required if one wishes to return to the gauge (2.17) and check the new values of the parameters η_α . Since no operator corresponds to the unit operator, the dimensionality of the space spanned by (2.24) is only $n-1$.

In the next section these methods are applied to the case of a topological sigma model with a target space which is a $g=0$ Riemann surface: the sphere.

3. THE TOPOLOGICAL SIGMA MODEL OF THE SPHERE

The TSM of a sphere has *ab initio* two coupling constants: one measures the relative weight of each world-sheet genus and is denoted by λ (the "dilaton") and the other measures the relative weight of each map according to its winding number; it is denoted by β (the "antisymmetric tensor"). The first step in setting up the data is to identify the observables. For a TSM, the BRST-invariant observables are in a one-to-one correspondence with the cohomology classes of the target space.

The sphere has two cohomology classes corresponding to the zero-form, represented by the identity operator, and a two-form represented by an operator denoted O . The next step is to calculate the metric $\eta_{\alpha\beta}(\alpha, \beta = 1, 2)$ corresponding to 1 and O respectively.

To obtain

$$(1 \cdot 1)_0 = \eta_{11} \quad (3.1)$$

it was noted [1] that to 1 corresponds the homology class, which is the full target space sphere.

For instanton configurations in the trivial class, the intersection of the moduli space is the intersection of the homologies (2.2), that is a sphere. As the dimension of a sphere is non-zero, the contribution of these maps to the correlation function is zero. To higher-winding-number instantons will correspond even larger intersections, and thus

$$\eta_{11} = 0. \quad (3.2)$$

In this simple case all n -point functions of O on the world-sheet sphere can be calculated together. The moduli space of an instanton of winding number k has real dimension $4k + 2$. This follows from the general form of such an instanton $w(x)$

$$w(x) = a \frac{\prod_{i=1}^k (x - b_i)}{\prod_{j=1}^k (x - c_j)}, \quad (3.3)$$

which is defined by the position of its k poles (c_j), k zeroes (b_i), and an overall complex normalization a in the projective coordinates.

The O is related to the homology class which is a point on the sphere. The dimension of the intersection space corresponding to an n -point function of O is thus $4k + 2 - 2n$. Only an instanton of winding number $k = (n-1)/2$ will contribute to the n -point function. Giving such a configuration a weight β^k results in

$$\langle O(x_1) \dots O(x_n) \rangle_0 = 0 \text{ if } n = 2k \text{ and } \langle O(x_1) \dots O(x_n) \rangle_0 = \beta^k \text{ if } n = 2k+1. \quad (3.4)$$

From this, it follows that the $\eta_{\alpha\beta}$ is given by the matrix

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.5)$$

Finally one needs to calculate $C_{\alpha\beta}^\gamma$. From the results

$$1 \cdot 1 = 1 \text{ and } 1 \cdot O = O \quad (3.6)$$

one has (3.7a). The remaining $C_{\alpha\beta}^\gamma$ are listed in (3.7b)

$$C_{11}^1 = 1, \quad C_{11}^2 = 0, \quad C_{21}^1 = C_{12}^1 = 0, \quad C_{12}^2 = C_{21}^2 = 1 \quad (3.7a)$$

$$C_{22}^1 = \beta, \quad C_{22}^2 = 0 \quad (3.7b)$$

The OPE $O \times O$ was determined using (3.4). On the one hand

$$C_{22}^2 = \langle O(x_1) O(x_2) O(x_3) \rangle_0 = \beta; \quad (3.8)$$

on the other hand, using (3.5):

$$C_{22}^2 = C_{22}^1 \langle 1(x_2) O(x_3) \rangle_0 + C_{22}^2 \langle O(x_2) O(x_3) \rangle_0 = C_{22}^1; \quad (3.9)$$

C_{22}^2 is found by considering $\langle O(x_1) O(x_2) \rangle_0$ in a similar fashion $0 = \langle O(x_1) O(x_2) \rangle_0 = C_{22}^1 \langle 1 \rangle_0 + C_{22}^2 \langle O \rangle_0 = C_{22}^2$,

completing the determination of $C_{\alpha\beta}^\gamma$ as listed in (3.7). The operator W used to relate correlation functions on different world-sheet genera is given by (2.10), (3.5), (3.7) to be

