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THE INFLUENCE OF A VERTICAL MAGNETIC FIELD ON OSCILLATIONS IN AN ISOTHERMAL STRATIFIED ATMOSPHERE

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Abstract

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The Influence of a Vertical Magnetic Field on Oscillations in an Isothermal Stratified Atmosphere

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Abstract. We examine the effect of a uniform vertical magnetic field on the modes of an isothermal stratified atmosphere. The general solutions of the wave equation for an isothermal magnetized medium are first given. We present asymptotic expansions of these solutions in the strong- and weak-field limits. In the latter case, an analytical expression for the dispersion relation is derived, which allows the effect of a weak magnetic field on the modes to be studied. For a weak field we find that, to lowest order in our perturbation expansion, the oscillation spectrum can be analysed in terms of (a) p - and g -like modes; (b) magnetic Lamb modes; and (c) magnetic or slow modes. Our approximation is valid as long as the frequencies of the individual modes are well separated and not too low. The frequency corrections for each of the modes due to coupling with the remaining modes are calculated. It is explicitly demonstrated that when the frequencies of different modes are almost identical, strong mode coupling occurs and the waves acquire a mixed character. At such frequencies, the asymptotic behaviour cannot be treated in terms of elementary modes, but has to be modified to include the effects of degeneracy. An analysis of the solutions in the vicinity of the degenerate frequencies is carried out. Detailed results are presented in the form of diagnostic diagrams, showing the variation of frequency with horizontal wave number. The solutions clearly reveal the presence of *avoided crossings*, which occur at the otherwise degenerate frequencies. Examples of different types of avoided crossings are given. We also present results for the moderate to strong field case and identify the solutions of various orders. A comparison is made with previous studies.

1. INTRODUCTION

Magnetic fields dominate the structure of the solar surface layers, and the existence of magnetic fields in other stars has been confirmed by several observations. For instance, it is well known that the solar photosphere is permeated with vertical magnetic fields, usually in the form of flux tubes. Examples of these range from fibrils or intense flux tubes, which constitute the small-scale magnetic field, to sunspots, which play an important role in global solar activity. A study of wave motions can reveal useful information on the structure of magnetic elements and thus serve as a powerful diagnostic tool. Indeed, oscillations with periods in a fairly broad range of frequencies have been widely reported in magnetic elements. Examples of these are umbral oscillations (Beckers and Schultz 1972; Bhatnagar *et al.* 1972; Rice and Gaizauskas 1973; Schröter and Soltau 1976; von Uexküll *et al.* 1983; Lites 1984; Lites and Thomas 1985; Abdelatif *et al.* 1984, 1986; Baltasar *et al.* 1987; Gurman 1987; see also the review by Moore and Rabin 1985). Apart from the observations of Giovanelli (1975) and Giovanelli *et al.* (1978) few data are available on oscillations in fibrils, due to their small horizontal dimensions.

Given the large amount of observational evidence on the existence of oscillations in magnetic elements, it is natural to seek a theoretical explanation of the various wave modes that can occur in them. Any theory has to take into account the variable structure of the

atmosphere in the vertical direction. As a first step, one can use local dispersion relations (*e.g.*, McLellan and Winterberg 1968; Bel and Mein 1971; Nagakawa *et al.* 1973), which provide a qualitative description of the oscillations. In particular, this method demonstrates the existence of cut-offs in the propagation frequency. However, for comparing theoretical results with observations, global dispersion relations must be used. Indeed, it has been found that the local approach, based upon the WKB approximation, can give misleading results (Thomas 1982).

Thus, what is required is a theory of wave propagation in a stratified atmosphere with a vertical magnetic field. The theoretical understanding of the nature of wave modes in such a medium is currently far from complete, despite the existence of a large number of papers on the subject (*e.g.*, Ferraro and Plumpton 1958, hereafter FP; Uchida and Sakurai 1975; Antia and Chitre 1978; Zhugzhda 1979; Scheuer and Thomas 1981; Leroy and Schwartz 1982; Zhugzhda and Dzhalilov 1982, hereafter ZD, 1984a, 1984b; Hasan and Sobouti 1987; Hasan and Abdelatif 1990; Hasan 1991; Abdelatif 1991; see also the reviews by Thomas 1983; Roberts 1985; Campos 1987; Hollweg 1990). This is principally because the theory of magneto-acoustic-gravity (hereafter abbreviated as MAG) waves is rather complicated. Analytical progress has generally been confined to special cases, such as assuming an isothermal stratification. This case was examined in some detail by FP, who obtained asymptotic solutions of the wave equations in the limits of strong and weak magnetic fields for an isothermal plasma. The analysis was carried further by Zhugzhda (1979) and ZD to treat the more general case of arbitrary field strength. The latter authors were able to obtain the exact solution of the wave equation in terms of Meijer functions. Despite this mathematical progress, the physical properties of the solutions have not been fully explored. The stratification allows mutual transformation of waves, so that in general the problem cannot be examined in terms of the elementary modes of a non-stratified magnetized atmosphere.

In this investigation, we attempt to analyse the physical nature of MAG oscillations, by examining how the normal modes of an unmagnetized stratified atmosphere are modified by the introduction of a small vertical magnetic field. Some aspects of this problem have been studied in the past amongst others by Ledoux and Simon (1957), Goossens (1972), Goossens, Smeysers and Denis (1976), Biront *et al.* (1982), Roberts and Soward (1983). These investigations used a perturbation approach to treat the effect of a magnetic field on radial oscillations. In the present study, we shall be concerned primarily with non-radial modes (*i.e.*, modes with finite horizontal wave number). New features present in our analysis include a calculation of the frequency shift of the p - and g -modes of an unmagnetized atmosphere due to the presence of a vertical magnetic field, the explicit demonstration of the existence of a magnetic Lamb mode and a treatment of the interaction of magnetic modes with non-magnetic modes. The calculation of the frequency shifts of p -modes due to a *horizontal* (or toroidal) magnetic field has been carried out by several authors such as Dziembowski and Goode (1984), Gough and Taylor (1984), Roberts and Campbell (1986), Gough and Thompson (1988, 1990), Vorontsov (1988), and Campbell and Roberts (1989). However, an investigation of the frequency shifts of p -modes for a vertical field in a stratified atmosphere has so far received practically no attention, largely due to the fact that the order of the relevant differential equation changes from two (for an

unmagnetized or horizontal field) to four (for a vertical field). For vertical fields, the effect on p -mode frequencies due to scattering off flux tubes has been addressed by Zweibel and Bogdan (1987), Bogdan and Zweibel (1987), and Bogdan and Cattaneo (1989), but the stratification was neglected in these studies.

Strictly speaking, the present analysis pertains to homogeneous fields and not to flux tubes. Unfortunately, even the construction of an equilibrium solution for a stratified flux tube is extremely complicated, let alone the question of wave propagation. Most of the analyses of waves in flux tubes have either neglected gravity or used the thin flux tube approximation (see Hollweg 1990; Roberts 1990; and Thomas 1990 for recent reviews on flux tube waves). The former approach, which neglects the stratification, is inappropriate for application to photospheric flux tubes. The thin flux tube approximation, on the other hand, provides a more reasonable framework for treating wave propagation in intense flux tubes, where the tube radius is typically smaller than the pressure scale height in the vertical direction. However, in addition to its somewhat restrictive scope, this approximation has the limitation that it effectively eliminates the fast mode from the analysis. In the present investigation, it is assumed that the flux tubes in question are sufficiently thick, so that the field can be regarded to have infinite horizontal extent.

The plan of the paper is as follows: in Section 2, the basic wave equations are presented for an isothermal stratified atmosphere in a vertical magnetic field. This is followed in Sections 3 and 4 by an analysis of the asymptotic solutions in the strong- and weak-field limits respectively. In Section 4 we determine the normal-mode spectrum for a weak field. It turns out that the normal-mode frequency can be expressed as the sum of an elementary-mode frequency and a first-order correction, due to coupling between the different modes. We also examine the behaviour of the solutions close to the degenerate frequencies, *i.e.*, when the different elementary modes have the same frequency. In Sections 5 and 6, numerical results are presented showing the variation of the eigenfrequencies with horizontal wave number, for both weak and moderate field strengths. A discussion of the results and a comparison with previous studies is taken up in Section 7. Finally, the salient features of this investigation and the main conclusions are summarized in Section 8.

2. WAVE EQUATIONS IN A UNIFORM VERTICAL FIELD

We consider an isothermal stratified atmosphere, with a uniform vertical magnetic field B , which is unbounded in the horizontal direction. In cartesian geometry, the linearized equations for MAG waves can be written in terms of the Lagrangian displacement $\xi \sim e^{i(\omega t - kx)}$ as (Ferraro and Plumpton, 1958)

$$\left[v_A^2 \frac{d^2}{dz^2} - (c_s^2 + v_A^2) k^2 + \omega^2 \right] \xi_x - ik \left(c_s^2 \frac{d}{dz} - g \right) \xi_z = 0, \quad (1)$$

$$\left[c_s^2 \frac{d^2}{dz^2} - \gamma g \frac{d}{dz} + \omega^2 \right] \xi_z - ik \left[c_s^2 \frac{d}{dz} - (\gamma - 1)g \right] \xi_x = 0, \quad (2)$$

$$(\omega^2 + v_A^2 \frac{d^2}{dz^2}) \xi_y = 0, \quad (3)$$

where z , the vertical co-ordinate, is measured positive upwards, k is the horizontal wave number, ω is the frequency, g is the acceleration due to gravity, γ is the ratio of specific heats, c_s is the sound speed and v_A is the Alfvén speed. The sound and Alfvén speeds are defined respectively as

$$c_s = \sqrt{\frac{\gamma p}{\rho}} \quad \text{and} \quad v_A = \frac{B}{\sqrt{4\pi\rho}},$$

where p is the gas pressure and ρ is the mass density. Equations (1) and (2) govern the propagation of MAG waves in a stratified atmosphere.

Equation (3), which, incidentally, is decoupled from the other equations and describes the purely transverse Alfvén waves, will not be considered in the present investigation. Detailed solutions for Alfvén waves in an isothermal plasma can be found in the paper by FP. We have implicitly assumed that the propagation and motions of the MAG modes are confined to the $x - z$ plane. This involves no loss of generality.

For simplicity, let us assume that the equilibrium atmosphere is isothermal. This implies that c_s is constant with z and that ρ has the following height dependence

$$\rho = \rho_0 e^{-z/H}, \quad (4)$$

where H is the scale height of the atmosphere.

