Analytic Image Reconstruction in PVI Using the 3D Radon Transform

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Abstract

Algorithms have been derived for three dimensional image reconstruction in positron volume imaging (PVI) using the inversion of the three dimensional Radon Transform (RT) for a truncated cylindrical detector geometry. The RT is formed by histogramming events according to the planes in which they lie, with a weight determined by an angular acceptance function. We describe a fast back-projection method for the RT in which the computation time is found to be up to 20 times faster with the new algorithm compared to the traditional X-ray Transform (XT) backprojection method. Monte Carlo simulations show that statistical noise levels in images reconstructed from complete projections with the new RT algorithm are comparable to those obtained using the Fourier Transform (FT) inversion of the XT [1].

I. INTRODUCTION

The RT of a three dimensional activity distribution is the set of plane integrals of that distribution. It has been shown that events in PVI can be histogrammed into the 3D RT [2]. Reconstruction of an image proceeds by the well-known inversion formula [3-5] for the RT. It has been demonstrated that most oblique events can be included in a RT histogram for the reconstruction of an image in a single pass [6]. This is an advantage over the XT inversion methods that have a spatial invariance requirement [7,1] and therefore include these oblique events in an iteration step [8].

In this paper, we derive reconstruction formulas for the truncated cylindrical detector commonly used in PVI cameras. Two types of RT histogram are described: a 4D histogram that retains the original event-line direction information, and a more concise 3D histogram which records only the planes each event lies in.

We also describe a technique for speeding the voxel driven backprojection of the 3D RT by an order of magnitude over that of the XT. Using Monte Carlo simulations, the point response, the noise properties and the reconstruction times of the new RT algorithms are compared to the XT inversion method of Colsher [1].

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II. THEORY

A. Introduction

The RT of an activity distribution $f(\mathbf{x})$ in three dimensions is the two dimensional projection of $f(\mathbf{x})$. The RT, denoted by $\tilde{f}(p, \hat{\mathbf{n}})$, represents the integrals of $f(\mathbf{x})$ over the planes $p = \mathbf{x} \cdot \hat{\mathbf{n}}$ [3]

$$\tilde{f}(p, \hat{\mathbf{n}}) = \int d\mathbf{x} f(\mathbf{x}) \delta(p - \mathbf{x} \cdot \hat{\mathbf{n}}) \quad (1)$$

where δ is the Dirac delta function and $\hat{\mathbf{n}}$ is a unit direction vector. The function $\tilde{f}(p, \hat{\mathbf{n}})$ is a function of three variables, the distance of a plane from the origin p and the two spherical polar angles required to specify the vector $\hat{\mathbf{n}}$: θ_n , and φ_n .

An unfiltered backprojection image $\sigma_R(\mathbf{x})$ can be formed by integrating $\tilde{f}(p, \hat{\mathbf{n}})$ over all orientations $\hat{\mathbf{n}}$ in a complete hemisphere of directions $\mathbf{H}$,

$$\sigma_R(\mathbf{x}) = \int_{\mathbf{H}} d\hat{\mathbf{n}} \tilde{f}(p, \hat{\mathbf{n}}) \quad (2)$$

where $p = \mathbf{x} \cdot \hat{\mathbf{n}}$. This smoothed image $\sigma_R(\mathbf{x})$ is inverted by [3]

$$f(\mathbf{x}) = -\frac{1}{(2\pi)^2} \nabla^2 \sigma_R(\mathbf{x}) \quad (3)$$

where ∇^2 is the 3D Laplacian operator.

To compare with the RT, the XT of a 3D function $f(\mathbf{x})$, is the one dimensional projection of $f(\mathbf{x})$ given by

$$g(\mathbf{x}_\tau; \hat{\boldsymbol{\tau}}) = \int_{-\infty}^{\infty} dt f(\mathbf{x}_\tau + t\hat{\boldsymbol{\tau}}) \quad (4)$$

which is a function of the two dimensional event-line direction vector $\hat{\boldsymbol{\tau}}$ and of the two dimensional position vector $\mathbf{x}_\tau$, normal to $\hat{\boldsymbol{\tau}}$. The vector $\mathbf{x}_\tau$ is the projection of the 3D vector $\mathbf{x}$ onto the plane normal to $\hat{\boldsymbol{\tau}}$, (i.e. $\mathbf{x}_\tau = \mathbf{x} - (\mathbf{x} \cdot \hat{\boldsymbol{\tau}})\hat{\boldsymbol{\tau}}$), where $\hat{\boldsymbol{\tau}}$ is the unit vector specifying the orientation of the direction of integration and is parameterized by the two spherical polar angles $\theta_\tau, \varphi_\tau$.

