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Abstract

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SEISMIC INVESTIGATION OF THE SOLAR INTERIOR

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ABSTRACT. Observation of a large number of modes of solar oscillation has permitted detailed investigation of the solar interior. To illustrate the diagnostic potential of the frequencies, some properties of the observed p -modes are discussed in terms of a simple ray picture of the oscillations. Solar models and their frequencies are used to illustrate how the frequencies depend on the physics of the solar interior. From inverse analyses of the frequencies, one may determine, *e.g.*, the variation of sound speed with position. Solar rotation causes fine structure in the frequencies. By inverting the observations, it is possible to infer the angular velocity as a function of depth and latitude in much of the Sun. Recent observations have given detailed information about frequency changes during the solar cycle. In future, greatly expanded sets of observations will result from networks of observing stations around the Earth and from space. Also, it is hoped that investigations of this nature can be extended to other stars.

1. Introduction

The Sun is observed to oscillate simultaneously in a very large number of individual modes. These oscillations typically have periods between about 15 and 3 minutes; they range in spatial scale from radially symmetric modes to modes whose wavelength on the solar surface is only a few thousand kilometers. The amplitudes for each mode in Doppler velocity is generally well below 1 m sec^{-1} , corresponding to a relative variation in intensity of order 10^{-6} . The frequencies of thousands of modes have been determined, with relative precisions that, in some cases, are better than 10^{-5} . These frequencies provide detailed and accurate information about the properties of the solar interior. *Helioseismology* is concerned with the utilization of these data.

Modes of solar oscillation are characterized by their degree l , azimuthal order m , and radial order n . The Sun can support three essentially different types of oscillations: standing acoustic waves (or p -modes), standing internal gravity waves (or g -modes), and surface gravity waves (or f -modes). At the lowest degrees, the observed periods correspond to high-order p -modes, whereas at moderate or high degrees both p -modes of low and moderate order and f -modes are observed.

Observation of solar oscillations is complicated by their very small amplitudes and by the complexity of the oscillation spectrum. On the other hand, the very long phase stability of the modes means that data can be combined coherently over long timeseries, bringing out the oscillations amongst the incoherent solar "noise." The modes have been observed by means of a variety of techniques, the most extensive sets of data having been obtained

by means of observations of Doppler velocity. From spatially resolved observations, one can achieve considerable separation of the modes by means of suitable spatial transforms. Observations in light integrated over the solar disk, corresponding to observations of stars, are primarily sensitive to modes of very low degree. Problems are introduced by gaps in the timeseries, which give rise to sidebands surrounding the real peaks in the power spectra. These complicate the mode identification and may introduce systematic errors in the frequencies or contribute to the noise. To get long continuous time strings, projects are under way to combine observations from several sites.

For a given solar model, it is relatively straightforward to compute the frequencies numerically. However, to understand the behaviour of the modes and the relation between the structure of the model and the frequencies, simplified asymptotic descriptions are very useful. Through its effect on the structure of the model, the physics used in the computation plays an important role in determining the oscillation frequencies. This can be illustrated by comparing frequencies of models computed with different physics. It is found that there is a substantial sensitivity to the assumed equation of state and opacity. Indeed, one may hope from helioseismic data to obtain information about the properties of matter in the solar interior, although this requires that the effects of the physics can be separated from other uncertainties in the models, including departures from the assumptions underlying "standard" solar model computations.

In a non-rotating star, the frequencies are independent of the azimuthal order m . The rotation of the Sun introduces a splitting of the frequencies according to m , analogous to the Zeeman splitting of the energy levels of an atom in a magnetic field, with different modes sampling the rotation in different parts of the Sun. Frequency splitting, although with a different dependence on m , would also be introduced by a large-scale magnetic field, or by departures from spherical symmetry of the structure of the Sun.

The frequencies can be regarded as integrals over the solar interior structure or angular velocity. In this sense, the observed frequencies provide a set of integral equations for the properties of the solar interior. The solution of integral equations of this nature is usually known as *inverse analysis*. In the case of the observed solar acoustic oscillations, approximate inverse techniques can be based on the asymptotic properties of the modes. These have been used to determine the sound speed in much of the Sun. More general inversions for solar structure that do not rely on the asymptotic approximation have also been performed, leading to fairly accurate inferences about the density and sound speed in most of the Sun, including much of the energy-generating core. In addition, extensive inverse analyses have enabled inferences about the angular velocity in the solar interior.

Dramatic improvements in the helioseismic data may be expected over the next decade. These include the Global Oscillation Network Group (GONG) project to observe solar oscillations from a network of six stations, and helioseismic instruments on the *SOHO* satellite (for Solar and Heliospheric Observatory), to be launched in 1995. Finally, several efforts are under way or being planned to search for solar-like oscillations of other stars. Altogether, the field of seismological investigations of the Sun and other stars promises to be very active and dynamic in the coming years.