It is convenient to work in terms of the following dimensionless parameters

$$K = kH, \quad (5a)$$

$$\Omega = \frac{\omega H}{c_s}, \quad (5b)$$

and the dimensionless vertical coordinate

$$\theta = \frac{\omega H}{v_A} = \frac{c_s}{v_{A,0}} \Omega e^{-z/(2H)}, \quad (5c)$$

where $v_{A,0}$ is the Alfvén speed at $z = 0$. In terms of the variables defined by equations (5a) – (5c), equations (1) – (2) can be combined into the following fourth-order differential equation for ξ_z (Zhugzhda 1979)

$$\left\{ \theta^4 \frac{d^4}{d\theta^4} + 4\theta^3 \frac{d^3}{d\theta^3} + [1 + 4(\Omega^2 - K^2) + 4\theta^2] \theta^2 \frac{d^2}{d\theta^2} - [1 - 4(\Omega^2 + K^2) - 12\theta^2] \theta \frac{d}{d\theta} + 16[\Omega^2 + K^2(\frac{\gamma-1}{\gamma^2\Omega^2} - 1)]\theta^2 - \Omega^2 K^2 \right\} \xi_z = 0. \quad (6)$$

The solution of equation (6) can be expressed in terms of Meijer functions as follows (ZD)

$$\xi_z^{(h)} = G_{2,4}^{12} \left(\begin{matrix} a_1 \\ \mu_1, \dots, \mu_4 \end{matrix} \middle| \theta^2 \right) \quad (i, h = 1, \dots, 4; i \neq h), \quad (7)$$

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where

$$\mu_{1,2} = \frac{(1 \pm i\alpha)}{2}, \quad \mu_{3,4} = \pm K, \quad (8)$$

$$a_{1,2} = \frac{(1 \pm \phi)}{2},$$

$$\alpha = \sqrt{4\Omega^2 - 1},$$

$$\phi = \sqrt{-\alpha^2 + 4K^2(1 - \Omega_{BV}^2/\Omega^2)},$$

and $\Omega_{BV}^2 = (\gamma - 1)/\gamma^2$ is the Brunt-Väisälä frequency (in dimensionless units).

Once $\xi_z^{(h)}$ is known, it is fairly straightforward to determine the corresponding solutions $\xi_z^{(h)}$ from either of equations (1) or (2). The complete solutions satisfying the required boundary conditions can be built up as linear combinations of $\xi_z^{(h)}$ and $\xi_z^{(h)}$.

3. STRONG FIELD SOLUTIONS

In the limit of a strong field, for which $c_s/v_A \ll 1$ and $\theta \ll 1$, the solutions are to lowest order in θ

$$\xi_z^{(h)} = R_h \theta^{2\mu_h} \quad (h = 1, \dots, 4), \quad (9)$$

$$\xi_z^{(h)} = \begin{cases} R_h \frac{4i\Omega^2(\mu_h^2 - K^2)}{K(1/\gamma + \mu_h - 1)} \frac{\theta^{2\mu_h}}{4\theta^2} & (h = 1, 2), \\ -R_h \frac{iK[(\gamma - 1)/\gamma + \mu] \theta^{2\mu_h}}{(\mu^2 + \mu + \Omega^2)} & (h = 3, 4), \end{cases} \quad (10)$$

where

$$R_h = \frac{\prod_{j=1}^4 \Gamma(1 + \mu_h - a_j)}{\prod_{j \neq h}^4 \Gamma(1 + \mu_h - \mu_j)}.$$

When $\alpha^2 < 0$ ($\Omega^2 < 1/4$), the amplitudes given by equations (9) – (10) either fall off or increase exponentially with height. Purely growing solutions are unphysical and can be discarded. The only permitted solutions correspond to evanescent waves. Hence, the atmosphere has a cut-off frequency, which is the same as the usual acoustic cut-off frequency ($\Omega_a = 1/2$) in an unmagnetized atmosphere. On the other hand, for $\Omega > \Omega_a$, the solutions represent propagating waves.

In order to get a physical picture of the solutions, let us consider the case of upward travelling waves, excited from below at $z = 0$, in an atmosphere that is not necessarily isothermal. To the extent that their local behaviour can be described in terms of the

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isothermal solutions, the waves are evanescent or propagating for $\Omega < \Omega_a$ and $\Omega > \Omega_a$, respectively. Even in the latter case, standing wave solutions are possible if Ω_a increases with height. A wave propagating upward will be reflected down when its frequency becomes equal to the local acoustic cut-off value.

If $z = z_t$ denotes this height, we find that for $z \leq z_t$ a standing wave can develop due to the superposition of waves propagating in opposite directions.

The asymptotic solutions, just presented, are valid only in regions of the atmosphere where $v_A \gg c_s$. Typically, this is the case in the upper layers of the solar photosphere. In the sub-photospheric layers, the opposite inequality holds.

4. WEAK FIELD SOLUTIONS

We now focus our attention on the opposite limit of a weak field, for which $\frac{c_s}{v_A} \gg 1$ and $\theta \gg 1$. This corresponds to the limit of small ϵ , with

$$\epsilon \equiv \frac{v_{A,0}}{c_s}. \quad (11)$$

However, in addition we must require that the vertical extension d of the layer is not too large (cf. equation [5c]) (for $d > 0$). Since the density increases for negative z , the above restriction, of course, does not apply if d lies below the plane $z = 0$.

When $\theta \gg 1$, the Meijer function in equation (7) has the following asymptotic expansion (ZD):

$$G_{2,4}^{12}(\mu_h|\theta^2) = \frac{1}{2\sqrt{\pi\theta}} \left[e^{i(2\theta - \epsilon_h)} \left(1 + i\frac{M}{\theta} - \frac{M_1}{\theta^2} \right) + e^{-i(2\theta - \epsilon_h)} \left(1 - i\frac{M}{\theta} - \frac{M_1}{\theta^2} \right) \right] \\ + \frac{1}{\theta} \left[S_h \left(1 + \frac{L_1^+}{\theta^2} \right) \theta^{2iK_z} + T_h \left(1 + \frac{L_1^-}{\theta^2} \right) \theta^{-2iK_z} \right] + O\left(\frac{1}{\theta^3}\right), \quad (12)$$

where

$$K_z^2 = \Omega^2 - K^2 \left(1 - \frac{\Omega_{BV}^2}{\Omega^2} \right) - \frac{1}{4} = \frac{\phi}{2\Omega}, \quad (13)$$

$$\delta_h = \pi \left(\frac{1}{4} + \mu_h \right),$$

$$M = K^2 \frac{\Omega_{BV}^2}{\Omega^2} - \frac{1}{16}, \quad M_1 = \frac{K^2}{2} - \frac{1}{32} + \frac{M}{4} (2M - 3),$$

$$L_1^\pm = - \frac{[w_\pm(w_\pm + 1) + \Omega^2](w_\pm^2 - K^2)}{2w_\pm}, \quad r = K \left(1 - \frac{\Omega_{BV}^2}{\Omega^2} \right),$$

$$w_\pm = \frac{1}{2} \mp iK_z = a_{2,1},$$

$$S_h = \frac{\Gamma(\phi) \Gamma(1 + \mu_h - a_1)}{\prod_{j \neq h}^4 \Gamma(a_1 - \mu_h)},$$

$$T_h = \frac{\Gamma(-\phi) \Gamma(1 + \mu_h - a_2)}{\prod_{j \neq h}^4 \Gamma(a_2 - \mu_h)}.$$

It should be borne in mind that the expansion for the Meijer given by equation (12) is valid as long as $M/\theta \ll 1$, which restricts K to values such that $K^2 \ll \Omega^3 v_A/c_s$. Thus, for $K \sim 1$, we have the condition $v_A/c_s \ll \Omega^3$, which places a more severe restriction on the permitted field strengths than the condition $\theta \gg 1$.

Substituting equation (12) in equation (7), we find that the solution can be expressed in a compact form as follows

$$\xi_z^{(h)} = \frac{1}{\theta} \sum_{j=1}^2 C_j^{(h)} f_j(\theta) + \frac{1}{\sqrt{\theta}} \sum_{j=3}^4 C_j^{(h)} f_j(\theta), \quad (14)$$

where

$$C_1^{(h)} = T_h + S_h, \quad C_2^{(h)} = T_h - iS_h, \quad (15)$$

$$C_3^{(h)} = \frac{1}{\sqrt{\pi}} \sin \delta_h, \quad C_4^{(h)} = \frac{1}{\sqrt{\pi}} \cos \delta_h.$$

Here

$$f_1 = (q_1 + \frac{q'_1}{\theta^2}) \cos K_z Z - (q_2 + \frac{q'_2}{\theta^2}) \sin K_z Z + O\left(\frac{1}{\theta^3}\right), \quad (16)$$

$$f_2 = (q_1 + \frac{q'_1}{\theta^2}) \sin K_z Z + (q_2 + \frac{q'_2}{\theta^2}) \cos K_z Z + O\left(\frac{1}{\theta^3}\right), \quad (17)$$

$$f_3 = \left(1 - \frac{M_1}{\theta^2} \right) \cos 2\theta - \frac{M}{\theta} \sin 2\theta + O\left(\frac{1}{\theta^3}\right), \quad (18)$$

$$f_4 = \left(1 - \frac{M_1}{\theta^2} \right) \sin 2\theta + \frac{M}{\theta} \cos 2\theta + O\left(\frac{1}{\theta^3}\right), \quad (19)$$

where $Z = z/H$ and

$$g_1 = \frac{1}{2} \left[\left(\frac{\Omega}{\epsilon} \right)^{-2iK_z} + \left(\frac{\Omega}{\epsilon} \right)^{2iK_z} \right], \quad g_2 = \frac{1}{2i} \left[\left(\frac{\Omega}{\epsilon} \right)^{-2iK_z} - \left(\frac{\Omega}{\epsilon} \right)^{2iK_z} \right],$$

$$g'_1 = \frac{1}{2} \left[L_1^- \left(\frac{\Omega}{\epsilon} \right)^{-2iK_z} + L_1^+ \left(\frac{\Omega}{\epsilon} \right)^{2iK_z} \right], \quad (20)$$

$$g'_2 = \frac{1}{2i} \left[L_1^- \left(\frac{\Omega}{\epsilon} \right)^{-2iK_z} - L_1^+ \left(\frac{\Omega}{\epsilon} \right)^{2iK_z} \right].$$

It follows that to this asymptotic order the general solution ξ_z , which is a linear combination of the $\xi_z^{(h)}$, may equally well be expressed as a linear combination of the functions f_j ($j = 1, \dots, 4$). Physically, $f_{1,2}$ and $f_{3,4}$ correspond to the non-magnetic and magnetic contributions to the solution, respectively. In order to calculate $\xi_z^{(h)}$, we assume a series expansion in ascending powers of $1/\theta$, similar to $\xi_z^{(h)}$ but with different coefficients. It turns out that $\xi_z^{(h)}$ can be expressed as

$$i\xi_z^{(h)} = \frac{1}{\theta} \sum_{j=1}^2 C_j^{(h)} g_j(\theta) + \frac{1}{\theta^{3/2}} \sum_{j=3}^4 C_j^{(h)} g_j(\theta). \quad (21)$$

