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PARAMETERS IN THE ELECTROWEAK THEORY
I. POINT SPLITTING AS REGULARIZATION

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ABSTRACT

This series of papers deals with the problem of finding relations between the parameters of the Standard Model of Glashow, Weinberg and Salam. In this first paper, the method of point splitting, suitably modified, is proposed as a useful regularization. When applied to quantum electrodynamics, this point-split regularization is indeed found to have desirable properties. More specifically, to one loop, the photon is shown to remain massless and the photon-photon scattering amplitude gauge invariant. In the following papers, this point-split regularization is applied to the electroweak theory. In the second paper, a first mass relation is obtained from the quadratic divergences. In the third paper, a second mass relation is found from the Higgs interactions. From these two relations, the masses of the Higgs boson and the top quark are determined theoretically for the Standard Model as $m_H = 190$ GeV and $m_t = 120$ GeV.

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1. Introduction

The Standard Model of Glashow, Weinberg, and Salam [1] has recently been verified at LEP to high accuracy [2]. Thus most of particle physics can be said to be understandable in terms of the parameters of the Standard Model, nearly twenty of them.

It is perhaps reasonable to believe that these parameters are not independent of each other. The purpose of the present series of papers is to follow one specific line of investigation to find possible relations. Before motivating this line of investigation, it may be worthwhile to recall some of the wisdom about quantum field theory that has been accumulated during the last half a century.

- (1) Beginning in the nineteenforties, renormalization was invented and perfected by Tomonaga, Schwinger, Feynman, and Dyson [3]. A remarkable feature of the work of Dyson, who was the first one to carry renormalization to all orders, is that no regularization or cutoff is ever introduced. This is also true of the other papers of that era [4].
- (2) The Standard Model [1] has the nice feature of the cancellation [5] of anomalies [6]. If the anomalies did not cancel, then renormalization would be spoiled. This is an example where renormalization requires certain equalities to be satisfied.
- (3) Supersymmetry [7] is based on the idea that quadratic divergences can be eliminated by arranging fermions and bosons into pairs. Theories with supersymmetry have attracted a great deal of attention because of the well-justified belief that divergences, especially quadratic ones, are undesirable.
- (4) In the early seventies, 't Hooft and Veltman [8] succeeded in showing that Yang-Mills non-Abelian gauge theories [9] can be renormalized to one-loop. In this work, the dimensional regularization of Giambiagi and Bollini [10] is used.

From these four examples (and there are many more), it is clear that there are many possible views for quantum field theories. In particular, it is striking that the philosophy about regularization has changed so dramatically between the two eras roughly a quarter of a century apart. If the introduction of regularization is to be justified, it must accomplish something that has not been accomplished without it. What can this be?

With the question phrased this way, the following answer suggests itself. In the work on the renormalization of quantum electrodynamics in the forties and the fifties, it was difficult to justify the massless nature of the photon. Indeed, if one computes the one-loop correction to the photon mass by applying the Feynman rules in the most straightforward way, the answer is infinity rather than zero. [Remember that the photon mass does not have a counter term.] On the other hand, if dimensional regularization is used, this one-loop correction to the photon mass is indeed zero.

The present line of investigation consists of the following:

- Choose a suitable regularization;
- Calculate, to one-loop order in the Standard Model, some of the divergent terms that are gauge invariant; and
- Set these divergent terms to zero.

The last step is justified by the observations (2) and (3) above, and leads to relations between various parameters of the Standard Model. In the first step, the regularization must be chosen such that it is suitable for the Standard Model. We do not deem the dimensional regularization to be a good choice. The most important reason is that, in the dimensional regularization, the dimension is considered to be a continuous parameter, while γ_5 , which plays a fundamental role in the Standard Model, is defined only for four dimensions. This problem was first pointed out by 't Hooft and Veltman [10] and is discussed extensively by Breitenlohner and Maison [11]; many attempts have been made to circumvent it. Nevertheless, it is fair to say that none of these attempts can be considered to be completely successful. Dimensional regularization also has other problems, such as the anomaly discussed by Speer [12]. Since it seems unlikely that these deep difficulties can be resolved soon, we choose to use a different regularization, which is a modification of the method of point splitting [13].

In this first paper, this version of point splitting is discussed and applied to the case of quantum electrodynamics. In the second paper, this method is applied to the Standard Model to get the first mass relation; in the third paper, the second mass relation is similarly derived. With the present experimental knowledge, there are only two unknown masses, namely those of the top quark and the Higgs boson. Since there are two mass relations, they can be used to solve for these two unknown masses. The resulting masses will also be discussed in the third paper.

2. Point Splitting

In quantum electrodynamics, the current is given by

$$J_\mu(x) = e\bar{\psi}(x)\gamma_\mu\psi(x) \quad (2.1)$$

where the $\bar{\psi}(x)$ and the $\psi(x)$ for the electron field are at the same space-time point x . The appearance of ultraviolet divergences in quantum electrodynamics is intimately related to this having field operators at the same space-time point. Thus point splitting, i.e., the assignment of slightly different space-time points to the $\bar{\psi}$ and the ψ , removes the

ultraviolet divergence. However, it is not admissible to use the naive current

$$e\bar{\psi}(x - \tfrac{1}{2}\delta)\gamma_\mu\psi(x + \tfrac{1}{2}\delta) \quad (2.2)$$

because of the lack of gauge invariance. Instead, it is necessary to use

$$\mathcal{J}_\mu(x) = e\bar{\psi}(x - \tfrac{1}{2}\delta)\gamma_\mu\psi(x + \tfrac{1}{2}\delta)\exp\left[-ie\int_{x-\frac{1}{2}\delta}^{x+\frac{1}{2}\delta}d\xi^\nu A_\nu(\xi)\right]. \quad (2.3)$$

The use of such exponential factors because of gauge invariance has a long history, going back over fifty years [13]. The prescription is to use eq. (2.3) for the calculation of various physical quantities of interest, and to take the limit $\delta \rightarrow 0$ afterwards.

Even with the inclusion of the exponential factor, the current (2.3) is not conserved before the limit $\delta \rightarrow 0$ is taken. Therefore the Lagrangian consistent with this current (2.3) is not gauge invariant. The important point to remember in this connection is that point splitting cannot be applied to the quadratic kinetic terms. The conclusion cannot be avoided that it is by no means straightforward to ascertain the Feynman rules with point splitting.

As a preparation for use in perturbation theory, let the point-split current (2.3) be expanded into a power series in the coupling constant e . Thus

$$\begin{aligned} \mathcal{J}_\mu(x) = i \bigg\{ & -ie\bar{\psi}(x - \tfrac{1}{2}\delta)\gamma_\mu\psi(x + \tfrac{1}{2}\delta) \\ & + (-ie)^2\bar{\psi}(x - \tfrac{1}{2}\delta)\gamma_\mu\psi(x + \tfrac{1}{2}\delta)\left[\int_{x-\frac{1}{2}\delta}^{x+\frac{1}{2}\delta}d\xi^\nu A_\nu(\xi)\right] \\ & + \frac{(-ie)^3}{2!}\bar{\psi}(x - \tfrac{1}{2}\delta)\gamma_\mu\psi(x + \tfrac{1}{2}\delta)\left[\int_{x-\frac{1}{2}\delta}^{x+\frac{1}{2}\delta}d\xi^\nu A_\nu(\xi)\right]^2 \\ & + \frac{(-ie)^4}{3!}\bar{\psi}(x - \tfrac{1}{2}\delta)\gamma_\mu\psi(x + \tfrac{1}{2}\delta)\left[\int_{x-\frac{1}{2}\delta}^{x+\frac{1}{2}\delta}d\xi^\nu A_\nu(\xi)\right]^3 + \dots \bigg\}. \end{aligned} \quad (2.4)$$

Since the current is related to the Lagrangian through the relation

$$\mathcal{J}_\mu(x) = \frac{\delta L}{\delta A^\mu(x)}. \quad (2.5)$$

it seems reasonable to take the following interacting Lagrangian density with point splitting

$$\mathcal{L}(x) = i \bigg\{ -ie\bar{\psi}(x - \tfrac{1}{2}\delta)A\psi(x + \tfrac{1}{2}\delta)$$

$$\begin{aligned}
& + \frac{(-ie)^2}{2!} \bar{\psi}(x - \tfrac{1}{2}\delta) A \psi(x + \tfrac{1}{2}\delta) \left[\int_{x-\frac{1}{2}\delta}^{x+\frac{1}{2}\delta} d\xi^\nu A_\nu(\xi) \right] \\
& + \frac{(-ie)^3}{3!} \bar{\psi}(x - \tfrac{1}{2}\delta) A \psi(x + \tfrac{1}{2}\delta) \left[\int_{x-\frac{1}{2}\delta}^{x+\frac{1}{2}\delta} d\xi^\nu A_\nu(\xi) \right]^2 \\
& + \frac{(-ie)^4}{4!} \bar{\psi}(x - \tfrac{1}{2}\delta) A \psi(x + \tfrac{1}{2}\delta) \left[\int_{x-\frac{1}{2}\delta}^{x+\frac{1}{2}\delta} d\xi^\nu A_\nu(\xi) \right]^3 + \dots \} \quad (2.6)
\end{aligned}$$

In this Lagrangian density, the first term gives an $ee\gamma$ vertex, the second one an $ee\gamma\gamma$ vertex, the third an $ee\gamma\gamma\gamma$ vertex, the fourth an $ee\gamma\gamma\gamma\gamma$ vertex etc. Only these first four vertices are to be used. While the first $ee\gamma$ vertex is the familiar one, all the others are due to the point splitting, and they are proportional to δ , δ^2 , and δ^3 respectively. We shall refer to these additional vertices as “peculiar vertices”, and the Feynman diagrams with such peculiar vertices as “peculiar diagrams”.

This is not yet the desired interacting Lagrangian density. There are two general approaches to studying the problem of determining the Feynman rules:

- (1) Point splitting is applied at only one of the vertices for each Feynman diagram; and
- (2) Point splitting is used at every vertex of every Feynman diagram.

While the approach (1) may have its advantages, it is not to be pursued here. The reasons are twofold: first, such a procedure seems to be inconsistent with the basic idea of using Feynman rules; and secondly, there seems to be no reasonable way to decide which vertex of a Feynman diagram is to be chosen as the special one where point splitting is applied, especially for multi-loop diagrams. Therefore the second procedure is to be followed in this paper.

Whether a particular regularization scheme is acceptable depends on the maintenance of gauge invariance. One simple way to verify gauge invariance is to contract with an external four-momentum. For definiteness, let us consider the case of the one-loop photon self-energy. In this case, there are two Feynman diagrams: In addition to the usual one with an electron loop, there is a second diagram with a peculiar vertex of $ee\gamma\gamma$. Now imagine that the usual diagram is contracted with an external momentum. This contraction, through the usual manipulation, cancels one of the electron propagators. Since this diagram has two $ee\gamma$ vertices, there are two δ 's. In order for the second, peculiar diagram to restore gauge invariance, it is necessary to use 2δ , instead of δ , at the peculiar vertex. For the explicit calculation of this peculiar diagram, see Section 4.

