VARIATIONAL FORMULATION FOR LINEAR OPTICS IN A PERIODIC FOCUSING SYSTEM

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We derive the equation for the betatron function \( \beta(s) \) from a variational principle; the corresponding stationary value of the action integral equals the betatron phase advance \( \mu \). This permits us to obtain an accurate value of \( \mu \) even for a simple trial function \( \beta(s) \) and simplifies the derivation of the change in \( \mu \) caused by errors in the focussing gradient \( K(s) \). We also present a variational form for the dispersion function \( D(s) \) and apply our results to the example of alternating gradients (FD lattice), obtaining accurate estimates for the phase advance and the transition energy. These variational forms are special cases of a general principle for the eigenvalues of a (stable) symplectic matrix.

1 INTRODUCTION

The equation for the betatron oscillations in a linear periodic focussing system has been analyzed in detail by Courant and Snyder\(^1\). In particular, for a decoupled ideal machine, the oscillations are governed by the Hill equation

\[
x'' + K(s)x = 0,
\]

where the prime stands for \( d/ds \) and the focussing gradient \( K(s) \) is a periodic function of \( s \) with period \( L \).

For stable oscillations, the solution of Eq. (1) can be written as

\[
x(s) = A \sqrt{\beta(s)} \exp \left( \pm i \int_{0}^{s} \frac{ds}{\beta(s)} \right),
\]

where the lattice function \( \beta(s) \) is a periodic solution of the differential equation

\[
\frac{1}{2} \beta \beta'' - \frac{1}{4} \beta'^2 + K(s)\beta^2 = 1.
\]

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The phase advance corresponding to the Floquet solution in Eq. (2) is

\[ \mu = \int_0^L \frac{ds}{\beta(s)}. \]  

(4)

The relation between the envelope function \( a(s) \), which satisfies

\[ a'' + K(s)a = \frac{1}{a^3}, \]  

(5)

and the betatron function \( \beta(s) \) is

\[ a(s) = \sqrt{\beta(s)}. \]  

(6)

In the next section we present a variational form for \( \mu \) in terms of a trial function \( \beta(s) \). This permits us to obtain an accurate value of \( \mu \) even for a simple trial function \( \beta(s) \) and simplifies the derivation of the change in \( \mu \) caused by errors (or chromatic variations) in \( K(s) \). In Section 3, we also present a variational form for the transition energy \( \gamma_t \) in terms of a trial dispersion function \( D(s) \). Finally, in Section 4, we give a numerical example for an FD lattice with constant bending radius. These two variational forms are special cases of a general variational principle, discussed in the Appendix, relating the eigenvalues and eigenvectors of the single-turn transfer matrix.

2 VARIATIONAL FORM FOR THE BETATRON PHASE ADVANCE

We start with the Lagrangian \( \mathcal{L}(a, a'; s) \) corresponding to Eq. (5)

\[ \mathcal{L}(a, a'; s) = \frac{1}{2} \left[ a'^2 - K(s)a^2 - \frac{1}{a^2} \right] \]  

(7)

and write the action as \( -\int_0^L \mathcal{L} \, ds \). This is now expressed in terms of \( \beta \) and \( \beta' \) as

\[ \mathcal{A}_1 = -\int_0^L \mathcal{L}(\beta, \beta'; s) \, ds = \frac{1}{2} \int_0^L \frac{ds}{\beta} \left[ 1 - \frac{1}{4} \beta'^2 + K(s)\beta^2 \right]. \]  

(8)

It is easy to verify that the Euler equation corresponding to this action is equivalent to Eq. (3). Moreover, if one puts the correct solution \( \beta(s) \) into Eq. (8) and discards the term \( \int_0^L \beta''(s) \, ds \) since \( \beta(s) \) is periodic, the action \( \mathcal{A}_1 \) takes the stationary value \( \mu \). Thus

\[ \mu = \frac{1}{2} \int_0^L \frac{ds}{\beta} \left[ 1 - \frac{1}{4} \beta'^2 + K(s)\beta^2 \right] \]  

(9)

is a variational formulation for the phase advance in terms of the trial periodic function \( \beta(s) \).