* Actually only the absolute value of k is fixed by n . Allowing however both values of k to contribute to eq. (3.4) leads to correlation functions in which eq. (2.4) is not realized. Thus b cannot be viewed as a real "torsion", which should lead to periodicity.

$$W = 2O \quad (3.10)$$

For non-zero β , W is invertible as from (3.7)

$$O \cdot O = \beta I, \quad (3.11)$$

leading to

$$W^{-1} = (2\beta)^{-1} O. \quad (3.12)$$

The exact zero-point correlation function is given by

$$\langle 1 \rangle_{\text{exact}} = \frac{1}{\lambda} \text{Tr} \left(W^{-1} \frac{1}{1 - \lambda W} \right) = \frac{1}{2\lambda\beta} \text{Tr} \left(O_\omega \frac{1}{1 - 2\lambda O_\omega} \right). \quad (3.13)$$

The matrix $(O_\omega)_{\alpha\beta}$ is defined by $(O_\omega)_{\alpha\beta} = C_{2\alpha}^\beta$. Thus

$$O_\omega = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix} \quad (3.14)$$

and

$$\langle 1 \rangle_{\text{exact}} = \sum_{g=0}^{\infty} \sum_{k=0}^{\infty} \delta_{g,2k+1} (2\beta^{1/2}\lambda)^{g-1} = \frac{2}{1 - 4\lambda^2\beta}. \quad (3.15)$$

only odd genera contribute to $\langle 1 \rangle_{\text{exact}}$, and the contribution is $2(4\beta)^{\frac{g-1}{2}}$. Note that the exact zero-point function depends only on the scaling combination

$$\tilde{\lambda} = 2\lambda\beta^{1/2}. \quad (3.16)$$

This is true on any genus g world-sheet surface.

The other correlation functions break into two groups according to their parity, $O_{\omega\omega}^{2n} = \beta^n$ from (3.11), thus

$$\begin{aligned} \langle O^{2n} \rangle_{\text{exact}} &= \beta^n \langle 1 \rangle_{\text{exact}} = \frac{2\beta^n}{1 - 4\lambda^2\beta} \\ \text{and} \\ \langle O^{2n+1} \rangle_{\text{exact}} &= \frac{\beta^n}{2\lambda} \langle 1 \rangle_{\text{exact}} = \frac{\beta^n}{\lambda(1 - 4\lambda^2\beta)}. \end{aligned} \quad (3.17)$$

Note also that rescaling the operator O and defining

$$\tilde{O} = 2\lambda O, \quad (3.18)$$

one has

$$\langle \tilde{O}^{2n+1} \rangle_{\text{exact}} = \langle \tilde{O}^{2n} \rangle_{\text{exact}} = \frac{2\tilde{\lambda}^{2n}}{1 - \tilde{\lambda}^2}. \quad (3.19)$$

Let us modify the theory in the way described in (2.21). We add to the Lagrangian of the model a term of the form $(\mu_1 + \mu_2 O)\sqrt{g}Rd^2x$. Clearly the coupling μ_1 just changes the relative weight of different genera renormalizing λ into λe^{μ_1} . Denoting $\mu = \mu_2$, we study the effect of the coupling μ on the model.

The modified action is now

$$S(\lambda, \beta, \mu) = S_0(\lambda, \beta) + \mu(1 - g)O. \quad (3.20)$$

Expectation values of an operator A in the theory $S(\lambda, \beta, \mu)$ will be denoted by $\langle\langle A \rangle\rangle$. The defining data for the modified theory, i.e. the genus-zero two- and three-point functions, are calculated using (3.20) and (3.4) to give:

$$\begin{aligned} C_{111}^\mu &= \eta_{11}^\mu = \langle\langle 1 \rangle\rangle_0 = \langle e^{\mu O} \rangle_0 = \beta^{-\frac{1}{2}} \sinh \mu\beta^{\frac{1}{2}} \\ C_{112}^\mu &= \eta_{12}^\mu = \langle\langle O \rangle\rangle_0 = \langle e^{\mu O} O \rangle_0 = \cosh \mu\beta^{\frac{1}{2}} \\ C_{122}^\mu &= \eta_{22}^\mu = \langle\langle O^2 \rangle\rangle_0 = \langle e^{\mu O} O^2 \rangle_0 = \beta^{\frac{1}{2}} \sinh \mu\beta^{\frac{1}{2}} \\ C_{222}^\mu &= \langle\langle O^3 \rangle\rangle_0 = \langle e^{\mu O} O^3 \rangle_0 = \beta \cosh \mu\beta^{\frac{1}{2}} \end{aligned} \quad (3.21)$$

