String Theories on the Asymmetric Orbifolds

with Twist-Untwist Intertwining Currents *

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Abstract

We give various examples of asymmetric orbifold models to possess intertwining currents which convert untwisted string states to twisted ones, and vice versa, and see that such asymmetric orbifold models are severely restricted. The existence of the intertwining currents leads to the enhancement of symmetries in asymmetric orbifold models.

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1. Introduction

String theory has been regarded as a candidate for the unified theory including gravity and extensively been studied to construct phenomenologically realistic models. Orbifold compactification [1] is one of the most promising methods to build them and the search for realistic orbifold models has been continued by many groups and various models have been proposed [2-5]. However, we have not yet found any satisfactory orbifold models to describe our real world. Therefore, it would be of great importance to classify all the compactifications of string theories on orbifolds thoroughly. In this paper, we shall investigate symmetries between untwisted and twisted string states on asymmetric orbifolds [6].

Suppose that there exists an intertwining current operator which converts string states in the untwisted sector to string states in the twisted sector in an asymmetric orbifold model. This current operator will correspond to a state of the conformal weight $(1,0)$ (or $(0,1)$) in the twisted sector and connect the ground state of the untwisted sector to the $(1,0)$ (or $(0,1)$) twisted state. Therefore, the existence of a $(1,0)$ (or $(0,1)$) twisted state implies the appearance of a symmetry between the untwisted and twisted sectors. It leads to the enhancement of symmetries in asymmetric orbifold models. It should be emphasized that it does not occur in the case of symmetric orbifolds because the left- and right-conformal weights, $h$ and $\bar{h}$, of a ground state of any twisted sector are both positive for symmetric orbifolds and hence no $(1,0)$ or $(0,1)$ state appears in any twisted sector.

In the next section, we briefly review general properties of asymmetric orbifolds and give the modular invariance conditions of the one-loop partition function. In section 3, we investigate $\mathbb{Z}_N$-automorphisms of the lattice defining the orbifolds. In section 4, we prove the “torus-orbifold equivalence” [1,6,13-15]. This equivalence is used to determine the symmetries of orbifold models. In section 5, we give various examples of asymmetric $\mathbb{Z}_2$, $\mathbb{Z}_3$- and $\mathbb{Z}_N$-orbifold models which possess $(1,0)$ twisted states and show the enhancement of symmetries in the asymmetric orbifold models.

‡1 Some examples have been discussed in refs.[1,7,8] and in our previous paper [9].
‡2 This current is a twisted state emission vertex operator [7,8,10-12] with the conformal weight $(1,0)$ or $(0,1)$, whose explicit construction is not easy in general.
Finally in section 6, we present our conclusion.

2. Asymmetric $\mathbb{Z}_N$-Orbifolds

In the construction of toroidal orbifolds, we start with a $D$-dimensional toroidally compactified closed bosonic string theory which is specified by a $D + D$-dimensional Lorentzian even self-dual lattice $\Gamma^{D,D}$ [16]. The left- and right-moving momentum $(p^I_L, p^I_R)$ $(I = 1, \cdots, D)$ lies on the lattice $\Gamma^{D,D}$. For simplicity, we will restrict our considerations to asymmetric $\mathbb{Z}_N$-orbifolds and choose $\Gamma^{D,D}$ to be of the form

$$\Gamma^{D,D} = \{ (p^I_L, p^I_R) \mid p^I_L - p^I_R \in \Lambda \quad \text{and} \quad p^I_L, p^I_R \in \Lambda^* \}, \quad (2.1)$$

where $\Lambda$ is a $D$-dimensional Euclidean lattice and $\Lambda^*$ is the dual lattice of $\Lambda$. Note that $\Gamma^{D,D}$ is even self-dual if $\Lambda$ is even integral. Let $g$ be a group element which generates the asymmetric $\mathbb{Z}_N$-transformation. The $g$ is defined by

$$g : (X^I_L, X^I_R) \rightarrow (U^{IJ}X^J_L, X^I_R), \quad I, J = 1, \ldots, D, \quad (2.2)$$

where $X^I_L, (X^I_R)$ is a left- (right-) moving string coordinate and $U$ is a rotation matrix which satisfies $U^N = 1$. The $\mathbb{Z}_N$-transformation has to be an automorphism of the lattice $\Gamma^{D,D}$, i.e.,

$$(U^{IJ}p^I_L, p^I_R) \in \Gamma^{D,D} \quad \text{for all} \quad (p^I_L, p^I_R) \in \Gamma^{D,D}. \quad (2.3)$$

In the untwisted sector, the boundary condition of the string coordinate is the same as the torus case, so that the left- and right-moving string coordinate will be expanded as

$$X^I_L(z) = x^I_L - ip^I_L \ln z + i \sum_{n \neq 0} \frac{1}{n} \alpha^I_{L,n} z^{-n},$$

$$X^I_R(\bar{z}) = x^I_R - ip^I_R \ln \bar{z} + i \sum_{n \neq 0} \frac{1}{n} \alpha^I_{R,n} \bar{z}^{-n}. \quad (2.4)$$

Now we will introduce the operator $R_{(0)}$, which induces the $\mathbb{Z}_N$-transformation, i.e.,

$$R_{(0)} (X^I_L(z), X^I_R(\bar{z})) R_{(0)}^{-1} = (U^{IJ}X^J_L(z), X^I_R(\bar{z})). \quad (2.5)$$
To explicitly construct \( R_{(0)} \) in terms of the operators of the mode expansion of the string coordinate, it will be convenient to use a complex coordinate. Since \( U \) is an orthogonal matrix, it can be diagonalized by a unitary matrix \( M \):

\[
M U M^\dagger = U_{\text{diag}}.
\]

(2.6)

Since \( U^N = 1 \), we may write

\[
U_{\text{diag}} = \begin{pmatrix}
\omega^{r_1} & 0 & \cdots & 0 \\
0 & \omega^{r_2} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \omega^{r_D}
\end{pmatrix}.
\]

(2.7)

where \( 0 \leq r_I \leq N - 1 \) \((r_I \in \mathbb{Z})\) and \( \omega = e^{2 \pi i / N} \). The set of eigenvalues \( \{ \omega^{r_I} \} \) is identical to the set of \( \{ \omega^{-r_I} \} \) because \( U \) is an orthogonal matrix. Thus, we may write the eigenvalues of \( U \) as \( \{ \omega^{r_I} \text{ and } \omega^{-r_I}, \quad I = 1, 2, \ldots, \frac{D}{2} \} \).