In this paper, the relationship between the RT and the XT will be shown for the case of complete [9] (i.e. spatially invariant) XT projections, and then the extension of the RT method to the spatially varying case will be discussed.

The recovery of $f(\mathbf{x})$ from $g(\mathbf{x}_\tau; \hat{\boldsymbol{\tau}})$ was described by Orlov [7] for the case of reconstruction from complete XT projections. Consider the region $\mathbf{G}$ on the unit sphere mapped out by event vectors $\hat{\boldsymbol{\tau}}$ such that $\hat{\boldsymbol{\tau}} \cdot \hat{\mathbf{z}} < \sin(\psi_{\max})$,

and $\psi_{\max}$ is the maximum event inclination angle (measured from the transaxial plane) such that no projection is truncated. Orlov showed that the following expression inverts the XT:

$$f(\mathbf{x}) = -\frac{1}{(2\pi)^2} \nabla^2 \int_{\mathbf{G}} d\hat{\tau} \tilde{g}(\mathbf{x}_r; \hat{\tau}). \quad (5)$$

In (5), $d\hat{\tau}$ is a solid angle element on the unit sphere and the backprojection integral can be explicitly written

$$\int_{\mathbf{G}} d\hat{\tau} = \int_0^\pi d\varphi_\tau \int_{\theta_{\min}}^{\theta_{\max}} d\theta_\tau \sin \theta_\tau \quad (6)$$

and

$$\tilde{g}(\mathbf{x}_r; \hat{\tau}) = \int_{\Sigma_\tau} dS' \frac{g(\mathbf{x}_r - \mathbf{x}_r'; \hat{\tau})}{|\mathbf{x}_r'| L(\hat{\tau} \times \mathbf{x}_r')}. \quad (7)$$

In (7), Σ_τ is the XT projection plane normal to the projection vector $\hat{\tau}$, and dS' is an area element on Σ_τ corresponding to the integration variable $\mathbf{x}_r'$. For a two dimensional region $\mathbf{G}$,

$$L_{2D} \equiv L(\hat{\tau} \times \mathbf{x}_r) = \int_{\mathbf{G}} dG_{r'} \delta(\hat{\tau}' \cdot (\hat{\tau} \times \hat{\mathbf{x}}_r)). \quad (8)$$

The word definition of L is given by Orlov: " L is a function only of the direction of its argument. L_{2D} is equal to the length of the arc of the great semicircle lying in the region $\mathbf{G}$ and passing through the ends of the vectors $\hat{\tau}$ and $\hat{\mathbf{x}}_r$." This function is just the angular extent of the region $\mathbf{G}$ in the plane with normal $\hat{\mathbf{n}} = \hat{\tau} \times \mathbf{x}_r$, and is calculated in Section IIE.

The filtered backprojection of (5) has been implemented [10] in object space as the backprojection of the filtered projections formed by the convolution

$$\tilde{g}^*(\mathbf{x}_r; \hat{\tau}) = g(\mathbf{x}_r; \hat{\tau}) * \nabla^2 \frac{1}{|\mathbf{x}_r| L(\hat{\tau} \times \mathbf{x}_r)}. \quad (9)$$

This object space recovery operator has been shown [10] to be equivalent to the FT inversion method of Colsher [1].

B. The Orlov Recovery Operator Related to the Radon Transform

The inversion of the 3D RT for a truncated detector can be derived from the Orlov recovery operator.