2. Properties of Solar Oscillations

To get an impression of the possibilities of helioseismology and evaluate the results, some understanding of the properties of solar oscillations is required. Since the oscillations have small amplitudes, they can be described in terms of linear perturbation theory. Then, for a single mode, the dependence on colatitude θ and longitude ϕ is given by a *spherical harmonic* $Y_l^m(\theta, \phi)$. Here, the *degree* l determines the overall complexity of the perturbation over spherical surfaces. More precisely, the horizontal wavelength λ_h and the magnitude k_h of the horizontal component of the wavenumber at a distance r from the centre of the Sun are

$$k_h = \frac{2\pi}{\lambda_h} = \frac{L}{r} \quad (1)$$

(cf. section 2.2), where $L \equiv \sqrt{l(l+1)}$. The *azimuthal order* m determines the number of nodes around the equator. For a spherically symmetric star, the orientation of the coordinate system used to describe the oscillation can have no effect on the properties of the modes, including their frequencies. Since the definition of m is related to the location of the equator, it follows that frequencies of a spherical star must be independent of m . However, as discussed in section 2.4, the presence of rotation (or any other symmetry-breaking agent) lifts this degeneracy and causes a splitting according to m .

For each l and m , there is a spectrum of modes, distinguished by the dependence of the perturbation on r and characterized by the *radial order* n , which measures the number of zeros in the radial direction in, say, the radial component of the displacement associated with the perturbation.

As discussed in section 2.1, the excitation and damping of the modes is neglected in this Chapter; then, as a function of time t , the modes are purely harmonic functions $\cos \omega t$, where ω is the *angular frequency* of oscillation. For a theoretical description, the frequency is most simply characterized by ω ; on the other hand, observed frequencies are normally given in terms of the *cyclic frequency*

$$\nu = \frac{\omega}{2\pi} = \frac{1}{P}, \quad (2)$$

where P is the oscillation period.

To illustrate the basic properties of the spectrum of possible oscillations, Figure 1 shows computed frequencies for a typical solar model. It is immediately obvious that there are two distinct classes of modes, which are conventionally distinguished by the sign of n . Those with $n > 0$ are normally called *p-modes*; they have frequencies that, for large degree, increase with l roughly as $l^{1/2}$. For modes with $n < 0$, normally called *g-modes*, the frequencies tend to a limit as l increases. The modes with $n = 0$ form an intermediate case, despite the similarity of the behaviour of their frequencies to that of the *p-modes*; these modes are called *f-modes*.

It should be noted that Figure 1 shows the modes that are *possible* in a solar model. What is of interest, from the point of view of helioseismology, are the modes that are actually excited in the Sun. The only modes for which definite data are available are the so-called *five-minute oscillations*, which occupy the range in period between about 15 and about 3 minutes, at all degrees between 0 and an upper limit of $l \simeq 1500$ which is mainly set by

Figure 1. Adiabatic oscillation frequencies for a normal model of the present Sun, as functions of the degree l . For clarity, points corresponding to modes with a given radial order have been connected by straight lines. Only g -modes with radial order less than 40 have been included.

observational constraints. Figure 1 shows that they correspond to p -modes and, at higher degree, to f -modes. In the following, I am concerned only with these modes.

THE ADIABATIC APPROXIMATION

In most of the Sun, the typical thermal timescale τ_{th} is much longer than typical pulsation periods (in particular, τ_{th} for the entire Sun is roughly the Kelvin-Helmholtz time t_{KH} , of order 3×10^7 years; cf. Chapter 2). The only exception is very near the solar surface, where τ_{th} is comparable with the oscillation periods. As a result, it is generally a good approximation to neglect loss or gain of energy during a pulsation period. This corresponds to assuming that the oscillations occur *adiabatically*, so that

$$\frac{\delta p}{p} = \Gamma_1 \frac{\delta \rho}{\rho}, \quad (3)$$

where p is pressure, ρ is density, $\Gamma_1 = (\partial \ln p / \partial \ln \rho)_s$, (the derivative being at constant specific entropy s), and δ indicates the *Lagrangian* perturbation, *i.e.*, the perturbation following the motion.

The adiabatic approximation is evidently highly restrictive. In particular, the motion is conservative, and hence the total energy is constant. Thus, when considering a single mode of oscillation in the adiabatic approximation, the amplitude of the mode is constant and, therefore, it is not possible to treat its excitation or damping. Also, close to the surface and in the atmosphere, the thermal time scale is comparable with or smaller than the periods and there are large departures from adiabaticity. Nonadiabatic effects modify the dynamical properties of the motion, and hence affect the oscillation frequencies; however, by far the largest fraction of the mass involved in the oscillations is in regions where τ_{th} is large and the dynamics is barely affected by nonadiabaticity. Thus, although the nonadiabatic effects on the frequencies are not negligible, they are generally small.