In terms of \( \gamma^I_{Ln} \equiv M^{IJ} \alpha^I_{Ln} \) \((n \in \mathbb{Z} > 0)\), the operator \( R_{(0)} \) is given by

\[
R_{(0)} = \omega^{- \sum_{I=1}^{D} \sum_{n=1}^{\infty} \frac{r_I}{n} \gamma^I_{Ln} \gamma^I_{Ln} } \sum_{(p^I_L, p^I_R) \in \Gamma} |p^I_L, p^I_R \rangle < U^{IJ} p^I_L, p^I_R |. 
\]

(2.8)

The one-loop partition function in the untwisted sector is given by

\[
Z^{(0)}(\tau) = \frac{1}{N} \sum_{m=0}^{N-1} Z(1, U^m; \tau),
\]

(2.9)

where

\[
Z(1, U^m; \tau) = \text{Tr}[(R_{(0)})^m q^{L_0 - \frac{D}{2}} q^{\bar{L}_0 - \frac{D}{2}}],
\]

\[
q = e^{i2\pi \tau},
\]

\[
L_0 = \sum_{I=1}^{D} \left\{ \frac{1}{2} (p^I_L)^2 + \sum_{n=1}^{\infty} \alpha^I_{Ln} \alpha^I_{Ln} \right\},
\]

\[
\bar{L}_0 = \sum_{I=1}^{D} \left\{ \frac{1}{2} (p^I_R)^2 + \sum_{n=1}^{\infty} \alpha^I_{Rn} \alpha^I_{Rn} \right\}.
\]

\( ^{\dagger3} \) Here we assumed that the dimension \( D \) is even integer. In fact, \( D \) is even for all the models we consider.
Calculating the trace of each terms, we have

$$Z(1,1;\tau) = \frac{1}{|\eta(\tau)|^{2D}} \sum_{(p'_L,p'_R) \in \Gamma \delta, D} q^{\frac{1}{2}(p'_L)^2} q^{\frac{1}{2}(p'_R)^2},$$

$$Z(1,U^m;\tau) = \frac{1}{|\eta(\tau)|^{2D}} \prod_{l=1}^{D/2} \left[ \frac{-2\sin(\pi \frac{m}{N} |\tau|)^3}{\vartheta_1^{-}(\frac{m}{N} |\tau|)} \right] \sum_{(p'_L,p'_R) \in \Gamma_{inv}^D} q^{\frac{1}{2}(p'_L)^2} q^{\frac{1}{2}(p'_R)^2},$$

for \( m = 1, 2, \ldots, N - 1, \)

where \( \eta(\tau) \) is the Dedekind \( \eta \)-function and \( \vartheta_1(\nu|\tau) \) is the Jacobi theta function:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

$$\vartheta_1(\nu|\tau) = \sum_{n=-\infty}^{\infty} \exp\{i\pi(n + \frac{1}{2})^2\tau + i2\pi(n + \frac{1}{2})(\nu + \frac{1}{2})\},$$

\([x]\) denotes that \([x] = x \mod N \) and \( 0 \leq [x] < N \) for any integer \( x \) and

$$\Gamma_{inv}^D = \{(p_L^I, p_R^I) \in \Gamma^D, D | (p_L^I, p_R^I) = (\{U^m\}^I_{JI} p_L^J, p_R^I)\}. \quad (2.12)$$

Note that modular invariance of \( Z(1,1;\tau) \) requires that the lattice \( \Gamma^D, D \) must be a \((D + D)\)-dimensional Lorentzian even self-dual lattice.

In the \( U^\ell \)-twisted sector \((\ell = 1, 2, \ldots, N - 1)\), the string coordinate will obey the following \( U^\ell \)-twisted boundary condition:

$$X_L^I(e^{2\pi i z}) = (U^\ell)^{IJ} X_L^J(z) + (shift),$$

$$X_R^I(e^{-2\pi i \bar{z}}) = X_R^I(\bar{z}) + (shift). \quad (2.13)$$

Thus \( X_L^I(z) \) and \( X_R^I(\bar{z}) \) will be expanded as

$$X_L^I(z) = x_L^I + i \sum_{n_j \in \mathbb{Z} + \frac{(\nu_j)}{N} > 0} \frac{1}{n_j} \{ (M^{iJ}_L \gamma^J_{L,n_j} z^{-n_j}) - M^{Jl}_L \gamma^J_{L,n_j} z^{n_j} \},$$

$$X_R^I(\bar{z}) = x_R^I - ip_R^I \ln \bar{z} + i \sum_{n \neq 0} \frac{1}{n} \alpha_R^I z^{-n}. \quad (2.14)$$

\( \dagger \dagger \) In this paper, we will consider \( Z_N \)-orbifold models, in which \( Z_N \)-transformation leaves only the origin fixed. Therefore \( U^\ell \) has no eigenvalue of one.
As in the untwisted sector, the partition function of the $U^\ell$-twisted sector consists of $N$ parts:

$$Z^{(\ell)}(\tau) = \frac{1}{N} \sum_{m=0}^{N-1} Z(U^\ell, U^m; \tau), \quad \ell = 1, 2, ..., N - 1,$$

(2.15)

where

$$Z(U^\ell, U^m; \tau) = \text{Tr}[(R_{(\ell)})^m q^{L_0 - \frac{B}{2\pi}} \bar{q}^{\bar{L}_0 - \frac{B}{2\pi}}]$$

and $R_{(\ell)}$ is the $Z_N$-transformation operator in the $U^\ell$-twisted sector:

$$R_{(\ell)}(X^I_L(z), X^I_R(\bar{z})) R_{(\ell)}^{-1} = (U^{IJ} X^J_L(z), X^J_R(\bar{z})).$$

However the twisted Hilbert space is not obvious in the case of asymmetric orbifolds. In the following, to understand this Hilbert space, we will use the modular transformation properties of the one-loop partition function as a guiding principle. In order that the partition function is modular invariant, each term in the partition function should transform as

$$Z(U^\ell, U^m; \tau + 1) = Z(U^\ell, U^{m+\ell}; \tau),$$

$$Z(U^\ell, U^m; -1/\tau) = Z(U^{-m}, U^\ell; \tau).$$

(2.16)