The convolution in (7) over the XT plane Σ_τ can be commuted as follows:

$$\tilde{g}(\mathbf{x}_r; \hat{\tau}) = \int_{\Sigma_\tau} dS' \frac{g(\mathbf{x}_r'; \hat{\tau})}{|\mathbf{x}_r - \mathbf{x}_r'| L(\hat{\tau} \times (\mathbf{x}_r - \mathbf{x}_r'))}. \quad (10)$$

Taking the polar coordinates r' and θ_r' with their origin at $\mathbf{x}_r$, (Figure 1), as the primed integration variables on the plane Σ_τ , that is,

$$\mathbf{r}' = \mathbf{x}_r' - \mathbf{x}_r, \quad dS' = r' dr' d\theta_r',$$

Figure 1. The coordinates of the two dimensional RT on Σ_τ .

we have, from (10)

$$\tilde{g}(\mathbf{x}_r; \hat{\tau}) = \int_{-\pi/2}^{\pi/2} d\theta_r' \int_{-\infty}^{\infty} dr' \frac{g(\mathbf{x}_r + \mathbf{r}'; \hat{\tau})}{L(\hat{\tau} \times \mathbf{r}')}. \quad (11)$$

noting that $L(\hat{\tau} \times \mathbf{r}')$ is an even function of its argument.

Changing the angular variable θ_r' to ξ , which specifies the angle to the direction of the normal to $\mathbf{r}'$, denoted by $\hat{\mathbf{n}}$, (Figure 1)

$$\xi = \theta_r' + \frac{\pi}{2}, \quad \hat{\mathbf{n}} = \hat{\mathbf{n}}(\xi) = (\cos \xi, \sin \xi)$$

$$\hat{\mathbf{n}}_\perp = \hat{\mathbf{n}}_\perp(\xi) = (\sin \xi, -\cos \xi), \quad \mathbf{r}' = r' \hat{\mathbf{n}}_\perp.$$

the integral becomes

$$\tilde{g}(\mathbf{x}_r; \hat{\tau}) = \int_0^\pi d\xi \int_{-\infty}^{\infty} dr' \frac{g(\mathbf{x}_r + r' \hat{\mathbf{n}}_\perp(\xi); \hat{\tau})}{L(\hat{\tau} \times \mathbf{r}')}. \quad (12)$$

L is a function only of the direction of its argument, so that $L(\hat{\tau} \times \mathbf{r}') = L(\hat{\tau} \times \hat{\mathbf{n}}_\perp) = L(\hat{\mathbf{n}}(\xi))$. Making the substitution $s = r' + \mathbf{x}_r \cdot \hat{\mathbf{n}}_\perp$ (Figure 1), and using $\hat{\mathbf{x}}_r = p \hat{\mathbf{n}} + (\mathbf{x}_r \cdot \hat{\mathbf{n}}_\perp) \hat{\mathbf{n}}_\perp$, we have

$$\tilde{g}(\mathbf{x}_r; \hat{\tau}) = \int_0^\pi \frac{d\xi}{L(\hat{\mathbf{n}}(\xi))} h(p, \xi; \hat{\tau}) \quad (13)$$

where $p = \mathbf{x}_r \cdot \hat{\mathbf{n}}(\xi)$ and where the function

$$h(p, \xi; \hat{\tau}) = \int_{-\infty}^{\infty} ds g(p \hat{\mathbf{n}}(\xi) + s \hat{\mathbf{n}}_\perp(\xi); \hat{\tau}) \quad (14)$$

is the two dimensional RT of the XT $g(\mathbf{x}_r; \hat{\tau})$ on Σ_τ [3]. Equation (14) expresses the line integrals of the function $g(\mathbf{x}_r; \hat{\tau})$ parameterized by the angle ξ and the distance p .

The fact that (14) is the explicit two dimensional RT of $g(\mathbf{x}_r; \hat{\tau})$ is not used in the reconstruction. Instead it is noted that the function $h(p, \xi; \hat{\tau})$ is a redundant set of

plane integrals of the activity distribution of the object. To make this explicit we substitute (4) into (14)

$$h(p, \xi; \hat{\tau}) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt f(p\hat{n} + s\hat{n}_{\perp} + t\hat{\tau}) \quad (15)$$

and $p = \mathbf{x} \cdot \hat{n} = \mathbf{x} \cdot \hat{n}$ since $\hat{\tau} \cdot \hat{n} = 0$.