Similarly, by considering the gauge invariance of the diagrams for photon-photon scattering as to be discussed in Sec. 8, it is seen that for the $ee\gamma\gamma\gamma\gamma$ vertex 3δ should be

used, and for the $ee\gamma\gamma\gamma$ vertex 4δ be used. Therefore, the interacting Lagrangian density (2.6) must be modified to be

$$\begin{aligned}\mathcal{L}(x) = i \Big\{ & -ie\bar{\psi}(x - \tfrac{1}{2}\delta)A\psi(x + \tfrac{1}{2}\delta) \\ & + \frac{(-ie)^2}{2!}\bar{\psi}(x - \delta)A\psi(x + \delta) \left[\int_{x-\delta}^{x+\delta} d\xi^\nu A_\nu(\xi) \right] \\ & + \frac{(-ie)^3}{3!}\bar{\psi}(x - \tfrac{3}{2}\delta)A\psi(x + \tfrac{3}{2}\delta) \left[\int_{x-\frac{3}{2}\delta}^{x+\frac{3}{2}\delta} d\xi^\nu A_\nu(\xi) \right]^2 \\ & + \frac{(-ie)^4}{4!}\bar{\psi}(x - 2\delta)A\psi(x + 2\delta) \left[\int_{x-2\delta}^{x+2\delta} d\xi^\nu A_\nu(\xi) \right]^3 + \dots \Big\} \quad (2.7)\end{aligned}$$

This is the interacting Lagrangian density that is to be used in this paper. Note that (2.7), unlike (2.6), cannot be summed up into an exponential.

If point splitting is to be used successfully as a regularization, at least the following results must be verified:

- (1) To one loop, the radiative correction to the photon mass must be zero; and
- (2) To one loop, the matrix element for photon-photon scattering must be zero when all four external momenta are zero.

Neither statement is true in the naive calculation without regularization.

These two statements are to be verified in this paper using the point-splitting Lagrangian density (2.7).

3. Photon self energy—the electron loop

As just mentioned, a first step in justifying the point-split regularization is to apply it to the photon self energy. The vacuum polarization tensor is of the form

$$\Pi_{\mu\nu}(p) = (p_\mu p_\nu - p^2 g_{\mu\nu})A(p^2) + g_{\mu\nu}B(p^2), \quad (3.1)$$

with p the photon momentum. If an unacceptable regularization procedure is used, some non-zero B will be induced, whereby current conservation is no longer manifest, and the photon mass becomes non-zero. We shall here show that this problem does not arise with point splitting. In order to test for the presence of the B -term, we shall contract the vacuum polarization tensor with the photon momentum. If the B -term is absent, this contraction yields zero.

With point splitting, there are two one-loop Feynman diagrams contributing to the photon self energy, as shown in fig. 1 and fig. 2. In this section, the diagram of fig. 1 is studied. Here, with point splitting, the two segments of the fermion loop are actually separated at the end points by some small amounts δ .

The amplitude involves the two propagators

$$S_c(y - x + \delta) \quad \text{and} \quad S_c(x - y + \delta), \quad (3.2)$$

where the integration over x and y yield momentum conservation at the two vertices in the usual way. The contribution from this electron-loop diagram to the photon self energy is then

$$\begin{aligned} I_{\mu\nu} &= \frac{-1}{(2\pi)^4} \int d^4k e^{2ik \cdot \delta} \text{Tr} \left[\frac{i}{\not{k} + \frac{1}{2}\not{p} - m + i\epsilon} (-ie\gamma_\mu) \frac{i}{\not{k} - \frac{1}{2}\not{p} - m + i\epsilon} (-ie\gamma_\nu) \right] \\ &= \frac{-4e^2}{(2\pi)^4} \int d^4k e^{2ik \cdot \delta} \frac{2k_\mu k_\nu - \frac{1}{2}p_\mu p_\nu - (k^2 - \frac{1}{4}p^2 - m^2)g_{\mu\nu}}{[(k + \frac{1}{2}p)^2 - m^2 + i\epsilon][(k - \frac{1}{2}p)^2 - m^2 + i\epsilon]} \\ &= \frac{-4e^2}{(2\pi)^4} \int_0^1 d\alpha e^{-i(1-2\alpha)p \cdot \delta} I_{0\mu\nu}, \end{aligned} \quad (3.3)$$

with

$$\begin{aligned} I_{0\mu\nu} &= \int d^4l e^{2il \cdot \delta} [l^2 + \alpha(1-\alpha)p^2 - m^2 + i\epsilon]^{-2} \\ &\quad \times \{2[l_\mu - (\frac{1}{2} - \alpha)p_\mu][l_\nu - (\frac{1}{2} - \alpha)p_\nu] - \frac{1}{2}p_\mu p_\nu - [(l - (\frac{1}{2} - \alpha)p)^2 - \frac{1}{4}p^2 - m^2]g_{\mu\nu}\}. \end{aligned} \quad (3.4)$$

Without the factor $e^{2il \cdot \delta}$, this integral would be quadratically divergent for large values of $|l|$. We therefore expect it to behave like $1/\delta^2$.

In order to check gauge invariance, we are interested in the contraction of this quantity $I_{\mu\nu}$ with the photon momentum, p^ν ,

$$J_\mu \equiv p^\nu I_{\mu\nu}. \quad (3.5)$$

By eq. (3.3),

$$J_\mu = \frac{-4e^2}{(2\pi)^4} \int_0^1 d\alpha e^{-i(1-2\alpha)p \cdot \delta} \bar{J}_\mu, \quad (3.6)$$

where

$$\begin{aligned} \bar{J}_\mu &= \int d^4l e^{2il \cdot \delta} [l^2 + \alpha(1-\alpha)p^2 - m^2 + i\epsilon]^{-2} \\ &\quad \times \{2[l_\mu - (\frac{1}{2} - \alpha)p_\mu][(l \cdot p) - (\frac{1}{2} - \alpha)p^2] - \frac{1}{2}p^2 p_\mu - [(l - (\frac{1}{2} - \alpha)p)^2 - \frac{1}{4}p^2 - m^2]p_\mu\} \\ &= \int d^4l e^{2il \cdot \delta} \frac{2l_\mu[(l \cdot p) - (\frac{1}{2} - \alpha)p^2] + p_\mu[-l^2 - \alpha(1-\alpha)p^2 + m^2]}{[l^2 + \alpha(1-\alpha)p^2 - m^2 + i\epsilon]^2}. \end{aligned} \quad (3.7)$$

Explicitly,

$$\begin{aligned}
J_\mu &= \frac{-4e^2}{(2\pi)^4} \int_0^1 d\alpha e^{-i(1-2\alpha)p \cdot \delta} \\
&\times \int d^4l e^{2il \cdot \delta} [l^2 + \alpha(1-\alpha)p^2 - m^2 + i\epsilon]^{-2} \\
&\times \{2l_\mu[(l \cdot p) - (\tfrac{1}{2} - \alpha)p^2] - p_\mu[l^2 + \alpha(1-\alpha)p^2 - m^2]\}. \tag{3.8}
\end{aligned}$$

If we define now the integrals,

$$I^{(1)}(\delta) = \int d^4l e^{il \cdot \delta} \frac{1}{l^2 - M^2 + i\epsilon}, \tag{3.9}$$

$$I_{\mu\nu}^{(2)}(\delta) = \int d^4l e^{il \cdot \delta} \frac{l_\mu l_\nu}{(l^2 - M^2 + i\epsilon)^2}, \tag{3.10}$$

$$I_\mu^{(3)}(\delta) = \int d^4l e^{il \cdot \delta} \frac{l_\mu}{(l^2 - M^2 + i\epsilon)^2}, \tag{3.11}$$

with

$$M^2 = m^2 - \alpha(1-\alpha)p^2, \tag{3.12}$$

then the result (3.8) can be expressed as

$$J_\mu = \frac{-4e^2}{(2\pi)^4} \int_0^1 d\alpha e^{-i(1-2\alpha)p \cdot \delta} [2p^\nu I_{\mu\nu}^{(2)}(2\delta) - 2(\tfrac{1}{2} - \alpha)p^2 I_\mu^{(3)}(2\delta) - p_\mu I^{(1)}(2\delta)]. \tag{3.13}$$

These integrals (3.9)–(3.11) are evaluated in appendices A, B, and C, respectively. The former two are quadratically divergent, in the sense that if we set $\delta = 0$, they depend quadratically on the cut-off in $|l|$. In terms of δ , this means that as $\delta \rightarrow 0$, they behave like $1/\delta^2$. The third one, (3.11), is linearly divergent.

Substituting eqs. (A.14), (B.19) and (C.7) for the integrals $I^{(1)}$, $I_{\mu\nu}^{(2)}$, and $I_\mu^{(3)}$, respectively into eq. (3.13), we obtain, with δ time-like,

$$\begin{aligned}
J_\mu &= \frac{-4e^2}{(2\pi)^4} \int_0^1 d\alpha e^{-i(1-2\alpha)p \cdot \delta} \\
&\times \left[-2\pi^2 i \frac{(p \cdot \delta)\delta_\mu}{\delta^4} (1 + M^2 \delta^2) + \pi^2 (1 - 2\alpha)p^2 \frac{\delta_\mu}{\delta^2} \right]_{M^2=m^2-\alpha(1-\alpha)p^2}, \tag{3.14}
\end{aligned}$$

where we have used the small- δ expansions. If we also expand the exponent for small values of δ , the integration over the Feynman parameter is straight-forward,

$$J_\mu = \frac{e^2 i}{2\pi^2} \frac{(p \cdot \delta)\delta_\mu}{\delta^2} \left[\frac{1}{\delta^2} + m^2 - \frac{(p \cdot \delta)^2}{6\delta^2} \right]. \tag{3.15}$$

4. Photon self energy—The peculiar diagram

Because of the additional vertex of the type $ee\gamma\gamma$ introduced by point splitting and given by the Lagrangian in (2.7), there is a second Feynman diagram as shown in fig. 2. Actually there are two such diagrams with x and ξ reversed, i. e., another one with the incoming photon line connected to ξ and the outgoing photon line to x . Such further diagrams will not be shown explicitly. The contribution to the photon self-energy is

$$I'_{\mu\nu} = \frac{-1}{(2\pi)^4} \int d^4k e^{2ik\cdot\delta} \text{Tr} \frac{i}{\not{k} - m + i\epsilon} \int_{-\delta}^{\delta} d(\xi - x)_\nu \frac{(-ie)^2}{2} \gamma_\mu e^{ip\cdot(\xi-x)} + (\mu \leftrightarrow \nu). \quad (4.1)$$

As already explained in Sec. 2 and shown in fig. 2, the amount of point splitting is here 2δ , not δ .

The first step is to carry out the $\xi - x$ integration. Let $\bar{\xi} = \xi - x$, then clearly, for $p = 0$,

$$\int_{-\delta}^{\delta} d\bar{\xi}_\nu = 2\delta_\nu. \quad (4.2)$$

More generally, for any p , only the component of p along δ contributes,

$$\int_{-\delta}^{\delta} d\bar{\xi}_\nu e^{ip\cdot\bar{\xi}} = \frac{2 \sin(p\cdot\delta)}{p\cdot\delta} \delta_\nu. \quad (4.3)$$

Therefore,

$$I'_{\mu\nu} = \frac{\sin(p\cdot\delta)}{p\cdot\delta} (-ie)^2 \frac{-1}{(2\pi)^4} \int d^4k e^{2ik\cdot\delta} \text{Tr} \frac{i}{\not{k} - m + i\epsilon} (\gamma_\mu \delta_\nu + \gamma_\nu \delta_\mu), \quad (4.4)$$

or

$$I'_{\mu\nu} = \frac{4ie^2}{(2\pi)^4} \frac{\sin(p\cdot\delta)}{p\cdot\delta} \int d^4k e^{2ik\cdot\delta} \frac{k_\mu \delta_\nu + k_\nu \delta_\mu}{k^2 - m^2 + i\epsilon}. \quad (4.5)$$

This integral is seen to diverge cubically, if we formally set $\delta = 0$. We therefore have to handle it very carefully.