The adiabatic approximation simplifies considerably the discussion of the properties of the oscillations. Thus, in the following, I shall assume that the oscillations are adiabatic, keeping in mind that I thereby introduce errors in the computed frequencies.

Additional complications are introduced by the instability of the outer layers to convection, which gives rise to convective motion. An obvious effect of this is the convective flux, which in most of the convection zone dominates the energy transport; however, convection also gives rise to a “turbulent pressure” which affects the hydrostatic structure of the model and the dynamics of the oscillations. No satisfactory procedure exists for dealing with the convection in a static star, let alone in the presence of pulsations. Balmforth and Gough (1990a) and Balmforth (1992) used a time-dependent generalization of mixing-length theory to treat the effect of convection on the oscillations, whereas Stein, Nordlund, and Kuhn (1988) and Stein and Nordlund (1991) studied such effects by means of detailed hydrodynamic models of convection; these calculations showed that convection may introduce significant, and still somewhat uncertain, shifts in the frequencies. Here, I shall ignore effects of convection; the resulting errors in the properties of the oscillations, in particular the frequencies, must be kept in mind when computed frequencies are compared with the observations.

2.2. BASIC OSCILLATION EQUATIONS

A complete derivation of the equations of linear stellar oscillation is beyond the scope of this Chapter; however, for the subsequent discussion, it is convenient to include a brief summary. (More detailed discussions are given by, for example, Unno *et al.* 1989 and Christensen-Dalsgaard and Berthomieu 1991). The derivation starts from the general hydrodynamical equations of motion and continuity and Poisson’s equation; viscosity is neglected. The equations are linearized; this is done most conveniently in terms of the *Eulerian* perturbations, *i.e.*, the perturbations at a given point in space, which are denoted by a prime. Also, the adiabatic relation, Equation (3), is applied. Because of the spherical symmetry and time independence of the equilibrium model, one can separate out the dependence of the perturbation quantities on co-latitude θ , longitude ϕ , and time, using

spherical polar coordinates (r, θ, ϕ) . The relevant functions of θ and ϕ are eigenfunctions of the tangential Laplace operator

$$r^2 \nabla_t^2 \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

These may be chosen as spherical harmonics

$$Y_l^m(\theta, \phi) = (-1)^m c_{lm} P_l^m(\cos \theta) e^{im\phi}, \quad (4)$$

where P_l^m is a Legendre function, and the normalization constant c_{lm} is determined such that the integral of $|Y_l^m|^2$ over the unit sphere is unity. Similarly, the time dependence can be expressed in terms of a harmonic function. Thus, *e.g.*, the pressure perturbation is written as

$$p'(r, \theta, \phi, t) = \sqrt{4\pi} Re [p'(r) Y_l^m(\theta, \phi) e^{-i\omega t}] , \quad (5)$$

where, for simplicity, I use p' to denote both the full perturbation and the amplitude function. Also, it follows from the momentum equation that the displacement vector can be written as

$$\delta \mathbf{r}(r, \theta, \phi, t) = \sqrt{4\pi} Re \left\{ \left[\xi_r(r) Y_l^m \mathbf{a}_r + \xi_h(r) \left(\frac{\partial Y_l^m}{\partial \theta} \mathbf{a}_\theta + \frac{1}{\sin \theta} \frac{\partial Y_l^m}{\partial \phi} \mathbf{a}_\phi \right) \right] e^{-i\omega t} \right\} , \quad (6)$$

where $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ are unit vectors in the r , θ , and ϕ directions. It may be noted that, with this separation of variables, the effect of ∇_t^2 on any scalar variable corresponds to multiplication by $-k_h^2$, where k_h is given by Equation (1). The root mean squares, over a spherical surface at radius r and time, of the vertical and horizontal components of the displacement are given by

$$\langle \delta r \rangle_{rms} = \frac{1}{\sqrt{2}} \xi_r(r), \quad \langle \delta h \rangle_{rms} = \frac{L}{\sqrt{2}} \xi_h(r). \quad (7)$$