The coefficients $g_{3,4}$ and $g_{1,2}$ in equation (21) are the corresponding magnetic and non-magnetic contributions to ξ_z respectively. These can be determined from equations (1) and (2) respectively, rewritten as

$$[4(\Omega^2 - K^2)\theta^2 - 4\Omega^2 K^2 + \Omega^2 \theta \frac{d}{d\theta} + \Omega^2 \theta^2 \frac{d^2}{d\theta^2}] \xi_z = -4iK\theta^2 \left(\frac{1}{\gamma} + \frac{\theta}{2} \frac{d}{d\theta} \right) \xi_z, \quad (22)$$

and

$$\left[\Omega^2 + \frac{3}{4} \theta \frac{d}{d\theta} + \frac{\theta^2}{4} \frac{d^2}{d\theta^2} \right] \xi_z = -iK \left(\frac{\gamma-1}{\gamma} + \frac{\theta}{2} \frac{d}{d\theta} \right) \xi_z. \quad (23)$$

The functions g_j , ($j = 1, \dots, 4$) are

$$g_1 = (p_1 + \frac{p_1}{\theta^2}) \cos K_z Z - (p_2 + \frac{p_2}{\theta^2}) \sin K_z Z + O\left(\frac{1}{\theta^3}\right), \quad (24)$$

$$g_2 = (p_1 + \frac{p_1}{\theta^2}) \sin K_z Z + (p_2 + \frac{p_2}{\theta^2}) \cos K_z Z + O\left(\frac{1}{\theta^3}\right), \quad (25)$$

$$g_3 = K(\sin 2\theta + \frac{N}{\theta} \cos 2\theta) + O\left(\frac{1}{\theta^2}\right), \quad (26)$$

$$g_4 = K(-\cos 2\theta + \frac{N}{\theta} \cos 2\theta) + O\left(\frac{1}{\theta^2}\right). \quad (27)$$

Here

$$p_1 = \frac{K}{2} \left[Q_0^- \left(\frac{\Omega}{\epsilon} \right)^{-2iK_z} + Q_0^+ \left(\frac{\Omega}{\epsilon} \right)^{2iK_z} \right], \quad (28)$$

$$p_2 = \frac{K}{2i} \left[Q_0^- \left(\frac{\Omega}{\epsilon} \right)^{-2iK_z} - Q_0^+ \left(\frac{\Omega}{\epsilon} \right)^{2iK_z} \right],$$

$$p'_1 = \frac{K}{2} \left[Q_1^- \left(\frac{\Omega}{\epsilon} \right)^{-2iK_z} + Q_1^+ \left(\frac{\Omega}{\epsilon} \right)^{2iK_z} \right],$$

$$p'_2 = \frac{K}{2i} \left[Q_1^+ \left(\frac{\Omega}{\epsilon} \right)^{-2iK_z} - Q_1^- \left(\frac{\Omega}{\epsilon} \right)^{2iK_z} \right],$$

where

$$N = M + \frac{1}{\gamma} + \frac{1}{4} - \frac{\Omega^2}{K^2}, \quad (29)$$

$$Q_0^\pm = -\frac{1}{rK} t_\pm, \quad t_\pm = \pm K_z + i\left(\frac{1}{\gamma} - \frac{1}{2}\right),$$

$$Q_1^\pm = \frac{(w_\pm + 1/\gamma)(w_\pm^2 - K^2)}{2w_\pm}.$$

Before examining the general solutions, it is instructive to consider the limiting cases $K \rightarrow 0$ and $K \rightarrow \infty$.

4.1 Solutions for $K \rightarrow 0$

From equations (22) - (23), we find that in the limit $K = 0$, ξ_z and ξ_z become decoupled. It is fairly straightforward to see that as $K \rightarrow 0$, there are two sets of solutions

$$\xi_z = c_1 J_0(2\theta) + c_2 Y_0(2\theta), \quad \xi_z = 0, \quad (30)$$

$$\xi_z = c_3 \theta^{-1+i\alpha} + c_4 \theta^{-1-i\alpha}, \quad \xi_z = 0, \quad (31)$$

where c_i ($i = 1, 2, \dots, 4$) are constants, and J_0 and Y_0 are the Bessel functions. It should be noted that the above solutions are valid for arbitrary field strengths. In the limit of a weak magnetic field, for which $\theta \gg 1$, the asymptotic form of the solution in equation (30) essentially corresponds to the leading-order terms in f_3 and f_4 (cf. equations [18] and [19]). Similarly, in this limit the solution in equation (31) corresponds to the leading-order term of f_1 and f_2 as defined in equations (16) and (17); this solution is recognized as the usual p - or acoustic mode in an unmagnetized plasma and represents a vertically propagating wave when $\Omega > \Omega_a$. It is unaffected by the magnetic field.

4.2 Solutions for $K \rightarrow \infty$

In the limit $K \rightarrow \infty$, it can be easily seen that equations (1) and (2) reduce to the following differential equation (Moreno-Insertis and Spruit 1989; see also the derivation in Roberts and Small 1990)

$$\frac{d^2 \xi_z}{dz^2} - \frac{1}{H} \frac{c_T^2}{c_S^2} \frac{d \xi_z}{dz} + \frac{1}{H^2} \frac{c_T^2}{c_S^2} \left[\Omega^2 - \frac{c_T^2}{v_A^2} (\Omega_{BV}^2 + \frac{c_T^2}{\gamma c_S^2}) \right] \xi_z = 0, \quad (32)$$

where c_T denotes the tube speed given by

$$c_T^2 = \frac{c_S^2 v_A^2}{c_S^2 + v_A^2}.$$

When $c_S/v_A \gg 1$, $c_T \rightarrow v_A$ and equation (32), becomes

$$\frac{d^2 \xi_z}{dz^2} + \frac{1}{H^2} \frac{c_S^2}{v_A^2} (\Omega^2 - \Omega_{BV}^2) \xi_z = 0. \quad (33)$$

Equation (33) admits the solutions

$$J_0(\psi) \quad \text{and} \quad Y_0(\psi),$$

where

$$\psi = \frac{2}{\epsilon} e^{-z/2H} \sqrt{\Omega^2 - \Omega_{BV}^2}.$$

For propagating wave solutions, $\Omega > \Omega_{BV}$. These solutions are essentially slow MHD waves, modified by gravity. They will be discussed in greater detail in Section 4.4.3. From equation (1), we find that as $K \rightarrow \infty$, $\xi_x \rightarrow 0$. It should also be noted that taking the limit $K \rightarrow \infty$ effectively removes the fast mode.

4.3 Normal Modes

We now examine the normal modes of a stratified atmosphere with a weak magnetic field, in order to delineate the influence of the field on the oscillation spectrum. It was argued in Section 3, that for upward wave propagation in a stratified atmosphere with a strong field, standing wave solutions are possible. These arise due to a change of Ω_a with height. For a weak field, the normal modes (as we shall see) can be analysed, to lowest order in θ , in terms of the modes of a non-magnetic atmosphere and a magnetic mode. The latter is essentially a slow mode. It is well known that non-magnetic modes are easily reflected if the temperature of the medium changes with height (for a good discussion see Leibacher and Stein 1981). The slow mode can be reflected due to the increasing Alfvén speed with height from layers where $v_A \sim c_S$, through conversion into a fast mode (*e.g.* Zhugzhda 1984). Although our analysis pertains to an isothermal medium with a weak

field, we implicitly assume that the properties of the atmosphere change abruptly at the top boundary, resulting in downward reflection of the waves. This is, of course, an idealization made purely for mathematical convenience.

For simplicity, we therefore assume rigid boundary conditions such that

$$\xi_x = \xi_z = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = d, \quad (34a,b)$$

where d is the height of the top boundary boundary; we also introduce the dimensionless height $D = d/H$.

Substituting equations (34a) – (34b) into equations (14) and (21) yields the following dispersion relation (for details see the Appendix)

$$\begin{aligned} (\Omega^2 - K^2) \sin \tilde{\theta} \sin(K_z D) &= 2 \frac{\epsilon}{\Omega} e^{D/4} \left\{ K_z K^2 \left[\cosh(D/4) \cos \tilde{\theta} \cos(K_z D) - 1 \right] \right. \\ &\quad + \sinh(D/4) \cos \tilde{\theta} \sin(K_z D) \left[M(\Omega^2 - K^2) - K^2 \left(\frac{1}{\gamma} - \frac{1}{2} \right) \right] \Big\} \\ &\quad + \frac{\epsilon^2}{\Omega^2} e^{D/2} \sin \tilde{\theta} \left\{ \left[M^2 (K^2 - \Omega^2) + 2MK^2 \left(\frac{1}{\gamma} - \frac{1}{2} \right) - rK^3 \right] \sin(K_z D) \right. \\ &\quad + 2K^2 (u_- + NK_z) \sinh(D/2) \cos(K_z D) \\ &\quad + 2 \left[M_1 (\Omega^2 - K^2) - iK^2 u_+ \right] \cosh(D/2) \sin(K_z D) \Big\} \\ &\quad + O\left(\frac{\epsilon^3}{\Omega^3}\right) = 0, \end{aligned} \quad (35)$$

where

$$\begin{aligned} \theta_0 &= \theta(0), & \theta_D &= \theta(D), & \tilde{\theta} &= 2(\theta_0 - \theta_D), \\ u_{\pm} &= \frac{1}{2}(t - Q_1^+ \pm t_+ Q_1^-). \end{aligned}$$

For $\epsilon \ll \Omega$, the dispersion relation to lowest order in ϵ/Ω becomes

$$(\Omega^2 - K^2) \sin \tilde{\theta} \sin(K_z D) = 0. \quad (36)$$

Equation (36) admits the following solutions

$$\sin(K_z D) = 0, \quad (37)$$

$$\Omega = K, \quad (38)$$

$$\sin \tilde{\theta} = 0. \quad (39)$$

We first consider the solution given by equation (37) which implies that $K_z D = n\pi$, where n is an integer and denotes the order of the mode. Using equation (13) yields the usual relation for p - and g -modes:

$$\Omega_i^4 - \Omega_i^2(K_i^2 + \frac{1}{4}) + K^2\Omega_{BV}^2 = 0 \quad (i = p, g), \quad (40)$$

where $K_i^2 = K_z^2 + K^2$.

The solution corresponding to equation (38) can easily be recognised as the Lamb wave in an unmagnetized atmosphere (Lamb 1932). We denote its frequency by Ω_L . This is simply a horizontally-propagating sound wave, which is evanescent in the vertical direction. Strictly speaking, the Lamb solution is not a normal mode of the atmosphere in the absence of a magnetic field, since it does not satisfy the boundary conditions (34a,b). However, in the presence of a weak field, this can be achieved through a slight coupling with the magnetic modes.

Turning our attention to the solutions given by equation (39), we find that these modes arise solely due to the presence of the magnetic field. The magnetic modes, hereafter referred to as m -modes, have frequencies

$$\Omega_m = \frac{\epsilon l \pi}{2s}, \quad (l = 1, 2, \dots), \quad (41)$$

where $s = (1 - e^{-D/2})$. The frequencies given by equation (41) agree with those obtained from the exact solution, i.e., equation (30) valid for $K = 0$, in the limit $\theta \gg 1$. These modes are approximately transverse, since it can be shown from equation (14) and (21) that

$$\frac{\xi_z^{(1,2)}}{\xi_z^{(1,2)}} \sim O(\theta).$$

Physically, these modes can be interpreted as gravity-modified slow modes in a weak magnetic field.