Again, we need to contract this contribution $I'_{\mu\nu}$ to the vacuum polarization, with the photon momentum. Let

$$\begin{aligned} J'_\mu &= p^\nu I'_{\mu\nu} \\ &= \frac{4e^2 i}{(2\pi)^4} \frac{\sin(p\cdot\delta)}{p\cdot\delta} \int d^4k e^{2ik\cdot\delta} \frac{k_\mu (p\cdot\delta) + \delta_\mu (k\cdot p)}{k^2 - m^2 + i\epsilon}. \end{aligned} \quad (4.6)$$

We here require the integral

$$I_\mu^{(5)}(\delta) = \int d^4l e^{il\cdot\delta} \frac{l_\mu}{l^2 - M^2 + i\epsilon}, \quad (4.7)$$

which is evaluated in appendix C. [$I^{(4)}(\delta)$ is defined below in (7.2).]

In terms of this integral, we find

$$J'_\mu = \frac{4e^2 i}{(2\pi)^4} \frac{\sin(p \cdot \delta)}{p \cdot \delta} \cdot \left\{ (p \cdot \delta) I_\mu^{(5)}(2\delta) + \delta_\mu p^\nu I_\nu^{(5)}(2\delta) \right\}. \quad (4.8)$$

If we substitute for $I_\mu^{(5)}$ from (C.20), and expand the factor $\sin(p \cdot \delta)$, we obtain

$$J'_\mu = -\frac{e^2 i}{2\pi^2} \frac{(p \cdot \delta) \delta_\mu}{\delta^2} \left[\frac{1}{\delta^2} + m^2 - \frac{(p \cdot \delta)^2}{6\delta^2} \right]. \quad (4.9)$$

This is precisely the opposite of J_μ of eq. (3.15). Thus,

$$J_\mu + J'_\mu = 0. \quad (4.10)$$

This is the desired result, showing the absence of the $B(p^2)$ term in eq. (3.1). Therefore

$$\Pi_{\mu\nu}(p) = (p_\mu p_\nu - p^2 g_{\mu\nu}) A(p^2). \quad (4.11)$$

5. Direct evaluation of $I_{\mu\nu}$ and $I'_{\mu\nu}$

Returning to the $I_{\mu\nu}$ and $I_{0\mu\nu}$ of eqs. (3.3), and (3.4), we note that

$$\begin{aligned} I_{0\mu\nu} = & 2I_{\mu\nu}^{(2)}(2\delta) - (1 - 2\alpha)[I_\mu^{(3)}(2\delta)p_\nu + p_\mu I_\nu^{(3)}(2\delta)] - 2\alpha(1 - \alpha)p_\mu p_\nu I^{(4)}(2\delta) \\ & - I^{(1)}(2\delta)g_{\mu\nu} + (1 - 2\alpha)I_\rho^{(3)}(2\delta)p^\rho g_{\mu\nu} + 2\alpha(1 - \alpha)p^2 I^{(4)}(2\delta)g_{\mu\nu}, \end{aligned} \quad (5.1)$$

where the integrals $I^{(1)}, \dots, I^{(4)}$ are given in appendices A, B, and C.

With δ time-like, we find

$$\begin{aligned} I_{0\mu\nu} = & -\frac{2\pi^2 i}{\delta^2} \frac{\delta_\mu \delta_\nu}{\delta^2} (1 + M^2 \delta^2) + \pi^2 (1 - 2\alpha) \frac{\delta_\mu p_\nu + p_\mu \delta_\nu - (p \cdot \delta) g_{\mu\nu}}{\delta^2} \\ & + 4\pi^2 i \alpha (1 - \alpha) (p_\mu p_\nu - p^2 g_{\mu\nu}) \left[\ln(M\delta) + \gamma + \frac{i\pi}{2} \right]. \end{aligned} \quad (5.2)$$

Substituting into eq. (3.3), and integrating over α , we obtain

$$\begin{aligned} I_{\mu\nu} = & \frac{-2e^2 i}{(2\pi)^2} \left\{ -\frac{\delta_\mu \delta_\nu}{\delta^2} \left[\frac{1}{\delta^2} + m^2 - \frac{(p \cdot \delta)^2 + p^2 \delta^2}{6\delta^2} \right] \right. \\ & - \frac{1}{6} \frac{\delta_\mu p_\nu + p_\mu \delta_\nu - (p \cdot \delta) g_{\mu\nu}}{\delta^2} (p \cdot \delta) \\ & \left. + 2(p_\mu p_\nu - p^2 g_{\mu\nu}) \int_0^1 d\alpha \alpha (1 - \alpha) \left[\ln \left(\sqrt{m^2 - \alpha(1 - \alpha)p^2} \delta \right) + \gamma + \frac{i\pi}{2} \right] + \mathcal{O}(\delta^2) \right\}. \end{aligned} \quad (5.3)$$

If we similarly substitute for $I_\mu^{(3)}$ into eq. (4.5) for $I'_{\mu\nu}$, and expand the sine, we obtain

$$I'_{\mu\nu} = \frac{-2e^2 i}{(2\pi)^2} \frac{\delta_\mu \delta_\nu}{\delta^2} \left[\frac{1}{\delta^2} + m^2 - \frac{(p \cdot \delta)^2}{6\delta^2} \right]. \quad (5.4)$$

Adding these two contributions, we get

$$\begin{aligned} \Pi_{\mu\nu} &= I_{\mu\nu} + I'_{\mu\nu} \\ &= \frac{-e^2 i}{\pi^2} \left\{ -\frac{1}{12} \frac{(p \cdot \delta)[\delta_\mu p_\nu + p_\mu \delta_\nu] - (p \cdot \delta)^2 g_{\mu\nu} - p^2 \delta_\mu \delta_\nu}{\delta^2} \right. \\ &\quad \left. + (p_\mu p_\nu - p^2 g_{\mu\nu}) \int_0^1 d\alpha \alpha(1-\alpha) \left[\ln \left(\sqrt{m^2 - \alpha(1-\alpha)p^2} \delta \right) + \gamma + \frac{i\pi}{2} \right] \right\}. \end{aligned} \quad (5.5)$$

If we here average over directions of δ_μ , according to

$$\langle \delta_\mu \delta_\nu \rangle = \frac{1}{4} \delta^2 g_{\mu\nu}, \quad (5.6)$$

we obtain

$$\begin{aligned} \langle \Pi_{\mu\nu} \rangle &= (p_\mu p_\nu - p^2 g_{\mu\nu}) \frac{-e^2 i}{\pi^2} \\ &\quad \times \left\{ -\frac{1}{24} + \int_0^1 d\alpha \alpha(1-\alpha) \left[\ln \left(\sqrt{m^2 - \alpha(1-\alpha)p^2} \right) + \gamma + \frac{i\pi}{2} \right] \right\}. \end{aligned} \quad (5.7)$$

6. Photon-photon scattering—The problem

We next study the problem of gauge-noninvariance in photon-photon scattering. There are a total of six diagrams, but their contributions are pairwise identical. It suffices to consider the three diagrams of fig. 3,

$$I = 2(I_1 + I_2 + I_3), \quad (6.1)$$

with

$$\begin{aligned} I_1 &= -\frac{1}{(2\pi)^4} \int d^4 k \operatorname{Tr} \left[\frac{i}{\not{k} - m + i\epsilon} (-ie\gamma_\mu) \frac{i}{\not{k} - p_1 - m + i\epsilon} (-ie\gamma_\nu) \right. \\ &\quad \left. \times \frac{i}{\not{k} + p_3 + p_4 - m + i\epsilon} (-ie\gamma_\rho) \frac{i}{\not{k} + p_3 - m + i\epsilon} (-ie\gamma_\sigma) \right] \\ &= -\frac{e^4}{(2\pi)^4} \int d^4 k \operatorname{Tr} \left[\frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - p_1 - m + i\epsilon} \gamma_\nu \right. \\ &\quad \left. \times \frac{1}{\not{k} + p_3 + p_4 - m + i\epsilon} \gamma_\rho \frac{1}{\not{k} + p_3 - m + i\epsilon} \gamma_\sigma \right], \end{aligned} \quad (6.2)$$

where for brevity we write $I_1 = I_{1\mu\nu\rho\sigma}$. Similarly,

$$I_2 = -\frac{e^4}{(2\pi)^4} \int d^4k \operatorname{Tr} \left[\frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - \not{p}_1 - m + i\epsilon} \gamma_\nu \right. \\ \left. \times \frac{1}{\not{k} + \not{p}_3 + \not{p}_4 - m + i\epsilon} \gamma_\sigma \frac{1}{\not{k} + \not{p}_4 - m + i\epsilon} \gamma_\rho \right], \quad (6.3)$$

$$I_3 = -\frac{e^4}{(2\pi)^4} \int d^4k \operatorname{Tr} \left[\frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - \not{p}_1 - m + i\epsilon} \gamma_\rho \right. \\ \left. \times \frac{1}{\not{k} + \not{p}_2 + \not{p}_3 - m + i\epsilon} \gamma_\nu \frac{1}{\not{k} + \not{p}_3 - m + i\epsilon} \gamma_\sigma \right]. \quad (6.4)$$

All momenta flow inward,

$$p_1 + p_2 + p_3 + p_4 = 0. \quad (6.5)$$

We shall contract these quantities (6.2)–(6.4) with one of the photon momenta. Let

$$J_1 = p_4^\rho I_1, \\ J_2 = p_4^\rho I_2, \\ J_3 = p_4^\rho I_3. \quad (6.6)$$

The result of this contraction simplifies if we use relations like

$$\frac{1}{\not{k} + \not{p}_4 - m + i\epsilon} \not{p}_4 \frac{1}{\not{k} - m + i\epsilon} = \frac{1}{\not{k} - m + i\epsilon} - \frac{1}{\not{k} + \not{p}_4 - m + i\epsilon}. \quad (6.7)$$

We may graphically represent the result of this contraction as in fig. 4, where dashed lines indicate absence of a propagator, and absence of γ_ρ . Several terms are then seen to cancel, but the remaining ones are linearly divergent and have to be treated very carefully [14]. We find

$$J = 2(J_1 + J_2 + J_3) \\ = -\frac{2e^4}{(2\pi)^4} \int d^4k \left\{ -\operatorname{Tr} \left[\frac{1}{\not{k} + \not{p}_4 - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} + \not{p}_2 + \not{p}_3 + \not{p}_4 - m + i\epsilon} \gamma_\nu \right. \right. \\ \left. \left. \times \frac{1}{\not{k} + \not{p}_3 + \not{p}_4 - m + i\epsilon} \gamma_\sigma \right] \right. \\ \left. + \operatorname{Tr} \left[\frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} + \not{p}_2 + \not{p}_3 - m + i\epsilon} \gamma_\nu \frac{1}{\not{k} + \not{p}_3 - m + i\epsilon} \gamma_\sigma \right] \right\}. \quad (6.8)$$

The two terms in the integrand coincide, if we in the former shift the variable of integration, k , by the amount $-p_4$. However, since the integral is linearly divergent, we are not allowed

to do this. Since the possible contribution due to shifting a linearly divergent integral comes from the region of large k , we may neglect the masses and the fixed momenta p_2 and p_3 . Thus,

$$J = -\frac{2e^4}{(2\pi)^4} \int d^4k \left\{ -\text{Tr} \left[\frac{1}{\not{k} + \not{p}_4 + i\epsilon} \gamma_\mu \frac{1}{\not{k} + \not{p}_4 + i\epsilon} \gamma_\nu \frac{1}{\not{k} + \not{p}_4 + i\epsilon} \gamma_\sigma \right] \right. \\ \left. + \text{Tr} \left[\frac{1}{\not{k} + i\epsilon} \gamma_\mu \frac{1}{\not{k} + i\epsilon} \gamma_\nu \frac{1}{\not{k} + i\epsilon} \gamma_\sigma \right] \right\}. \quad (6.9)$$

Here, both terms in the integrand have the structure

$$\begin{aligned} & \text{Tr} \left[\frac{1}{\not{k} + i\epsilon} \gamma_\mu \frac{1}{\not{k} + i\epsilon} \gamma_\nu \frac{1}{\not{k} + i\epsilon} \gamma_\sigma \right] \\ &= \frac{1}{(k^2 + i\epsilon)^3} \text{Tr}[\not{k} \gamma_\mu \not{k} \gamma_\nu \not{k} \gamma_\sigma] \\ &= \frac{4}{(k^2 + i\epsilon)^3} [4k_\mu k_\nu k_\sigma - k^2(k_\mu g_{\nu\sigma} + k_\nu g_{\mu\sigma} + k_\sigma g_{\mu\nu})]. \end{aligned} \quad (6.10)$$

Therefore,

$$J = -\frac{8e^4}{(2\pi)^4} \int d^4k \left[\frac{4k_\mu k_\nu k_\sigma - k^2(k_\mu g_{\nu\sigma} + k_\nu g_{\mu\sigma} + k_\sigma g_{\mu\nu})}{(k^2 + i\epsilon)^3} \right. \\ \left. - \text{same with } (k \rightarrow k + p_4) \right]. \quad (6.11)$$

This integral is symmetric in the three indices μ , ν , and σ , and has the dimension of momentum. Therefore it must be of the form

$$J = C(p_{4\mu} g_{\nu\sigma} + p_{4\nu} g_{\mu\sigma} + p_{4\sigma} g_{\mu\nu}), \quad (6.12)$$

where C is a constant. The question is: Is C zero or not?