The function $h(p, \xi; \hat{\tau})$ is the integral of $f(\mathbf{x})$ over the plane given by $p = \mathbf{x} \cdot \hat{n}$ as in (1). If we consider a fixed plane normal $\hat{n}$ and move the event direction vector $\hat{\tau}$ through $\mathbf{G}$ such that $\hat{\tau} \cdot \hat{n} = 0$, the orientation angle ξ of the RT plane in the XT projection, Σ_{τ} , changes with $\hat{\tau}$. At each orientation $\hat{\tau}$, an independent measurement of the plane integral is provided according to (14). These measurements can be combined to form the 3D RT of the object. The recovery operator (5) is written

$$f(\mathbf{x}) = -\frac{1}{(2\pi)^2} \nabla^2 \int_{\mathbf{G}} d\hat{\tau} \int_0^{\pi} d\xi \frac{h(\mathbf{x} \cdot \hat{n}, \xi; \hat{\tau})}{L(\hat{n}(\xi); \hat{\tau})} \quad (16)$$

where the function L weights the redundant plane integral measurements of $h(p, \xi; \hat{\tau})$ to form the plane integrals of the RT. The triple angular integral over θ_{τ} , φ_{τ} , and ξ is the backprojection over the weighted planes of the RT. This backprojected image is inverted to form the true image by the application of the Laplacian in image space.

To implement the inversion as a filtered backprojection (i.e. to move the 3D Laplacian filtering in image space to the one dimensional filtering of the projections), the order of integration and differentiation may be interchanged. Note that for any function $u(\mathbf{x}) = u(\mathbf{x} \cdot \hat{n})$, the Laplacian takes the form

$$\nabla^2 u(\mathbf{x} \cdot \hat{n}) = \frac{\partial^2}{\partial p^2} u(p), \quad p = \mathbf{x} \cdot \hat{n}. \quad (17)$$

Taking the Laplacian of the inner integrand of (16), the backprojection formula becomes

$$f(\mathbf{x}) = -\frac{1}{(2\pi)^2} \int_{\mathbf{G}} d\hat{\tau} \tilde{g}^*(\mathbf{x}_{\tau}; \hat{\tau}) \quad (18)$$

where

$$\tilde{g}^*(\mathbf{x}_{\tau}; \hat{\tau}) = \int_0^{\pi} \frac{d\xi}{L(\hat{n}(\xi); \hat{\tau})} \frac{\partial^2}{\partial p^2} h(p, \xi; \hat{\tau}) \quad (19)$$

with $p = \mathbf{x} \cdot \hat{n}$.

The RT integrated in this manner is equivalent to the filtered XT plane, Σ_{τ} , of (9). Using the fact that the RT and the FT are related by a one dimensional FT with the transform variables p and ν along $\hat{n}$ [3], it has been shown that the inversion method of (18) and that of Colsher [1] are analytically equivalent [11].

Equation (16) or equivalently (18), is one analytic form of the backprojection of the 3D RT for the two dimensional region of complete projections $\mathbf{G}$. The expressions represent the inversion of the 3D RT without the explicit formation of the three parameter RT. The redundant angular dimension is used to retain the original event direction information. This formulation has the feature that

Figure 2. The limits of α on $\mathbf{G}$ in the plane normal to $\hat{n}$.

the backprojection can be performed along lines (rather than on planes) by first performing the integration over the variable ξ in (13) on a regular tessellation in the Σ_{τ} plane [11]. This implementation reduces the backprojection time over that required to backproject on planes and is shown in this paper to also speed the backprojection of the 3D RT as well. Results of the implementation of (18) were initially given in [11] and then in [12] as a comparison with other reconstruction methods.

C. The Explicit Radon Transform

It is useful to rewrite (16) to show the explicit form of the 3D RT, as it makes clear the relationship of the Orlov XT inversion formula to the RT inversion formula used elsewhere [4,5,2]. The integration in (16) is over all plane normals $\hat{n}$ perpendicular to a given vector $\hat{\tau}$ and then over all vectors $\hat{\tau}$ in the region $\mathbf{G}$.

The plane integral $h(p, \xi; \hat{\tau})$ has an equivalent representation $h(p, \alpha; \hat{n})$ where the event orientation vector $\hat{\tau}$ and the plane orientation angle ξ are replaced by the event orientation angle α and the plane orientation vector $\hat{n}$, where α is measured about $\hat{n}$ (Figure 2).