The amplitude ξ_h of the horizontal component of displacement is related to p' and the perturbation Φ' in the gravitational potential by

$$-\omega^2 \rho \xi_h = -\frac{1}{r} (p' - \rho \Phi'). \quad (8)$$

The equations may now be arranged into the following fourth-order set of ordinary differential equations for the amplitude functions $(\xi_r(r), \xi_h(r), \Phi'(r))$:

$$\frac{d\xi_r}{dr} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1} \frac{d \ln p}{dr}\right) \xi_r + \frac{r\omega^2}{c^2} \left(\frac{S_l^2}{\omega^2} - 1\right) \xi_h - \frac{1}{c^2} \Phi' \quad (9)$$

$$\frac{d\xi_h}{dr} = \frac{1}{r} \left(1 - \frac{N^2}{\omega^2}\right) \xi_r + \left(\frac{1}{\Gamma_1} \frac{d \ln p}{dr} - \frac{d \ln \rho}{dr} - \frac{1}{r}\right) \xi_h + \frac{N^2}{rg\omega^2} \Phi' \quad (10)$$

$$\frac{d^2 \Phi'}{dr^2} = -\frac{2}{r} \frac{d\Phi'}{dr} - 4\pi G \rho \left(\frac{N^2}{g} \xi_r + \frac{r\omega^2}{c^2} \xi_h\right) + \left(\frac{L^2}{r^2} - \frac{4\pi G \rho}{c^2}\right) \Phi'. \quad (11)$$

Here c is the adiabatic sound speed,

$$c^2 = \frac{\Gamma_1 p}{\rho} \simeq \frac{\Gamma_1 k_B T}{\mu m_u}, \quad (12)$$

where the last approximation is valid for an ideal gas; here, T is temperature, k_B is Boltzmann's constant, m_u is the atomic mass unit, and μ is the mean molecular weight. Also, I have introduced the Lamb frequency

$$S_l = \frac{Lc}{r}, \quad (13)$$

and the buoyancy frequency N defined by

$$N^2 = g \left(\frac{1}{\Gamma_1} \frac{d \ln p}{dr} - \frac{d \ln \rho}{dr} \right), \quad (14)$$

where g is the gravitational acceleration.

The amplitude functions must satisfy boundary conditions at the centre and surface of the Sun. The centre is a regular singular point of Equations (9) – (11). The behaviour of the solution near $r = 0$ can be analyzed by expansion in r , using the expansion of the equilibrium quantities; in this way, boundary conditions that isolate the regular solutions can be established. At the surface, one boundary condition is obtained by requiring that Φ' and its first derivative match continuously onto the decreasing solution to Laplace's equation outside the Sun. The dynamical surface boundary condition should, in principle, take into account the detailed behaviour of the oscillation in the solar atmosphere. In the idealized case, where the model is assumed to have a free outer surface, the absence of forces on the surface corresponds to demanding that the pressure perturbation vanish on the perturbed surface, *i.e.*,

$$\delta p = p' + \frac{dp}{dr} \xi_r = 0, \quad \text{at } r = R. \quad (15)$$

In practice, a somewhat more realistic condition is applied at a suitable point within the solar atmosphere (*e.g.* Unno *et al.* 1989). This condition is qualitatively similar to Equation (15), particularly at fairly low frequencies; furthermore, the solution is relatively insensitive to the details of the boundary condition, as long as it is applied sufficiently high in the atmosphere (see also Ulrich and Rhodes 1983; Noels, Scuflaire, and Gabriel 1984).

By using Equation (8) and the equation of hydrostatic equilibrium, Equation (15) can also be written as

$$\xi_h = \frac{GM}{R^3 \omega^2} \xi_r - \frac{1}{R \omega^2} \Phi', \quad (16)$$

where M is the total mass of the Sun. As argued below, Φ' is often small. Hence, Equations (16) and (7) show that the ratio between the root-mean-squares of the horizontal and vertical components of the surface velocity is given approximately by

$$\frac{\langle V_h \rangle_{rms}}{\langle V_r \rangle_{rms}} \simeq \frac{GM}{R^3 \omega^2} L, \quad (17)$$

and hence is a function of the frequency and the degree, but not of the detailed nature of the mode. Also, it is evident that for high-frequency modes of low degree the surface velocity is predominantly in the radial direction.

It should be noted that Equations (9) – (11), as well as the boundary conditions, are independent of the azimuthal order m ; it follows that the frequency and the eigenfunctions do not depend on m . This is a consequence of the assumed spherical symmetry: m is related to the choice of polar axis in the coordinate system used to describe the oscillations, and that choice can have no physical effects in a spherically symmetric system. As discussed in section 2.4, rotation introduces a preferred axis, and hence a dependence of the frequencies on m .

The differential equations and boundary conditions constitute a boundary value problem. This is in principle straightforward to solve numerically; in practice, the high radial order of the observed modes means that some care is required in the calculations to match the precision of the observed frequencies. A summary of computational techniques and remarks about their precision was given by Christensen-Dalsgaard and Berthomieu (1991).