If there are field errors \( \delta K(s) \), the first order change in \( \mu \) will correspond to the change in Eq. (9) due to \( \delta K(s) \) as well as to the induced change \( \delta \beta \). But the first order change
in $\mu$ due to $\delta \beta$ will vanish, because $\mu$ is the stationary value of expression (9). Thus we obtain directly the well known result:

$$\delta \mu = \frac{1}{2} \int_0^L ds \beta(s) \delta K(s).$$ (10)

### 3 Dispersion

In a circular accelerator, off-momentum particles oscillate around a displaced equilibrium orbit given by the equation

$$x'' + K(s)x = G(s) \frac{\Delta p}{p},$$ (11)

where $G(s) = 1/\rho(s)$ and $\rho(s)$ is the radius of curvature of the unperturbed orbit as a function of position around the ring. We define the periodic solution to Eq. (11) as

$$x_{co}(s) = \frac{\Delta p}{p} D(s),$$ (12)

where the dispersion $D(s)$ is the periodic solution of

$$D'' + K(s)D = G(s).$$ (13)

Once again, a Lagrangian corresponding to Eq. (13) is

$$\mathcal{L}(D, D'; s) = \frac{1}{2} \left[ D'^2 - K(s)D^2 \right] + G(s)D.$$ (14)

We form the action

$$\mathcal{A}_2 = 2 \int_0^L ds \mathcal{L}(D, D'; s) = \int_0^L ds \left[ D'^2 - K(s)D^2 + 2G(s)D \right]$$ (15)

and note that, when $D(s)$ is a periodic solution of Eq. (13), the action takes the stationary value

$$\mathcal{A}_2 = \int_0^L G(s)D(s)ds.$$ (16)

The fractional change in orbit period due to a momentum deviation $\Delta p$ is

$$\frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta \nu}{\nu} = \frac{\Delta L}{L} - \frac{1}{\gamma^2} \frac{\Delta p}{p},$$ (17)

where $\nu$ is the particle velocity and $\gamma$ is the total particle energy in units of its rest mass (Lorentz factor). But the increase in circumference is determined by the equilibrium orbit displacement $D(s)$ as

$$\Delta L = \frac{\Delta p}{p} \int_0^L G(s)D(s)ds.$$ (18)
Thus
\[ \frac{\Delta T}{T} = \frac{\Delta p}{p} \left( \frac{1}{\gamma_t^2} - \frac{1}{\gamma_i^2} \right), \]  
(19)
where the transition energy $\gamma_t$ is defined by
\[ \frac{1}{\gamma_t^2} = \frac{1}{L} \int_0^L G(s)D(s)ds. \]  
(20)

Then, from Eqs. (15) and (16), we see that
\[ \frac{1}{\gamma_t^2} = \frac{1}{L} \int_0^L ds \left[ D'^2 - K(s)D^2 + 2G(s)D \right] \]  
(21)
is a variational form for the transition energy (in fact for the momentum compaction factor $\alpha_p = 1/\gamma_t^2$).

It is also a simple matter to determine the change in transition energy due to a gradient error $\delta K(s)$ and/or a magnetic curvature error $\delta G(s)$. Specifically
\[ \delta \left( \frac{1}{\gamma_t^2} \right) = -\frac{1}{L} \int_0^L ds \left[ D^2(s) \delta K(s) - 2D(s) \delta G(s) \right]. \]  
(22)

4 NUMERICAL EXAMPLE

We illustrate the use of Eq. (9) by writing the trial betatron function
\[ \beta(s) = \frac{L}{2\pi} b \left[ 1 + \epsilon \cos \left( \frac{2\pi s}{L} \right) \right], \]  
(23)
for a periodic FD lattice with focussing gradient $K(s) = k \text{ sign}[\cos(2\pi s/L)]$, alternating between the two opposite values $\pm k$. The variable parameters are $b$ and $\epsilon$.