Notice that the operator product coefficients $C_{\alpha\beta}^\gamma$ remain the same as in (3.7) and are not affected by the parameter μ . In the modified theory we can further sum over genera with weight λ and get its "exact" matrix elements. For example,

$$\begin{aligned} \langle\langle 1 \rangle\rangle_{\text{exact}} &= \langle \exp(\mu(1 - g)O) \rangle_{\text{exact}} \\ &= \frac{1}{2\lambda\beta^{1/2}} \left(\frac{\exp(\mu\beta^{1/2})}{1 - 2\lambda\beta^{1/2} \exp(-\mu\beta^{1/2})} - \frac{\exp(-\mu\beta^{1/2})}{1 + 2\lambda\beta^{1/2} \exp(\mu\beta^{1/2})} \right) \end{aligned} \quad (3.22)$$

$$\langle\langle O \rangle\rangle_{\text{exact}} = \frac{1}{2\lambda} \left(\frac{\exp(\mu\beta^{1/2})}{1 - 2\lambda\beta^{1/2} \exp(-\mu\beta^{1/2})} + \frac{\exp(-\mu\beta^{1/2})}{1 + 2\lambda\beta^{1/2} \exp(\mu\beta^{1/2})} \right) \quad (3.23)$$

and in general

$$\langle\langle O^{2n} \rangle\rangle_{\text{exact}} = \beta^n \langle\langle O \rangle\rangle_{\text{exact}} = \frac{\beta^n}{\lambda} \left(\frac{\exp(\tilde{\mu})}{1 - \tilde{\lambda} \exp(-\tilde{\mu})} - \frac{\exp(-\tilde{\mu})}{1 + \tilde{\lambda} \exp(\tilde{\mu})} \right) \quad (3.24)$$

$$\langle\langle O^{2n+1} \rangle\rangle_{\text{exact}} = \beta^{n+1/2} \langle\langle O \rangle\rangle_{\text{exact}} = \frac{\beta^{n+1/2}}{\tilde{\lambda}} \left(\frac{\exp(\tilde{\mu})}{1 - \tilde{\lambda} \exp(-\tilde{\mu})} + \frac{\exp(-\tilde{\mu})}{1 + \tilde{\lambda} \exp(\tilde{\mu})} \right) \quad (3.25)$$

with

$$\tilde{\lambda} = 2\lambda\beta^{1/2}, \quad \tilde{\mu} = \mu\beta^{1/2}. \quad (3.26)$$

One may rescale again, the correlation of the operators $\tilde{O} = 2\lambda O$ depends only on $\tilde{\lambda}$ and $\tilde{\mu}$:

$$\begin{aligned} \langle\langle \tilde{O}^{2n} \rangle\rangle_{\text{exact}} &= \frac{2\tilde{\lambda}^{2n-1}}{(1 - \tilde{\lambda} \exp(-\tilde{\mu})) (1 + \tilde{\lambda} \exp(\tilde{\mu}))} (\sinh(\tilde{\mu}) + \tilde{\lambda} \cosh(2\tilde{\mu})) \\ \langle\langle \tilde{O}^{2n+1} \rangle\rangle_{\text{exact}} &= \frac{2\tilde{\lambda}^{2n}}{(1 - \tilde{\lambda} \exp(-\tilde{\mu})) (1 + \tilde{\lambda} \exp(\tilde{\mu}))} (\cosh(\tilde{\mu}) + \tilde{\lambda} \sinh(2\tilde{\mu})) \end{aligned}$$

The CP1 sigma model is defined in Ref. [1] by a BRST procedure. Therefore we can also study modifications of the theory of the type shown in (2.24). Since for the unit operator $1^1 = 1^2 = 0$, we have in this mode of modification only one parameter at our disposal: the coupling constant t_2 , multiplying O^2 . As explained in Ref. [5], integrating O^2 over the world sheet amounts to integrating the pull back of the corresponding target space two-form A_2 . This form can be chosen to have a δ function support at an arbitrary point of target space. Integrating it over the world-sheet counts the number of times this point of target space is covered by the world sheet, i.e. it measures the winding number. Hence the coupling t_2 just rescales the parameter β into βe^{t_2} .