To determine C , consider

$$J p_4^\mu g^{\nu\sigma} = 6C p_4^2. \quad (6.13)$$

Thus,

$$C = \frac{1}{6p_4^2} \frac{-8e^4}{(2\pi)^4} \int d^4k \left[\frac{-2k^2(p_4 \cdot k)}{(k^2 + i\epsilon)^3} - \frac{-2(k + p_4)^2(p_4 \cdot (k + p_4))}{[(k + p_4)^2 + i\epsilon]^3} \right]. \quad (6.14)$$

This integral is evaluated in appendix D. The result is

$$C = -\frac{ie^4}{3(2\pi)^2}. \quad (6.15)$$

Thus,

$$J = -\frac{ie^4}{3(2\pi)^2}(p_{4\mu}g_{\nu\sigma} + p_{4\nu}g_{\mu\sigma} + p_{4\sigma}g_{\mu\nu}), \quad (6.16)$$

Returning now to eqs. (6.1) and (6.6), we conclude that

$$I \rightarrow -\frac{ie^4}{3(2\pi)^2}(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}), \quad (6.17)$$

as all $p_j \rightarrow 0$. In other words, the photon-photon scattering amplitude is not gauge invariant.

This result can alternatively be obtained by setting all external momenta to zero,

$$p_1 = p_2 = p_3 = p_4 = 0, \quad (6.18)$$

(keeping the electron mass finite). Consider eq. (6.2), with all external momenta equal to zero,

$$\begin{aligned} I_1 &= -\frac{e^4}{(2\pi)^4} \int d^4k \operatorname{Tr} \left[\frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - m + i\epsilon} \gamma_\nu \right. \\ &\quad \left. \times \frac{1}{\not{k} - m + i\epsilon} \gamma_\rho \frac{1}{\not{k} - m + i\epsilon} \gamma_\sigma \right] \\ &= -\frac{e^4}{(2\pi)^4} \int d^4k \frac{\mathcal{T}}{(k^2 - m^2 + i\epsilon)^4}, \end{aligned} \quad (6.19)$$

with

$$\begin{aligned} \mathcal{T} &= \operatorname{Tr}[(\not{k} + m)\gamma_\mu(\not{k} + m)\gamma_\nu(\not{k} + m)\gamma_\rho(\not{k} + m)\gamma_\sigma] \\ &= 4 \left\{ 8k_\mu k_\nu k_\rho k_\sigma - 2(k^2 - m^2)[k_\mu k_\nu g_{\rho\sigma} + k_\nu k_\rho g_{\mu\sigma} + k_\rho k_\sigma g_{\mu\nu} + k_\sigma k_\mu g_{\nu\rho}] \right. \\ &\quad \left. + (k^2 - m^2)^2[g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}] \right\}. \end{aligned} \quad (6.20)$$

By eqs. (6.1)–(6.4), with $I_2 = I_1(\rho \leftrightarrow \sigma)$, and $I_3 = I_1(\rho \leftrightarrow \nu)$,

$$\begin{aligned} I &= -\frac{8e^4}{(2\pi)^4} \int d^4k (k^2 - m^2 + i\epsilon)^{-4} \\ &\quad \times \left\{ 24k_\mu k_\nu k_\rho k_\sigma - 4(k^2 - m^2)[k_\mu k_\nu g_{\rho\sigma} + k_\mu k_\sigma g_{\nu\rho} + k_\mu k_\rho g_{\nu\sigma} \right. \\ &\quad \left. + k_\nu k_\sigma g_{\mu\rho} + k_\nu k_\rho g_{\mu\sigma} + k_\sigma k_\rho g_{\mu\nu}] \right. \\ &\quad \left. + (k^2 - m^2)^2[g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}] \right\}. \end{aligned} \quad (6.21)$$

Therefore, at this point (6.18),

$$I = C'(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}). \quad (6.22)$$

Contracting with $g^{\mu\nu}g^{\rho\sigma}$, we obtain

$$g^{\mu\nu}g^{\rho\sigma}I = 24C', \quad (6.23)$$

or

$$\begin{aligned} C' &= \frac{1}{24} \frac{-8e^4}{(2\pi)^4} \int d^4k (k^2 - m^2 + i\epsilon)^{-4} \\ &\quad \times 24 \left\{ (k^2)^2 - 2(k^2 - m^2)k^2 + (k^2 - m^2)^2 \right\} \\ &= \frac{-8e^4}{(2\pi)^4} \int d^4k \frac{m^4}{(k^2 - m^2 + i\epsilon)^4}. \end{aligned} \quad (6.24)$$

By eq. (D.8),

$$C' = -\frac{ie^4}{3(2\pi)^2}, \quad (6.25)$$

in agreement with the result (6.15) for C .

7. Photon-photon scattering—Point splitting

We shall now regularize the amplitude for photon-photon scattering, using point splitting. Since the quantity C of eq. (6.15) is independent of the photon momentum, we shall again take all the external photon momenta to be zero.

The quantity that we need to compute is [cf. eqs. (3.3) and (6.21)]

$$\begin{aligned} I &= -\frac{8e^4}{(2\pi)^4} \int d^4k e^{4ik \cdot \delta} (k^2 - m^2 + i\epsilon)^{-4} \\ &\quad \times \left\{ 24k_\mu k_\nu k_\rho k_\sigma - 4(k^2 - m^2)[k_\mu k_\nu g_{\rho\sigma} + k_\mu k_\rho g_{\nu\sigma} + k_\mu k_\sigma g_{\nu\rho} \right. \\ &\quad \left. + k_\nu k_\sigma g_{\mu\rho} + k_\nu k_\rho g_{\mu\sigma} + k_\rho k_\sigma g_{\mu\nu}] \right. \\ &\quad \left. + (k^2 - m^2)^2[g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}] \right\}, \end{aligned} \quad (7.1)$$

where 4δ is the sum of the point splittings at the four vertices.

We here need the following integrals,

$$I^{(4)}(\delta) = \int d^4l e^{il \cdot \delta} \frac{1}{(l^2 - M^2 + i\epsilon)^2}, \quad (7.2)$$

$$I_{\mu\nu}^{(6)}(\delta) = \int d^4l e^{il \cdot \delta} \frac{l_\mu l_\nu}{(l^2 - M^2 + i\epsilon)^3}, \quad (7.3)$$

$$I_{\mu\nu\rho\sigma}^{(7)}(\delta) = \int d^4l e^{il \cdot \delta} \frac{l_\mu l_\nu l_\rho l_\sigma}{(l^2 - M^2 + i\epsilon)^4}. \quad (7.4)$$

The first one is evaluated in appendix C, the second one in appendix B, and the third one in appendix E.

In terms of these integrals, we have, with $M = m$,

$$\begin{aligned} I = -\frac{8e^4}{(2\pi)^4} \{ & 24I_{\mu\nu\rho\sigma}^{(7)}(4\delta) - 4[I_{\mu\nu}^{(6)}(4\delta)g_{\rho\sigma} + I_{\mu\rho}^{(6)}(4\delta)g_{\nu\sigma} + I_{\mu\sigma}^{(6)}(4\delta)g_{\nu\rho} \\ & + I_{\nu\sigma}^{(6)}(4\delta)g_{\mu\rho} + I_{\nu\rho}^{(6)}(4\delta)g_{\mu\sigma} + I_{\rho\sigma}^{(6)}(4\delta)g_{\mu\nu}] \\ & + I^{(4)}(4\delta)[g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}] \}. \end{aligned} \quad (7.5)$$

Substituting for the integrals, expanded for small values of δ , we obtain

$$I = -\frac{8ie^4}{(2\pi)^4} \frac{\delta_\mu \delta_\nu \delta_\rho \delta_\sigma}{\delta^4}. \quad (7.6)$$

All terms involving $g_{\mu\nu}$, $g_{\mu\rho}$, etc. cancel.

8. Photon-photon scattering—Peculiar diagrams

In addition to the diagrams discussed in the previous section, there are for photon-photon scattering also four “peculiar” diagrams, originating from the expansion of the exponential in the current (2.3). These follow more explicitly from the Lagrangian (2.7), and are given in fig. 5. We denote their contributions I' , I'' , I''' , and I'''' .

Unlike the case of the photon self energy, the factors $\sin(p_j \cdot \delta)/(p_j \cdot \delta)$ of eq. (4.3) and similar ones may here be replaced by 1. Thus,

$$\begin{aligned} I' = & \frac{2ie^4}{(2\pi)^4} \frac{1}{2} \int d^4k \exp\left\{\frac{i}{2}[2k \cdot \delta + 3(k + p_2) \cdot \delta + 3(k - p_1) \cdot \delta]\right\} \\ & \times \text{Tr} \left[\frac{1}{\not{k} - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - \not{p}_1 - m + i\epsilon} [2\gamma_\rho \delta_\sigma + 2\delta_\rho \gamma_\sigma] \frac{1}{\not{k} + \not{p}_2 - m + i\epsilon} \gamma_\nu \right], \end{aligned} \quad (8.1)$$

where the factor $1/2$ in front of the integral is the coefficient in eq. (2.7). Similarly, we obtain

$$\begin{aligned}
I'' &= \frac{e^4}{(2\pi)^4} \left(\frac{1}{2}\right)^2 \int d^4k \exp\left\{2i\left[\left(k + \frac{p_1 + p_2}{2}\right) \cdot \delta + \left(k - \frac{p_1 + p_2}{2}\right) \cdot \delta\right]\right\} \\
&\times \text{Tr}\left[\frac{1}{\not{k} + \frac{1}{2}(\not{p}_1 + \not{p}_2) - m + i\epsilon} [2\gamma_\mu \delta_\nu + 2\delta_\mu \gamma_\nu] \right. \\
&\quad \left. \times \frac{1}{\not{k} - \frac{1}{2}(\not{p}_1 + \not{p}_2) - m + i\epsilon} [2\gamma_\rho \delta_\sigma + 2\delta_\rho \gamma_\sigma] \right],
\end{aligned} \tag{8.2}$$

$$\begin{aligned}
I''' &= \frac{e^4}{(2\pi)^4} \frac{1}{3} \int d^4k \exp\{2i[(k + \frac{1}{2}p_1) \cdot \delta + (k - \frac{1}{2}p_1) \cdot \delta]\} \\
&\times \text{Tr}\left[\frac{1}{\not{k} + \frac{1}{2}\not{p}_1 - m + i\epsilon} \gamma_\mu \frac{1}{\not{k} - \frac{1}{2}\not{p}_1 - m + i\epsilon} \right. \\
&\quad \left. \times 9[\gamma_\rho \delta_\sigma \delta_\nu + \delta_\rho \gamma_\sigma \delta_\nu + \delta_\rho \delta_\sigma \gamma_\nu] \right],
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
I'''' &= \frac{-ie^4}{(2\pi)^4} \frac{1}{4} \int d^4k \exp\{4ik \cdot \delta\} \\
&\times \text{Tr}\left[\frac{1}{\not{k} - m + i\epsilon} 4^3 [\gamma_\mu \delta_\nu \delta_\rho \delta_\sigma + \delta_\mu \gamma_\nu \delta_\rho \delta_\sigma + \delta_\mu \delta_\nu \gamma_\rho \delta_\sigma + \delta_\mu \delta_\nu \delta_\rho \gamma_\sigma] \right].
\end{aligned} \tag{8.4}$$