With this change of coordinates, (16) can be rewritten

$$f(\mathbf{x}) = -\frac{1}{(2\pi)^2} \nabla^2 \int_{\mathbf{H}} d\hat{n} \int_{\mathbf{G}} d\alpha \frac{h(\mathbf{x} \cdot \hat{n}, \alpha; \hat{n})}{L(\hat{n})} \quad (20)$$

where $\mathbf{H}$ is the hemispherical region on the unit sphere that defines the complete set of directions of the normals $\hat{n}$ to the plane integrals of the RT and

$$\int_{\mathbf{H}} d\hat{n} = \int_{-\pi/2}^{\pi/2} d\varphi_n \int_{-\pi/2}^{\pi/2} d\theta_n \sin \theta_n. \quad (21)$$

The limits of α on $\mathbf{G}$ are shown in Figure 2. These limits are independent of p as the region $\mathbf{G}$ has been assumed spatially invariant. The value of a given plane integral, $h(p, \alpha; \hat{n})$, for a given p and $\hat{n}$, is, for each α , an independent measurement of $f(p, \hat{n})$, the 3D RT of the function $f(\mathbf{x})$. Note that

$$\int_{\alpha_1}^{\alpha_2} d\alpha = \int_{\mathbf{G}} d\hat{\tau}' \delta(\hat{\tau}' \cdot \hat{n}) \quad (22)$$

is just the definition of the Orlov L_{2D} function given in (8). A measured plane integral of the RT, $\tilde{f}(p, \hat{n})$, is obtained by the weighted integration of the redundant measurements of that plane integral, $h(p, \alpha; \hat{n})$, over the angle α [6]:

$$\tilde{f}(p, \hat{n}) = \frac{1}{L(\hat{n})} \int_{\alpha_1}^{\alpha_2} d\alpha h(p, \alpha; \hat{n}) \quad (23)$$

For a cylindrically symmetrical detector and central field-of-view (FOV), the limits of α are

$$-\alpha_1 = \alpha_2 = \frac{L(\hat{n})}{2}.$$

In the absence of statistical errors, $\tilde{f}(p, \hat{n}) = h(p, \alpha; \hat{n})$ is independent of α for a given p and $\hat{n}$. The inner integral over $\mathbf{G}$ in (20) is the statistical estimate of the 3D RT formed by averaging the independent statistical measurements of the plane integrals of (15). The weight function $L(\hat{n})$ is the detector acceptance angle for measuring a given plane integral of $f(\mathbf{x})$.

Then (20) is just the inversion of the 3D RT, $\tilde{f}(p, \hat{n})$, formed by the statistical average of its measurements. Using the result of (17),

$$f(\mathbf{x}) = -\frac{1}{(2\pi)^2} \int_{\mathbf{H}} d\hat{n} \frac{\partial^2}{\partial p^2} \tilde{f}(p, \hat{n}), \quad p = \mathbf{x} \cdot \hat{n}.$$

Explicitly

$$f(\mathbf{x}) = -\frac{1}{(2\pi)^2} \int_{\mathbf{H}} d\hat{n} \int_{-\frac{L(\hat{n})}{2}}^{\frac{L(\hat{n})}{2}} \frac{d\alpha}{L(\hat{n})} \frac{\partial^2}{\partial p^2} h(p, \alpha; \hat{n}) \quad (24)$$

with $p = \mathbf{x} \cdot \hat{n}$.

An algorithm for image reconstruction in PVI using this formula has been described and implemented [6]. The integral over α is performed numerically event-by-event during histogramming of the RT. This histogramming results in a three dimensional representation of the RT in terms of the variables p , φ_n , and θ_n .

D. Fast Backprojection of the 3D Radon Transform

The 3D histogram $\tilde{f}(p, \hat{n})$ that results from the integration over α in (23) allows for the implementation of a fast backprojection using (5) or (18) and a reinterpreted (13) or (19). The 3D histogram $\tilde{f}(p, \hat{n})$, parameterized by the variables p , φ_n , θ_n , and containing all orientations of events, can be integrated as in (13) into a set of transaxial XT projections:

$$\tilde{g}(\mathbf{x}_\tau, \varphi_\tau, \pi/2) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta_n}{L(\hat{n})} \tilde{f}(p, \varphi_n, \theta_n) \quad (25)$$

with $p = \mathbf{x}_\tau \cdot \hat{n}$ and where $\hat{n}$ has been restricted to the transaxial directions, ($\theta_\tau = \pi/2$), and we have taken $\xi = \theta_n + \pi/2$ and $\varphi_\tau = \varphi_n + \pi/2$ in (13). In this manner, we have transformed the 3D RT into a transaxial 3D XT.