The preferred basis satisfying the gauge condition (2.17) consists in our CP1 TSM case of the two operators:

$$O_{\pm} = \frac{1}{2} \left(1 \pm \beta \frac{1}{2} O \right) \quad (3.27)$$

and the parameters η_{α} are:

$$\eta_{\pm} = \frac{\pm 1}{2\beta^2} e^{\pm i\mu\beta^2}. \quad (3.28)$$

The "exact", summed over genera, correlation functions are, in these variables,

$$\langle\langle O_{\pm} \rangle\rangle_{\text{exact}} = \frac{(\eta_{\pm})^2}{\eta_{\pm} - 1}. \quad (3.29)$$

The present models were described as topological CP1 sigma models. Note however that topological models with two operators can arise also in different contexts. For example the coset model of $SU(2)/SU(2)$ level 1 is a topological model with two operators [6], which should also correspond to some point in the two-dimensional moduli space considered here.

4. THE EFFECTIVE TARGET SPACE THEORY

4.1 Construction of an effective target space theory

We now turn to the search for a field theory in target space from which to reproduce all physical correlations as given in (3.17). We propose to call this the effective topological target space theory. As the target space (TS) is a two-dimensional sphere, one should search among theories of two-dimensional TS gravity for candidates for such a theory. The correlation functions in (3.17) contain no poles as long as $\tilde{\lambda} < 1$, thus the exact TSM does not give rise to particles in TS. This suggests that the two-dimensional TS gravity theory it is describing should be topological as well. We expect, then, generally for TSM in which a sum over genera has a non-zero radius of convergence, that topological theories on the world sheet describe topological TS theories. Let us check this assumption for our particular case of the world-sheet CP1 model. Now that the search has been narrowed to topological theories of two-

dimensional TS gravity, the question is what fields to introduce in the TS theory. Actually in a conventional TS gravity theory one would need indeed to specify the full particle content (such as graviton, tachyon, etc.), but for a topological TS theory the natural question is what target space to use for the effective target space theory. Such a target space, we call target target space (TTS). For the present case, since there are two operators at hand, an immediate candidate is a CP1 TTS. Recall from the previous section that a typical topological theory with two operators is a CP1 TSM, possibly modified by the μ parameter of (3.20). We claim that the TS correlation functions obtained by the instructions of string theory, i.e. as a weighted sum over genera of world-sheet correlators, are themselves $g = 0$ (since TS is a sphere) correlators of some TFT. That is

$$\langle O_{\alpha_1} O_{\alpha_2} \dots O_{\alpha_n} \rangle_{\text{exact}} = \langle O_{\alpha_1} O_{\alpha_2} \dots O_{\alpha_n} \rangle_{g=0}^{\text{TS}} \quad (4.1)$$

for any $\alpha_1 \dots \alpha_n$, where $\langle \dots \rangle^{\text{TS}}$ stands for correlation function in some effective TS topological theory. This claim is non-trivial since if the r.h.s. of (4.1) is a topological theory, then an unlimited number of $g = 0$ correlation functions are actually determined by a small number of OP coefficients, because of the requirement that, for any observable A , $\langle O_{\alpha} O_{\beta} A \rangle_g^{\text{TS}} = C_{\alpha\beta}^{\text{TSY}} \langle O_{\gamma} A \rangle_g^{\text{TS}}$. This strong requirement is, however, trivially fulfilled by (4.1), since

$$\langle O_{\alpha} O_{\beta} A \rangle_0^{\text{TS}} = \sum_g \lambda^g \langle O_{\alpha} O_{\beta} A \rangle_g = \sum_g \lambda^{g-1} C_{\alpha\beta}^{\gamma} \langle O_{\gamma} A \rangle_g = C_{\alpha\beta}^{\gamma} \langle O_{\gamma} A \rangle_0^{\text{TS}}. \quad (4.2)$$

Thus the correlators of the effective TS theory do satisfy the conditions required of $g = 0$ correlators of a topological theory with operator product coefficients that are identical to those of the world sheet topological theory: $C_{\alpha\beta}^{\text{TSY}} = C_{\alpha\beta}^{\gamma}$.