Like in the previous section, we may set all external momenta to zero. Then,

$$\begin{aligned}
I' &= \frac{2ie^4}{(2\pi)^4} \int \frac{d^4k e^{4ik \cdot \delta}}{(k^2 - m^2 + i\epsilon)^3} \\
&\quad \times \text{Tr}\{(\not{k} + m)\gamma_\mu(\not{k} + m)[\gamma_\rho \delta_\sigma + \delta_\rho \gamma_\sigma](\not{k} + m)\gamma_\nu\},
\end{aligned} \tag{8.5}$$

$$\begin{aligned}
I'' &= \frac{e^4}{(2\pi)^4} \int \frac{d^4k e^{4ik \cdot \delta}}{(k^2 - m^2 + i\epsilon)^2} \\
&\quad \times \text{Tr}\{(\not{k} + m)[\gamma_\mu \delta_\nu + \delta_\mu \gamma_\nu](\not{k} + m)[\gamma_\rho \delta_\sigma + \delta_\rho \gamma_\sigma]\},
\end{aligned} \tag{8.6}$$

$$\begin{aligned}
I''' &= \frac{3e^4}{(2\pi)^4} \int \frac{d^4k e^{4ik \cdot \delta}}{(k^2 - m^2 + i\epsilon)^2} \\
&\quad \times \text{Tr}\{(\not{k} + m)\gamma_\mu(\not{k} + m)[\gamma_\rho \delta_\sigma \delta_\nu + \delta_\rho \gamma_\sigma \delta_\nu + \delta_\rho \delta_\sigma \gamma_\nu]\},
\end{aligned} \tag{8.7}$$

$$I'''' = \frac{-16ie^4}{(2\pi)^4} \int \frac{d^4k e^{4ik \cdot \delta}}{k^2 - m^2 + i\epsilon} \times \text{Tr}\{(\not{k} + m)[\gamma_\mu \delta_\nu \delta_\rho \delta_\sigma + \delta_\mu \gamma_\nu \delta_\rho \delta_\sigma + \delta_\mu \delta_\nu \gamma_\rho \delta_\sigma + \delta_\mu \delta_\nu \delta_\rho \gamma_\sigma]\}. \quad (8.8)$$

For I' , we need the trace

$$\begin{aligned} & \text{Tr}[(\not{k} + m)\gamma_\mu(\not{k} + m)(\gamma_\rho \delta_\sigma + \delta_\rho \gamma_\sigma)(\not{k} + m)\gamma_\nu] \\ &= 4\delta_\rho [4k_\mu k_\nu k_\sigma - (k^2 - m^2)(k_\mu g_{\nu\sigma} + k_\nu g_{\mu\sigma} + k_\sigma g_{\mu\nu})] + (\rho \leftrightarrow \sigma). \end{aligned} \quad (8.9)$$

In addition to the $I_\mu^{(3)}$ of appendix C, we need the integral

$$I_{\mu\nu\rho}^{(8)}(\delta) = \int d^4l e^{il \cdot \delta} \frac{l_\mu l_\nu l_\rho}{(l^2 - M^2 + i\epsilon)^3}. \quad (8.10)$$

This is given in appendix B. Substituting into the expression for I' , we find

$$I' = \frac{4ie^4}{(2\pi)^2} \frac{\delta_\mu \delta_\nu \delta_\rho \delta_\sigma}{\delta^4}. \quad (8.11)$$

Note that all the $g_{\mu\nu}$ and similar terms are cancelled.

For I'' , we need the trace

$$\text{Tr}[(\not{k} + m)\gamma_\mu(\not{k} + m)\gamma_\rho] = 8k_\mu k_\rho - 4(k^2 - m^2)g_{\mu\rho}. \quad (8.12)$$

Thus,

$$\begin{aligned} I'' = \frac{4e^4}{(2\pi)^4} \int d^4k e^{4ik \cdot \delta} \left\{ \left[\left(\frac{2k_\mu k_\rho}{(k^2 - m^2 + i\epsilon)^2} - \frac{g_{\mu\rho}}{k^2 - m^2 + i\epsilon} \right) \delta_\nu \delta_\sigma \right. \right. \\ \left. \left. + (\mu \leftrightarrow \nu) \right] + (\rho \leftrightarrow \sigma) \right\}. \end{aligned} \quad (8.13)$$

Substituting for the integrals $I^{(1)}(4\delta)$ and $I_{\mu\rho}^{(2)}(4\delta)$ of appendices A and B, respectively, we obtain

$$I'' = \frac{-2ie^4}{(2\pi)^2} \frac{\delta_\mu \delta_\nu \delta_\rho \delta_\sigma}{\delta^4} \quad (8.14)$$

The I''' is essentially the same as I'' , except that there are three terms instead of four. Therefore,

$$I''' = \frac{-9ie^4}{2(2\pi)^2} \frac{\delta_\mu \delta_\nu \delta_\rho \delta_\sigma}{\delta^4}. \quad (8.15)$$

Finally, we have

$$I'''' = \frac{-ie^4}{(2\pi)^4} \int d^4k e^{4ik \cdot \delta} \frac{k_\mu \delta_\nu \delta_\rho \delta_\sigma + \delta_\mu k_\nu \delta_\rho \delta_\sigma + \delta_\mu \delta_\nu k_\rho \delta_\sigma + \delta_\mu \delta_\nu \delta_\rho k_\sigma}{k^2 - m^2 + i\epsilon}. \quad (8.16)$$

With the integral $I^{(1)}$ of appendix A, we have

$$\begin{aligned} I'''' &= \frac{-4ie^4}{(2\pi)^4} 16 \left[\left(\frac{1}{4i} \frac{\partial}{\partial \delta^\mu} I^{(1)}(4\delta) \right) \delta_\nu \delta_\rho \delta_\sigma + \text{three similar terms} \right] \\ &= \frac{8ie^4}{(2\pi)^2} \frac{\delta_\mu \delta_\nu \delta_\rho \delta_\sigma}{\delta^4}. \end{aligned} \quad (8.17)$$

We need to add the various contributions I , I' , I'' , I''' , and I'''' , including their multiplicative factors $N', \dots, N''''$ for the numbers of diagrams. These considerations simplify if we introduce the abbreviation

$$K \equiv -\frac{8ie^4}{(2\pi)^2} \frac{\delta_\mu \delta_\nu \delta_\rho \delta_\sigma}{\delta^4}. \quad (8.18)$$

Furthermore, the number of diagrams of the different classes are given in fig. 5.

Then

$$\begin{aligned} I &= K, \\ I' &= -\frac{1}{2}K, \\ I'' &= \frac{1}{4}K, \\ I''' &= \frac{9}{16}K, \\ I'''' &= -K. \end{aligned} \quad (8.19)$$

It is now easily seen that

$$\begin{aligned} I + N' I' + N'' I'' + N''' I''' + N'''' I'''' \\ = K[1 - 6 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{9}{16} - 1] = 0. \end{aligned} \quad (8.20)$$

Thus, with point-split regularization, the photon-photon scattering amplitude is explicitly gauge invariant.

9. Discussions

It has been shown that point splitting, suitably modified, can serve as a useful regularization for gauge field theory. The modification consists of the proper choices of the

amount of point splitting. Point-splitting regularization has all the desirable properties, especially the maintainance of gauge invariance. It does not have any of the difficulties of dimensional regularization.

In the next paper of the series, Paper II, this point-splitting regularization is to be applied to obtain a mass relation for the Standard Model. In Paper III, it is used to get a second mass relation. Since there are only two masses that are not known experimentally, those of the Higgs boson and the top quark, these two mass relations can be solved to yield the following theoretical prediction for the Standard Model:

$$m_H = 190 \text{ GeV} \quad \text{and} \quad m_t = 120 \text{ GeV}. \quad (9.1)$$

These predictions, together with some of their experimental consequences, will be further discussed in Paper III.

The technology of carrying out the calculation of Feynman diagrams with point-splitting regularization has only been partially developed. A great deal remains to be done, including especially their application to multi-loop diagrams. This application to multi-loop diagrams probably needs to be studied in conjunction with a systematic treatment of renormalization. On the practical side, it is hoped that an analysis of multi-loop Feynman diagrams may lead to a deeper understanding of the Standard Model.

Appendix A. The integral $I^{(1)}$

Consider the integral

$$I^{(1)} = \int d^4 l e^{i l \cdot \delta} \frac{1}{l^2 - M^2 + i\epsilon}. \quad (\text{A.1})$$

We need to consider two cases, depending on whether δ is space-like or time-like.

Consider first δ space-like. Then $l \cdot \delta = -\vec{l} \cdot \vec{\delta}$, and we perform a Wick rotation, with

$$i l^4 = l^0.$$

Thus, as an integral over the Euclidean variable l_E , we have

$$I^{(1)} = i \int d^4 l_E e^{-i \vec{l}_E \cdot \vec{\delta}} \frac{-1}{l_E^2 + M^2}. \quad (\text{A.2})$$

In an attempt to keep the presentation simple, we shall occasionally change notation. Thus, we omit the subscript E, and denote the moduli of l_E and $\vec{\delta}$ simply as l and δ . Then,

$$\begin{aligned} I^{(1)} &= -i \int_0^\infty l^3 dl \int_0^\pi \sin^2 \theta d\theta \int_0^\pi \sin \phi_1 d\phi_1 \int_0^{2\pi} d\phi_2 e^{-i l \delta \cos \theta} \frac{1}{l^2 + M^2} \\ &= -4\pi i \int_0^\infty l^3 dl \frac{1}{l^2 + M^2} \int_0^\pi \sin^2 \theta d\theta e^{-i l \delta \cos \theta}. \end{aligned} \quad (\text{A.3})$$

With the Poisson integral representation [15]

$$\int_0^\pi d\theta \sin^2 \theta e^{-i l \delta \cos \theta} = \frac{\pi}{l \delta} J_1(l \delta), \quad (\text{A.4})$$

we obtain

$$\begin{aligned} I^{(1)} &= -4\pi^2 i \frac{1}{\delta} \int_0^\infty l^2 dl \frac{1}{l^2 + M^2} J_1(l \delta) \\ &= -4\pi^2 i M^2 \frac{K_1(M \delta)}{M \delta}, \end{aligned} \quad (\text{A.5})$$

where in the last step we have used [see ref. [15], p. 95, eq. (51)]

$$\int_0^\infty l^2 dl \frac{1}{l^2 + M^2} J_1(l \delta) = M K_1(M \delta), \quad (\text{A.6})$$

with K_1 a modified Bessel function. For small values of δ , we may expand the Bessel function. This yields

$$I^{(1)} \Big|_{\text{space-like}} = -4\pi^2 i M^2 \left\{ \frac{1}{M^2 \delta^2} + \frac{1}{2} \left[\ln \frac{M \delta}{2} + \gamma - \frac{1}{2} \right] + \mathcal{O}(\delta^2) \right\}, \quad (\text{A.7})$$

where γ is the Euler constant, and $\delta = |\vec{\delta}|$, hence $\delta^2 > 0$.