Therefore, we define $Z(U^\ell, 1; \tau)$ as follows:

$$Z(U^\ell, 1; \tau) \equiv Z(1, U^{-\ell}; -1/\tau)$$

$$= \frac{\sqrt{\text{det}(1 - U^\ell)}}{V_{\Gamma_{\text{inv}}^{D,D}}} \left[ \frac{(\eta(\tau))^{3}}{\eta(\tau)} \prod_{I=1}^{D/2} \left[ \frac{q^{\frac{1}{2}(\ell r_I)}}{1 - q^{\frac{\ell r_I}{N}}} \right] \right] \sum_{(p^I_L, p^I_R) \in \Gamma_{\text{inv}}^{D,D*}} q^{\frac{1}{2}(p^I_L)^2} q^{\frac{1}{2}(p^I_R)^2},$$

(2.17)

where $V_{\Gamma_{\text{inv}}^{D,D}}$ is the volume of the unit cell of $\Gamma_{\text{inv}}^{D,D}$, $h_\ell$ is the conformal weight of $U^\ell$-twisted vacuum:

$$h_\ell = \frac{1}{4} \sum_{I=1}^{D} \frac{[\ell r_I]}{N} (1 - \frac{[\ell r_I]}{N})$$

(2.18)

and $\Gamma_{\text{inv}}^{D,D*}$ is the dual lattice of $\Gamma_{\text{inv}}^{D,D}$. 

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The remaining parts of the partition function are defined as follows \(^\dagger\dagger\):

\[
Z(U^\ell, U^{\ell m}; \tau) \equiv Z(U^\ell, 1; \tau + m) = \frac{\sqrt{\det(1 - U^\ell)}}{V_{\Gamma_{D,D}^{inv}}} e^{\frac{2\pi im h_\ell}{\ell}} q^{\ell} \prod_{I} \left[ \frac{1}{\eta(\tau)^{2D}} \right] \left[ \frac{(\eta(\tau))^3}{\eta_i^{[\ell]} N} q^{\ell} \right] \frac{\vartheta_4^{[\ell]}(\tau + m)}{\vartheta_4^{[\ell]}(\tau)} \]

\[
= \sqrt{\det(1 - U^\ell)} e^{\frac{2\pi im h_\ell}{\ell}} q^{\frac{\ell}{2}} q^{\frac{1}{2} (p_L^I)^2} q^{\frac{1}{2} (p_R^I)^2} \sum_{(p_L^I, p_R^I) \in \Gamma_{D,D}^{inv}^*} e^{\pi m ((p_L^I)^2 - (p_R^I)^2)} q^{\frac{1}{2} (p_L^I)^2} q^{\frac{1}{2} (p_R^I)^2}. \tag{2.19}
\]

As stated above, we have obtained \(Z(U^\ell, U^{\ell m}; \tau) (\ell \neq 0)\) from \(Z(1, U^{-\ell}; \tau)\) using the modular transformation properties. Considering these partition functions, we now find that \(Z(U^\ell, U^{\ell m}; \tau) (\ell \neq 0)\) given in eq.(2.19) can also be obtained from the trace formula in the operator formalism:

\[
Z(U^\ell, U^{\ell m}; \tau) = \text{Tr} [(R_{(\ell)})^{\ell m} q^{L_0 - \frac{D}{2}} \bar{q}^{\bar{L}_0 - \frac{D}{2}}], \tag{2.20}
\]

where

\[
L_0 = \sum_{I=1}^{D} \sum_{n_I \in \mathbb{Z} + \frac{[\ell]}{N} > 0} \gamma^I_{Ln_I} \gamma^I_{Ln_I} + h_\ell,
\]

\[
\bar{L}_0 = \sum_{I=1}^{D} \left( \frac{1}{2} (p_R^I)^2 + \sum_{n=1}^{\infty} \alpha^I_{R-n} \alpha^I_{Rn} \right),
\]

\[
(R_{(\ell)})^\ell = e^{2\pi i (L_0 - \bar{L}_0)}.
\]

Note that the trace of momentum \((p_L^I, p_R^I)\) is over \(\Gamma_{D,D}^{inv}^*\) and the number of degeneracy of the ground states in the \(U^\ell\)-twisted sector is given by

\[
n = \frac{\sqrt{\det(1 - U^\ell)}}{V_{\Gamma_{D,D}^{inv}}} \tag{2.21}
\]

Suppose that \((N, \ell) = d\), where \((N, \ell)\) denotes the greatest common divisor of \(N\) and \(\ell\). Since \(Z(U^\ell, U^{\ell m}; \tau)\) has to be invariant under the modular transformation \(\tau \to \tau + N/d\) because of \(U^N = 1\), the necessary condition for the modular invariance is

\[
N_d (L_0 - \bar{L}_0) = 0 \text{ mod } 1. \tag{2.22}
\]

\(^\dagger\dagger\) If \((N, \ell) \neq 1\), undetermined parts still remain, where \((N, \ell)\) denotes the greatest common divisor of \(N\) and \(\ell\). These are defined by other partition functions \(Z(U^\ell, U^{\ell m}; \tau)\) using the modular transformation: \(\tau \to -1/\tau\).
This is called the left-right level matching condition and it has been proved that this condition is also a sufficient condition for modular invariance [6,17]. This condition can equivalently be rewritten as follows:

\[
\frac{N}{d} h_\ell = 0 \mod 1, \tag{2.23}
\]

\[
\frac{N}{d} ((p^I_L)^2 - (p^I_R)^2) = 0 \mod 2 \quad \text{for all } (p^I_L, p^I_R) \in \Gamma^{D,D}_{\text{inv}}. \tag{2.24}
\]

### 3. Automorphism of \( \Gamma^{D,D} \)

In the following we consider asymmetric \( \mathbb{Z}_N \)-orbifolds, where \( \Lambda \) in eq.(2.1) is a root lattice of a simply-laced Lie group \( G \) (i.e., \( \Lambda = \Lambda_R(G) \)) and the squared length of the root is normalized to two. For simplicity, we will restrict our considerations to the case that \( \mathbb{Z}_N \)-transformation leaves only the origin fixed. We will first investigate the automorphisms of \( \Gamma^{D,D} \) \((D = \text{rank} G)\). The group of “asymmetric” automorphisms of \( \Gamma^{D,D} \), \( \text{Aut} \Gamma^{D,D} \), is defined by

\[
(U^{IJ} p^I_L, p^I_R) \in \Gamma^{D,D} \quad \text{for all } (p^I_L, p^I_R) \in \Gamma^{D,D}, \tag{3.1}
\]

where \( U \in \text{Aut} \Gamma^{D,D} \). This means that \( \text{Aut} \Gamma^{D,D} \) must be contained in the group of automorphisms of the root lattice \( \Lambda_R(G) \), \( \text{Aut} \Lambda_R(G) \).