When using observed frequencies to probe the structure of the solar interior, it is crucially important how the frequencies depend on the properties of the solar model. In fact, the coefficients in Equations (9) – (11) are essentially completely determined if density ρ and the adiabatic exponent Γ_1 are given as functions of r , assuming that the model is in hydrostatic equilibrium. Given ρ , the interior mass $m(r)$ is obtained by integration, with the obvious boundary condition $m(0) = 0$. Then $p(r)$ can be obtained from Equation (2.1) of Chapter 2 by integrating from the surface; this requires an assumption about the surface pressure which, for example, can be obtained from semi-empirical models of the solar atmosphere. Given $p(r)$, $\rho(r)$, and $\Gamma_1(r)$, the coefficients can be evaluated. Instead of ρ and Γ_1 , one may choose any other pair of variables from which ρ and Γ_1 can be derived; given that acoustic modes are predominantly influenced by the sound speed, a natural choice is to use $c(r)$ and $\Gamma_1(r)$.

2.3. ASYMPTOTIC THEORY OF p -MODES

Numerical solution of the equations of adiabatic oscillations is straightforward. However, the understanding of the numerical results and the interpretation of the observed oscillations have been greatly assisted by asymptotic analyses of the oscillation equations. The usefulness of asymptotics is, to a large extent, due to the fact that the observed acoustic modes in the five-minute region have high radial order or high degree. Hence, asymptotics can provide relations of acceptable accuracy between the properties of the Sun and the properties of the oscillation frequencies.

The asymptotic analysis is simplified considerably by noting that at high degree or radial order the perturbation Φ' in the gravitational acceleration can be neglected. This approximation, known as the *Cowling approximation*, was first suggested by Cowling (1941). It may be justified by noting that at high degree or order a mode gives rise to many regions of alternating sign in the density perturbation, the effects of which largely cancel in the perturbation of the gravitational potential.

When Φ' is neglected, the system (9) – (11) of equations is reduced to a second-order system, which is amenable to JWKB analysis (for Jeffreys, Wentzel, Kramers, Brillouin; see, for example, Gough 1986a, Unno *et al.* 1989). However, the asymptotic behaviour of p -modes can, in fact, be understood very simply in terms of the propagation of sound waves (*e.g.*, Christensen-Dalsgaard *et al.* 1985). The wavelength of high-order modes is small

compared with the typical scale over which the equilibrium structure changes; furthermore, the frequencies of the modes are high compared with N and, therefore, effects of buoyancy can be neglected. Hence, the modes can be approximated locally by plane sound waves with the dispersion relation

$$k^2 \equiv k_r^2 + k_h^2 = \frac{\omega^2}{c^2}, \quad (18)$$

where k_r and k_h are the radial and horizontal components of the wave vector. For a wave corresponding to a mode of oscillation, k_h is given by Equation (1). From Equation (18), one then obtains

$$k_r^2 = \frac{\omega^2}{c^2} - \frac{L^2}{r^2}. \quad (19)$$

Close to the surface, c is small, and hence k_r is large. Here, the wave propagates almost vertically. With increasing depth, c increases and k_r decreases (see Figure 2) until $k_r = 0$ and the wave propagates horizontally. This happens at the so-called *lower turning point* $r = r_t$ where

$$\frac{r_t}{c(r_t)} = \frac{L}{\omega}. \quad (20)$$

Thus, the location of the lower turning point is a function of ω/L alone. The reflection of the wave at the surface is not immediately contained in this simple description; however, it is at least plausible that it results from the steep density gradient. A more detailed asymptotic analysis shows that the wave is reflected when its frequency is less than the acoustical cutoff frequency, ω_{ac} which, for the approximately isothermal solar atmosphere, is given by

$$\omega_{ac} = \frac{c}{2H_p} \quad (21)$$

(Lamb 1909), where H_p is the pressure scale height (in the solar atmosphere, $\omega_{ac}/2\pi$ is around $5300 \mu\text{Hz}$). Thus, at frequencies below ω_{ac} the wave propagates in a series of “bounces” between the surface and the turning point (*cf.* Figure 2). A mode of oscillation is formed as the interference pattern between such bouncing waves.

A more direct derivation of this behaviour, which avoids explicit separation of the spherical harmonics, can be carried out on the basis of ray theory (Gough 1984, 1986a).

The behaviour of r_t determines the regions over which the modes propagate and which, therefore, affect their frequencies. Figure 3 shows how r_t varies with l and the frequency for a normal solar model. Low-degree modes extend almost to the centre of the Sun, whereas modes at the highest degrees observed are confined to the outer fraction of a per cent of the solar radius.