Once the consistency of our assumption is settled, we shall identify which target space TSM corresponds to the original world-sheet TSM of (3.4)–(3.12). In other words one searches for coupling constants $\lambda_{\text{TS}} \beta_{\text{TS}}$ such that

$$\langle O^n \rangle_{\text{exact, ws}} = \langle \langle O^n \rangle \rangle_{g=0}^{\text{TS}}. \quad (4.3)$$

Two conclusions may be drawn already. Since the operator product coefficients of the world-sheet theory are the same as those of the TS theory, Eq. (3.11) implies that

$$\beta_{\text{TS}} = \beta_{\text{ws}}. \quad (4.4)$$

From (3.4) applied to the TS theory, it is clear that μ_{TS} must be different from zero as the exact correlation functions on the world sheet are non-zero for both even and odd values of n , while a $\mu = 0$ theory leads, on a sphere, only to non-zero odd-point functions. We thus attempt to find an effective theory in a CP1 TFT with a non-zero μ modification. As the dependence on μ has emerged, let us consider already the more general problem where μ_{ws} was introduced as an additional world sheet parameter. In that case one should obtain

$$\langle O^n \rangle_{\text{exact, ws}}(\lambda_{\text{ws}}, \beta_{\text{ws}}, \mu_{\text{ws}}) = \langle \langle O^n \rangle \rangle_{g=0, \text{TS}}(\lambda_{\text{TS}}, \beta_{\text{TS}}, \mu_{\text{TS}}). \quad (4.5)$$

Using (4.2) it will be possible to satisfy the full set of relations (4.5) if one has satisfied it for the two cases $n = 0, 1$.

More explicitly, one should have

$$\frac{1}{\lambda_{\text{ws}}} \left(\frac{\exp(\tilde{\mu}_{\text{ws}})}{1 - \tilde{\lambda}_{\text{ws}} \exp(-\tilde{\mu}_{\text{ws}})} + \frac{\exp(-\tilde{\mu}_{\text{ws}})}{1 + \tilde{\lambda}_{\text{ws}} \exp(\tilde{\mu}_{\text{ws}})} \right) = \frac{2 \sinh(\tilde{\mu}_{\text{TS}})}{\tilde{\lambda}_{\text{TS}}}, \quad (4.6)$$

$$\frac{\beta^{1/2}}{\lambda_{\text{ws}}} \left(\frac{\exp(\tilde{\mu}_{\text{ws}})}{1 - \tilde{\lambda}_{\text{ws}} \exp(-\tilde{\mu}_{\text{ws}})} + \frac{\exp(-\tilde{\mu}_{\text{ws}})}{1 + \tilde{\lambda}_{\text{ws}} \exp(\tilde{\mu}_{\text{ws}})} \right) = \frac{2\beta^{1/2} \cosh(\tilde{\mu}_{\text{TS}})}{\tilde{\lambda}_{\text{TS}}}, \quad (4.7)$$

where $\tilde{\lambda}, \tilde{\mu}$ were defined in (3.26). It will be instructive to cast the l.h.s. of (4.6) in the form

$$\langle \langle 1 \rangle \rangle_{\text{exact, ws}} = \frac{2 \sinh \left(\tilde{\mu}_{\text{ws}} + \frac{1}{2} \ln \left(\frac{1 + \tilde{\lambda}_{\text{ws}} \exp(\tilde{\mu}_{\text{ws}})}{1 - \tilde{\lambda}_{\text{ws}} \exp(-\tilde{\mu}_{\text{ws}})} \right) \right)}{\tilde{\lambda}_{\text{ws}} \left(1 - \tilde{\lambda}_{\text{ws}} \exp(-\tilde{\mu}_{\text{ws}}) (1 + \tilde{\lambda}_{\text{ws}} \exp(\tilde{\mu}_{\text{ws}})) \right)}. \quad (4.8)$$

The solution thus is

$$\begin{aligned}\bar{\lambda}_{TS} &= \bar{\lambda}_{WS} \sqrt{\frac{\beta_{TS} = \beta_{WS}}{(1 - \bar{\lambda}_{WS} \exp(-\bar{\mu}_{WS})) (1 + \bar{\lambda}_{WS} \exp(\bar{\mu}_{WS}))}} \\ \bar{\mu}_{TS} &= \bar{\mu}_{WS} + \frac{1}{2} \ln \left(\frac{1 + \bar{\lambda}_{WS} \exp(\bar{\mu}_{WS})}{1 - \bar{\lambda}_{WS} \exp(-\bar{\mu}_{WS})} \right).\end{aligned}\quad (4.9)$$

The CP1 TSM has passed the test of being itself a product of string theory. We have shown that any CP1 TSM is an effective theory of some string theory. Moreover the exact topological string theory is itself a string theory of a CP1 TSM, albeit with different couplings (backgrounds).