\( \text{Aut} \Lambda_R(G) \) is semi-direct product of two groups [18]:

\[
\text{Aut} \Lambda_R(G) = W_G \rtimes \Gamma_G, \tag{3.2}
\]

where \( W_G \) is the Weyl group of the root system of \( G \), i.e., the group generated by the Weyl reflection of simple roots, and \( \Gamma_G \) is \( ^{\dagger 6} \)

\[
\Gamma_G = \{ \sigma \in \text{Aut} \Lambda_R(G) | \sigma(\Delta) = \Delta \}. \tag{3.3}
\]

Here \( \Delta \) is a fixed basis of \( \Lambda_R(G) \), i.e., \( \Delta = \{ \alpha_1, \alpha_2, \cdots, \alpha_D \} \) and \( \alpha_i \) \((i = 1, 2, \cdots, D)\) is a simple root of \( G \). \( \Gamma_G \) corresponds to the group of symmetries of the Dynkin diagram of \( G \). Any element of \( W_G \) transforms \( p^I_L \) \(( \in \Lambda_R(G)^* = \Lambda_W(G) \)) to \( U^{IJ} p^I_L \)

\(^{\dagger 6} \) See ref.[18] for detail.
in the same conjugacy class of $G$ but an element in $\Gamma_G$ does not. Therefore from eq. (3.1) $\text{Aut}^{D,D} \Delta$ must be in $W_G$ not in full $\text{Aut} \Lambda R(G)$.

Every element in the Weyl group, $w \in W_G$, can be written as a product of rank $G$ or less Weyl reflections of linearly independent roots [19,20], i.e.,

$$w = w_1w_2...w_k, \quad 1 \leq k \leq \text{rank}G,$$

(3.4) where $w_i$ denotes a Weyl reflection with respect to a root $\alpha_i$ which needs not to be a simple root. Reflection of $k$ ($\leq \text{rank}G$) linearly independent roots in a rank $G$-dimensional vector space leaves a $(\text{rank}G - k)$-dimensional subspace fixed. Hence Weyl elements to leave only the origin fixed must be reflections of $k = \text{rank}G$ linearly independent roots. These elements are given in ref.[19] and are summarized as follows:

In the case of $G = SU(\ell + 1)$, a Weyl element leaves only the origin fixed only if the order of the element is $\ell + 1$ and is prime. This element is given by

$$w = w_1w_2...w_\ell,$$

(3.5) where $w_i$ is the Weyl reflection of a simple root $\alpha_i$. Under this element the simple root transforms as follows:

$$\alpha_i \to \alpha_{i+1}, \quad i = 1, 2, \cdots, \ell,$$

and

$$\alpha_{\ell+1} \equiv -(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell) \to \alpha_1.$$

In the case of $G = SO(2\ell)$, the order of allowed Weyl elements is

$$2^N, 2^{N-1}, \ldots, 2 \quad \text{for } \ell = 2^Np \quad (p = \text{odd integer}).$$

The root vectors of $SO(2\ell)$ will be given by $\pm e_a \pm e_b$ ($a \neq b$), where $e_a$ ($a = 1, 2, \cdots, \ell$) is an orthogonal unit vector. A cyclic transformation of $e_{k_1}, e_{k_2}, \ldots, e_{k_m}$ such that

$$e_{k_1} \to e_{k_2} \to \cdots \to e_{k_m} \to -e_{k_1} \to -e_{k_2} \to \cdots \to -e_{k_m} \to e_{k_1}$$

is denoted by $[\mathcal{M}]$, then the transformation of the order $2^i$ ($i = 1, \ldots, N$) is expressed by

$$2^i : \{2^{i-1}, 2^{i-1}, \ldots, 2^{i-1}\},$$

$2^{N+1-i}p$ times.
For example, the transformations of order 2 and $2^2$ are given by
\[
e_1 \to -e_1 \to e_1, \quad e_2 \to -e_2 \to e_2, \quad \ldots, \quad e_\ell \to -e_\ell \to e_\ell
\]
and
\[
e_1 \to e_2 \to -e_1 \to -e_2 \to e_1, \\
e_3 \to e_4 \to -e_3 \to -e_4 \to e_3, \\
\ldots
\]
\[
e_{\ell-1} \to e_\ell \to -e_{\ell-1} \to -e_\ell \to e_{\ell-1},
\]
respectively.

In the case of $G = E_6, E_7, E_8$, the order of allowed Weyl elements is
\[
E_6 : \quad 3, 9, \\
E_7 : \quad 2, \\
E_8 : \quad 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30.
\]

4. Torus-Orbifold Equivalence

In this paper we shall investigate the symmetries of asymmetric $\mathbb{Z}_N$-orbifold models with intertwining currents. However the explicit construction of these currents is not easy in general. Therefore we may rewrite the orbifold model into an equivalent torus model using the “torus-orbifold equivalence” [1,6,13-15] and investigate symmetries of this torus model instead of the orbifold model. The “torus-orbifold equivalence” tells us that any compactified closed bosonic string theory on a $\mathbb{Z}_N$-orbifold is equivalent to that on a torus if the rank of the gauge symmetry of strings on the orbifold is equal to the dimension of the orbifold. It may be instructive to give a proof of the “torus-orbifold equivalence” here. The proof will follow ref.[15].