This description of the p -modes also yields the asymptotic dispersion relation for their frequencies. To get a standing wave, the change in radial phase between the lower turning point and the surface must be an integral multiple of π , apart from contributions coming from the turning points. It follows from Equation (19) that this condition may be expressed as

$$\omega \int_{r_t}^R \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{1/2} \frac{dr}{c} \simeq \pi(n + \alpha). \quad (22)$$

Here, α describes the phase change at the turning points; a more careful analysis shows that $\alpha = \alpha(\omega)$ is, in general, a function of frequency.

Figure 2. Schematic illustration of the propagation of sound waves in a star. Due to the increase of the sound speed with depth, the deeper parts of the wave fronts move faster. This causes the refraction of the wave described by Equation (19). Notice that waves with a smaller wavelength, corresponding to a higher value of the degree l , penetrate less deeply.

Figure 3. The turning point radius r_t (a) and the penetration depth $R - r_t$ (b), in units of the solar radius R , as a function of degree l for three values of the frequency ν . The curves have been calculated from Equation (20) for a normal model of the present Sun.

As r_t is a function of ω/L , Equation (22) may be written as

$$\frac{\pi[n + \alpha(\omega)]}{\omega} = F\left(\frac{\omega}{L}\right), \quad (23)$$

where

$$F(w) = \int_{r_t}^R \left[1 - \left(\frac{c}{rw} \right)^2 \right]^{1/2} \frac{dr}{c}. \quad (24)$$

A relation like Equation (23) was first found by Duvall (1982) for observed frequencies.

It is of interest to use Equations (23) and (24) to estimate the effects on the frequencies of *changes* in the equilibrium model. If δc denotes the difference in c between two solar models at fixed r , and $\delta\alpha$ denotes the difference in α at fixed frequency, it follows (by linearization), assuming δc and $\delta\alpha$ to be small, that

$$S \frac{\delta\omega}{\omega} \simeq \int_{r_t}^R \left(1 - \frac{L^2 c^2}{r^2 \omega^2}\right)^{-1/2} \frac{\delta c}{c} \frac{dr}{c} + \pi \frac{\delta\alpha}{\omega}, \quad (25)$$

where

$$S = \int_{r_t}^R \left(1 - \frac{L^2 c^2}{r^2 \omega^2}\right)^{-1/2} \frac{dr}{c} - \pi \frac{d\alpha}{d\omega} \quad (26)$$

(Christensen-Dalsgaard, Gough, and Pérez Hernández 1988). Since α is assumed to be a function of ω alone, and r_t is determined by ω/L , the scaled frequency difference $S \delta\omega/\omega$ is predicted to be of the form

$$S \frac{\delta\omega}{\omega} \simeq H_1 \left(\frac{\omega}{L}\right) + H_2(\omega), \quad (27)$$

where the two functions H_1 and H_2 are determined by Equation (25). Furthermore, the last term on the right-hand side of Equation (26) is, in general, relatively small compared with the first, and hence S is approximately a function of ω/L . Equation (27) evidently describes a very special dependence of $\delta\omega$ on ω and L .

Since c/r decreases quite rapidly with increasing r , $(Lc/r\omega)^2 \ll 1$ except near the turning point r_t ; hence, as a rough approximation, $1 - L^2 c^2/r^2 \omega^2$ may be replaced by 1 in the integrals in Equations (25) and (26). If, furthermore, the term in $\delta\alpha$ can be neglected, the result is the very simple relation between the changes in sound speed and frequency:

$$\frac{\delta\omega}{\omega} \simeq \frac{\int_{r_t}^R \frac{\delta c}{c} \frac{dr}{c}}{\int_{r_t}^R \frac{dr}{c}}. \quad (28)$$

This shows that the change in sound speed in a region of the Sun affects the frequency with a weight determined by the time the mode, regarded as a superposition of traveling waves, spends in that region. Thus changes near the surface, where the sound speed is low, have relatively large effects on the frequencies.

As discussed in section 5.3.1, the asymptotic relations (22) and (25) permit an almost direct, and surprisingly precise, inversion for the solar sound speed or the sound-speed difference between the Sun and a model.

When l is small, $(Lc/\omega r)^2 \ll 1$ except close to the centre of the Sun. Also, it is found that $L = \sqrt{l(l+1)}$ should be replaced by $l + 1/2$ (e.g. Brodsky and Vorontsov 1987). By expanding Equation (22) to take out the dependence on l , one then obtains

$$\nu_{nl} = \frac{\omega_{nl}}{2\pi} \simeq \left(n + \frac{l}{2} + \frac{1}{4} + \alpha\right) \Delta\nu, \quad (29)$$

where

$$\Delta\nu = \left[2 \int_0^R \frac{dr}{c}\right]^{-1} \quad (30)$$

is the inverse of twice the sound travel time between the centre and the surface (see also Vandakurov 1967, Tassoul 1980). Thus, to this asymptotic order, there is a uniform spacing $\Delta\nu$ between modes of same degree but different order. In addition, Equation (29) predicts the approximate equality

$$\nu_{nl} \simeq \nu_{n-1, l+2} . \quad (31)$$

This frequency pattern has been observed for the solar five-minute modes of low degree (*cf.* section 3.3) and may be used in the search for stellar oscillations of solar type.