Figure 3. Determination of the Orlov L function from the detector geometry.

The backprojection time is then reduced to the sum of the integration time over the variable θ_n and the backprojection time for the resulting transaxial set of XT projections. This will be shown explicitly in Section IV to reduce the backprojection time by more than an order of magnitude compared to the FT method [1].

E. The Orlov L Function

For a truncated cylindrical or spherical detector geometry, symmetric on reflection through the origin, the function $L(\hat{n})$ is only a function of θ_n , that is $L(\hat{n}) = L(\theta_n)$. On this symmetric two dimensional region $\mathbf{G}$, and with an acceptance angle limited to $\psi_{\max}$, (8) becomes

$$L_{2D} = L(\hat{n}) = L(\theta_n) = \int_{\alpha_1}^{\alpha_2} d\alpha = 2\alpha_2 = 2 \arcsin d \quad (26)$$

where d (as shown in Figure 3) is the projected distance of the extent of the plane, with normal $\hat{n}$, from the origin to the edge of $\mathbf{G}$ when viewed from the projection where $\theta_\tau = \pi/2$. If in Figure 3, $|\theta_n| \leq \psi_{\max}$, then $\alpha_1 = \pi/2$ and

$$L(\theta_n) = \pi.$$

If $|\theta_n| > \psi_{\max}$, then from Figure 3,

$$|\sin \theta_n| = d \sin \psi_{\max}$$

so that

$$L(\theta_n) = L(\theta_n; \psi_{\max}) = 2 \arcsin \frac{\sin \psi_{\max}}{|\sin \theta_n|},$$

and therefore

$$L(\theta_n) = \begin{cases} \pi, & \text{if } |\theta_n| \leq \psi_{\max} \\ 2 \arcsin \frac{\sin \psi_{\max}}{|\sin \theta_n|}, & \text{otherwise} \end{cases} \quad (27)$$

The function $L(\theta_n)$ is the angular factor in the Colsher frequency space filter [1].

When $\mathbf{G}$ is a one dimensional region Γ , L has a different functional form [7]. For the transaxial directions, $\theta_\tau = \pi/2$, the expression given by Orlov reduces to

$$L_{1D} = L(\xi) = |\cos \xi|^{-1}. \quad (28)$$

This function is the solid angle correction in the integral over the region $\mathbf{H}$ in (21), with a change of coordinates.

As we have discussed in [6], in forming the 3D RT from events, it is possible to include oblique events from truncated XT projections. Recognizing that the integral in (22) over the angle α is over the extent of the measurement of complete RT plane integrals within the region $\mathbf{G}$ (of all spatially invariant or complete XT projections), it is a simple matter to extend that integral to accept a wider range in α for a given plane ($p = \mathbf{x} \cdot \hat{\mathbf{n}}$), namely to all α 's that allow the complete measurement of the plane integral within the dimensions of the detector.

The resulting angular acceptance function $A(p, \hat{\mathbf{n}})$, from reference [6], takes the role of the function $L(\hat{\mathbf{n}})$ in (16),(19),(20),(24). Results of such an implementation have been given in [6] for (20) and have shown that an improvement in statistical noise occurs when this wider range of α is used. In this paper we implement reconstructions only from complete XT projections; however, all the reconstruction formulas and the fast backprojection technique apply equally well to a 3D RT which includes events from partially measured XT projections.

III. HISTOGRAMMING

A 3D histogram in the discrete coordinates (p, φ_n, θ_n) is formed by binning detected events into planes. Each event represents part of the statistical measurement of the many plane integrals in all planes that contain the event line. To form the discrete histogram, a set of N_ξ plane normals, $\hat{\mathbf{n}}(\xi)$, are generated for each event direction $\hat{\mathbf{r}}$. Those normals satisfy $\hat{\mathbf{n}}(\xi) \cdot \hat{\mathbf{r}} = 0$. For each normal, the distance of the event line to each plane is computed by $p = \mathbf{x}_0 \cdot \hat{\mathbf{n}}(\xi)$, where $\mathbf{x}_0$ is some point on the event line (e.g., a detected end point). In the case of complete projections, all N_ξ histogram locations at the discrete coordinates p, φ_n, θ_n are incremented. (If we were to include events from incomplete projection planes, the oblique events would be added only to those RT histogram locations that correspond to completely measured RT plane integrals [6].)