Let us consider a $D$-dimensional torus model associated with the root lattice $\Lambda_R(G)$ of a simply-laced Lie group $G$ ($D = \text{rank} G$). Then this model has the affine Kac-Moody algebra $\hat{g} + \hat{\hat{g}}$, which can be constructed in the vertex operator representation à la Frenkel and Kac [21]:

\[
P_L^I(z) \equiv i \partial_z X_L^I(z), \\
V_L(\alpha; z) \equiv: \exp\{i\alpha \cdot X_L(z)\}:
\]

(4.1)
and
\[ P_R^I(\bar{z}) \equiv i\partial_{\bar{z}}X_R^I(\bar{z}), \]
\[ V_R(\alpha; \bar{z}) \equiv: exp\{i\alpha \cdot X_R^I(\bar{z})\}: . \]  

A $\mathbb{Z}_N$-orbifold model is obtained by modding out of this torus model by a $\mathbb{Z}_N$-rotation which is an automorphism of the lattice defining the torus. Since every physical string state on the $\mathbb{Z}_N$-orbifold is invariant under the $\mathbb{Z}_N$-transformation, the gauge symmetry $G_0$, which is the invariant subalgebra of $G$ under $\mathbb{Z}_N$-transformation, will appear in the spectrum.

In the case of rank $G_0 = D$, the $\mathbb{Z}_N$-invariant operator $P_L^I(z)$, $P_R^I(\bar{z})$ ($I = 1, 2, \cdots, D$) can be constructed from suitable linear combinations of $P_L^I(z)$, $V_L(\alpha; z)$ and $P_R^I(\bar{z})$, $V_R(\alpha; \bar{z})$ such that
\[ R_{(\ell)}(P_L^I(z), P_R^I(\bar{z}))R_{(\ell)}^{-1} = (P_L^I(z), P_R^I(\bar{z})) \]  

and
\[ P_L^I(w)P_L^J(z) = \frac{\delta^{IJ}}{(w - z)^2} + \cdots, \]
\[ P_R^I(\bar{w})P_R^J(\bar{z}) = \frac{\delta^{IJ}}{(\bar{w} - \bar{z})^2} + \cdots, \]  

where the operator $R_{(\ell)}$ is the $\mathbb{Z}_N$-transformation operator in the $U^\ell$-sector ($\ell = 0$ for the untwisted sector and $\ell = 1, 2, \cdots, N - 1$ for the $U^\ell$-twisted sector). It follows from (4.4) that $P_L^I(z)$, $P_R^I(\bar{z})$ can be expanded as
\[ P_L^I(z) \equiv i\partial_z X_L^I(z) \equiv \sum_{n \in \mathbb{Z}} \alpha_{Ln}^I z^{-n-1}; \]
\[ P_R^I(\bar{z}) \equiv i\partial_{\bar{z}} X_R^I(\bar{z}) \equiv \sum_{n \in \mathbb{Z}} \alpha_{Rn}^I \bar{z}^{-n-1} \]  

with
\[ [\alpha_{Ln}^I, \alpha_{Ln}^{IJ}] = m\delta_{m+n,0}^J, \]
\[ [\alpha_{Rm}^I, \alpha_{Rn}^{IJ}] = m\delta_{m+n,0}^J. \]  

In this basis the vertex operator will be written as
\[ V'(k_L, k_R; z) =: exp\{ik_L \cdot X_L^I(z) + ik_R \cdot X_R^I(\bar{z})\} :, \quad (k_L, k_R) \in \Gamma_{G,D}^D. \]
Since the operators $P^{\ell I}_L(z)$, $P^{\ell I}_R(\bar{z})$ are invariant under the $\mathbb{Z}_N$-transformation, the vertex operator will transform as

$$R_{(\ell)} V'(k_L, k_R, z) R_{(\ell)}^{-1} = e^{i 2\pi (k_L \cdot v_L - k_R \cdot v_R)} V'(k_L, k_R, z), \quad (4.7)$$

where $(v_L, v_R)$ is some constant vector. Therefore, $R_{(\ell)}$ will be given by

$$R_{(\ell)} = \eta_{(\ell)} \exp\{i 2\pi (p_L' \cdot v_L - p_R' \cdot v_R)\}, \quad (4.8)$$

where $\eta_{(\ell)}$ is a constant phase. Thus, the string coordinate in the new basis transforms as

$$(R_{(\ell)})^\ell (X^I_L(z), X^I_R(\bar{z})) (R_{(\ell)}^{-1})^\ell = (X^I_L(z) + 2\pi \ell v^I_L, X^I_R(\bar{z}) - 2\pi \ell v^I_R). \quad (4.9)$$

This implies that the string coordinate $(X^I_L(z), X^I_R(\bar{z}))$ obeys the boundary condition

$$(X^I_L(e^{2\pi i z}), X^I_R(e^{-2\pi i \bar{z}})) = (X^I_L(z) + 2\pi \ell v^I_L, X^I_R(\bar{z}) - 2\pi \ell v^I_R) + (torus \ shift), \quad (4.10)$$

and hence that

$$(p^I_L, p^I_R) \in \Gamma^D_G + \ell (v^I_L, v^I_R). \quad (4.11)$$

In the new basis, $(R_{(\ell)})^\ell$ will be given by

$$(R_{(\ell)})^\ell = e^{i 2\pi (L'_0 - \bar{L}'_0)} \quad (4.12)$$

because of the relation (2.16). Here $L'_0$ and $\bar{L}'_0$ are

$$L'_0 = \sum_{I=1}^D \left\{ \frac{1}{2} (p^I_L)^2 + \sum_{n=1}^\infty \alpha^I_{L-n} \alpha^I_{L+n} \right\},$$

$$\bar{L}'_0 = \sum_{I=1}^D \left\{ \frac{1}{2} (p^I_R)^2 + \sum_{n=1}^\infty \alpha^I_{R-n} \alpha^I_{R+n} \right\}. \quad (4.13)$$

Then it follows from eqs.(4.8),(4.11),(4.12) and (4.13) that $\eta_{(\ell)}$ is given by

$$\eta_{(\ell)} = \exp\{-i \pi \ell ((v^I_L)^2 - (v^I_R)^2)\}.$$

Since $(v^I_L, v^I_R)$ will correspond to one of momentum eigenvalues of the ground state in the $U$-twisted sector[15] and $(R_{(\ell)})^N = 1$, $(v^I_L, v^I_R)$ must satisfy

$$\frac{1}{2} (v^I_L)^2 = h_1,$$

$$\frac{1}{2} (v^I_R)^2 = \bar{h}_1,$$

$$N(v^I_L, v^I_R) \in \Gamma^D_G, \quad (m(v^I_L, v^I_R) \notin \Gamma^D_G, m = 1, 2, \ldots, N - 1).$$
where $h_1$ ($\tilde{h}_1$) is the conformal weight of the ground state of left- (right-) mover in $U$-twisted sector.