4.4 Asymptotic Frequency Corrections

We now consider the corrections to the lowest-order frequencies obtained from equation (36).

4.4.1 Frequency Correction to the p - and g -modes

We write

$$\Omega = \Omega_i + \delta\Omega_i, \quad (42)$$

where Ω_i ($i = p, g$) is the solution of equation (37), and $\delta\Omega_i$ denotes the first-order correction to Ω due to the magnetic field. Substituting equation (42) in the dispersion relation (equation [35]) yields an expression for $\delta\Omega_i$, which is

$$\delta\Omega_i = \frac{2\epsilon e^{D/4} K_z^2 K^2}{D \Omega_i^2 (\Omega_i^2 - K^2) \sin \tilde{\theta}} \left(1 - \frac{K^2 \Omega_{BV}^2}{\Omega_i^4}\right)^{-1} [\cosh(D/4) \cos \tilde{\theta} - (-1)^n], \quad (43)$$

where $K_z = n\pi/D$.

It is evident that equation (43) is singular at frequencies such that $\sin \tilde{\theta} = 0$, i.e., where the frequencies of the p - or g -modes cross the frequencies of the m -modes. As the frequency spacing between magnetic modes is proportional to ϵ (cf. equation [41]), mode crossing will occur frequently when ϵ is small. As an example, panel (a) of Figure 1 shows the frequency of the p -mode with $n = 1$ evaluated from equations (42) and (43), in an atmosphere with $\epsilon = 0.01$ and $D = 1$. To illustrate more clearly the behaviour near the singularity, panel (b) of Figure 1 shows $\delta\Omega_p$. At relatively low wavenumber, the region in K which is strongly influenced by the singularities is relatively small compared to the region where they have marginal influence, and where therefore equation (43) can be used to determine the frequency corrections. On the other hand, for $K \gtrsim 1.5$ the singularities dominate the behaviour of $\delta\Omega_p$. In the vicinity of the singularity the asymptotic description must take into account the presence of both types of modes. We return to the treatment of this case in section 4.5.1; also, Section 7 presents a comparison between the asymptotic behaviour and the numerical results.

In the limit $K \rightarrow 0$, $\Omega_g \rightarrow 0$ and $\Omega_p \rightarrow n\pi/D$ (for $n\pi/D \gg 1$). The correction to Ω_p in this limit is

$$\delta\Omega_p = \frac{2\epsilon D e^{D/4} K^2}{n^2 \pi^2 \sin \tilde{\theta}} [\cosh(D/4) \cos \tilde{\theta} - (-1)^n]. \quad (44)$$

The frequency correction, for small K varies as K^2 and n^{-2} , becoming negligibly small for high orders. In the limit $K \rightarrow 0$, equation (43) cannot be used for the g -modes since these have very small frequencies ($\Omega_g \sim K$).

Consider now the limit of large K , so that $K \gg \Omega$ and $K \gg K_z$. In this limit, $\Omega_g \rightarrow \Omega_{BV}$ and

$$\delta\Omega_g = \frac{2\epsilon e^{D/4} n^2 \pi^2}{D^3 K^2 \sin \tilde{\theta}} [\cosh(D/4) \cos \tilde{\theta} - (-1)^n]. \quad (45)$$

Equation (45) hold for K in the range $\Omega \ll K \ll \Omega_{BV}^{3/2} \epsilon^{-1/2}$. The frequency correction, for large K varies as K^{-2} and n^2 , and hence increases with mode order. For larger values of K , the asymptotic solution to equation (33) can be used. This will be considered in Section 4.4.3.

The correction to Ω_p cannot be obtained from equation (43) in this limit, since here $\Omega_p \sim K$.

4.4.2 Frequency Correction to the Lamb Mode

Let us now consider the frequency correction to the Lamb mode with frequency $\Omega_L = K$. The frequency correction to this mode is

$$\delta\Omega_L = \frac{\epsilon e^{D/4} \tilde{K}_z}{\sin \theta \sinh(\tilde{K}_z D)} \{ \cos \theta \cosh[(3/4 - 1/\gamma)D] - 1 \} + O(\frac{\epsilon}{K}), \quad (46)$$

where $\tilde{K}_z = (1/\gamma - 1/2)$. Equation (46) is valid for $K \gg \epsilon$ for small or moderate K . However, when K becomes large, the second order correction instead varies as $\epsilon^2 K$. Thus, equation (46) is valid in the range $\epsilon^{-1} \gg K \gg \epsilon$.

As in the case of the p - and g -modes, the correction is singular at the intersections with m -modes. Figure 2 shows $\Omega_L + \delta\Omega_L$ for the same model parameters as in Figure 1. It is evident that for the Lamb mode the singularities dominate essentially everywhere. Hence equation (46), while formally correct, may be of little practical use. An asymptotic description which takes into account the interaction with the magnetic modes is discussed in section 4.5.2.

4.4.9 Frequency Correction to the Magnetic Mode

The frequency correction to Ω_m can be expressed as

$$\delta\Omega_m = \frac{2\epsilon e^{D/4}}{i\pi(\Omega_m^2 - K^2)} \left\{ \frac{K_z K^2}{\sin(K_z D)} [\cosh(D/4) \cos(K_z D) - (-1)^l] \right. \\ \left. + \sinh(D/4) [M(\Omega_m^2 - K^2) - K^2(\frac{1}{\gamma} - \frac{1}{2})] \right\}, \quad (47)$$

where Ω_m is given by equation (41), and K_z by equation (14). Note that when K_z is imaginary, K_z should be replaced by $\tilde{K}_z = |K_z|$, $\sin(K_z D)$ by $\sinh(\tilde{K}_z D)$ and $\cos(K_z D)$ by $\cosh(\tilde{K}_z D)$ in equation (47).

In this case there are two types of singularities: one, where $\sin(K_z D) = 0$ and $K_z \neq 0$, corresponding to the crossing with the p - or g -modes; the other, where $\Omega_m = K$, corresponding to the crossing with the Lamb mode. This behaviour is illustrated in Figure 3 which shows $\delta\Omega_m$ (again with $\epsilon = 0.01$ and $D = 1$) for the mode with $l = 90$, corresponding to an uncorrected magnetic frequency $\Omega_m \approx 3.6$. It is evident that only a relatively small part of the region is influenced by the singularities; the behaviour of the modes in the vicinity of the singularities is described in sections 4.5.1 and 4.5.2.

In the limit $K \ll \Omega_m$, we find

$$\delta\Omega_m = \frac{\epsilon e^{D/4}}{i\pi} \left\{ -\frac{\sinh(D/4)}{8} + \frac{2K^2}{\Omega_m^2} \left[\frac{K_z}{\sin(K_z D)} (\cosh(D/4) \cos(K_z D) - (-1)^l) \right. \right. \\ \left. \left. + \sinh(D/4) (\frac{1}{2} - \frac{1}{\gamma^2}) \right] \right\}, \quad (48)$$

where $K_z = \sqrt{\Omega_m^2 - 1/4}$. The first term on the right-hand side of equation (48), which is independent of K , can be traced to the asymptotic expansions of the solutions of equation

(30). The second term, which varies as K^2 , gives the frequency correction due to a finite horizontal wave number.

In the opposite limit, when $K \gg \Omega_m$, we have

$$\delta\Omega_m = \frac{\epsilon^2 e^{D/2} K^2 \Omega_{BV}^2}{\Omega_m^3}. \quad (49)$$

The frequency correction given by equation (49) holds as long as $\delta\Omega_m/\Omega_m \ll 1$. This restricts the validity of equation (49) to K in the range $\Omega_m \ll K \ll \Omega_m^2 \epsilon^{-1}$ or roughly when $n\pi\epsilon \ll K \ll n^2\pi^2\epsilon$.

When $K \rightarrow \infty$, the frequencies can be obtained by applying the boundary conditions to the solutions of equation (33). Assuming that $\Omega \gg \epsilon$, we obtain

$$\Omega = \sqrt{\Omega_{BV}^2 + \Omega_m^2}. \quad (50)$$

Thus, for $\Omega_m \gg \Omega_{BV}$, the solutions essentially resemble m -waves modified by gravity. In this case, we have

$$\delta\Omega_m = \frac{\Omega_{BV}^2}{2\Omega_m}. \quad (51)$$

In the opposite limit, when $\Omega_{BV} \gg \Omega_m$, the character of the solutions is a gravity wave with frequency Ω_{BV} , modified by the magnetic field. The frequency correction to the g -mode is thus

$$\delta\Omega_g = \frac{\Omega_m^2}{2\Omega_{BV}}. \quad (52)$$

4.5 Behaviour in the Vicinity of Degenerate Modes

As discussed above, the expressions for the frequency corrections derived in the preceding sections break down where frequencies of different types of modes are nearly degenerate. Thus, for example, in the vicinity of an m -mode, $\sin \theta \approx 0$ and the expression (43) for the p - or g -mode frequency correction is singular. At such points an asymptotic description is required which takes into account the presence of both types of modes.

We again start from the dispersion relation (35). As discussed in Section 4.3, to lowest order the eigenfrequencies are determined by equation (36) (i.e., by neglecting the right-hand side in equation [35]), which leads to the solutions given by equations (37) - (39). Since the frequency of the m -mode is independent of K , whereas the frequencies of both p -, g - and Lamb modes increase with K , it is evident that the m -mode frequencies will cross the frequencies of the other modes. The actual behaviour in the vicinity of such points is determined by the right-hand side of equation (35), which provides a coupling between the different types of modes.