The integrals in Eq. (9) may be evaluated, leading to
\[ \frac{\mu}{\pi} = \frac{1}{b \sqrt{1 - \epsilon^2}} + \frac{b}{4} \left[ \lambda \epsilon - 1 + \sqrt{1 - \epsilon^2} \right], \]  
(24)
where
\[ \lambda = \frac{2}{\pi^3} kL^2. \]  
(25)
Minimization of Eq. (24) with respect to $b$ leads to
\[ \left( \frac{\mu}{2\pi} \right)^2 = \frac{1}{4} \left[ \lambda \epsilon - 1 + \sqrt{1 - \epsilon^2} + 1 \right], \]  
whose stationary value with respect to $\epsilon$ occurs at $\epsilon = \lambda$. The variational result for the phase advance is given by
\[ \mu_{\text{var}} = \pi \left[ 1 - \sqrt{1 - \lambda^2} \right]^{1/2}, \]  
(26)
whereas the exact value is given in this case by

\[
\cos (\mu_{\text{exact}}) = \cos \left( \sqrt{\frac{\pi^3}{8} \lambda} \right) \cosh \left( \sqrt{\frac{\pi^3}{8} \lambda} \right).
\] (27)

Comparison of the two results is shown in Fig. 1: they are in excellent agreement over most of the stability range (and well beyond the region \( \lambda \ll 1 \) where the so-called smooth approximation applies).

We similarly use a trial dispersion function

\[
D(s) = \frac{L}{2\pi} b \left[ 1 + \epsilon \cos \left( \frac{2\pi s}{L} \right) \right]
\] (28)

in Eq. (21), assuming a circular accelerator consisting of \( N \) magnet periods with constant bending radius \( \rho = NL/2\pi \). This leads to

\[
\frac{1}{\gamma_r^2} = \frac{2b}{N} + \frac{b^2}{2} (\epsilon^2 - \lambda \epsilon),
\] (29)

where \( \lambda \) is given again by Eq. (25). The stationary value occurs for \( b = 8/N \lambda^2 \), \( \epsilon = \lambda/2 \) and leads to the variational result

\[
(\gamma_r^2)_{\text{var}} = \frac{N^2 \lambda^2}{8}.
\] (30)

Using the corresponding result for the tune, which is here given by \( \nu = N \mu/2\pi \), we find

\[
(\gamma_r^2)_{\text{var}} = \nu_{\text{var}}^2 \left( 1 - 2 \frac{\nu_{\text{var}}^2}{N^2} \right).
\] (31)

For \( \nu^2 \ll N^2 \), we obtain the usual approximation

\[
\gamma_r \simeq \nu.
\]

The exact value of the momentum compaction factor is

\[
\alpha_p = \left( \frac{1}{\gamma_r^2} \right)_{\text{exact}} = \frac{4}{\Phi k \rho^2 \left[ \coth \frac{\Phi}{2} - \cot \frac{\Phi}{2} \right]},
\] (32)

where

\[
\Phi = \sqrt{k} \frac{L}{2} = \sqrt{\frac{\pi^3}{8} \lambda}.
\]

Therefore we have

\[
\left( \frac{\gamma_r}{N} \right)_{\text{exact}} = \frac{\pi^3}{4} \left( \frac{\pi}{8} \lambda \right)^{3/2} \left[ \coth \left( \frac{\pi}{4} \sqrt{\frac{\pi}{2} \lambda} \right) - \cot \left( \frac{\pi}{4} \sqrt{\frac{\pi}{2} \lambda} \right) \right].
\] (33)

Figure 2 shows the variational and exact results for \( \gamma_r/N \) as a function of \( \lambda \). For this configuration (and this choice of trial function), the results are again very close to one another throughout the whole stability range.
FIGURE 1: Phase advance for an FD-cell of length $L$ and focusing gradient $\pm k$ (the dimensionless parameter $\lambda$ is defined by Eq. (25)).

