Corrections $\alpha^6 \log \alpha$ in the Two-Body QED Problem

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Abstract

The corrections in the QED two-body problem are calculated in a perturbation theory for the non-relativistic Schrödinger equation. Perturbation operators are derived by computing on-mass-shell Feynman diagrams in non-covariant approach. For positronium, the calculated logarithmic correction does not vanish only in n^3S_1 states and is equal to $\frac{5}{24}m\alpha^6 \log \frac{1}{\alpha}/n^3$.

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1 Introduction

The problem of calculating corrections in QED two-body systems is of a considerable theoretical interest in itself. On the other hand, the increasing precision of experiments demands the corresponding accuracy of theoretical calculations.

The generally accepted theoretical approach to this problem goes back to Refs. [1, 2, 3]. It is based on a relativistic two-body wave equation, which can be solved exactly, and, in the non-relativistic limit, reduces to the Schrödinger equation. The next step of those calculations consists in developing a perturbation series about the exact solution.

Our approach is different. In order to calculate corrections that are logarithmic in α , all we have to know about a bound state wave function is its value at the origin in the non-relativistic limit. Such corrections originate from the operators which can be considered local with the logarithmic accuracy and can thus be computed in the standard perturbation theory. These operators are derived via the expansion of Feynman diagrams in v/c . The use of non-covariant techniques enables us to handle the expansion with ease. Some contributions can be calculated directly via the Breit equation. It elucidates the relativistic nature of the corrections: they originate in fact from the expansion in v/c . A more formal argument in favour of this origin is that the discussed terms of high order in α do not have any power of π in the denominator, which distinguishes them from the usual QED expansion. This is why we avoid in our paper to add the common adjective "radiative" to the noun "corrections".

This approach in its simplest variant was previously used by two of us to calculate the corrections of relative order $\alpha^2 \log \alpha$ to the para- and orthopositronium decay rates [4]. In the present work we address the more complicated problem of the correction of relative order $\alpha^4 \log \alpha$ to the energy levels. The main object of our interest here is again positronium, where many transition frequencies have been measured with high precision and new experiments are being planned. However, at least to have an extra check on our results, we consider the more general case of particles with different masses, M and m .

2 Method and Results of the Calculation

2.1 Idea of the Calculation

We are calculating the terms of order $\alpha^6 \log \alpha$, emphasized by the large parameter $\log \alpha$ in comparison with $\mathcal{O}(\alpha^6)$ terms. In perturbation expansion, these corrections originate from logarithmic integrals which are cut off at momenta about μ from above and at $\mu\alpha$ from below. The upper limit is the scale where the non-relativistic approximation breaks down, while the lower one is the characteristic relative momentum in QED bound systems. As usual, $\mu = mM/(m + M)$ is the reduced mass.

The logarithmic nature of the corrections we are looking for greatly simplifies the problem. Since momenta in logarithmic integrals obey the condition $|\vec{p}| \gg \mu\alpha$, we can merely

calculate (quasi-)local operators of the kind

$$V(q) = (A + B\vec{\sigma}\vec{\sigma}') \frac{\pi\alpha^3}{mM} \log \frac{\mu}{q}, \quad (1)$$

where $q \equiv |\vec{q}|$, $\vec{\sigma}$ and $\vec{\sigma}'$ are spin operators of the particles. The corresponding energy shift is just the mean value of this operator over the non-relativistic wave function. In the logarithmic approximation, the energy shift of a level with quantum numbers n, l is reduced to

$$V(\mu\alpha) |\psi_{n,l}(0)|^2 = (A + B\vec{\sigma}\vec{\sigma}') \frac{\mu^3}{mM} \alpha^6 \log \frac{1}{\alpha} \frac{\delta_{l,0}}{n^3}, \quad (2)$$

where $\psi_{n,l}(0)$ is the value of the non-relativistic wave function at the origin.

To find such operators it is enough to take the $\mathcal{O}(\alpha^3)$ two-particle on-mass-shell scattering amplitude for non-relativistic particles and then to extract its part logarithmically divergent for the vanishing momenta of interacting particles.