Every physical state in the $U^\ell$-sector must obey the condition $R^\ell = 1$ because it must be invariant under the $\mathbb{Z}_N$-transformation. Thus the allowed momentum eigenvalues $(p'_L, p'_R)$ of the physical state in the $U^\ell$-sector are restricted to

$$ (p'_L, p'_R) \in \Gamma^{D,D}_G + \ell(v'_L, v'_R) \quad \text{with} \quad p'_L \cdot v_L - p'_R \cdot v_R - \frac{1}{2} \ell((v'_L)^2 - (v'_R)^2) \in \mathbb{Z} $$

for the $U^\ell$-sector. (4.15)

The total physical Hilbert space $\mathcal{H}$ of strings on the $\mathbb{Z}_N$-orbifold is the direct sum of the physical space $\mathcal{H}^\ell$ in each sector:

$$ \mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \cdots \oplus \mathcal{H}^{(N-1)}. $$

(4.16)

In the above consideration we have shown that $\mathcal{H}$ is equivalent to

$$ \mathcal{H} = \{ \alpha'_{L-n} \cdots \alpha'_{R-m} \cdots |p'_L, p'_R| > n, m, \cdots \in \mathbb{Z} > 0, \ (p'_L, p'_R) \in \Gamma^{D,D}_G \}, $$

(4.17)

where

$$ \Gamma^{D,D}_G = \bigcup_{\ell=0}^{N-1} (\Gamma^{D,D}_G + \ell(v'_L, v'_R)) \quad |p'_L \cdot v_L - p'_R \cdot v_R - \frac{1}{2} \ell((v'_L)^2 - (v'_R)^2) \in \mathbb{Z} \}. $$

(4.18)

From eq.(4.14), $\Gamma^{D,D}_G$ is Lorentzian even self-dual lattice if $\Gamma^{D,D}_G$ is. Therefore, the total physical Hilbert space of strings on the $\mathbb{Z}_N$-orbifold is nothing but that of the strings on the torus associated with the Lorentzian even self-dual lattice $\Gamma^{D,D}_G$. This means that the symmetries of the $\mathbb{Z}_N$-orbifold models are the same as the torus model with $\Gamma^{D,D}_G$.

5. Examples of Asymmetric $\mathbb{Z}_N$-Orbifolds

(1) Examples of Asymmetric $\mathbb{Z}_2$-Orbifolds

Let us first consider asymmetric $\mathbb{Z}_2$-orbifolds. The $\mathbb{Z}_2$-transformation is defined by

$$ (X^I_L, X^I_R) \rightarrow (-X^I_L, X^I_R), \quad (I = 1, \cdots, D). $$

(5.1)
In this case, the necessary and sufficient conditions for modular invariance (2.23), (2.24) are
\[ D = 0 \mod 8, \]  
\[ p_R^2 = 0 \mod 1 \quad \text{for all } p_R \in \Gamma_0^*, \]  
where
\[ \Gamma_0 = \{ p_R \mid (p_L = 0, p_R) \in \Gamma^{D,D} \}. \]

In this paper we will investigate the models with the conformal weight (1,0) states in the twisted sector. The conformal weight of the ground state in the twisted sector is given by \((\frac{D}{16}, 0)\), so that it will be sufficient to consider only the cases of \(D = 8\) and \(16\). For \(D = 16\), the ground state in the twisted sector has the conformal weight (1,0). For \(D = 8\), the ground state has the conformal weight \((\frac{1}{2}, 0)\) but the first excited state has the conformal weight (1,0) because the left-moving oscillators are expanded in half-odd-integral modes in the twisted sector.

Let us take \(\Lambda\) to be the root lattice of a simply-laced Lie group \(G\) having the \(\mathbb{Z}_2\)-automorphism shown in section 3. However, modular invariance puts severe restrictions on \(G\) or \(\Lambda\) and all the possibilities of \(G\) are listed in Table 1. In the following, we will concentrate only on the left-movers. The \(G_0\) in Table 1 denotes the \(\mathbb{Z}_2\)-invariant subgroup of \(G\), which is the “unbroken” symmetry in each (untwisted or twisted) left-moving Hilbert space. However, since there appear twisted states with the conformal weight (1,0), the symmetry will be enhanced: The physical (i.e., \(\mathbb{Z}_2\)-invariant) (1,0) states in the untwisted sector correspond to the adjoint representation of \(G_0\). Each (1,0) state in the twisted sector corresponds to an intertwining current which converts untwisted states to twisted ones, and vice versa. Thus, the (1,0) states in the untwisted sector together with the (1,0) states in the twisted sector will form an adjoint representation of a larger group \(G'\) than \(G_0\), which is the full symmetry of the total Hilbert space (\(G'\) is not necessarily the same as \(G\)).

To investigate the enhanced symmetry \(G'\), we rewrite the \(\mathbb{Z}_2\)-orbifold model into an equivalent torus model using the “torus-orbifold equivalence”. The equivalent torus model to the \(\mathbb{Z}_2\)-orbifold model is specified by the following lattice:

\[ \Gamma^{D,D}_G = \{(p'_L, p'_R) \in \bigcup_{\ell=0}^{1} (\Gamma^{D,D}_G + \ell(v_L, 0)) \mid p'_L \cdot v_L - \frac{1}{2} \ell v_L^2 \in \mathbb{Z}\}, \]  

- 14 -
where the shift vectors \((v_L, 0)\) are

\[
v_L = (1, 0, 0, 0, 0, 0, 0, 0) \quad \text{for } G = E_8, \\
v_L = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0 \right) \quad \text{for } G = SO(16), \\
v_L = \left( \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, 0 \right) \quad \text{for } G = [SO(8)]^2, \\
v_L = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \quad \text{for } G = [E_8]^2, \\
v_L = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right) \quad \text{for } G = Spin(32)/Z_2, \\
v_L = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0 \right) \quad \text{for } G = SO(32), \\
v_L = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right) \quad \text{for } G = SO(24) \times SO(8), \\
v_L = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \quad \text{for } G = E_8 \times SO(16), \\
v_L = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right) \quad \text{for } G = [SO(16)]^2, \\
v_L = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \quad \text{for } G = E_8 \times [SO(8)]^2, \\
v_L = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right) \quad \text{for } G = SO(16) \times [SO(8)]^2, \\
v_L = \left( \frac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right) \quad \text{for } G = [SO(8)]^4.
\]

in the orthogonal basis. From \(\Gamma_D^{G,D}\), we will find what is the enhanced symmetry \(G'\). The results are summarized in Table 1.