FIGURE 2: Transition energy for a circular accelerator consisting of $N$ identical FD-cells with constant bending radius $\rho = NL/2\pi$. 
5 SUMMARY AND CONCLUSIONS

Particle trajectories in a periodic focusing system obey the well known principle of least action. We have shown that also the optical functions for a linear lattice can be derived from a variational principle. In the Appendix we shall prove that the two variational principles are in fact the same, with the only difference that the optical functions (related to the eigenvectors of the single-turn transfer matrix) are constrained to be periodic and of unit symplectic norm.

An interesting generalization of this idea would be the derivation of nonlinear optical distortions, such as ‘smear’ and nonlinear chromaticity, from a variational principle. The advantage of a variational formulation is twofold: as shown in Section 4, it allows an accurate numerical solution by means of suitable trial functions and simplifies the derivation of the change in the stationary action integral (namely tune or transition energy, in the linear case) caused by magnetic errors.

Two possible fields of application are the design of a final focus for linear colliders and a ‘thermodynamic’ approach to the equilibrium emittances in electron storage rings. In the case of a final focus, a variational formulation would be very useful to minimize chromatic or geometric aberrations by varying the quadrupole strengths.

As for the beam relaxation associated with synchrotron radiation in electron storage rings, a variational principle has already been established for the beam envelope matrix\(^3\), expressed as a linear combination of Twiss matrices. The coefficients of this linear combination are the beam emittances, but no variational form could be found for the Twiss matrices. A straightforward application of our results should lead to a unified variational principle for the equilibrium emittances, involving directly the eigenvectors of the single-turn transfer matrix (besides the damping and noise matrices). This would simplify the derivation and possibly help interpreting the result as a generalized principle of ‘minimum entropy production’.

REFERENCES

APPENDIX A Variational form for the eigenvalues of a symplectic matrix

In this appendix we discuss a general variational principle relating the eigenvalues and eigenvectors of a stable symplectic matrix. As shown in the next two appendices, this reduces to Eq. (9) for one-dimensional betatron oscillations, while the variational form for the synchrotron phase advance embeds Eq. (21) and leads to the related formula (22) for the change of transition energy.

The linearized equations of motion for the oscillations around the synchronous particle
in a circular accelerator can be written in the form

\[ y' = Ly, \quad \text{with} \quad L = JH, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad y = \begin{pmatrix} q \\ p \end{pmatrix}. \] (34)

Here \( y(s) \) is the (six-dimensional) phase-space vector corresponding to small deviations of the particle coordinates and momenta from those of the synchronous particle, \( J \) is the unit symplectic matrix and \( H(s) \) the symmetric matrix associated with the Hamiltonian quadratic form for linearized oscillations. In the case of one-dimensional betatron oscillations, the phase-space variables reduce to \( q = x \) and \( p = x' \) while the matrix \( L(s) \) corresponding to the Hill equation (1) is

\[ L(s) = \begin{pmatrix} 0 & 1 \\ -K(s) & 0 \end{pmatrix}. \] (35)

The action integral associated with the linearized equations of motion can be written

\[ A = \int_{s_1}^{s_2} ds \langle y|J|(y' - Ly)\rangle, \] (36)

where we have used Dirac’s notation for complex ‘bra’ and ‘ket’ vectors (namely \( \langle z | = z^* \) \) is the complex conjugate of the row vector obtained by transposing the column vector \( |z\rangle = z \), the standard hermitian product being \( \langle z |w\rangle = \bar{z}w \)). This means that

\[ \delta A = 0 \quad \iff \quad y' - Ly = 0 \]

for any variation \( \delta y \) such that \( \delta y(s_1) = \delta y(s_2) = 0 \). The symplectic product \( \langle x|J|y\rangle \) is invariant under the Hamiltonian evolution of any pair of phase-space vectors \( x \) and \( y \). Therefore, if we consider a complex solution \( z = x + iy \) of the equations of motion, the corresponding Courant–Snyder invariant \( W \) is given by \( \langle z|J|z\rangle = iW \).