In terms of the parameters $\eta_{\pm}$ of (3.28), the relation between TS and WS parameters is, by (3.29),

$$\eta_{\pm}^{TS} = \frac{\eta_{\pm}^{WS^2}}{\eta_{\pm}^{WS} - 1} \quad (4.10)$$

or, in terms of $\frac{1}{\eta_{\pm}}$,

$$\frac{1}{\eta_{\pm}^{TS}} = \frac{1}{\eta_{\pm}^{WS}} \left(1 - \frac{1}{\eta_{\pm}^{WS}} \right). \quad (4.11)$$

Notice that according to (2.20), $(1/\eta_{\pm})^{g-1}$ is the weight of the genus g contribution to the correlation function of $O_{\pm}$. Equation (4.11) gives $1/\eta_{\pm}^{TS} \leq 1/\eta_{\pm}^{WS}$.

Thus the mapping (4.11) has only a trivial fixed point, that is only the $g=0$ classical limit reproduces itself numerically. A weaker "bootstrap" exists however, as the TFT CP1 theory reproduces a CP1-like TFT. If for some reason one would wish to iterate the procedure one would obtain a convergence of any initial data to the classical limit. Note that that in Eq. (4.10) there is a complete definition of the effective TS theory. The Lagrangian realization of the theory is given by Eqs. (2.21) and (3.20).

4.2 More effective field theories

The TS theory was defined in (4.1) by identifying its $g=0$ correlation functions with the exact ones of the WS theory. The special role of $g=0$ comes from the natural identification of the target space of CP1 as a sphere. Note, however, that if one chooses the TS to be, less

naturally, a genus- g Riemann surface, $g \neq 1$, a consistent topological effective TS theory can still be defined. For an even g , one just replaces the $g=0$ correlation function in the r.h.s. of (4.5) by a genus- g correlation, identifying the effective TS correlators at genus g with the exact WS ones. One then obtains, for the parameters of this TS theory,

$$\begin{aligned}\bar{\lambda}_{TS} &= \left(\bar{\lambda}_{WS} \sqrt{\frac{1}{(1 - \bar{\lambda}_{WS} \exp(-\bar{\mu}_{WS})) (1 + \bar{\lambda}_{WS} \exp(\bar{\mu}_{WS}))}} \right)^{1-g} \\ \bar{\mu}_{TS} &= \frac{1}{1-g} \left(\bar{\mu}_{WS} + \frac{1}{2} \ln \left(\frac{1 + \bar{\lambda}_{WS} \exp(\bar{\mu}_{WS})}{1 - \bar{\lambda}_{WS} \exp(-\bar{\mu}_{WS})} \right) \right).\end{aligned}\quad (4.12)$$

Note that for large g , λ_{TS} tends to 1 (singularity) and μ_{TS} tends to zero. For an odd $g \neq 1$ this procedure leads to a similar result, provided we identify odd-TS correlations with even-WS correlations and vice versa. In particular

$$\begin{aligned}\langle 1 \rangle_{\text{exact, WS}} &= \langle O \rangle_{TS, g \text{ odd}} \\ \langle O \rangle_{\text{exact, WS}} &= \langle 1 \rangle_{TS, g \text{ odd}}.\end{aligned}\quad (4.13)$$

Otherwise, without this switch, $\bar{\mu}_{TS}$ will turn out to be complex, which may also be interesting. From the world-sheet point of view the above results mean that the set of correlation functions given by eqs. (3.24), (3.25) do not fix the genus of the two-dimensional theory of the type (3.20). For any genus g there exists a g -dependent set of couplings to be used in (3.20) which reproduce the same correlation functions. All these different theories are equivalent. This result is true for any TFT with n members of the ring. The correlation functions (2.20) are reproduced by the Lagrangian (2.21). All theories with a constant $[\eta_{\alpha}(g)]^{1-g}$ are equivalent. In terms of the couplings appearing in (2.21) they are related through (2.22)

$$\left[\eta_{\alpha}^{TS} \exp(\mu_{\alpha}^{TS}(g_1)) \right]^{1-g_1} = \left[\eta_{\beta}^{TS} \exp(\mu_{\beta}^{TS}(g_2)) \right]^{1-g_2}$$

where α and β belong to any permutations of $1, 2, \dots, n$.