The *deviations* from the simple relation (29) have considerable diagnostic potential. By taking the expansion of Equation (22) to the next order (Gough 1986a), or from direct JWKB analysis of the oscillation equations (Tassoul 1980), one obtains

$$\delta\nu_{nl} \equiv \nu_{nl} - \nu_{n-1, l+2} \simeq -(4l+6) \frac{\Delta\nu}{4\pi^2\nu_{nl}} \int_0^R \frac{dc}{dr} \frac{dr}{r} . \quad (32)$$

Thus $\delta\nu_{nl}$ is determined predominantly by conditions in the solar core. This may be understood physically by noting that only near the centre is k_h comparable with k_r . Elsewhere, the wave vector is almost vertical and the dynamics of the oscillations are largely independent of their horizontal structure, *i.e.*, of l ; therefore, at given frequency the contributions of these layers to the frequency are nearly the same, and hence almost cancel in the difference $\delta\nu_{nl}$.

It should be noted that the accuracy of Equation (32) is questionable: the derivation ignores effects of the perturbation in the gravitational potential, which are not negligible for low-degree modes; in addition, the asymptotic description assumes that the equilibrium model varies on a scale that is large compared with the wavelength of the modes, and this condition is not satisfied in the core of the model. Largely fortuitously, these two errors almost cancel in models of the present Sun where, consequently, Equation (32) agrees with computed frequencies; it is less successful for models of different ages or masses (Christensen-Dalsgaard 1988, 1992a). However, the general form of the dependence of $\langle\delta\nu_{nl}\rangle$ on l shown in Equation (32), as well as the argument that this quantity is most sensitive to conditions in stellar cores, has a broader range of validity.

In comparisons between computed and observed values of $\delta\nu_{nl}$, it is convenient to use a parametrization in terms of a small number of parameters. Equation (32) suggests that one consider the scaled separation

$$d_{nl} = \frac{3}{2l+3} \delta\nu_{nl} , \quad (33)$$

thus effectively reducing the separation to the value for $l=0$; the dependence of d_{nl} on l can then be used as an indication of departures from the asymptotic behaviour. Elsworth *et al.* (1990a) proposed that d_{nl} be analyzed in terms of a linear fit of the form

$$d_{nl} = \bar{d}_l + s_l(n - n_1) , \quad (34)$$

where n_1 is a suitable reference value of the order n . Examples of such fits to observed and computed frequencies are discussed in section 4.2.

Asymptotic analysis also shows the existence of modes whose frequencies are given by

$$\omega^2 = \frac{g_s L}{R} , \quad (35)$$

where g_s is the surface gravity. For these modes, $\text{div} \delta \mathbf{r}$ vanishes and ξ_r decreases with increasing depth as $\exp[-k_h(R-r)]$; they are entirely equivalent to surface gravity waves at a free surface (for example on a Scottish loch). They correspond to the f -modes at moderate or high degree in Figure 1.

2.4. ROTATIONAL SPLITTING

I assume that rotation is sufficiently slow to allow neglect of the rotational distortion of the star or, more generally, of terms of order Ω^2 , where Ω is the angular velocity. Then, the structure of the star is unchanged and the effects of rotation can be treated as a small perturbation in the equations describing the oscillations.

That rotation causes a splitting of the frequencies can be seen from a purely geometrical argument. Assume the angular velocity Ω to be uniform, and consider an oscillation with a frequency ω_0 , independent of m , in the frame rotating with the star. I introduce a coordinate system in this frame, with coordinates (r', θ', ϕ') which are related to the coordinates (r, θ, ϕ) in an inertial frame through

$$(r', \theta', \phi') = (r, \theta, \phi - \Omega t) . \quad (36)$$

It follows from Equation (6) that, in the rotating frame, the perturbations depend on ϕ' and t as $\cos(m\phi' - \omega_0 t)$; hence, the dependence in the inertial frame is $\cos(m\phi - \omega_m t)$, where

$$\omega_m = \omega_0 + m\Omega . \quad (37)$$

Thus an observer in the inertial frame finds that the frequency is split uniformly according to m .

This description is obviously incomplete. Even in the case of uniform rotation, the effects of the Coriolis force must be taken into account in the rotating frame, causing a contribution to the frequency splitting (Cowling and Newing 1949; Ledoux 1949). Furthermore, in general the angular velocity is a function $\Omega(r, \theta)$ of position. Nevertheless, as shown below, the effect of the Coriolis force is often small and Equation (37) is approximately correct if Ω is replaced by a suitable average of the position-dependent angular velocity.