4.5.1 Crossings with p - or g -modes.

We first consider a crossing between a p - or g -mode of order n , characterized by $K_z D = n\pi$, and a magnetic mode of order l with frequency $\Omega_m = \epsilon\pi l/2s$. From equation (14) it follows that this takes place at the point (K_{nl}, Ω_m) in the (K, Ω) diagram, where

$$K_{nl}^2 = \left(\Omega_m^2 - \frac{n^2 \pi^2}{D^2} - \frac{1}{4} \right) \left(1 - \frac{\Omega_{BV}^2}{\Omega_m^2} \right)^{-1} \quad (33)$$

(obviously crossings are only possible where $K_{nl}^2 > 0$). We now consider the behaviour of the full dispersion relation (35) near this point, by setting

$$K = K_{nl} + \delta K \quad \Omega = \Omega_m + \delta\Omega, \quad (34)$$

where δK and $\delta\Omega$ are regarded as small quantities. Hence we may expand $\tilde{\theta}$ and K_z , to obtain

$$\begin{aligned} \tilde{\theta} &= l\pi + \frac{2s}{\epsilon} \delta\Omega \\ K_z &\approx \frac{n\pi}{D} - \frac{DK_{nl}}{n\pi} \left(1 - \frac{\Omega_{BV}^2}{\Omega_m^2} \right) \delta K + \frac{D\Omega_m}{n\pi} \left(1 - K_{nl}^2 \frac{\Omega_{BV}^2}{\Omega_m^4} \right) \delta\Omega. \end{aligned} \quad (35)$$

These expansions are substituted in the left-hand side of equation (35). The right-hand side and the factor $(\Omega^2 - K^2)$ are evaluated at (K_{nl}, Ω_m) . Neglecting the term in ϵ^2 , we are left with

$$\begin{aligned} &(-1)^{n+l} \frac{2s}{\epsilon} (\Omega_m^2 - K_{nl}^2) \frac{D^2 \Omega_m}{n\pi} \left[\left(1 - K_{nl}^2 \frac{\Omega_{BV}^2}{\Omega_m^4} \right) \delta\Omega - \left(1 - \frac{\Omega_{BV}^2}{\Omega_m^2} \right) \delta K \right] \delta\Omega \\ &= \frac{\epsilon}{\Omega_m} e^{D/4} \frac{2n\pi}{D} K_{nl}^2 [(-1)^{n+l} \cosh(D/4) - 1]. \end{aligned} \quad (36)$$

Equation (36) provides a quadratic equation for $\delta\Omega$ as a function of δK . We write it as

$$(\delta\Omega)^2 - A \delta K \delta\Omega - 1/4 (\delta\Omega_{\min})^2 = 0, \quad (37)$$

with the solution

$$\delta\Omega = 1/2 A \delta K \pm 1/2 (A^2 \delta K^2 + 4\delta\Omega_{\min}^2)^{1/2}. \quad (38)$$

Here

$$A = \frac{K_{nl} \Omega_m (\Omega_m^2 - \Omega_{BV}^2)}{\Omega_m^4 - K_{nl}^2 \Omega_{BV}^2}, \quad (39)$$

and

$$(\delta\Omega_{\min})^2 = \frac{32 \sinh(D/4) \eta^2}{D^3} \frac{n^2 K_{nl}^2 [\cosh(D/4) - (-1)^{n+l}]}{(\Omega_m^2 - K_{nl}^2)(\Omega_m^2 - K_{nl}^2 \Omega_{BV}^2 / \Omega_m^2)}, \quad (40)$$

where $\eta = \epsilon\pi/(2s)$ is the separation between adjacent magnetic modes (cf. equation [41]).

It is straightforward to show that for all p - and g -modes $(\delta\Omega_{\min})^2$ is positive, so that the two roots in equation (38) are always real and distinct. Thus, the coupling between the modes effectively removes the singular behaviour, which arises when the frequencies of different modes become identical. This is characteristic of the so-called *avoided crossings*, which is a common phenomenon amongst the eigenvalues of problems having different classes of eigenfunctions; a general analysis of the behaviour near such an avoided crossing was given by von Neuman & Wigner (1929). Evidently $\delta\Omega_{\min}$ is the minimum separation between the two branches, at $\delta K = 0$. In the limit of large $|\delta K|$, $\delta\Omega \approx 0$ on one branch, corresponding to the unperturbed magnetic mode, whereas on the second branch $\delta\Omega \approx A\delta K$. The latter behaviour can be understood by noting that $A = d\Omega/dK$, the derivative being taken along the unperturbed p - or g -mode ridge; hence the solution in equation (38) simply approximates the linear expansion of that ridge.

From equation (60) it follows that the minimum separation in the avoided crossings alternates between even and odd $n+l$. Neglecting the variation in K_{nl} and Ω_m , the ratio between the minimum separations is

$$\frac{\delta\Omega_{\min}^{(-)}}{\delta\Omega_{\min}^{(+)}} = \left[\frac{\cosh(D/4) - 1}{\cosh(D/4) + 1} \right]^{1/2} = \tanh(D/8). \quad (41)$$

Hence for small or moderate D avoided crossings with even $n+l$ are much narrower than crossings with odd $n+l$. It should also be noted that in terms of $\delta\tilde{K} \equiv K/\eta$ and $\delta\tilde{\Omega} \equiv \Omega/\eta$, the behaviour of the avoided crossing is independent of ϵ , to this level of approximation. In particular, $\delta\Omega_{\min}$ is a fixed fraction of the separation η between adjacent magnetic modes. The behaviour of minimum separation is illustrated in panel (a) of Figure 4, which shows $\delta\Omega_{\min}^{(+)} \eta$ as a function of K for $D = 1, \dots, 3$, in the case when $\epsilon = 0.01$ and $\gamma = 5/3$. Only odd values for $n+l$ are included (cf. equation [60]). We find that $\delta\Omega_{\min}^{(+)} \eta$ increases sharply at small values of K , but saturates to values where the minimum frequency separation becomes comparable to or even larger than the adjacent magnetic mode frequencies. This behaviour evidently reflects the fact that the validity of the asymptotic expansions becomes questionable for large D and K .

4.5.2 Crossings with the Lamb mode.

In the vicinity of the unmodified Lamb mode, with $\Omega = K$, K_z is imaginary. Hence in equation (35) we replace K_z by $\tilde{K}_z = |K_z| \sin(\tilde{K}_z D)$ by $\sinh(\tilde{K}_z D)$ and $\cos(K_z D)$ by $\cosh(\tilde{K}_z D)$. The frequencies of the unmodified magnetic modes are still given by equation (41), and hence the crossing takes place at $K = K_l = \Omega_m$; at this point $\tilde{K}_z = 1/\gamma - 1/2$. Expanding again around this point, by writing

$$K = K_l + \delta K \quad \Omega = \Omega_m + \delta\Omega, \quad (42)$$

and replacing $\tilde{K}_z$ and the right-hand side by the values at the crossing, we obtain the approximate dispersion relation

5. WEAK-FIELD SOLUTIONS

We now present numerical results for the weak-field case. Unless otherwise stated, the atmosphere is characterized by the default values $D = 1$, $\epsilon = 0.01$ and $\gamma = 5/3$. Figure 5 shows the variation of frequency with horizontal wave number (the solid curves) as a function of K , obtained by solving the exact equation (35) numerically, using a fourth-order Newton-Raphson-Kantorovich scheme (Cash and Moore 1980). For comparison, we also show results (long dashed curves), using the dispersion relation given by equation (35). The short dashed line corresponds to the Lamb solution $\Omega = K$ and the dotted curves correspond to $g_1, \dots, g_3$ modes in a non-magnetic atmosphere.

Before discussing the nature of the solutions, let us first compare the numerical results with those obtained using the asymptotic expansions for the weak-field case. The two solutions are almost identical for $K < 1$. However, as K increases, the accuracy of the asymptotic solutions begins to deteriorate, the largest error occurring for the modes with lowest frequency. For $K \geq 3$, the error in the asymptotic solutions becomes fairly large for modes of low order. This is consistent with the fact that for such modes the restriction $K^2 \ll \Omega^2 v_A/c_s$ (cf. the discussion before Section 4.1) no longer holds when K gets large.

We now turn to the behaviour of the solutions. To lowest order in ϵ , the only allowed solutions for $\Omega < 0.5$ (which is the region shown in Figure 5) consist of magnetic or m -modes, g -modes and Lamb modes. In the absence of a magnetic field, it is well known that the frequencies of gravity waves extend down to zero at all K . Physically, this is because the effect of buoyancy, driving the g -modes, vanishes for purely vertical propagation. The frequency of the Lamb waves also go to zero at $K = 0$. However, in the presence of a finite magnetic field, the situation is modified. Our calculations reveal that there is a lower threshold frequency for g - and Lamb-like modes in the magnetic case, which turn out to be the frequency of the m_1 mode (the lowest-order magnetic mode). The reason for this is that the magnetic pressure driving the magnetic modes does not vanish for $K = 0$, hence adding “stiffness” to the system and preventing the frequencies from decreasing below that limit. Thus, the entire low-frequency part of the g -mode spectrum is effectively eliminated by the presence of the field. It follows that at small values of K and frequencies below those of the p -modes only m -modes are found. As K increases, however, the purely magnetic modes become modified by gravity. At frequencies at which $\Omega \approx K$, this occurs through an interaction with the Lamb modes (modified by the magnetic field), through *avoided crossings* which we shall discuss in greater detail when considering Figure 7. The m -modes begin to acquire a g -like character for $K \gg \Omega$. For example, the frequency of the g_1 mode exceeds that of the m_1 mode for $K \approx 0.8$, while for the higher-order g -modes, this occurs for even larger values of K . In this situation, the various modes, corresponding to equations (37) – (39), become coupled. As a result, the solutions for moderate to large K neither resemble pure m -modes nor pure g -modes, but a combination of the two. There is a curious tendency, which we have not explored in detail, for the two lowest modes to run in parallel with a small frequency separation as K is increased. This trend continues towards values of K beyond what has been included in the figure, and a similar trend develops for at least the next pair of modes.

We now consider a part of the diagnostic diagram where there are p -like modes. Panel a) of Figure 6 shows the variation of Ω with K using the default values for ϵ , D and γ .

$$\begin{aligned}
 & (-1)^l \frac{1}{\epsilon} (\Omega_m \delta\Omega - K_1 \delta K) \sinh[(1/\gamma - 1/2)D] \\
 &= \frac{\epsilon}{\Omega_m} e^{D/4} \left\{ 2 \left(\frac{1}{\gamma} - \frac{1}{2} \right) K_l^2 \{ (-1)^l \cosh(D/4) \cosh[(1/\gamma - 1/2)D] - 1 \} \right. \\
 &\quad \left. - 2 \sinh(D/4) \sinh[(1/\gamma - 1/2)D] K_l^2 \left(\frac{1}{\gamma} - \frac{1}{2} \right) \right\} \\
 &= (-1)^l \frac{2\epsilon}{\Omega_m} e^{D/4} \left(\frac{1}{\gamma} - \frac{1}{2} \right) K_l^2 \{ \cosh[(3/4 - 1/\gamma)D] - (-1)^l \},
 \end{aligned} \tag{63}$$

where terms of order ϵ^2 have again been neglected. As before, this gives a quadratic equation for $\delta\Omega$, which may be written

$$(\delta\Omega)^2 - \delta K \delta\Omega - 1/4 (\delta\Omega_{\min})^2 = 0, \tag{64}$$

with the solution

$$\delta\Omega = 1/2 \delta K \pm 1/2 [\delta K^2 + (\delta\Omega_{\min})^2]^{1/2}. \tag{65}$$

In this case

$$(\delta\Omega_{\min})^2 = 16 \frac{\pi^2}{\gamma^2} \frac{\sinh(D/4)}{\sinh[(1/\gamma - 1/2)D]} \left(\frac{1}{\gamma} - \frac{1}{2} \right) \{ \cosh[(3/4 - 1/\gamma)D] - (-1)^l \}, \tag{66}$$

which again is non-negative. Hence the two roots of equation (65) are real.