We now look for a stationary value of our action integral \( A \) in the space of the complex vectors \( e(s) \) periodic in \( s \) (with period \( L \)) and with unit symplectic norm:

\[ A = \int_0^L ds \langle e|J|(e' - Le)\rangle, \quad \delta A = 0 \quad \text{with} \quad \langle e|J|e\rangle = i. \] (37)

Since \( A \) is real* while the constraint \( \langle e|J|e\rangle = i \) is purely imaginary, using a Lagrange multiplier \( \mu \) we can write

\[ \delta A + i \mu \delta (\langle e|J|e\rangle) = \delta \left( A + i \int_0^L ds \langle e|J|\Phi e\rangle \right) = 0, \] (38)

where \( \Phi(s) \) is an arbitrary function of \( s \) such that \( \int_0^L ds \Phi'(s) = \mu \). The corresponding Euler equation for \( e(s) \) is therefore

\[ e' = Le - i\Phi' e \] (39)

* This follows from the antisymmetry of the unit symplectic matrix \( J \).
and the stationary value $A_{\text{var}}$ of the action is

$$A_{\text{var}} = \int_0^L ds \langle e | J | (-i \Phi' e) \rangle = -i \mu \langle e | J | e \rangle = \mu. \quad (40)$$

For a small variation $\delta L$ of the focussing or of the RF parameters, the variation of $\mu$ is therefore

$$\delta \mu = - \int_0^L ds \langle e | J | \delta L e \rangle. \quad (41)$$

It is now simple to show that $e(s)$ is a periodic eigenvector of the single-turn transfer matrix $\tilde{M}(s)$, belonging to the eigenvalue $e^{i \mu}$: therefore, for stable betatron and synchrotron oscillations, there are in general three solutions corresponding to the normal modes of the system. Indeed, from Eq. (39), we get

$$|e(s)\rangle = e^{i(\Phi(s_o) - \Phi(s))} M(s|s_o)|e(s_o)\rangle, \quad (42)$$

where $M(s|s_o)$ denotes the transfer matrix from $s_o$ to $s$, satisfying the equation

$$\frac{d}{ds} M(s|s_o) = L(s)M(s|s_o), \quad \text{with} \quad M(s_o|s_o) = I. \quad (43)$$

For a complete machine revolution, we have $M(s + L|s) = \tilde{M}(s)$ and

$$|e(s + L)\rangle = e^{-i \mu} \tilde{M}(s)|e(s)\rangle = |e(s)\rangle. \quad (44)$$

The last equality follows from the required periodicity of $|e(s)\rangle$ and we have made use of $\Phi(s + L) - \Phi(s) = \mu$. Therefore

$$\tilde{M}(s)|e(s)\rangle = e^{i \mu}|e(s)\rangle, \quad (45)$$

showing that $|e(s)\rangle$ is an eigenvector of $\tilde{M}(s)$ belonging to the eigenvalue $e^{i \mu}$. This means that the Lagrange multiplier $\mu$ is the phase advance of the corresponding normal mode.

We now apply this general result to the case of one-dimensional betatron oscillations, thus recovering Eq. (9), and to the coupled horizontal and longitudinal oscillations in a machine with vanishing dispersion at the RF-cavities.

**APPENDIX B** One-dimensional betatron oscillations

In the case of one-dimensional betatron oscillations, the matrix $L(s)$ is given by Eq. (35) and the normalized, periodic eigenvector $|e(s)\rangle$ can be parametrized as follows

$$\langle e | J | e \rangle = i \quad \iff \quad |e\rangle = \frac{e^{i \theta}}{\sqrt{2}} \left( \begin{array}{c} a \\ b + i \\ d \end{array} \right), \quad (46)$$
where \( a(s), b(s) \) and \( \theta(s) \) are real, periodic functions of \( s \). From Eq. (37), we see that the Lagrangian \( \mathcal{L}(a, a', b, b'; s) \) is therefore