The formation of the 3D RT histogram is more time consuming than for the XT because for each detected event many increments are made to the histogram rather than just one in the case of the XT. The number of increments, N_ξ , is chosen to adequately sample the RT according to the Nyquist criterion. Typically $N_\xi \sim 100$.

IV. BACKPROJECTION AND FILTERING

The implementation of (20) was performed by voxel driven backprojection. For the case of the planar backprojection of the RT (i.e. without the fast backprojection), the distance p of each of the N_v^3 image voxels from the origin is calculated for the N_n discrete planes directions $\hat{\mathbf{n}}$. The image voxel is incremented by the value obtained from the 1D interpolation of the histogram in the p dimension at the discrete plane coordinates corresponding to $\hat{\mathbf{n}}$.

The number of operations required for this planar backprojection is

$$M_{RT} = N_v^3 N_n N_{1D} \quad (29)$$

where N_{1D} is the number of operations required for the 1D interpolation. N_n is the number of angular bins needed to tessellate the unit hemisphere uniformly in solid angle [6].

By comparison, the voxel driven backprojection of the XT requires a 2D interpolation in each of the Σ_r planes and

$$M_{XT} = N_v^3 N_{\theta r} N_{\phi r} N_{2D}. \quad (30)$$

For bins of approximately the same solid angle in both methods and for small detector ring angles, ψ_{det} , the total number of bins in the XT is a few times less than for the RT. This difference is compensated for by the larger number of operations required for the 2D interpolation in the case of the XT. As such, both methods take approximately the same backprojection time. For larger ψ_{det} , $N_{\theta r}$ increases whereas N_n remains constant, and the RT backprojection time will be faster than that of the XT.

However, as mentioned in Section IID, we have described a method to speed the backprojection by forming a 3D XT from the 3D RT. The RT can be backprojected by first integrating the 3D RT (23) into a transaxial set of XT projections according to (25). The transaxial Σ_r planes are first tessellated into N_{tx}^2 discrete points $\mathbf{x}_r$. At each of these points, the distance p of each plane with normal orientation ξ is computed (Figure 1), and a 1D interpolation in p is used to add the RT contribution to that XT plane position. There are then only $N_{\phi r}$ XT projections to backproject. The number of operations required in this fast backprojection method for the RT is then

$$M'_{RT} = N_v^3 N_{\phi r} N_{2D} + N_{tx}^2 N_\xi N_{\phi r} N_{1D}. \quad (31)$$

Depending on the dimensions, this backprojection method can be more than an order of magnitude faster than either the planar backprojection of the 3D RT or the backprojection along event-lines of the XT.

In our previous paper [6], we presented a fast method of implementing the Laplacian post-filtering of the backprojected image. In this paper, the filtering of the RT was accomplished as in (24) for the direct planar backprojection method and as in (19) for the fast backprojection method. An apodized second derivative operator was applied with a 1D convolution along the p dimension. The apodizing window used was the image space form of the von Hann window with a cutoff frequency of $\nu_c = 1/2$ [2]. The frequency space form of this same window was applied to the Colsher filter for the XT reconstructions by the FT method.

V. RESULTS AND DISCUSSION

Monte Carlo simulations of point sources and of a uniform spherical source were used to study the new algorithms. The simulations did not include scatter, attenuation or detector resolution effects. Complete projections were generated for event inclination angles up to

Figure 4. Profiles through reconstructed point sources for (a) the planar 3D RT backprojection, (b) the 3D RT using the fast backprojection, and (c) the XT.

$\psi_{\max} = 20^\circ$. For each simulation, two histograms were generated: one for the discrete representation of the 3D RT and the other for the 4D XT. Images were reconstructed from the RT histogram using both the direct planar backprojection method and the fast backprojection method discussed in Section IV. These images were compared to reconstructions of the XT using the 2D Colsher [1] filter applied to the FT of the XT projections. In the latter case, the FT was implemented using the FFT [13].

The point response of the new algorithm was studied with a phantom made up of several point sources. The profiles through an image containing a central and a transversally displaced point source are shown in Figure 4 for the three reconstruction methods.