(2) Examples of Asymmetric \(\mathbb{Z}_3\)-Orbifolds

Next we will discuss the asymmetric \(\mathbb{Z}_3\)-orbifolds. The \(\mathbb{Z}_3\)-transformation is defined by

\[
(X_L^I, X_R^I) \rightarrow (U^{IJ} X_L^J, X_R^I), \quad I = 1, 2, \cdots, D,
\]

where \(U\) is the \(\mathbb{Z}_3\)-rotation matrix, whose diagonalized matrix is expressed by

\[
U_{\text{diag}} = \text{diag}(\omega, \omega^2, \omega^2 \cdots \omega, \omega^2), \quad \omega = e^{2\pi i/3}.
\]

In this case, the modular invariance conditions are

\[
D = 0 \mod 6, \tag{5.9}
\]

\[
3p_R^2 = 0 \mod 2 \quad \text{for all } p_R \in \Gamma_0^*.
\]

\[\]
For our purpose it will be sufficient to consider only the case of $D = 6, 12, 18$, because the conformal weight of the ground state in the twisted sector is $(D/18, 0)$ and a conformal weight $(1, 0)$ state in the twisted sector appears only for $D \leq 18$. For $D = 18$, the ground state in the twisted sector has the conformal weight $(1, 0)$. For $D = 6, 12$, the ground state in the twisted sector has the conformal weight $(6/18, 0)$ and $(12/18, 0)$, respectively, but excited states may have the conformal weight $(1, 0)$. Although we take $\Lambda$ having $\mathbb{Z}_3$-automorphism, the modular invariance conditions restrict the allowed lattice $\Lambda$ or $G$. The possibilities of $G$ are given in Table 2. The $G_0$ is the $\mathbb{Z}_3$-invariant subgroup of $G$, which is the symmetry of each of the untwisted and twisted Hilbert spaces. However, there exist the $(1, 0)$ states in the twisted sector, so that the symmetry is enhanced and the full symmetry of the total Hilbert space is $G'$.

To determine $G'$, we consider the equivalent torus models to the $\mathbb{Z}_3$-orbifold models. The momentum lattices of the equivalent torus models are found as follows:

$$
\Gamma^D, D_G = \{(p'_L, p'_R) \in \bigcup_{\ell=0}^{2} (\Gamma^D, D_G + \ell(v_L, 0)) \mid p'_L \cdot v_L - \frac{1}{2} \ell v^2_L \in \mathbb{Z}\}
$$

and the shift vectors $(v_L, 0)$ are

- $v_L = v_2$, for $G = E_6$, 
- $v_L = (v_3, v_3, v_3)$, for $G = (SU(3))^3$, 
- $v_L = (v_2, v_2)$, for $G = (E_6)^2$, 
- $v_L = (v_2, v_3, v_3, v_3)$, for $G = E_6 \times (SU(3))^3$, 
- $v_L = (v_3, v_3, v_3, v_3, v_3, v_3)$, for $G = (SU(3))^6$, 
- $v_L = (v_1, v_3, v_3)$, for $G = E_8 \times (SU(3))^2$, 
- $v_L = (v_2, v_2, v_2)$, for $G = (E_6)^3$, 
- $v_L = (v_2, v_2, v_3, v_3, v_3)$, for $G = (E_6)^2 \times (SU(3))^3$, 
- $v_L = (v_2, v_3, v_3, v_3, v_3, v_3, v_3)$, for $G = E_6 \times (SU(3))^6$, 
- $v_L = (v_3, v_3, v_3, v_3, v_3, v_3, v_3)$, for $G = (SU(3))^9$, 
- $v_L = (v_1, v_2, v_3, v_3)$, for $G = E_8 \times E_6 \times (SU(3))^2$, 
- $v_L = (v_1, v_3, v_3, v_3, v_3, v_3)$, for $G = E_8 \times (SU(3))^5$, 
- $v_L = (v_1, v_1, v_3)$, for $G = (E_8)^2 \times SU(3)$. 


where
\[ v_1 \equiv \frac{1}{3}(2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 3\alpha_7 + \alpha_8), \]
\[ v_2 \equiv \frac{1}{3}(2\beta_2 + 2\beta_3 + 3\beta_4 + \beta_5 + \beta_6), \]
\[ v_3 \equiv \frac{1}{3}(\gamma_1 + \gamma_2), \]
and \( \alpha_i, \beta_i, \) and \( \gamma_i \) denote simple roots of \( E_8, E_6 \) and \( SU(3) \), respectively:

\[
\begin{align*}
E_8 : & \quad \alpha_2 \\
& \quad \circ \\
& \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\
& \quad \alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8 \\
E_6 : & \quad \beta_2 \\
& \quad \circ \\
& \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\
& \quad \beta_1 \quad \beta_3 \quad \beta_4 \quad \beta_5 \quad \beta_6 \\
SU(3) : & \quad \gamma_1 \quad \gamma_2
\end{align*}
\]

From this we will find what is the enhanced symmetry \( G' \). The results are summarized in Table 2.