\[
\mathcal{L} = \langle e | J | (e' - L e) \rangle = \frac{1}{2} \left( \begin{array}{c} a \ b - i \ a \ a' \ -1 \ 1 \ \left( \begin{array}{c} a' - \frac{b + i}{a} \\ \frac{b + i}{a} \end{array} \right) + K(s)a \end{array} \right)
\]

\[
= \frac{1}{2} \left[ b' + \frac{b}{a} \left( \frac{b}{a} - 2a' \right) + K(s)a^2 + \frac{1}{a^2} \right]. \tag{47}
\]

The corresponding Euler equations are

\[
\frac{d}{ds} \frac{\partial \mathcal{L}}{\partial a'} - \frac{\partial \mathcal{L}}{\partial a} = K(s)a - \frac{1}{a^3} + b' \left( \frac{a' - b}{a} \right),
\]

\[
\frac{d}{ds} \frac{\partial \mathcal{L}}{\partial b'} = 0 = \frac{\partial \mathcal{L}}{\partial b} = \frac{b}{a^2} - \frac{a'}{a} \quad \implies \quad \frac{b}{a} = a'.
\]

We can now insert \( b/a = a' \) in the original Lagrangian, thus obtaining a Lagrangian containing only \( a \) and \( a' \). Discarding the term \( b' \) (since it is a total derivative and does not change the Euler equation nor the value of the action integral over one turn), this yields

\[
\mathcal{L}(a, a'; s) = -\frac{1}{2} \left[ a'^2 - K(s)a^2 - \frac{1}{a^2} \right] \tag{48}
\]

which coincides, a part from the sign, with the Lagrangian Eq. (7) for the envelope function \( a(s) = \sqrt{\beta(s)} \). Therefore \( b = \frac{1}{2} \beta' \) and our general formula Eq. (40) reduces to the variational form (9) for the betatron phase advance.

**APPENDIX C  Coupled horizontal and longitudinal oscillations**

In a circular machine with plane reference orbit, the horizontal and longitudinal oscillations are coupled through a curvature term \( G(s) = 1/\rho(s) \). We have

\[
y = \left( \begin{array}{c} x \\ x' \\ z \\ \Delta p/p \end{array} \right), \quad L(s) = \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ -K(s) & 0 & 0 & G(s) \\ -G(s) & 0 & 0 & 1/\gamma^2 \\ 0 & 0 & -\epsilon(s) & 0 \end{array} \right), \tag{49}
\]

where \( z \) denotes the longitudinal displacement with respect to the synchronous particle and

\[
\epsilon = \frac{e\gamma}{cE},
\]

with \( \gamma \) the slope of the accelerating RF-voltage at the synchronous phase and \( E \) the nominal energy. Provided the RF-cavities are located in a dispersion-free region, the
normalized betatron and synchrotron eigenvectors can be written

\[ |e_\beta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\beta} \\ i - \alpha \sqrt{\beta} \\ i - \alpha D - \sqrt{\beta} D' \\ 0 \end{pmatrix}, \quad |e_s\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i - \alpha_s D \\ i - \alpha_s \sqrt{\beta_s} \\ i - \alpha_s \sqrt{\beta_s} D' \\ i - \alpha_s \sqrt{\beta_s} \end{pmatrix}. \]  

(50)

Here \( \alpha \) and \( \beta \) are Twiss parameters for the betatron oscillations while \( \alpha_s \) and \( \beta_s \) play the same role for synchrotron oscillations.