Since the momenta in logarithmic integrals satisfy the condition $p \ll \mu$, the most natural way to attack the problem is to work in the Coulomb gauge, using the non-covariant approach. Doing so, we get rid of the integration over relative energies on the one hand, and on the other we can easily make the non-relativistic expansion, thus classifying various contributions according to a power of v/c . For example, a zigzag in a graph brings an extra suppression due to a heavy intermediate state, and so on.

2.2 Pure Coulomb Contribution

Let us begin with the pure Coulomb graphs. The simplest one is shown in Fig. 1a. The upper solid line corresponds to a fermion of mass m propagating forward in time, the lower one to that of mass M , $\vec{q} = \vec{p}' - \vec{p}$, the dashed lines depict the instantaneous Coulomb interaction $V_C(\vec{q}) = -4\pi\alpha/\vec{q}^2 \beta \otimes \beta$. Thin vertical lines show quantum mechanical perturbation theory insertions

$$\sum_i \frac{|i\rangle\langle i|}{E - E_i},$$

which in this case are reduced to

$$\frac{\lambda_+(\vec{p})\beta \otimes \Lambda_+(-\vec{p})\beta}{E - \Omega_{\vec{p}} - \omega_{\vec{p}}}.$$

Here E is the total energy of the system (equal to $M + m$ with sufficient accuracy), $\lambda_+(\vec{p})$ and $\Lambda_+(\vec{p})$ are projectors on the positive-energy mass-shell, e.g.

$$\lambda_+(\vec{p}) = \frac{1}{2} \left(1 + \frac{\vec{\alpha}\vec{p} + \beta m}{\omega_{\vec{p}}} \right),$$

while $\omega_{\vec{p}} = \sqrt{m^2 + \vec{p}^2}$ and $\Omega_{\vec{p}} = \sqrt{M^2 + \vec{p}^2}$. We can neglect all external momenta, since in logarithmic integrals $p, p' \gg \mu\alpha$. So the potential corresponding to the graph in Fig. 1a is

$$V = - \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3} \frac{(4\pi\alpha)^3}{p^2 p'^2 (\vec{p} - \vec{p}')^2} \frac{\lambda_+(\vec{p}')\lambda_+(\vec{p}) \otimes \Lambda_+(-\vec{p}')\Lambda_+(-\vec{p})}{(E - \Omega_{\vec{p}'} - \omega_{\vec{p}})(E - \Omega_{\vec{p}} - \omega_{\vec{p}})}. \quad (3)$$

Since with the logarithmic accuracy one can assume $p, p' \ll \mu$, we expand the integrand in powers of $p(p')/m$ and $p(p')/M$. Some terms of this expansion are in one-to-one correspondence with a non-relativistic iteration, first of the Coulomb potential, then of the Breit corrections to it, and so on (we will discuss the physical origin of the corrections as well as details of the iteration procedure elsewhere [5]). In this way we obtain a sum of functions, homogeneous in p and p' . Corresponding integrals diverge in various ways. Those diverging at small momenta and converging at large ones correspond to a non-relativistic iteration of the Coulomb potential. They cannot contain the logarithm, being saturated by the region of integration $p, p' \sim \mu\alpha$. The integrals diverging at large momenta and converging at small ones are also non-logarithmic, since these integrals know nothing about the atomic scale. We are thus left with the integrals diverging logarithmically in both the infrared and ultraviolet regions. Being cut off at μ from above and at $\mu\alpha$ from below, they produce the following shift of energy levels:

$$\delta E_{1a}(n, l) = \mu\alpha^6 \log \frac{1}{\alpha} \left(1 - 3 \frac{\mu^2}{Mm} - \frac{3}{2} \frac{\mu^4}{M^2 m^2} + \frac{\vec{\sigma}\vec{\sigma}'}{6} \frac{\mu^4}{M^2 m^2} \right) \frac{\delta_{l,0}}{n^3}. \quad (4)$$