There is little difference between the images of Figure 4 (a) produced by the direct RT backprojection and of (c) produced by the XT inversion method implemented using the FT. Comparing the direct planar and fast backprojec-

tion RT images of Figure 4 (a) and (b), we find that for the fast RT in (b) the image has been slightly smoothed. This smoothing is expected because of the integration in (25) which requires a one dimensional interpolation in the p dimension of the histogram. A linear interpolation was used in this implementation.

For the study of noise propagation, a uniform spherical phantom of radius 5.2 cm was simulated using 8 million events. The binning was chosen to be fine enough so that binning artefacts were dominated by the statistical noise in the images. The linear dimensions were binned at 1.25 mm; the angular dimensions at 1.5° for the XT, and 0.6 msteradians for the RT. The RMS noise in the central 3.0 cm of the sphere is given in Table 1.

TABLE 1. Image Noise and Reconstruction Time		
	RMS Noise (%)	Reconstruction Time (sec)
3D RT	26	40K
FT	24	35K
3D RT (fast BP)	23	1.8K

For the above simulation, the histogram sizes were 91×9915 for the 3D RT and $91 \times 91 \times 126 \times 29$ for the XT. The images consisted of 1.3×10^6 voxels. The reconstructions were performed on a Sun SPARCstation 2.

The RMS noise level is similar in all the images. It is somewhat lower for the XT method than for the direct RT method. The noise is further reduced in the fast 3D RT method over the direct RT method because of the aforementioned smoothing.

The reconstruction times are also given in Table 1. For this simulation, the reconstruction time for the FT method was 12% faster than for the 3D RT. The fast backprojection method for the 3D RT was 22 times faster than the direct RT method and 20 times faster than the XT method. These times are for filtering and backprojection only, and do not include histogramming time. As discussed in Section III, the histogramming of the RT requires more computation than does the histogramming of the XT. In actual tomographs, the histogramming is performed by the front-end hardware. In our simulations, the software histogramming of the RT in the SPARC 2 took 10 times longer than for the XT.

This great advantage in backprojection time has been demonstrated in this paper in image reconstruction from complete projections. We have previously described [6] how events from incomplete XT projections can be included in the measurement of complete RT plane integrals. By the means of a spatially variant acceptance function $A(p, \hat{n})$,

some contribution from all oblique events is made to the 3D RT histogram. The fast backprojection technique can be applied to the statistically enhanced histogram formed by this method, the result being a fast filtered backprojection algorithm which includes all events in a single pass image. (The backprojection time is unaffected by including these oblique events as the histogram dimensions are unchanged.)

However, only for large detector angles, ψ_{det} , does the use of the spatially varying acceptance significantly reduce the RMS noise [6]. For detector rings with small ψ_{det} , the comparison between the FT method and the RT method should be made with the iteration technique used to complete projections [8]. In this case, the projection of events into the incomplete RT is more time consuming than the projection of the incomplete XT [11].

In this paper, we have transformed the 3D RT into a transaxial 3D XT using the fast backprojection technique. We note that the XT can be generated in this way along any curve defined by Orlov [14] as providing a necessary and sufficient set of XT projections. This method can be used to directly backproject an image at any orientation such as for sagittal or coronal sections. The transformation from the 3D RT to the XT also makes it possible to complete the missing data in incomplete XT projections directly from the RT histogram without the need for first reconstructing an image from the transaxial projections.

VI. CONCLUSIONS

We have presented algorithms for image reconstruction in PVI using the 3D RT. In Monte Carlo simulations, the noise performance of the algorithms is similar to that of the XT inversion method [1].

We have described a fast backprojection technique for the RT that is an order of magnitude faster than the XT method when backprojecting from spatially invariant projections. Further simulation studies are needed to compare processing times when using the reprojection technique [8] to include oblique events in either the RT or XT histograms.

For a given choice of bin sizes and when using the same apodizing filter, the images produced by the fast RT backprojection method are smoothed compared to those reconstructed using the planar backprojection of the 3D RT and compared to those of the XT inversion method. This is expected because of the intermediate integration step in the fast backprojection algorithm for the 3D RT.

Implementation of the RT method in an actual tomograph will require new front-end hardware to handle the greater histogramming computation required for the RT compared to the XT.

VII. ACKNOWLEDGEMENTS

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