Next, consider δ time-like. The result is then a little different. Basically, we get the above result (A.5) with δ replaced by $i\delta$. We take δ to lie along the positive time axis, and denote henceforth this time component (positive) by δ . Then,

$$I^{(1)} = \int d^3l \int dl_0 e^{il_0\delta} \frac{1}{l_0^2 - l^2 - M^2 + i\epsilon}, \quad (\text{A.8})$$

where l now refers to the length of the three-vector, $l = |\vec{l}|$. We close the contour in the upper half l_0 plane,

$$\begin{aligned} I^{(1)} &= 2\pi i \int d^3l \frac{e^{-i\sqrt{l^2+M^2}\delta}}{-2\sqrt{l^2+M^2}} \\ &= -\pi i (4\pi) \int_0^\infty l^2 dl \frac{e^{-i\sqrt{l^2+M^2}\delta}}{\sqrt{l^2+M^2}} \\ &= -4\pi^2 i \int_0^\infty M^2 \sinh^2 \theta d\theta e^{-iM\delta \cosh \theta}, \end{aligned} \quad (\text{A.9})$$

with

$$\cosh \theta = \sqrt{1 + l^2/M^2}. \quad (\text{A.10})$$

This is again a standard integral [see ref. [15], p. 82, eq. (20)],

$$\Gamma(\tfrac{1}{2} + \nu) K_\nu(z) = \pi^{1/2} (\tfrac{1}{2}z)^\nu \int_0^\infty e^{-z \cosh \theta} (\sinh \theta)^{2\nu} d\theta. \quad (\text{A.11})$$

Hence,

$$I^{(1)} = -4\pi^2 i M^2 \frac{K_1(iM\delta)}{iM\delta}. \quad (\text{A.12})$$

As already mentioned, this is the same result as was obtained for the case of a space-like δ , eq. (A.5), if one there substitutes

$$\delta \rightarrow i\delta. \quad (\text{A.13})$$

Expanding again the modified Bessel function for small values of the argument, we obtain

$$I^{(1)} \Big|_{\text{time-like}} = -4\pi^2 i M^2 \left\{ \frac{-1}{M^2 \delta^2} + \frac{1}{2} \left[\ln \frac{M\delta}{2} + \gamma - \frac{1}{2} \right] + i\frac{\pi}{4} + \mathcal{O}(\delta^2) \right\}. \quad (\text{A.14})$$

Appendix B. The integrals $I_{\mu\nu}^{(2)}$, $I_{\mu\nu}^{(6)}$ and $I_{\mu\nu\rho}^{(8)}$

This appendix deals with the three integrals (3.10), (7.3), and (8.10),

$$I_{\mu\nu}^{(2)} = \int d^4 l e^{il \cdot \delta} \frac{l_\mu l_\nu}{(l^2 - M^2 + i\epsilon)^2}, \quad (\text{B.1})$$

$$I_{\mu\nu}^{(6)} = \int d^4 l e^{il \cdot \delta} \frac{l_\mu l_\nu}{(l^2 - M^2 + i\epsilon)^3}, \quad (\text{B.2})$$

$$I_{\mu\nu\rho}^{(8)} = \int d^4 l e^{il \cdot \delta} \frac{l_\mu l_\nu l_\rho}{(l^2 - M^2 + i\epsilon)^3}. \quad (\text{B.3})$$

Consider first the integral $I_{\mu\nu}^{(2)}$. The other two are obtained by differentiation. Again, if we formally set $\delta = 0$, this integral diverges quadratically, like $I^{(1)}$ of eq. (3.9).

δ space-like

We consider first δ space-like, and perform a Wick rotation,

$$I_{\mu\nu}^{(2)} = i \int d^4 l e^{-i\vec{l} \cdot \vec{\delta}} \frac{l_\mu l_\nu}{(l^2 + M^2)^2}, \quad \mu, \nu = 1, 2, 3, 4, \quad (\text{B.4})$$

where we write l for the Euclidean four-vector l_E . The space-like δ may be chosen to lie in the direction of the positive fourth axis. The integral is then seen to vanish for $\mu \neq \nu$. There are two distinct non-trivial cases, $\mu = \nu = 1$, or $\mu = \nu = 4$.

a. δ space-like, $\mu = \nu = 1$

$$\begin{aligned} I_{11}^{(2)} &= i \int d^3 l \int dl_4 e^{-il_4 \delta} \frac{l_1^2}{(l_4^2 + l^2 + M^2)^2} \\ &= \frac{i}{3} \int d^3 l \int dl_4 e^{-il_4 \delta} \frac{l^2}{(l_4^2 + l^2 + M^2)^2} \\ &= \frac{4\pi i}{3} \int_0^\infty l^4 dl \int dl_4 e^{-il_4 \delta} \frac{1}{(l_4^2 + l^2 + M^2)^2}, \end{aligned} \quad (\text{B.5})$$

where now $l = |\vec{l}|$ denotes $\sqrt{l_1^2 + l_2^2 + l_3^2}$. For the purpose of integrating over l_4 it is convenient to rewrite this as

$$I_{11}^{(2)} = \frac{4\pi i}{3} \left(-\frac{\partial}{\partial M^2} \right) \int_0^\infty l^4 dl \int dl_4 e^{-il_4 \delta} \frac{1}{l_4^2 + l^2 + M^2}. \quad (\text{B.6})$$

The integral over l_4 may be expressed in terms of the residue at $l_4 = -i\sqrt{l^2 + M^2}$,

$$\begin{aligned} I_{11}^{(2)} &= \frac{4\pi i}{3} \left(-\frac{\partial}{\partial M^2} \right) (-2\pi i) \int_0^\infty l^4 dl e^{-\sqrt{l^2 + M^2} \delta} \frac{1}{-2i\sqrt{l^2 + M^2}} \\ &= \frac{4\pi^2 i}{3} \left(-\frac{\partial}{\partial M^2} \right) \int_0^\infty M^4 \sinh^4 \theta d\theta e^{-M\delta \cosh \theta}. \end{aligned} \quad (\text{B.7})$$

Invoking again eq. (A.11), we find

$$I_{11}^{(2)} = -4\pi^2 i \frac{\partial}{\partial M^2} M^4 \frac{K_2(M\delta)}{(M\delta)^2}. \quad (\text{B.8})$$

If we here expand K_2 for small arguments, we obtain

$$I_{11}^{(2)} \Big|_{\text{space-like}} = 2\pi^2 i \left\{ \frac{1}{\delta^2} + \frac{M^2}{2} \left[\ln \frac{M\delta}{2} + \gamma - \frac{1}{2} \right] + \mathcal{O}(\delta^2) \right\}. \quad (\text{B.9})$$

b. δ space-like, $\mu = \nu = 4$

We next need to evaluate

$$\begin{aligned} I_{44}^{(2)} &= i \int d^3 l \int dl_4 e^{-il_4 \delta} \frac{l_4^2}{(l_4^2 + l^2 + M^2)^2} \\ &= 4\pi i \left(-\frac{\partial}{\partial M^2} \right) \int_0^\infty l^2 dl \int dl_4 e^{-il_4 \delta} \frac{l_4^2}{l_4^2 + l^2 + M^2} \\ &= 4\pi i \frac{\partial}{\partial M^2} \int_0^\infty l^2 dl \int dl_4 e^{-il_4 \delta} \frac{l^2 + M^2}{l_4^2 + l^2 + M^2} \\ &= -3I_{11}^{(2)} + 4\pi i \frac{\partial}{\partial M^2} \int_0^\infty l^2 dl \int dl_4 e^{-il_4 \delta} \frac{M^2}{l_4^2 + l^2 + M^2} \\ &= -3I_{11}^{(2)} + 4\pi i \frac{\partial}{\partial M^2} (-2\pi i) \int_0^\infty l^2 dl e^{-\sqrt{l^2 + M^2} \delta} \frac{1}{-2i\sqrt{l^2 + M^2}} \\ &= -3I_{11}^{(2)} + 4\pi^2 i \frac{\partial}{\partial M^2} M^2 \int_0^\infty M^2 \sinh^2 \theta d\theta e^{-M\delta \cosh \theta} \\ &= -3I_{11}^{(2)} + 4\pi^2 i \frac{\partial}{\partial M^2} M^4 \frac{K_1(M\delta)}{M\delta}. \end{aligned} \quad (\text{B.10})$$

If we now expand K_1 for small arguments, we obtain

$$I_{44}^{(2)} \Big|_{\text{space-like}} = 2\pi^2 i \left\{ -\frac{1}{\delta^2} + \frac{M^2}{2} \left[\ln \frac{M\delta}{2} + \gamma + \frac{1}{2} \right] + \mathcal{O}(\delta^2) \right\}. \quad (\text{B.11})$$

The results (B.9) and (B.11) may be combined, in rotationally invariant form (with $\delta^2 > 0$) as

$$I_{\mu\nu}^{(2)} \Big|_{\text{space-like}} = 2\pi^2 i \left\{ \left[\frac{1}{\delta^2} + \frac{M^2}{2} \left(\ln \frac{M\delta}{2} + \gamma - \frac{1}{2} \right) \right] \delta_{\mu\nu} - \frac{2}{\delta^2} \left(1 - \frac{M^2 \delta^2}{4} \right) \frac{\delta_\mu \delta_\nu}{\delta^2} \right\}. \quad (\text{B.12})$$

δ time-like

In this case, we choose the time axis to coincide with the direction of δ . Then,

$$I_{\mu\nu}^{(2)} = \int d^3l \int dl_0 e^{il_0\delta} \frac{l_\mu l_\nu}{(l_0^2 - l^2 - M^2 + i\epsilon)^2}. \quad (\text{B.13})$$

Again $I_{\mu\nu}^{(2)}$ vanishes unless $\mu = \nu$. There are two distinct cases, $\mu = \nu = 1$ and $\mu = \nu = 0$.

c. δ time-like, $\mu = \nu = 1$

By symmetry,

$$I_{11}^{(2)} = \frac{1}{3} \int d^3l \int dl_0 e^{il_0\delta} \frac{l^2}{(l_0^2 - l^2 - M^2 + i\epsilon)^2}, \quad (\text{B.14})$$

where we use the notation $l = |\vec{l}| = \sqrt{l_1^2 + l_2^2 + l_3^2}$. We proceed in a way similar to case *a* above, and obtain

$$\begin{aligned} I_{11}^{(2)} &= \frac{4\pi}{3} \frac{\partial}{\partial M^2} \int_0^\infty l^4 dl \int dl_0 e^{il_0\delta} \frac{1}{l_0^2 - l^2 - M^2 + i\epsilon} \\ &= \frac{4\pi}{3} \frac{\partial}{\partial M^2} \int_0^\infty l^4 dl (2\pi i) \frac{e^{-i\sqrt{l^2 + M^2}\delta}}{-2\sqrt{l^2 + M^2}} \\ &= -\frac{4\pi^2 i}{3} \frac{\partial}{\partial M^2} \int_0^\infty M^4 \sinh^4 \theta d\theta e^{-iM\delta \cosh \theta} \\ &= -4\pi^2 i \frac{\partial}{\partial M^2} M^4 \frac{K_2(iM\delta)}{(iM\delta)^2}. \end{aligned} \quad (\text{B.15})$$