(3) Examples of Asymmetric \( \mathbb{Z}_N \)-Orbifolds

In this subsection, we shall discuss asymmetric \( \mathbb{Z}_N \)-orbifolds associated with the root lattice of \( (SU(N))^n \). The root lattice of \( SU(N) \) has the \( \mathbb{Z}_N \)-symmetry, i.e., the cyclic permutation symmetry: \( \alpha_i \rightarrow \alpha_{i+1} \) \( (i = 0, 1, \cdots, N-1) \) where \( \alpha_i \) \( (i = 1, \cdots, N-1) \) is a simple root of \( SU(N) \) and \( \alpha_0 = \alpha_N = -(\alpha_1 + \alpha_2 + \cdots + \alpha_{N-1}). \)

Then, we define the \( \mathbb{Z}_N \)-transformation by

\[
g : \quad (X^I_L, X^I_R) \rightarrow (U^{IJ}X^J_L, X^I_R), \quad (5.12)
\]

where \( U \) is the \( n(N-1) \times n(N-1) \) rotation matrix which generates the above \( \mathbb{Z}_N \)-cyclic permutation and its diagonalized matrix is

\[
U_{diag} = \text{diag}(\omega \ \omega^2 \ \cdots \ \omega^{N-1} \ \omega \ \omega^2 \ \cdots \ \omega^{N-1} \ \cdots \ \omega \ \omega^2 \ \cdots \ \omega^{N-1}), \quad \omega = e^{2\pi i/N}.
\]

\[
(5.13)
\]
The conditions for modular invariance and the existence of (1,0) twisted states put severe restrictions on the allowed values of $N$ and $n$. All the models we have to consider are shown in Table 3. The $G_0$ is the $\mathbb{Z}_N$-invariant subgroup of $G$, which is the symmetry in each of the untwisted and twisted Hilbert spaces. The full symmetry of the total Hilbert space is denoted by $G'$. To determine $G'$, we consider the equivalent torus model to the $\mathbb{Z}_N$-orbifold model. The momentum lattice of the equivalent torus model is found as follows:

$$\Gamma'^{D,D}_{(SU(N))} = \bigcup_{\ell=0}^{N-1} (\Gamma^{D,D}_{(SU(N))} + \ell(v_L,0)) \mid p'_L \cdot v_L - \frac{1}{2} \ell v_L^2 \in \mathbb{Z}, \quad (5.14)$$

where the shift vector $(v_L,0)$ is

$$v_L = (v_L^{(1)}, v_L^{(2)}, \ldots, v_L^{(n)}),$$

$$v_L^{(i)} = \frac{1}{2N} \sum_{j=1}^{N-1} j(N-j) \alpha_j \quad \text{for all } i.$$  

Thus we will find what is the enhanced symmetry $G'$. The results are summarized in Table 3.

6. Conclusion

We have shown various examples of $\mathbb{Z}_N$-asymmetric orbifold models to possess twist-untwist intertwining currents and investigated the symmetries of them. Since every physical string state must be invariant under the $\mathbb{Z}_N$-transformation, the $\mathbb{Z}_N$-invariant subgroup $G_0$ is the “unbroken” symmetry in each of the untwisted and twisted sectors. However, when there exist intertwining currents, the currents convert untwisted states to twisted ones and vice versa. Therefore the symmetry is enhanced to a larger group $G'$ than $G_0$. We have seen that the conditions for the lattice with $\mathbb{Z}_N$-automorphism, modular invariance and the existence of (1,0) twisted states put severe restrictions on such orbifold models.

We have here investigated $\mathbb{Z}_2$- and $\mathbb{Z}_3$-orbifold models associated with the root lattices of simply-laced Lie groups and $\mathbb{Z}_N$-orbifold models with the root lattices of $(SU(N))^n$, where the $\mathbb{Z}_N$-transformation leaves only the origin fixed. The remaining
task is to find all other $\mathbb{Z}_N$-orbifold models which possess intertwining currents. The complete list of such orbifold models will be reported elsewhere.

In this paper, we have discussed the case that rank of $G$ is equal to the dimension of the orbifold, i.e., the case that the $\mathbb{Z}_N$-transformation is an inner automorphism, and we have investigated the symmetry of the orbifold model by rewriting it into an equivalent torus model. In the previous paper [9], we have also found the asymmetric orbifold models which can probably be rewritten into torus models though the orbifolds are defined by outer automorphisms of the momentum lattices.

It will be straightforward to apply our analysis to the heterotic string theory [22]. We hope to get new phenomenologically interesting superstring models along this line.
References


[18] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory (Springer 1972);

    253; B267 (1986) 75.
Table Captions

Table 1. Examples of asymmetric $\mathbb{Z}_2$-orbifold models: $G_0$ denotes the $\mathbb{Z}_2$-invariant subgroup of $G$ which is the symmetry in each sector and $G'$ denotes the full symmetry of the total Hilbert space.

Table 2. Examples of asymmetric $\mathbb{Z}_3$-orbifold models: $G_0$ denotes the $\mathbb{Z}_3$-invariant subgroup of $G$ which is the symmetry in each sector and $G'$ denotes the full symmetry of the total Hilbert space.

Table 3. Examples of asymmetric $\mathbb{Z}_N$-orbifold models: $G_0$ denotes the $\mathbb{Z}_N$-invariant subgroup of $G$ which is the symmetry in each sector and $G'$ denotes the full symmetry of the total Hilbert space.
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<th>$D = 16$</th>
<th>$G'$</th>
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Table 2
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<td>$SU(5)$</td>
</tr>
<tr>
<td>$(SU(5))^2$</td>
<td>$(U(1))^8$</td>
<td>$(SU(5))^2$</td>
</tr>
<tr>
<td>$(SU(5))^3$</td>
<td>$(U(1))^{12}$</td>
<td>$(SU(4))^3 \times (U(1))^3$</td>
</tr>
<tr>
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<td>$(U(1))^{16}$</td>
<td>$(SU(2))^8 \times (U(1))^8$</td>
</tr>
<tr>
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<td>$(U(1))^{20}$</td>
<td>$(SU(2))^2 \times (U(1))^{18}$</td>
</tr>
<tr>
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<td>$(U(1))^6$</td>
<td>$SU(7)$</td>
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<tr>
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<td>$(U(1))^{12}$</td>
<td>$(SU(6))^2 \times (U(1))^2$</td>
</tr>
<tr>
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<td>$(SU(2))^9 \times (U(1))^9$</td>
</tr>
<tr>
<td>$SU(11)$</td>
<td>$(U(1))^{10}$</td>
<td>$SU(11)$</td>
</tr>
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<td>$SU(12) \times U(1)$</td>
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<tr>
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</tr>
<tr>
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<td>$(SU(6))^3 \times (U(1))^3$</td>
</tr>
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<td>$(U(1))^{22}$</td>
<td>$(SU(2))^{11} \times (U(1))^{11}$</td>
</tr>
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</table>

Table 3