One more pure Coulomb graph producing a logarithmic energy shift is shown in Fig. 1b. Following the same rules, we obtain an expression for the effective potential which differs from (3) only by the minus negative-energy projector for the upper particle propagating backward in time, instead of the positive-energy one, and by the first energy denominator, which now includes extra energy owing to the presence of a virtual pair in the intermediate state. Taking into account all possible locations of zigzag, and expanding the integrand in the same way as we did before, we arrive at the following expression for the shift of energy levels:

$$\delta E_{1b}(n, l) = -\mu\alpha^6 \log \frac{1}{\alpha} \left(1 - 3 \frac{\mu^2}{Mm} \right) \frac{\delta_{l,0}}{n^3}. \quad (5)$$

Simple power counting shows that all other graphs with pure Coulomb exchange are too singular in the ultraviolet region to produce integrals, which would logarithmically diverge at large distances. We thus obtain the following result for the pure Coulomb contribution in the order $\alpha^6 \log \alpha$:

$$\delta E_C(n, l) = \frac{\mu^5}{M^2 m^2} \alpha^6 \log \frac{1}{\alpha} \left(-\frac{3}{2} + \frac{\vec{\sigma}\vec{\sigma}'}{6} \right) \frac{\delta_{l,0}}{n^3}. \quad (6)$$

This correction, logarithmic in α , vanishes when one of the particles becomes infinitely heavy, in complete accord with the exact solution of the Dirac equation in the Coulomb field. Previously, the spin-dependent part of (6) has been found in Refs. [6, 3, 7].

2.3 Single-Transverse-Photon Contribution

Again, we begin with the simplest graph shown in Fig. 2a. Here the wavy line corresponds to a transverse photon, emitted by the lower particle and then absorbed by the upper one. We have to account for a finite time of propagation of a transverse photon.

The corresponding effective potential equals

$$V = - \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{p}'}{(2\pi)^3} \frac{(4\pi\alpha)^3}{p^2 p'^2 2q} \frac{\lambda_+(\vec{p}') \vec{\alpha} \vec{e} \lambda_+(\vec{p}) \otimes \Lambda_+(-\vec{p}') \vec{\alpha} \vec{e} \Lambda_+(-\vec{p})}{(E - \Omega_{\vec{p}'} - \omega_{\vec{p}})(E - \Omega_{\vec{p}'} - \omega_{\vec{p}} - q)(E - \Omega_{\vec{p}} - \omega_{\vec{p}})}. \quad (7)$$

Here $\vec{e}$ is polarization vector of the photon, while $(2q)^{-1}$ originates from its wave-function normalization factors. The sum over transverse photon polarizations is implicitly assumed:

$$e_i e_j \rightarrow \delta_{ij} - \frac{q_i q_j}{q^2}.$$

Proceeding in the same manner as before to extract the logarithm, we face an additional problem. In fact, the potential (7) gives rise to the $\mathcal{O}(\alpha^5 \log \alpha)$ correction as well [8]. In our approach the latter appears as the product of two integrals, one of which is logarithmic while the other linearly diverges in the infrared region. The linearly divergent integral could contain a non-zero constant, thus contributing to the $\mathcal{O}(\alpha^6 \log \alpha)$ correction. But this is not the case since the parameter of the non-relativistic expansion in its integrand is p^2/μ^2 , not p/μ . Hence, the $\mathcal{O}(\alpha^5)$ logarithm has a counterpart only in the order α^7 .

So, expanding (7) in $p(p')/\mu$ and leaving only logarithmic terms, we obtain

$$\delta E_{2a}(n, l) = \frac{\mu^3}{Mm} \alpha^6 \log \frac{1}{\alpha} \left(2 + 2 \frac{\mu^2}{Mm} + \frac{2}{3} \vec{\sigma} \vec{\sigma}' \right) \frac{\delta_{l,0}}{n^3}. \quad (8)$$

Of course, together with the contribution of the process represented by the diagram of Fig. 2a, we have included in (8) that of the process with the photon emitted by the upper fermion and absorbed by the lower one.