This result can be obtained from eq. (B.8), substituting there $\delta \rightarrow i\delta$. The expansion for small arguments yields

$$I_{11}^{(2)} \Big|_{\text{time-like}} = 2\pi^2 i \left\{ -\frac{1}{\delta^2} + \frac{M^2}{2} \left[\ln \frac{M\delta}{2} + \gamma - \frac{1}{2} + \frac{i\pi}{2} \right] + \mathcal{O}(\delta^2) \right\}. \quad (\text{B.16})$$

d. δ time-like, $\mu = \nu = 0$

This case is similar to case *b* above,

$$\begin{aligned} I_{00}^{(2)} &= \int d^3l \int dl_0 e^{il_0\delta} \frac{l_0^2}{(l_0^2 - l^2 - M^2 + i\epsilon)^2} \\ &= 3I_{11}^{(2)} + \int d^3l \int dl_0 e^{il_0\delta} \frac{l_0^2 - l^2}{(l_0^2 - l^2 - M^2 + i\epsilon)^2} \\ &= 3I_{11}^{(2)} + \int d^3l \int dl_0 e^{il_0\delta} \frac{\partial}{\partial M^2} \frac{M^2}{l_0^2 - l^2 - M^2 + i\epsilon} \end{aligned}$$

$$\begin{aligned}
&= 3I_{11}^{(2)} + 4\pi \frac{\partial}{\partial M^2} M^2 \int_0^\infty l^2 dl \int dl_0 e^{il_0 \delta} \frac{1}{l_0^2 - l^2 - M^2 + i\epsilon} \\
&= 3I_{11}^{(2)} + 4\pi \frac{\partial}{\partial M^2} M^2 \int_0^\infty l^2 dl (2\pi i) \frac{e^{-i\sqrt{l^2 + M^2} \delta}}{-2\sqrt{l^2 + M^2}} \\
&= 3I_{11}^{(2)} - 4\pi^2 i \frac{\partial}{\partial M^2} M^2 \int_0^\infty M^2 \sinh^2 \theta d\theta e^{-iM\delta \cosh \theta} \\
&= 3I_{11}^{(2)} - 4\pi^2 i \frac{\partial}{\partial M^2} M^4 \frac{K_1(iM\delta)}{iM\delta}.
\end{aligned} \tag{B.17}$$

The small-argument expansion yields

$$I_{00}^{(2)} \Big|_{\text{time-like}} = 2\pi^2 i \left\{ -\frac{1}{\delta^2} - \frac{M^2}{2} \left[\ln \frac{M\delta}{2} + \gamma + \frac{1}{2} + \frac{i\pi}{2} \right] + \mathcal{O}(\delta^2) \right\}. \tag{B.18}$$

We note that this result can be obtained from case *b* by the substitution (A.13), if we also change the over-all sign.

The results (B.16) and (B.18) may be combined, in Lorentz invariant form, as

$$I_{\mu\nu}^{(2)} \Big|_{\text{time-like}} = 2\pi^2 i \left\{ \left[\frac{1}{\delta^2} - \frac{M^2}{2} \left(\ln \frac{M\delta}{2} + \gamma - \frac{1}{2} + \frac{i\pi}{2} \right) \right] g_{\mu\nu} - \frac{2}{\delta^2} \left(1 + \frac{M^2 \delta^2}{4} \right) \frac{\delta_\mu \delta_\nu}{\delta^2} \right\}. \tag{B.19}$$

with $\delta^2 > 0$, and $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

$I_{\mu\nu}^{(6)}$ and $I_{\mu\nu\rho}^{(8)}$

The integral $I_{\mu\nu}^{(6)}$ of eq. (B.2) is related to this $I_{\mu\nu}^{(2)}$,

$$\begin{aligned}
I_{\mu\nu}^{(6)} &= \frac{1}{2} \frac{\partial}{\partial M^2} I_{\mu\nu}^{(2)} \\
&= -\frac{i\pi^2}{2} \left[\left(\ln \frac{M\delta}{2} + \gamma + \frac{i\pi}{2} \right) g_{\mu\nu} + \frac{\delta_\mu \delta_\nu}{\delta^2} \right],
\end{aligned} \tag{B.20}$$

whereas the evaluation of $I_{\mu\nu\rho}^{(8)}$ involves differentiation of $I_{\mu\nu}^{(6)}$ with respect to δ_ρ ,

$$\begin{aligned}
I_{\mu\nu\rho}^{(8)} &= -i \frac{\partial}{\partial \delta^\rho} I_{\mu\nu}^{(6)} \\
&= -\frac{\pi^2}{2} \left[\frac{\delta_\mu g_{\nu\rho} + \delta_\nu g_{\mu\rho} + \delta_\rho g_{\mu\nu}}{\delta^2} - \frac{2\delta_\mu \delta_\nu \delta_\rho}{\delta^4} \right].
\end{aligned} \tag{B.21}$$

Appendix C. The integrals $I_\mu^{(3)}$, $I^{(4)}$ and $I_\mu^{(5)}$

We consider here the integrals

$$I_\mu^{(3)} = \int d^4 l e^{il \cdot \delta} \frac{l_\mu}{(l^2 - M^2 + i\epsilon)^2}, \quad (\text{C.1})$$

$$I^{(4)} = \int d^4 l e^{il \cdot \delta} \frac{1}{(l^2 - M^2 + i\epsilon)^2}, \quad (\text{C.2})$$

$$I_\mu^{(5)} = \int d^4 l e^{il \cdot \delta} \frac{l_\mu}{l^2 - M^2 + i\epsilon}. \quad (\text{C.3})$$

The first two are related as follows,

$$I_\mu^{(3)} = -i \frac{\partial}{\partial \delta^\mu} I^{(4)}, \quad (\text{C.4})$$

and furthermore

$$I^{(4)} = \frac{\partial}{\partial M^2} I^{(1)}. \quad (\text{C.5})$$

Substituting from eqs. (A.7) and (A.14), we obtain

$$I^{(4)} = \begin{cases} -2\pi^2 i \left[\ln \frac{M\delta}{2} + \gamma \right] & \delta \text{ space-like,} \\ -2\pi^2 i \left[\ln \frac{M\delta}{2} + \gamma + \frac{i\pi}{2} \right] & \delta \text{ time-like.} \end{cases} \quad (\text{C.6})$$

Therefore, in both cases,

$$I_\mu^{(3)} = -2\pi^2 \frac{\delta_\mu}{\delta^2}, \quad (\text{C.7})$$

with $\delta^2 > 0$.

$I_\mu^{(5)}$, δ space-like

Let us now consider $I_\mu^{(5)}$. First, for δ space-like, we perform a Wick-rotation,

$$I_\mu^{(5)} = i \int d^4 l e^{-il \cdot \delta} \frac{-l_\mu}{l^2 + M^2}, \quad (\text{C.8})$$

where for brevity we write l for the Euclidean l_E . Choosing the direction of δ to coincide with the positive fourth axis, we have

$$I_\mu^{(5)} = 0, \quad \text{for } \mu = 1, 2, 3. \quad (\text{C.9})$$

Furthermore,

$$I_4^{(5)} = -i \int d^3 l \int dl_4 e^{-il_4 \delta} \frac{l_4}{l_4^2 + l^2 + M^2}, \quad (\text{C.10})$$

where l now denotes the length of the three-vector, $l = \sqrt{l_1^2 + l_2^2 + l_3^2}$. The integration over l_4 is performed by closing the contour in the lower half-plane,

$$I_4^{(5)} = -(2\pi)^2 \int_0^\infty l^2 dl e^{-\sqrt{l^2 + M^2} \delta}. \quad (\text{C.11})$$

Integrating by parts, we obtain

$$\begin{aligned} I_4^{(5)} &= -\frac{(2\pi)^2}{3} \int_0^\infty l^3 dl e^{-\sqrt{l^2 + M^2} \delta} \frac{l \delta}{\sqrt{l^2 + M^2}} \\ &= -\frac{(2\pi)^2}{3} \delta \int_0^\infty M^4 \sinh^4 \theta d\theta e^{-M \delta \cosh \theta}. \end{aligned} \quad (\text{C.12})$$

By eq. (A.11), we find

$$\begin{aligned} I_4^{(5)} &= -(2\pi)^2 M^3 \frac{K_2(M\delta)}{M\delta} \\ &= -2\pi^2 \left\{ \frac{4}{\delta^3} - \frac{M^2}{\delta} - \frac{M^4 \delta}{4} \left[\ln \frac{M\delta}{2} + \gamma - \frac{3}{4} \right] + \mathcal{O}(\delta^3) \right\}. \end{aligned} \quad (\text{C.13})$$

$I_\mu^{(5)}$, δ time-like

With δ time-like, we again choose the time axis to be along δ . Then

$$I_\mu^{(5)} = \int d^3 l \int dl_0 e^{il_0 \delta} \frac{l_\mu}{l_0^2 - l^2 - M^2 + i\epsilon}. \quad (\text{C.14})$$

Then

$$I_\mu^{(5)} = 0, \quad \text{for } \mu = 1, 2, 3, \quad (\text{C.15})$$

and

$$I_0^{(5)} = 4\pi \int_0^\infty l^2 dl \int dl_0 e^{il_0 \delta} \frac{l_0}{l_0^2 - l^2 - M^2 + i\epsilon}. \quad (\text{C.16})$$

Closing the contour in the upper half l_0 -plane, we get a contribution from the pole at $l_0 = -\sqrt{l^2 + M^2}$,

$$I_0^{(5)} = (2\pi)^2 i \int_0^\infty l^2 dl e^{-i\sqrt{l^2 + M^2} \delta} \quad (\text{C.17})$$

This integral is similar to the one in (C.11),

$$\begin{aligned} I_0^{(5)} &= (2\pi)^2 i M^3 \frac{K_2(iM\delta)}{iM\delta} \\ &= -2\pi^2 \left\{ \frac{4}{\delta^3} + \frac{M^2}{\delta} - \frac{M^4\delta}{4} \left[\ln \frac{M\delta}{2} + \gamma - \frac{3}{4} + \frac{i\pi}{2} \right] + \mathcal{O}(\delta^3) \right\}. \end{aligned} \quad (\text{C.18})$$

More generally,

$$I_\mu^{(5)} = -2\pi^2 \left\{ \frac{4}{\delta^4} + \frac{M^2}{\delta^2} - \frac{M^4}{4} \left[\ln \frac{M\delta}{2} + \gamma - \frac{3}{4} + \frac{i\pi}{2} \right] + \mathcal{O}(\delta^2) \right\} \delta_\mu. \quad (\text{C.19})$$

For use in eq. (4.8), the terms of orders $\ln \delta$ and 1 in the braces do not contribute. Therefore, for this purpose,

$$I_\mu^{(5)} = -2\pi^2 \left(\frac{4}{\delta^4} + \frac{M^2}{\delta^2} \right) \delta_\mu, \quad (\text{C.20})$$

which holds for both time-like and space-like δ .