As before, on one branch of the solution $\delta\Omega$ tends to 0 for large $|\delta K|$, whereas on the second $\delta\Omega \approx \delta K$ in that limit, corresponding to the unmodified Lamb mode. Furthermore, there is again an alternation between narrow avoided crossings for even l and wide crossings for odd l , the ratio between the minimum separations in the two cases being

$$\frac{\delta\Omega_{\min}^{(-)}}{\delta\Omega_{\min}^{(+)}} = |\tanh[1/2(3/4 - 1/\gamma)D]|. \tag{67}$$

For the normal case of $\gamma = 5/3$, and for small or moderate D , we obtain by expanding equation (67) to leading order that $\delta\Omega_{\min}^{(-)}/\delta\Omega_{\min}^{(+)} \approx 3D/40$; hence in this case also the “even” crossings are typically much narrower than the “odd” crossings. It should be noted that to this approximation $\delta\Omega_{\min}^{(-)} = 0$ for the particular case of $\gamma = 4/3$, suggesting that there may be an actual degeneracy of the eigenvalues.

Panel (b) of Figure 4 shows the minimum separation $\delta\Omega_{\min}^{(+)}/\eta$ in an avoided crossing between the Lamb mode and a magnetic mode of odd l , as a function of D (note that $\delta\Omega_{\min}^{(+)}$ is independent of K ; cf. equation [66]); as in panel (a) the separation has been normalized with the separation η between adjacent magnetic modes. The separation increases monotonically with D . Clearly, values of $\delta\Omega_{\min}^{(+)}/\eta$ exceeding unity indicate a breakdown of the asymptotic description, as in the case of the separation for p -modes illustrated in panel (a).

The dotted curve shows the pure p_1 mode and the dashed curve the Lamb mode ($\Omega = K$). Most of the diagnostic diagram is dominated by high-order ($l \sim 100$) magnetic modes (solid curves). However, in addition to the dense spectrum of magnetic modes, we can also discern the presence of the modified p_1 and Lamb modes. These undergo avoided crossings with various order m -modes, as discussed in greater detail in section 5.2.

Panel b) of Figure 6 shows the diagnostic diagram corresponding to the p_1 , p_2 and Lamb modes for $\epsilon = 0.02$ and $D = 2$. The non-magnetic p_1 and p_2 modes are shown as dotted lines, whereas the Lamb mode is shown as a dashed line. The behaviour is very similar to that in the previous panel.

5.1 Avoided Crossings Between the Lamb Mode and Magnetic Modes

Figure 7 shows in finer detail the avoided crossings between the Lamb mode and magnetic modes at low frequencies. In the limit of $K \rightarrow 0$, the frequencies of the magnetic modes are given by the solutions of equation (30), applying the boundary conditions (34a,b), the result is

$$\Omega_{m,l} = \frac{\epsilon}{2} f_{0,l}, \quad (68)$$

where $f_{0,l}$ denotes the zeros of

$$J_0(2\theta_0)Y_0(2\theta_D) - J_0(2\theta_D)Y_0(2\theta_0).$$

These modes have been labelled m_l ($l = 1, 2, \dots$). The Lamb mode frequency, as mentioned above, tends to zero for $K = 0$. However, for finite K , the Lamb mode frequency crosses those of the magnetic modes. To investigate the resulting behaviour, we first focus on the m_1 mode, as K increases, it begins to acquire the character of a magnetic Lamb mode with $\Omega \approx K$. The m_2 mode, on the other hand, with frequency larger than that of the Lamb mode, behaves as a pure magnetic mode, with no dependence on K . As K increases further, the frequencies of the two modes approach closely. However, instead of intersecting, the two curves undergo an avoided crossing. Physically, at the avoided crossing there is strong coupling between the two modes, and the character of the solution is a mixture of a pure magnetic-like and a pure Lamb-like mode. Increasing K further leads to mode transformation, with the Lamb mode being converted into a magnetic mode and the m_2 magnetic mode becoming Lamb-like. This phenomenon is repeated again between the “new” Lamb mode and the next order magnetic mode. It is also worth noting that the crossings alternate between narrow for even l and wide for odd l , in agreement with the discussion in Section 4.5.2. A similar behaviour at higher frequency is clearly visible in Figure 6.

5.2 Avoided Crossings between p -modes and Magnetic Modes

Figure 8 depicts in finer detail the avoided crossings between the p_1 mode and the magnetic modes. The heavy continuous curves show the numerically computed values,

whereas the asymptotic expression (58) is illustrated with thin dashed curves. We have adjusted the parameters K_{nl} and Ω_m in the asymptotic expression, in order to obtain a good fit to the numerical results. (The labelling of the computed curves is arbitrary and has been introduced for comparison with Figure 9). The nature of the crossings is similar to that discussed in the previous section. Also, the behaviour in the vicinity of the crossings is described quite accurately by the asymptotic expressions derived in Section 4.5.1. In particular, the alternation between the narrow and wide separation in the crossings for even and odd $n + l$ is evident.

To illustrate the behaviour of the eigenfunctions through the avoided crossings, it is instructive to consider the variation of the mode energy. We introduce

$$\tilde{E}_z = \int_0^d \xi_z^2 \exp(-z/H) dz, \quad (69)$$

$$\tilde{E}_z = \int_0^d \xi_z^2 \exp(-z/H) dz, \quad (70)$$

which are proportional to the contributions from the horizontal and the vertical velocity to the kinetic energy, respectively. Furthermore, let

$$E_z = \frac{\tilde{E}_z}{\tilde{E}_z + \tilde{E}_z}. \quad (71)$$

Hence E_z is the fraction of the mode energy contained in the vertical component of the velocity. For p -modes, E_z is large while it is very small for m -modes. Figure 9 shows the variation of E_z with K . The curves in the figure are labelled with the same numbering as the one used in Figure 8. For purposes of clarity each mode is distinguished by the line style in this diagram. The behaviour of E_z clearly reflects the changing nature of the eigenfunction as a mode passes through an avoided crossing. For example, the mode labelled (iii) starts out as a magnetic mode with very small E_z , changes into the p_1 mode after undergoing an avoided crossing with modes (ii), and changes back again into a magnetic mode after the crossing with mode (iv).

6. MODERATE- TO STRONG-FIELD SOLUTIONS

Let us now consider a situation which is more realistic as far as the solar photosphere is concerned. At the lower boundary, $v_A < c_S$, but at higher levels the Alfvén speed dominates over the sound speed. If we consider a vertical extent of several scale heights, then at the upper boundary the strong-field solutions of the wave equation corresponding to $\theta \ll 1$ given in Section 3 are applicable. Figure 10 shows the diagnostic diagram for $D = 20$. The solutions were obtained by solving equation (6) numerically. The solid and dashed curves correspond to $\epsilon = 0.5$ and $\epsilon = 0.25$ respectively. An inspection of the diagnostic diagram once again reveals the presence of avoided crossings, in this case due to the interaction of p -like modes with magnetic modes.

In order to understand this behaviour, we first consider the solutions near $K = 0$. We found in Section 4.1 that for $K = 0$, there are two family of solutions, corresponding to p -modes and m -modes. The frequencies of the m -modes are given by equation (68), while the frequencies of the p -modes are determined by equation (40) with $K = 0$, i.e.,

$$\Omega_{p,n} = \sqrt{\frac{n^2 \pi^2}{D^2} + \frac{1}{4}}, \quad (72)$$

where n denotes mode order. The solutions in equations (68) and (72) can be used to classify the modes close to $K = 0$ in the diagnostic diagram. In Figure 10, p_i and m_i , ($i = 1, 2, \dots$) denote the p - and m -modes of increasing order. The m -modes corresponding to $\epsilon = 0.25$ are denoted by $\tilde{m}_i$. For sufficiently small K , the solutions consist predominantly of p -modes, interlaced with m -modes. Apart from a small vertical layer of the atmosphere close to $z = 0$, $v_A/c_S \gg 1$, so that the spacing between the frequencies of the p -modes is smaller than that of the m -modes. For instance there are only two magnetic modes for $\epsilon = 0.5$ and four for $\epsilon = 0.25$ in the frequency range depicted in Figure 10.

The dependence of the magnetic-mode frequency on K can be understood from the so-called quasi-Alfvén approximation (Uchida and Sakurai 1975), as applied by Scheuer and Thomas (1981). It involves discarding the right-hand side of equation (22), which is justified on the basis that $\xi_z \gg \xi_x$ in the lower atmosphere and $v_A \gg c_S$ in the upper layers of the atmosphere. In the strong-field case, the frequencies of the m -modes are given approximately by (Scheuer and Thomas 1981)

$$\Omega_{m,l} = \sqrt{\frac{\epsilon^2}{4} j_{2K,l}^2 + K^2}, \quad (73)$$

where $j_{2K,l}$ denotes the l -th zero of J_{2K} . Hence, unlike the weak-field case, the m -mode frequencies increase roughly parabolically with K . On the other hand, the p -mode frequencies are essentially independent of K for small K . This is because, for these modes, $\xi_z \gg \xi_x$ (near $K \approx 0$), so that ξ_z can be determined from equation (23), with the right-hand side neglected. This solution is the same as equation (31) with eigenfrequencies given by equation (72), which are, of course, independent of K . For larger K , the frequencies decrease slightly with K . This behaviour was also found by Abdelatif (1991), and is related to the fact that as K increases, the mode takes on the character of a slow magneto-acoustic wave. The latter mode has a small negative slope in the diagnostic diagram. This behaviour leads to avoided crossings, which occur when the frequencies of the two sets of modes become close. Consider, for example, the m_1 mode. It undergoes an avoided crossing with the p_3 mode. As K increases further, a mode conversion occurs with a transformation of the p_3 mode into the m_1 mode and vice versa. The “new” m_1 mode undergoes avoided crossings with higher-order p -modes, as can be seen quite clearly in Figure 10.

The effect of varying the magnetic field can be seen from the solutions with $\epsilon = 0.25$, also shown in Figure 10. It is evident that at $K = 0$, where the acoustic and magnetic modes are decoupled, the p -mode frequencies are the same as before; however, there are

roughly twice as many magnetic modes as for $\epsilon = 0.5$. Otherwise the behaviour is quite similar in the two cases, although the avoided crossings tend to be less sharply defined for $\epsilon = 0.25$ than for $\epsilon = 0.5$.

Finally, let us consider the nature of the solutions in the limit of large K . As mentioned earlier, when $K \rightarrow \infty$, the frequency of the fast mode tends to infinity, so that the low-order modes in this limit are basically of the slow type, which travel roughly at the acoustic speed for $v_A/c_S \gg 1$. The latter condition is satisfied over most of the vertical extent, apart from the lower layers. When K is large, the two sets of curves practically coincide for low orders. These solutions can therefore be approximately regarded as slow modes. However, when n increases, the curves corresponding to $\epsilon = 0.25$ fall below those corresponding to $\epsilon = 0.5$, with the change in frequency increasing with n . This is because for large n the solutions are also influenced by physical conditions in the lower part of the atmosphere, where $v_A/c_S \sim 1$.