Quite a similar calculation, for those graphs shown in Figs. 2b–f, gives the following results:

$$\delta E_{2b}(n, l) = \frac{\mu^3}{Mm} \alpha^6 \log \frac{1}{\alpha} \left(1 - 4 \frac{\mu^2}{Mm} \right) \frac{2 \vec{\sigma} \vec{\sigma}'}{3} \frac{\delta_{l,0}}{n^3}, \quad (9)$$

$$\delta E_{2c}(n, l) = - \frac{\mu^3}{Mm} \alpha^6 \log \frac{1}{\alpha} \left(1 + \frac{2 \vec{\sigma} \vec{\sigma}'}{3} \right) \frac{\delta_{l,0}}{n^3}, \quad (10)$$

$$\delta E_{2d}(n, l) = - \frac{\mu^3}{Mm} \alpha^6 \log \frac{1}{\alpha} \frac{2 \vec{\sigma} \vec{\sigma}'}{3} \frac{\delta_{l,0}}{n^3}, \quad (11)$$

$$\delta E_{2e}(n, l) = \frac{\mu^3}{Mm} \alpha^6 \log \frac{1}{\alpha} \left(1 - \frac{2 \vec{\sigma} \vec{\sigma}'}{3} \right) \frac{\delta_{l,0}}{n^3}, \quad (12)$$

$$\delta E_{2f}(n, l) = \frac{\mu^3}{Mm} \alpha^6 \log \frac{1}{\alpha} \frac{2 \vec{\sigma} \vec{\sigma}'}{3} \frac{\delta_{l,0}}{n^3}. \quad (13)$$

All other contributions can be shown to be non-logarithmic. The total result for the contribution of the single-transverse-photon exchange is

$$\delta E_{1T}(n, l) = \frac{\mu^3}{Mm} \alpha^6 \log \frac{1}{\alpha} \left(2 + 2 \frac{\mu^2}{Mm} - \frac{8}{3} \frac{\mu^2}{Mm} \vec{\sigma} \vec{\sigma}' \right) \frac{\delta_{l,0}}{n^3}. \quad (14)$$

Its spin-dependent part was calculated previously [3]. The first term in the brackets agrees with the corresponding correction for hydrogen recently calculated in Ref. [9].

2.4 Double-Transverse-Photon Contribution

The typical diagrams with double-transverse-photon exchange, giving rise to logarithmic energy shifts, are shown in Figs. 3a–e. Together with these, we also include the ones that differ by permutation of the Coulomb exchange and transverse quanta, as well as by permutation of the fermions. Since there is nothing new in the way of calculations, we present only the results:

$$\delta E_{3a}(n, l) = \frac{\mu^5}{M^2 m^2} \alpha^6 \log \frac{1}{\alpha} \left(-\frac{3}{2} + \frac{5}{6} \vec{\sigma} \vec{\sigma}' \right) \frac{\delta_{l,0}}{n^3}, \quad (15)$$

$$\delta E_{3b}(n, l) = \frac{\mu^5}{M^2 m^2} \alpha^6 \log \frac{1}{\alpha} (1 + 3 \vec{\sigma} \vec{\sigma}') \frac{\delta_{l,0}}{n^3}, \quad (16)$$

$$\delta E_{3c}(n, l) = \frac{\mu^3}{M m} \alpha^6 \log \frac{1}{\alpha} \left(\frac{1}{2} - \frac{8}{\pi^2} \right) \frac{\delta_{l,0}}{n^3}, \quad (17)$$

$$\delta E_{3d}(n, l) = \frac{\mu^3}{M m} \alpha^6 \log \frac{1}{\alpha} \left(-\frac{1}{2} + \frac{8}{\pi^2} \right) \frac{\delta_{l,0}}{n^3}, \quad (18)$$

$$\delta E_{3e}(n, l) = -\frac{\mu^3}{M m} \alpha^6 \log \frac{1}{\alpha} \frac{\delta_{l,0}}{n^3}. \quad (19)$$

In the last three cases we also include graphs with crossed transverse quanta. The absence of spin-dependent terms in these contributions results from the cancellation between spin-dependent parts of graphs with crossed and uncrossed quanta.