4.3 A zero-point function which is not a TS cosmological constant

It has been suggested [7] that the observed strict upper bound for the cosmological constant actually follows from the latter's vanishing in

a finite scale-invariant theory. This result does not depend on the manner in which the underlying scale symmetry is realized.

It has become part of the string-theory culture to associate the world sheet zero-point function, evaluated at $g = 1$, with the TS cosmological constant. Topological field theories do have in quite a few cases [1, 2, 4], including the one discussed in this paper, a non-zero, zero-point function. Since in our case we identify the target space theory as a topological theory, it cannot have a cosmological constant. Recall that in ordinary string theory the zero-point function becomes a cosmological term, since after its calculation one has to integrate over the D constant zero modes of the string, which couples this zero-point function to $\int \sqrt{G} d^D x$. This is not the case for topological theory. Here we expect the zero-point function to couple to some topological D form in target space. For the case of two-dimensional TS, the only choice is to couple to the curvature form and thus to the Euler number of the surface. Since here only one target space with a fixed topology was studied, we cannot check this directly. We can, however, see that the genus-one contribution to the zero-point function of a general TSM is indeed proportional to the Euler number of the target space. For any theory, the $g = 1$ zero-point function is the number of physical states. As explained in Section 2, in a TSM there is a physical state, or a physical operator, for every element of the De Rahm cohomology of the target space. The $g = 1$ zero-point function is then the super trace over this De Rahm cohomology:

$$\langle 1 \rangle_{g=1} = \sum (-1)^i b_i, \quad (4.14)$$

which is the Euler number (b_i are the Betti numbers).

We wish to interpret the result in the following manner: $\langle 1 \rangle_1$ represents a topological term in the effective action of the TFT, the Euler number for the case of TS Riemann surfaces in general, and for CP1 in particular. For the CP1 case we have seen that the effective action contains indeed the topological term, the Euler number.

A non-zero value for $\langle 1 \rangle_1$ does not mean, then, that the effective TS theory has necessarily a non-zero cosmological constant. This

observation should hold also for non-topological theories: they may also have topological terms in their effective theories.

4.4 Non-perturbative duality

Duality relations connect apparently different points in the parameter space of a theory, which actually correspond to the same theory. In string theory one can distinguish between perturbative and non-perturbative duality. The former connects points which are equivalent, genus by genus: such is the familiar duality relation between a bosonic string moving on a circle of radius R and one moving on a circle of radius $1/R$. For a non-perturbative duality the equivalence is even more stringy, being true only for the sum over all genera. Our effective topological target space theories for world-sheet CP1 TSM provide examples for both kinds of duality. Using the parameters of the previous section define,

$$\eta^\pm = \frac{1}{\eta_\pm} \quad \eta_{TS}^\pm = \eta_{ws}^\pm (1 - \eta_{ws}^\pm). \quad (4.15)$$

Eq. (4.11) reads

Two types of duality relations arise. One can simply interchange the two parameters:

$$\eta_{ws}^+ \rightarrow \eta_{ws}^- \quad \eta_{ws}^- \rightarrow \eta_{ws}^+. \quad (4.16)$$

This is a perturbative duality, preserving the nature of the world-sheet theory genus by genus up to a transformation of the type (2.16), which interchanges O_+ and O_- . This transformation has a line of fixed points corresponding to $\eta^+ = \eta^-$. The target space theory is of course also preserved up to a similar transformation. Another duality transformation is suggested by (4.15).

$$\begin{aligned} \eta_{ws}^+ &\rightarrow 1 - \eta_{ws}^+ & \eta_{ws}^- &\rightarrow \eta_{ws}^- \\ \eta_{ws}^- &\rightarrow \eta_{ws}^- & \eta_{ws}^- &\rightarrow 1 - \eta_{ws}^- \end{aligned} \quad (4.17)$$

or both.