7. DISCUSSION

On the basis of quantitative calculations, the present analysis has brought out many interesting features of waves in stratified media. Inclusion of the variable structure of the atmosphere in the vertical direction is the key factor responsible for mode conversion, mode reflection as well as the presence of cut-off frequencies. In order to highlight the properties of the modes, it was found instructive to consider the solutions in the weak and strong field limits. The former condition is likely to be satisfied for instance in the deep sub-photospheric layers of the Sun where $c_S \gg v_A$. Thus, waves propagating in the deeper layers can be studied in the weak-field approximation. An upward propagating wave, excited from below, becomes increasingly dominated by magnetic effects, as a result of the sharp increase of v_A with z . In regions where $v_A/c_S \ll 1$, the character of the wave can be analysed in terms of elementary modes. For $K \sim 0$, the solutions consist of essentially acoustic and magnetic waves. At larger values of K , magnetically modified Lamb- and gravity-wave-like solutions are also possible. As the modes travel upwards, into layers of the atmosphere where $v_A \sim c_S$, the situation becomes very complicated (Zhugzhda 1984). Wave solutions in the strong field limit (corresponding to $v_A \gg c_S$) describe the modes in the higher layers, where only those waves, with frequencies greater than the acoustic cut-off value, can propagate. Therefore, an upward propagating wave, with $\Omega < \Omega_a$, will be reflected before it reaches the upper layers of the atmosphere. Thus, such layers act as a filter, allowing only those modes with frequencies greater than the acoustic cut-off to pass through.

The main emphasis of our work has been on the weak-field case, where $v_A \ll c_S$ throughout the layer. An important feature of our analysis is that it provides a fairly simple way to obtain the normal modes of an isothermal atmosphere with a vertical magnetic field. The general solutions involving Meijer functions are complicated and provide few clues about the physical nature of the waves. We have derived a dispersion relation which enables the normal modes of a system to be approximately determined, as long as $\theta > 1$. A characteristic feature is the presence of a spectrum of predominantly magnetic modes,

which becomes increasingly dense with decreasing field strength. In particular, there is the possibility of propagating waves and normal modes at frequencies intermediate between the acoustic cut-off frequency Ω_a and the buoyancy frequency Ω_{BV} . The frequency of the m_1 mode provides a lower limit on the possible frequencies of oscillation of the system. Thus the lowest-frequency g -modes are eliminated by the additional stiffness provided by the magnetic field.

Our analysis reveals the existence of a magnetic Lamb mode. In the absence of a magnetic field, it is straightforward to show from the linearized wave equations the existence of a solution for which $\Omega = K$, and with

$$\xi_z = C e^{(1-1/\gamma)z/H}, \quad \xi_r = 0,$$

where C is a constant. This solution, of course, does not satisfy our boundary conditions; but in the magnetic case the boundary conditions can be satisfied through the introduction of a small admixture of a magnetic mode. An analytic expression for the magnetic Lamb-mode frequency, showing the correction due to interaction with other modes does not appear to have been obtained before. However, the presence of this mode has been noted earlier by Antia and Chitre (1979), Cattaneo (1984) and Lou (1990, 1991).

We should also like to comment briefly on the fact that our solutions do not exhibit accidental degeneracy when the frequencies of different modes approach each other. The origin of this phenomenon can be traced to the dispersion relation. At such frequencies, the lowest-order approximation breaks down and the higher-order terms, which couple the modes, have to be considered. This coupling is responsible for avoided crossings. This type of behaviour has been known in atomic physics for a long time (*e.g.* von Neuman & Wigner 1929), and more recently it has been encountered in the study of solar and stellar global oscillations (*e.g.* Osaki 1975; Aizenmann *et al.* 1977; Christensen-Dalsgaard 1980) as well as in terrestrial problems (*e.g.* Jones 1970; Francis 1973); however, the presence of avoided crossings in connection with MAG waves appears to have been noticed only fairly recently (Hasan and Abdelatif 1990; Hasan 1991; Abdelatif 1991).

This behaviour is contained in the asymptotic dispersion relation, equation (35), derived in Section 4.3, and must be taken into account when evaluating asymptotic corrections to the frequencies. In particular, the simple expressions derived in Section 4.4, which neglected the presence of other modes, become singular in the vicinity of the avoided crossings. A more complete treatment was given in Section 4.5. To illustrate this asymptotic behaviour, Figure 11 shows numerically computed frequencies in a small region around the p_1 -mode for the default parameters ($\epsilon = 0.01$, $D = 1$, $\gamma = 5/3$), together with the two types of asymptotic frequencies. The singularities in the simple asymptotic frequency corrections near the avoided crossings are evident, as is the fact that the behaviour in these regions is reasonably well described by the extended asymptotic treatment accounting for the presence of other modes. On the other hand, the latter treatment in necessarily local in nature and hence does not capture the overall behaviour of the frequencies. Therefore, to obtain the complete picture both types of asymptotic description must be taken into account.

The asymptotic analysis shows that the behaviour of the frequencies near the avoided crossings, including the minimum separation between the two branches of frequencies,

scales with the frequency separation η between adjacent magnetic modes; η , in turn, is proportional to the magnetic field B . It follows that as B tends to zero the number of avoided crossings between any p -mode and the magnetic modes tends to infinity, whereas the fraction of the frequency interval which is affected by the avoided crossings is approximately constant. Hence the field-free case is not recovered in the limit $B \rightarrow 0$, which is therefore a singular limit. An analogous situation is obtained for the Lamb mode: here again the total extent of the region affected by avoided crossings tends to a constant as B tends to zero; but in the field-free case there is no Lamb mode satisfying the boundary conditions. That a singular limit should arise in this problem is hardly surprising, given that for $B = 0$ the order of the relevant differential equations is reduced from four to two. However, it would still be of some interest to investigate this behaviour in more detail.

In order to view the present work in perspective, let us compare our results with those of previous calculations on modes in an isothermal atmosphere with a vertical magnetic field. Although FP were the first to investigate this problem, their calculations did not include the determination of the normal mode frequencies. This was first carried out by Uchida and Sakurai (1975), under the “quasi-Alfvén” approximation and more rigorously by Schener and Thomas (1981). The latter authors were essentially attempting to use a simplified isothermal model to determine approximate mode frequencies in the umbra of a sunspot. They concentrated on the m -modes which behave like fast modes in the layers where $v_A \gg c_s$; these modes are localised mainly to the photosphere, because they suffer sharp downward reflection due to the increase of the Alfvén speed with height. Since the m -modes are mainly transverse, the sequence of mode numbers used by Schener and Thomas was with respect to the number of nodes in the horizontal displacement. They did not give the full frequency spectrum, in particular they omitted the p -mode like solutions, since they considered them irrelevant to the subject of their paper (J.H. Thomas, private communication). The complete set of modes for an isothermal atmosphere for the moderate to strong field case were obtained by Hasan and Abdelatif (1990), and Abdelatif 1991 (see also Wood 1990). In the latter paper only frequencies corresponding to a single value of K ($K = 0.124$) were given. The application of the isothermal calculations to umbral oscillations will not be taken up here, since since it has already been discussed in some detail by Hasan and Abdelatif (1990) and Abdelatif (1991). Avoided crossings, for an isothermal atmosphere with a strong field, were examined by Abdelatif (1991) in terms of coupling between fast and slow modes, but the weak-field case was not considered by him.

Lastly, we consider the question of wave classification. Our analysis has shown that in certain limits, it is possible to classify the wave solutions in terms of elementary modes of a stratified non-magnetic atmosphere or an unstratified atmosphere with a magnetic field. Our m -modes are simply the well-known slow modes in a weak field. For the most general case, the problem of mode classification is extremely difficult, especially when v_A/c_s changes by several orders of magnitudes over the vertical extent of the atmosphere. The stratification causes variation in the structure, which leads to mode conversion; hence, when analysing MAG waves in terms of elementary modes, one should bear in mind that their behaviour described in this way may be a local property. An alternative way of classifying modes, based on a Helmholtz decomposition of the displacements, was suggested by Hasan and Sobouti (1987) (see also Hasan and Abdelatif 1990, and Hasan 1991).

8. SUMMARY AND CONCLUSIONS

The main purpose of the present investigation was to extend the theory of MAG waves, by focusing on the effect of a vertical magnetic field on oscillations in a stratified atmosphere. Since the analysis of the general problem is rather complicated, it was found convenient, as a first step, to assume that the atmosphere is isothermal and that the magnetic field is uniform and of infinite horizontal extent. Although these assumptions are not generally valid, they nevertheless provide a starting point for a quantitative examination of MAG waves in stratified magnetized atmospheres. In order to obtain physical insight into the character of the solutions in regions where the Alfvén speed is much less than the sound speed, a weak-field approximation was used. The boundary conditions, though somewhat idealised, were made to simplify the mathematical analysis and to bring out the salient features associated with the effects of the magnetic field on the modal structure of the oscillations.

There are several interesting features to emerge from the calculations. The nature of the wave solutions depends upon the ratio of the Alfvén to sound speed in the atmosphere. It was found that in the weak-field limit, the oscillation spectrum could be analysed in terms of the modes of a non-magnetic stratified atmosphere and a magnetic mode (the slow MHD mode). For $K \sim 0$, only gravity-modified magnetic modes and p -modes are permitted. At moderate to large K , Lamb- and g -mode-like solutions are also possible. By using a perturbation expansion, we obtained the frequency corrections to the various modes due to mutual coupling with other modes. These corrections are valid as long as the frequencies of the individual modes are well separated. However, when the frequencies of two of the modes become close, there is strong mode coupling and *avoided crossings* occur. We examined the behaviour of the solutions in the vicinity of these crossings and obtained expressions for the frequency separation of the modes. Detailed calculations in the form of diagnostic diagrams were presented. The effect of changing the field strength on the oscillation spectrum was also demonstrated.