The total result for the contribution of the double-transverse-photon exchange is

$$\delta E_{2T}(n, l) = \frac{\mu^3}{M m} \alpha^6 \log \frac{1}{\alpha} \left(-1 - \frac{1}{2} \frac{\mu^2}{M m} + \frac{23}{6} \frac{\mu^2}{M m} \vec{\sigma} \vec{\sigma}' \right) \frac{\delta_{l,0}}{n^3}. \quad (20)$$

The spin-dependent part of (20) was found previously in Ref. [3]. The first term in the brackets, -1 , agrees with the result of Refs. [9, 10] for the contribution of the double-transverse-photon exchange to the Lamb shift in hydrogen.

3 Conclusions

Our total result for the $\mathcal{O}(\alpha^6 \log \alpha)$ contribution to the energy shift in the QED two-body system is the sum of (6), (14) and (20):

$$\delta E(n, l) = \frac{\mu^3}{M m} \alpha^6 \log \frac{1}{\alpha} \left(1 + \frac{4}{3} \frac{\mu^2}{M m} \vec{\sigma} \vec{\sigma}' \right) \frac{\delta_{l,0}}{n^3}. \quad (21)$$

As we mentioned earlier, Eq. (21) is in accord with the results for both $\mathcal{O}(\alpha^6 \log \alpha)$ contributions to the hyperfine splitting in muonium [3] and to the energy shift in hydrogen [9].

As for positronium, we should add to (21) the annihilation contribution [11],

$$\delta E_{ann}(n, l) = \frac{1}{96} m \alpha^6 \log \frac{1}{\alpha} (3 + \vec{\sigma} \vec{\sigma}') \frac{\delta_{l,0}}{n^3}. \quad (22)$$

This can be found easily either in the approach based on the Breit equation (see Ref. [4]) or by the method described in the present paper.

The final result for positronium is:

$$\delta E(n, l) = \frac{5}{96} m \alpha^6 \log \frac{1}{\alpha} (3 + \vec{\sigma} \vec{\sigma}') \frac{\delta_{l,0}}{n^3}, \quad (23)$$

which can be written also in a more compact form:

$$\delta E(n, l, s) = \frac{5}{24} m \alpha^6 \log \frac{1}{\alpha} \frac{\delta_{l,0} \delta_{s,1}}{n^3}. \quad (24)$$

In other words, this correction in positronium does not vanish only for triplet S -states.

Our result (23) differs from the recent one by Fell [12],

$$\delta E(n, l) = \frac{1}{96} m \alpha^6 \log \frac{1}{\alpha} (3 + 5 \vec{\sigma} \vec{\sigma}') \frac{\delta_{l,0}}{n^3}, \quad (25)$$

obtained via a relativistic two-particle equation for positronium. Unfortunately, the origin of the disagreement is not clear to us.

Adding (24) to the energy shifts of orders lower in α , we obtain the following values for transition frequencies (in MHz):

$$E(2^3S_1) - E(1^3S_1) = 1\,233\,607\,211.7, \quad (26)$$

$$E(2^3S_1) - E(2^3P_2) = 8\,627.7, \quad (27)$$

$$E(2^3S_1) - E(2^3P_1) = 13\,013.3, \quad (28)$$

$$E(2^3S_1) - E(2^3P_0) = 18\,498.5. \quad (29)$$

Correction (24) contributes -16.7 MHz to the first of these frequencies and 2.4 MHz to all the others.

A comparison with the experimental values

$$E(2^3S_1) - E(1^3S_1) = 1\,233\,607\,218.9 (10.9)[13], \quad (30)$$

$$E(2^3S_1) - E(2^3P_2) = 8\,628.4 (2.8)[14], \, 8\,619.6 (2.7)(0.9)[15], \quad (31)$$

$$E(2^3S_1) - E(2^3P_1) = 13\,001.3 (3.9)(0.9)[15], \quad (32)$$

$$E(2^3S_1) - E(2^3P_0) = 18\,504.1 (10.0)(1.7)[15], \quad (33)$$

demonstrates that the calculated correction will be quite essential for the next generation of experiments.

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Figure Captions

Figure 1: Pure Coulomb graphs giving rise to the logarithmic effective potential.

Figure 2: Typical graphs with single-transverse-photon exchange.

Figure 3: Typical graphs with double-transverse-photon exchange.

Figure 1

Figure 2

Figure 3