Appendix D. The coefficient C

We here evaluate the coefficient C given by eq. (6.14). Writing p instead of p_4 , we have

$$\begin{aligned} C &= \frac{1}{6p^2} \frac{-8e^4}{(2\pi)^4} \int d^4k \left[\frac{-2k^2(p \cdot k)}{(k^2 + i\epsilon)^3} - \frac{-2(k+p)^2(p \cdot (k+p))}{[(k+p)^2 + i\epsilon]^3} \right] \\ &= \frac{1}{6p^2} \frac{16e^4}{(2\pi)^4} \int d^4k \left[\frac{p \cdot k}{(k^2 + i\epsilon)^2} - \frac{p \cdot k + p^2}{[(k+p)^2 + i\epsilon]^2} \right] \\ &= \frac{1}{6p^2} \frac{16e^4}{(2\pi)^4} \int d^4k \frac{[2(p \cdot k) + p^2][2k^2 + 2(p \cdot k) + p^2](p \cdot k) - (k^2)^2 p^2}{(k^2 + i\epsilon)^2 [(k+p)^2 + i\epsilon]^2} \\ &= \frac{1}{6p^2} \frac{16e^4}{(2\pi)^4} 6 \int_0^1 d\alpha \int d^4k \frac{[2k^2 + 2(p \cdot k) + p^2][2(p \cdot k) + p^2](p \cdot k) - (k^2)^2 p^2}{[k^2 + 2\alpha(p \cdot k) + \alpha p^2 + i\epsilon]^4} \alpha(1-\alpha) \\ &= \frac{1}{p^2} \frac{16e^4}{(2\pi)^4} \int_0^1 d\alpha \int d^4l \frac{N(l, \alpha) \alpha(1-\alpha)}{[l^2 + \alpha(1-\alpha)p^2 + i\epsilon]^4}, \end{aligned} \quad (\text{D.1})$$

with

$$\begin{aligned} N(l, \alpha) &= [2l^2 + 2(1-2\alpha)(l \cdot p) + (1-2\alpha+2\alpha^2)p^2][2(l \cdot p) + (1-2\alpha)p^2] \\ &\quad \times [(l \cdot p) - \alpha p^2] - [l^2 - 2\alpha(l \cdot p) + \alpha^2 p^2]^2 p^2 \\ &= [2l^2 + (1-2\alpha+2\alpha^2)p^2][2(l \cdot p)^2 - \alpha(1-2\alpha)(p^2)^2] + 2(1-2\alpha)(1-4\alpha)(l \cdot p)^2 p^2 \\ &\quad + [-(l^2)^2 - 4\alpha^2(l \cdot p)^2 - 2\alpha^2 l^2 p^2 - \alpha^4(p^2)^2] p^2 + \text{terms odd under } (l \rightarrow -l). \end{aligned} \quad (\text{D.2})$$

Averaging over angles, according to

$$(l \cdot p)^2 \rightarrow \frac{1}{4} l^2 p^2, \quad (\text{D.3})$$

we obtain

$$N(l, \alpha) \rightarrow \frac{1}{2} p^2 \{ (2 - 12\alpha + 12\alpha^2) l^2 p^2 - 2\alpha(1 - 4\alpha + 6\alpha^2 - 3\alpha^3)(p^2)^2 \}. \quad (\text{D.4})$$

Therefore,

$$C = \frac{p^2}{2} \frac{16e^4}{(2\pi)^4} \int_0^1 d\alpha \int d^4 l \frac{(2 - 12\alpha + 12\alpha^2) l^2 - 2\alpha(1 - \alpha)(1 - 3\alpha + 3\alpha^2) p^2}{[l^2 + \alpha(1 - \alpha)p^2 + i\epsilon]^4} \alpha(1 - \alpha). \quad (\text{D.5})$$

In terms of the dimensionless four-vector q ,

$$l^2 = \alpha(1 - \alpha)p^2 q^2, \quad (\text{D.6})$$

we obtain

$$\begin{aligned} C &= \frac{16e^4}{(2\pi)^4} \int_0^1 d\alpha \int d^4 q \frac{(1 - 6\alpha + 6\alpha^2) q^2 - (1 - 3\alpha + 3\alpha^2)}{(q^2 + 1 + i\epsilon)^4} \\ &= \frac{-8e^4}{(2\pi)^4} \int d^4 q \frac{1}{(q^2 + 1 + i\epsilon)^4}. \end{aligned} \quad (\text{D.7})$$

But

$$\begin{aligned} \int d^4 q \frac{1}{(q^2 - m^2 + i\epsilon)^4} &= i \int d^4 q_E \frac{1}{(q_E^2 + m^2)^4} = 2\pi^2 i \int_0^\infty q_E^3 dq_E \frac{1}{(q_E^2 + m^2)^4} \\ &= \frac{i\pi^2}{m^4} \int_0^\infty \frac{x dx}{(x + 1)^4} = \frac{i\pi^2}{6m^4}. \end{aligned} \quad (\text{D.8})$$

Hence,

$$C = -\frac{e^4 i}{3(2\pi)^2}. \quad (\text{D.9})$$

Appendix E. The integral $I_{\mu\nu\rho\sigma}^{(\gamma)}$

We shall here evaluate the integral

$$I_{\mu\nu\rho\sigma}^{(\gamma)} = \int d^4 l e^{il \cdot \delta} \frac{l_\mu l_\nu l_\rho l_\sigma}{(l^2 - M^2 + i\epsilon)^4}, \quad (\text{E.1})$$

for δ time-like. There are then three cases.

δ time-like, μ, ν, ρ, σ all space-like

By symmetry, we can in this case write

$$I_{\mu\nu\rho\sigma}^{(7)} = I^{(7)}(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}). \quad (\text{E.2})$$

Contracting (E.1) with $g^{\mu\nu}g^{\rho\sigma}$, with

$$g^{\mu\nu}g^{\rho\sigma}(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}) = 3^2 + 3 + 3 = 15,$$

we obtain

$$I^{(7)} = \frac{1}{15} \int d^3l \int dl_0 e^{il_0\delta} \frac{l^4}{(l_0^2 - l^2 - M^2 + i\epsilon)^4}, \quad (\text{E.3})$$

where l denotes the length of the three-vector $\vec{l}$.

This integral can further be written as

$$\begin{aligned} I^{(7)} &= \frac{4\pi}{15} \int_0^\infty l^6 dl \int dl_0 e^{il_0\delta} \frac{1}{6} \left(\frac{\partial}{\partial M^2} \right)^3 \frac{1}{l_0^2 - l^2 - M^2 + i\epsilon} \\ &= -\frac{4\pi^2 i}{15 \cdot 6} \left(\frac{\partial}{\partial M^2} \right)^3 M^6 \int_0^\infty \sinh^6 \theta d\theta e^{-iM\delta \cosh \theta} \\ &= -\frac{4\pi^2 i}{6} \left(\frac{\partial}{\partial M^2} \right)^3 M^6 \frac{K_3(iM\delta)}{(iM\delta)^3} \\ &= -\frac{i\pi^2}{12} \left[\ln \frac{M\delta}{2} + \gamma + \frac{i\pi}{2} + \mathcal{O}(\delta^2) \right]. \end{aligned} \quad (\text{E.4})$$

δ time-like, $\rho = \sigma = 0$, μ and ν are $\neq 0$

In this case,

$$\begin{aligned} I_{\mu\nu 00}^{(7)} &= \int d^3l \int dl_0 e^{il_0\delta} \frac{l_\mu l_\nu l_0^2}{(l_0^2 - l^2 - M^2 + i\epsilon)^4} \\ &= -\frac{4\pi}{3} g_{\mu\nu} \int_0^\infty l^4 dl \int dl_0 e^{il_0\delta} \frac{l_0^2}{(l_0^2 - l^2 - M^2 + i\epsilon)^4} \\ &= -g_{\mu\nu} \left[5I^{(7)} + \frac{4\pi}{3} \int_0^\infty l^4 dl \int dl_0 e^{il_0\delta} \frac{l_0^2 - l^2}{(l_0^2 - l^2 - M^2 + i\epsilon)^4} \right] \\ &= -g_{\mu\nu} \left[5I^{(7)} + \frac{4\pi}{18} \left(\frac{\partial}{\partial M^2} \right)^3 M^2 \int_0^\infty l^4 dl \int dl_0 e^{il_0\delta} \frac{1}{l_0^2 - l^2 - M^2 + i\epsilon} \right] \\ &= -g_{\mu\nu} \left[5I^{(7)} - \frac{4\pi^2 i}{6} \left(\frac{\partial}{\partial M^2} \right)^3 \frac{K_2(iM\delta)}{(iM\delta)^2} \right] \\ &= -\frac{i\pi^2}{12} g_{\mu\nu} \left[\ln \frac{M\delta}{2} + \gamma + 1 + \frac{i\pi}{2} + \mathcal{O}(\delta^2) \right]. \end{aligned} \quad (\text{E.5})$$

δ time-like, $\mu = \nu = \rho = \sigma = 0$

Again with $l = |\vec{l}|$, we have

$$\begin{aligned}
I_{0000}^{(7)} &= \int d^3l \int dl_0 e^{il_0\delta} \frac{l_0^4}{(l_0^2 - l^2 - M^2 + i\epsilon)^4} \\
&= \int d^3l \int dl_0 e^{il_0\delta} \frac{(l_0^2 - l^2)^2 + 2l_0^2 l^2 - l^4}{(l_0^2 - l^2 - M^2 + i\epsilon)^4} \\
&= -15I^{(7)} + 2 \int d^3l \int dl_0 e^{il_0\delta} \frac{l_0^2 l^2}{(l_0^2 - l^2 - M^2 + i\epsilon)^4} \\
&\quad + \int d^3l \int dl_0 e^{il_0\delta} \frac{(l_0^2 - l^2)^2}{(l_0^2 - l^2 - M^2 + i\epsilon)^4} \\
&= -15I^{(7)} + 2g^{ij}I_{ij00}^{(7)} - \frac{4\pi^2 i}{6} \left(\frac{\partial}{\partial M^2} \right)^3 M^6 \frac{K_1(iM\delta)}{(iM\delta)} \\
&= -\frac{i\pi^2}{4} \left[\ln \frac{M\delta}{2} + \gamma + \frac{4}{3} + \frac{i\pi}{2} + \mathcal{O}(\delta^2) \right]. \tag{E.6}
\end{aligned}$$

δ time-like, Summary for $I_{\mu\nu\rho\sigma}^{(7)}$

The above results may be summarized as follows:

$$\begin{aligned}
I_{\mu\nu\rho\sigma}^{(7)} &= -\frac{i\pi^2}{12} \left[\ln \frac{M\delta}{2} + \gamma + \frac{i\pi}{2} \right] (g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}) \\
&\quad - \frac{i\pi^2}{12} \left[g_{\mu\nu} \frac{\delta_\rho\delta_\sigma}{\delta^2} + g_{\mu\sigma} \frac{\delta_\nu\delta_\rho}{\delta^2} + g_{\mu\rho} \frac{\delta_\nu\delta_\sigma}{\delta^2} + g_{\nu\sigma} \frac{\delta_\mu\delta_\rho}{\delta^2} + g_{\nu\rho} \frac{\delta_\mu\delta_\sigma}{\delta^2} + g_{\rho\sigma} \frac{\delta_\mu\delta_\nu}{\delta^2} \right] \\
&\quad + \frac{i\pi^2}{6} \frac{\delta_\mu\delta_\nu\delta_\rho\delta_\sigma}{\delta^4}. \tag{E.7}
\end{aligned}$$

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Figure captions

- Fig. 1. One-loop contribution to the photon self-energy.
- Fig. 2. One-loop “peculiar” diagram contributing to the photon self-energy.
- Fig. 3. The three basic diagrams describing photon-photon scattering.
- Fig. 4. Cancellations among various pieces of the three amplitudes after contraction with the photon momentum p_4 . Heavy arrows indicate that the terms cancel in the sum.
- Fig. 5. “Peculiar” diagrams for photon-photon scattering.

Figure 1

Figure 2

Figure 3

Figure 4

I'

6 diagrams

I''

3 diagrams

I'''

4 diagrams

I''''

1 diagram

Figure 5