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APPENDIX

DERIVATION OF DISPERSION RELATION (35)

Applying the boundary conditions given by equation (34a) and (34b) to equations (14) and (21) yields a set of four equations which can be written as

$$AX = 0, \quad (\text{A1})$$

where

$$X^T = (C_1 \quad C_2 \quad C_3 \quad C_4),$$

and

$$A = \begin{pmatrix} \frac{f_1(0)}{\theta_0} & \frac{f_2(0)}{\theta_0} & \frac{f_3(0)}{\sqrt{\theta_0}} & \frac{f_4(0)}{\sqrt{\theta_0}} \\ \frac{f_1(D)}{\theta_D} & \frac{f_2(D)}{\theta_D} & \frac{f_3(D)}{\sqrt{\theta_D}} & \frac{f_4(D)}{\sqrt{\theta_D}} \\ \frac{g_1(0)}{\theta_0} & \frac{g_2(0)}{\theta_0} & \frac{g_3(0)}{\theta_0^{3/2}} & \frac{g_4(0)}{\theta_0^{3/2}} \\ \frac{g_1(D)}{\theta_D} & \frac{g_2(D)}{\theta_D} & \frac{g_3(D)}{\theta_D^{3/2}} & \frac{g_4(D)}{\theta_D^{3/2}} \end{pmatrix}. \quad (\text{A2})$$

Here $\theta_0 = \theta(0)$ and $\theta_D = \theta(D)$. The condition for equation (A1) to have a non-trivial solution is

$$\det[A] = 0. \quad (\text{A3})$$

Expanding equation (A3), we have

$$\begin{aligned} & \frac{(f_{30}f_{4D} - f_{3D}f_{40})(g_{10}g_{2D} - g_{1D}g_{20})}{\sqrt{\theta_0}\theta_D} + \frac{(f_{3D}g_{4D} - g_{3D}f_{4D})(g_{10}f_{20} - g_{10}f_{20})}{\theta_D^2} + \\ & \frac{(f_{3D}g_{40} - g_{30}f_{4D})(f_{10}g_{2D} - g_{1D}f_{20})}{\theta_0^{3/2}\theta_D^{1/2}} + \frac{(f_{30}g_{40} - g_{30}f_{40})(g_{1D}f_{2D} - f_{1D}g_{2D})}{\theta_0^2} + \\ & \frac{(f_{30}g_{4D} - g_{3D}f_{40})(f_{1D}g_{20} - g_{10}f_{2D})}{\theta_0^{1/2}\theta_D^{3/2}} + \frac{(g_{30}g_{4D} - g_{3D}g_{40})(f_{10}f_{2D} - f_{1D}f_{20})}{\theta_0^{3/2}\theta_D^{3/2}} \\ & = 0, \end{aligned} \quad (\text{A4})$$

where $f_{10} \equiv f_1(0)$ and so on. Substituting for f_j and g_j ($j = 1, \dots, 4$) from equation (16) – (19) and (24) – (27) respectively, we have

$$f_{30}f_{4D} - f_{3D}f_{40} = -\sin\tilde{\theta} + \left(\frac{1}{\theta_D} - \frac{1}{\theta_0}\right)M\cos\tilde{\theta} - \frac{1}{\theta_0\theta_D}M^2\sin\tilde{\theta} + \left(\frac{1}{\theta_D^2} + \frac{1}{\theta_0^2}\right)M_1\sin\tilde{\theta} + O\left(\frac{1}{g^2}\right), \quad (\text{A5})$$

$$f_{3D}g_{4D} - g_{3D}f_{4D} = -K + O\left(\frac{1}{g^2}\right), \quad (\text{A6})$$

$$f_{30}g_{40} - g_{30}f_{40} = -K + O\left(\frac{1}{g^2}\right), \quad (\text{A7})$$

$$f_{3D}g_{40} - g_{30}f_{4D} = -K\left[\cos\tilde{\theta} + \left(\frac{M}{\theta_D} - \frac{N}{\theta_0}\right)\sin\tilde{\theta}\right] + O\left(\frac{1}{g^2}\right), \quad (\text{A8})$$

$$f_{30}g_{4D} - g_{3D}f_{40} = -K\left[\cos\tilde{\theta} - \left(\frac{M}{\theta_0} - \frac{N}{\theta_D}\right)\sin\tilde{\theta}\right] + O\left(\frac{1}{g^2}\right), \quad (\text{A9})$$

$$g_{30}g_{4D} - g_{3D}g_{40} = -K^2\sin\tilde{\theta} + O\left(\frac{1}{g}\right), \quad (\text{A10})$$

$$g_{10}g_{2D} - g_{1D}g_{20} = U\sin K_z D + \left(\frac{1}{\theta_D} - \frac{1}{\theta_0}\right)V_1\sin K_z D +$$

$$\left(\frac{1}{\theta_D^2} - \frac{1}{\theta_0^2}\right)W_1\cos K_z D + O\left(\frac{1}{g^4}\right), \quad (\text{A11})$$

$$g_{10}f_{20} - f_{10}g_{20} = W + O\left(\frac{1}{g^2}\right), \quad (\text{A12})$$

$$f_{10}g_{2D} - g_{1D}f_{20} = -W\cos K_z D + V\sin K_z D + O\left(\frac{1}{g^2}\right), \quad (\text{A13})$$

$$g_{1D}f_{2D} - f_{1D}g_{2D} = W + O\left(\frac{1}{g^2}\right), \quad (\text{A14})$$

$$f_{1D}g_{20} - g_{10}f_{2D} = -W\cos K_z D - V\sin K_z D + O\left(\frac{1}{g^2}\right), \quad (\text{A15})$$

$$f_{10}f_{2D} - f_{1D}f_{20} = \sin K_z D + O\left(\frac{1}{g^2}\right). \quad (\text{A16})$$

Here $\tilde{\theta} = \theta_D - \theta_0$ and M , M_1 and N are defined in equations (13) and (29); also

$$U = p_1^2 + p_2^2 = \frac{(\Omega^2 - K^2)}{rK}, \quad (\text{A17})$$

$$W = p_1q_2 - p_2q_1 = \frac{K_z}{r}, \quad (\text{A18})$$

$$V = p_1q_1 + p_2q_2 = \frac{1}{r}\left(\frac{1}{2} - \frac{1}{\gamma}\right), \quad (\text{A19})$$

$$W_1 = p_1p_2' - p_2p_1' = -\frac{K}{r}u_-, \quad (\text{A20})$$

$$V_1 = p_1p_2' - p_2p_1' = -\frac{K}{ir}u_+, \quad (\text{A21})$$

where $u_{\pm}$ is defined immediately after equation (35). Substituting equations (A5) – (A16) into equation (A4), we obtain

$$\begin{aligned} & -U\sin\tilde{\theta}\sin K_z D + \left(\frac{1}{\theta_D} - \frac{1}{\theta_0}\right)MU\sin K_z D\cos\tilde{\theta} - \frac{1}{\theta_0\theta_D}M^2U\sin\tilde{\theta}\sin K_z D \\ & + \left(\frac{1}{\theta_D^2} + \frac{1}{\theta_0^2}\right)(M_1U - V_1)\sin\tilde{\theta}\sin K_z D - \left(\frac{1}{\theta_D^2} - \frac{1}{\theta_0^2}\right)W_1\sin\tilde{\theta}\cos K_z D \\ & - \frac{2}{\sqrt{\theta_0\theta_D}}KW + \left(\frac{1}{\theta_D} + \frac{1}{\theta_0}\right)KW\cos K_z D\cos\tilde{\theta} + \left(\frac{1}{\theta_D} - \frac{1}{\theta_0}\right)KV\cos\tilde{\theta}\sin K_z D \\ & + \left(\frac{1}{\theta_D^2} - \frac{1}{\theta_0^2}\right)KW N\sin\tilde{\theta}\cos K_z D - \frac{2}{\theta_0\theta_D}KV M\sin\tilde{\theta}\sin K_z D \\ & + \left(\frac{1}{\theta_D^2} + \frac{1}{\theta_0^2}\right)KV N\sin\tilde{\theta}\sin K_z D - \frac{1}{\theta_0\theta_D}K^2\sin\tilde{\theta}\sin K_z D \\ & + O\left(\frac{1}{g^3}\right) = 0. \end{aligned} \quad (\text{A22})$$

Finally, the dispersion relation (35) in Section 4.3 follows by substituting equations (A17) – (A21) into equation (A22).

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Figure Captions

Figure 1. (a) Frequency of the p_1 -mode in an atmosphere with $\epsilon = 0.01$, $D = 1$ and $\gamma = 5/3$, evaluated from equation (42) and the simple asymptotic correction in equation (43). (b) Frequency correction $\delta\Omega_p$ for this mode, evaluated from equation (43).

Figure 2. Frequency of the Lamb mode in an atmosphere with $\epsilon = 0.01$, $D = 1$ and $\gamma = 5/3$, evaluated as $\Omega = K + \delta\Omega_L$, using the simple asymptotic expression in equation (46) for $\delta\Omega_L$.

Figure 3. Frequency correction $\delta\Omega_m$ for the magnetic mode of order $l = 90$ and frequency $\Omega \simeq 3.59$, in an atmosphere with $\epsilon = 0.01$, $D = 1$ and $\gamma = 5/3$; $\delta\Omega_m$ was evaluated from the simple asymptotic expression in equation (47).

Figure 4. Asymptotic values for the minimum separations between adjacent avoided crossings, in units of η for $\epsilon = 0.01$, and $\gamma = 5/3$. (a) $\delta\Omega_{\min}^{(+)}/\eta$ versus K for $D = 1, \dots, 3$ corresponding to the avoided crossings between the p_1 mode and magnetic modes. Only odd values of $n + l$ are included. (b) $\delta\Omega_{\min}^{(+)}/\eta$ versus D , for the avoided crossings between the Lamb mode and magnetic modes. Only odd values of l have been considered.

Figure 5. Variation of Ω with K in an isothermal atmosphere using the default values $\epsilon = 0.01$, $D = 1$ and $\gamma = 5/3$. The solid lines correspond to the numerical solution of equation (6), whereas the long-dashed curves are based upon the solution of the dispersion relation (35). The short-dashed curve represents the Lamb solution $\Omega = K$ and the dotted lines correspond to the g_1 , g_2 and g_3 modes in a non-magnetic atmosphere.

Figure 6. Regions in the diagnostic diagram where there are p - and Lamb-type solutions. Solid curves correspond to magnetic modes, whereas the dotted and dashed curves represent p - and Lamb modes respectively. (a) Ω versus K , using the default values for ϵ , D and γ . The m -modes have orders between 76 and 100. Only the lowest-order p -mode (p_1) is shown. (b) Ω versus K for $\epsilon = 0.02$ and $D = 2$. The m -modes have orders between 61 and 80. The dotted curves show the p_1 and p_2 modes for the corresponding non-magnetic atmosphere.

Figure 7. Details of the avoided crossings between the magnetic modes and the Lamb mode, in the diagnostic diagram shown in Figure 5. The magnetic modes, whose frequencies in the limit $K \rightarrow 0$ are given by the solutions of equation (30), have been labelled $m_1, m_2, \dots, m_6$.

Figure 8. Details of avoided crossings in the diagnostic diagram between the p_1 mode and the magnetic modes, for the case illustrated in Figure 6a. The heavy continuous curves show the numerical solutions, whereas the asymptotic values (based upon equation [58]) are shown by dashed curves. The curves have been labelled for purposes of comparison with Figure 9.

Figure 9. Fraction of the mode kinetic energy density contained in the vertical component of the motion versus K , using the default parameters. The curves have been labelled using the same numbering as Figure 8.

Figure 10. Variation of Ω with K in an isothermal atmosphere for $\epsilon = 0.5$ (solid lines) and $\epsilon = 0.25$ (dashed lines). The classification of the modes into m - and p -types refers to the solutions of equations (30) and (31), respectively, for $K = 0$. Note the avoided crossings between the p -like and magnetic modes.

Figure 11. Details of avoided crossings in the diagnostic diagram between the p_1 mode and the magnetic modes, in an atmosphere with $\epsilon = 0.01$, $D = 1$ and $\gamma = 5/3$. The heavy continuous curves show the numerical solutions; the result of using the simple frequency corrections in equations (43) and (47) are shown as dotted curves, whereas asymptotic values based on taking the avoided crossings into account, as in equation (58), are shown by dashed curves.

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Fig. 1a

Fig. 1b

Fig. 2

Fig. 3

Fig. 4a

Fig. 4b

Fig. 5

Fig. 6a

Fig. 6b

Fig. 7

Fig. 8

Fig. 9

Fig. 11

Fig. 10