We now consider a general variation of the focusing, curvature and RF parameters

\[ \delta L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\delta K & 0 & 0 & \delta G \\ -\delta G & 0 & 0 & \delta(1/\gamma^2) \\ 0 & 0 & -\delta \epsilon & 0 \end{pmatrix}. \]  

(51)

Then

\[ |\delta L e_\beta\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \sqrt{\beta} \delta K \\ \sqrt{\beta} \delta G \\ \left( \frac{i - \alpha}{\sqrt{\beta}} D - \sqrt{\beta} D' \right) \delta \epsilon \end{pmatrix}, \]

\[ |\delta L e_s\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i - \alpha_s \sqrt{\beta_s} (D \delta K - \delta G) \\ i - \alpha_s \sqrt{\beta_s} \left( D \delta G - \delta \left( \frac{1}{\gamma^2} \right) \right) \\ \sqrt{\beta_s} \delta \epsilon \end{pmatrix}. \]

(52)

and, using Eq. (41), the corresponding changes in the betatron phase advance \( \mu \) and in the synchrotron phase advance \( \mu_s \) are

\[ \delta \mu = \frac{1}{2} \int_0^L ds \left( \beta \delta K + \mathcal{H} \delta \epsilon \right), \]

(52)

\[ \delta \mu_s = \frac{1}{2} \int_0^L ds \left\{ \gamma_s [(D^2 \delta K - 2D \delta G) + \delta(1/\gamma^2)] + \beta_s \delta \epsilon \right\}. \]

(53)

Here \( \mathcal{H} \) is the dispersion pseudo-invariant and \( \gamma_s \) the third Twiss parameter for synchrotron oscillations:

\[ \mathcal{H} = \frac{D^2 + (\alpha D + \beta D')^2}{\beta}, \quad \gamma_s = \frac{1 + \alpha_s^2}{\beta_s}. \]  

(54)

From Eqs. (52) and (53), we see that the variation \( \delta \epsilon \) of the RF-parameter plays the same role of \( \delta K \) for synchrotron oscillations.
In order to identify the remaining contributions to $\delta \mu_s$, we consider the case of a single RF-cavity localized at $s = 0$ (where $D$ and $D'$ are assumed to vanish). The single-turn transfer matrix $\tilde{M}_s$ for synchrotron oscillations is then (at $s = 0$)

$$\tilde{M}_s = \begin{pmatrix} 1 & -\eta L \\ -\epsilon & 1 + \epsilon \eta L \end{pmatrix} = \begin{pmatrix} \cos \mu_s + \alpha_s \sin \mu_s & \beta_s \sin \mu_s \\ -\gamma_s \sin \mu_s & \cos \mu_s - \alpha_s \sin \mu_s \end{pmatrix},$$

where $\eta$ is the orbit dilation factor

$$\eta = \frac{1}{\gamma_i} - \frac{1}{\gamma^2}.$$

The value of $\gamma_s$ is constant, since the synchrotron transfer matrix in the arc of the machine corresponds to that of a drift space. Therefore we get

$$\gamma_s = \frac{\epsilon}{\sin \mu_s},$$

$$\cos \mu_s = \frac{1}{2} \text{Tr} \tilde{M}_s = 1 + \frac{1}{2} \epsilon \eta L.$$

Then the variation $\delta \mu_s$ caused by a variation $\delta \eta$ of the orbit dilation factor can be written

$$\delta \cos \mu_s = -\delta \mu_s \sin \mu_s = \frac{1}{2} \epsilon L \delta \eta$$

and, using Eq. (56), we obtain

$$\delta \mu_s = -\frac{1}{2} \frac{\epsilon}{\sin \mu_s} L \delta \eta = -\frac{1}{2} \gamma_s L \delta \eta.$$

By comparing Eqs. (53) and (58), we can identify $\delta \eta$ with $-1/L$ times the integral of the factor in square brackets that multiplies $\gamma_s$ in Eq. (53). Moreover,

$$\delta \eta = \delta \left( \frac{1}{\gamma_i^2} \right) - \delta \left( \frac{1}{\gamma^2} \right)$$

and thus we arrive at the following expression for the change in transition energy

$$\delta \left( \frac{1}{\gamma_i^2} \right) = -\frac{1}{L} \int_0^L ds \left( D^2 \delta K - 2 D \delta G \right).$$

This coincides with Eq. (22).