For n numbers of the chiral ring any permutation among the η_α is related to a generalized form of perturbative duality. Recall that by varying η_α one is also obtaining equivalent theories with different world-sheet topologies. Fixing $\eta_\alpha(0)$ for genus zero the equivalent theory on genus g is given by $\eta_\alpha(g) = \{\eta_\alpha(0)\}_{1-g}$ ($g=1$ is defined separately). The non-perturbative duality also persists with η_α going into $1 - \eta_\alpha$ and η_β into η_β for all possible partitions of the n -ring members into two groups. Had η been a periodic function of some meaningful parameter, the similarity with the extended duality of string background would be more apparent.

These transformations clearly change the world-sheet theory. Genus by genus, every correlation function is changed. However, (4.15) shows that the TS theory, resulting from summing all genera, is preserved. The transformation (4.17) is then an example, in the topological regime, of a non-perturbative duality. The target space couplings are manifestly invariant under this transformation, which has a fixed point at $\eta^\pm = 1/2$. Eq. (4.17) can be expressed in a somewhat more suggestive notation. Define $p^\pm$ by $\eta^\pm = 1/2(1 - p_\pm)$. Eq. (4.17) takes the form:

$$\begin{array}{ll} p_+ \rightarrow -p_+ & p_- \rightarrow p_- \\ \text{or,} & \\ p_+ \rightarrow p_+ & p_- \rightarrow -p_- \end{array}$$

This is reminiscent of duality transformations in bosonic conformal field theories where $p_L \rightarrow -p_L$ and $p_R \rightarrow p_R$, or vice versa, p_L and p_R being the left- and right-handed part of the target-space momentum. Note also that Eq. (3.29) for the TS n -point function of $O_\pm$ takes, in these variables the form

$$\langle (O_\pm)^n \rangle^{\text{TS}} = \frac{4}{1 - p_\pm^2},$$

which becomes singular when the $p_\pm$ "momentum" variables approach ± 1 , i.e. in the limit $\lambda = 1$, where the genus sum breaks down. We are tempted to conjecture that the topological phase, at such a pole, obtains features of the conventional phase of gravity.

5. DISCUSSION

We have shown that the correlation functions of the exact CP1 topological string theory can be derived from an effective Lagrangian which is itself in the same moduli space as the topological CP1 theory. Moreover, we have shown in Eq. (4.2) that the correlation functions of any topological string model with n observables can be expressed in terms of a $g = 0$ two-dimensional theory for which the values of the moduli are explicitly related to those of the string theory. In the case of CP1 this two-dimensional theory could well serve as the target space. For larger values of n we would need to find also an effective theory, defined generally on a higher-dimensional space, which would reproduce the same correlation functions. We would also need to face the fact that while, by definition, all topological theories are independent of the WS metric there exist theories which do depend on non-topological TS properties. We expect that this feature will be emulated by the TTS structure of the effective theory. We end by discussing a set of possible lessons.

The effective Lagrangian was constructed to reproduce the physical correlation functions, we note that if we would like to study an analog to Green's functions in the theory, we would need to relax the BRST invariance condition and, in a $N = 2$ SUSY formulation, we would have to deal with particles not obeying the spin-statistics theorem.

Assuming that we failed to find an effective model, we may try to check if restricting the topology of the target space was not a fault. We have tried to check this by considering as a target space two-dimensional surfaces with handles. It can be shown that as long as one is considering as observables just the cohomology classes of the surface, only the constant maps contribute to those correlation functions that are calculated using Eq. (2.2). In that sector we do not obtain an effective theory without allowing topological changes in the TS theory. The definition of the full topological theory when the target space is not simply connected is unknown at this stage [8]. The argument can be turned around and, for the CP1-type cases, even exact topological string theory does not require topological fluctuations in the target space.

We have also wondered if the appearance of a pole in correlation functions could be related to another phase of gravity: we note that in the topological phase the Wheeler-DeWitt (WD) equation holds; thus, if it would be related to another phase of general covariant systems, the WD equation would hold in that phase as well. Any restriction to positive metric would be a result of a spontaneous symmetry breaking.

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