The general case was considered by Hansen, Cox, and van Horn (1977), Cuypers (1980), and Gough (1981). The result is that the *rotational splitting*, i.e., the perturbation in the frequencies caused by rotation, can be written as

$$\omega_{nlm} - \omega_{nl0} = \delta\omega_{nlm} = m \frac{R_{nlm}}{I_{nlm}} , \quad (38)$$

where

$$R_{nlm} = \int_0^\pi \sin \theta d\theta \int_0^R \left\{ \xi_r^2 P_l^m(\cos \theta)^2 + \xi_h^2 \left[\left(\frac{dP_l^m}{d\theta} \right)^2 + \frac{m^2}{\sin^2 \theta} P_l^m(\cos \theta)^2 \right] \right. \\ \left. - 2P_l^m(\cos \theta)^2 \xi_r \xi_h - 2P_l^m(\cos \theta) \frac{dP_l^m}{d\theta} \frac{\cos \theta}{\sin \theta} \xi_h^2 \right\} \Omega(r, \theta) \rho(r) r^2 dr , \quad (39)$$

and

$$I_{nlm} = \frac{2}{2l+1} \frac{(l+|m|)!}{(l-|m|)!} \int_0^R [\xi_r^2 + l(l+1)\xi_h^2] \rho(r) r^2 dr . \quad (40)$$

It should be noted from Equations (38) – (40) and the properties of the P_l^m that $\delta\omega_{nlm}$ is an odd function of m , $\delta\omega_{nl-m} = -\delta\omega_{nlm}$; this is a consequence of the fact that rotation imparts orientation, and hence distinguishes between modes travelling East and West. In this, rotation differs from other effects, such as asphericity of solar structure or magnetic fields, that might cause frequency splitting according to m ; such splittings are even functions of m .

2.4.1. Splitting for spherically symmetric rotation. To proceed, an explicit assumption about the variation of Ω with θ is required. For simplicity, I shall assume first that Ω is independent of θ . In this case, the integrals over θ in Equation (39) only involve Legendre functions and may be evaluated analytically. The result is

$$\delta\omega_{nlm} = m \frac{\int_0^R \Omega(r) [\xi_r^2 + (L^2 - 1)\xi_h^2 - 2\xi_r\xi_h] r^2 \rho dr}{\int_0^R (\xi_r^2 + L^2\xi_h^2) r^2 \rho dr} . \quad (41)$$

It should be noticed that the integrands in Equation (41) are given solely in terms of ξ_r , ξ_h , and l , and hence are independent of m . Therefore, in the case of spherically symmetric rotation the splitting is proportional to m .

It is convenient to write Equation (41) as

$$\delta\omega_{nlm} = m\beta_{nl} \int_0^R K_{nl}(r) \Omega(r) dr , \quad (42)$$

where

$$K_{nl} = \frac{[\xi_r^2 + (L^2 - 1)\xi_h^2 - 2\xi_r\xi_h] r^2 \rho}{\int_0^R [\xi_r^2 + (L^2 - 1)\xi_h^2 - 2\xi_r\xi_h] r^2 \rho dr} , \quad (43)$$

and

$$\beta_{nl} = \frac{\int_0^R [\xi_r^2 + (L^2 - 1)\xi_h^2 - 2\xi_r\xi_h] r^2 \rho dr}{\int_0^R (\xi_r^2 + L^2\xi_h^2) r^2 \rho dr} . \quad (44)$$

This definition ensures that the *rotational kernel* K_{nl} is unimodular, i.e.,

$$\int_0^R K_{nl}(r) dr = 1 . \quad (45)$$

Hence for uniform rotation, where $\Omega = \Omega_s$ is constant,

$$\delta\omega_{nlm} = m\beta_{nl}\Omega_s . \quad (46)$$

Comparison with Equation (37) shows that, in this case, the effects of the detailed dynamics of the oscillations are contained in β_{nl} . For high-order or high-degree p -modes, the terms in ξ_r^2 and $L^2\xi_h^2$ dominate and $\beta_{nl} \simeq 1$. Thus, the rotational splitting between adjacent m -values is given simply by the angular velocity, as in Equation (37). Physically, the neglected terms in Equation (44) do indeed arise from the Coriolis force.

When Ω depends on r , the integral in Equation (42) provides a weighted average

$$\langle \Omega \rangle = \int_0^R K_{nl}(r) \Omega(r) dr \quad (47)$$

of $\Omega(r)$. For high-order p -modes, one obtains from the asymptotic behaviour of the eigenfunctions that

$$\delta\omega_{nlm} \simeq m\langle \Omega \rangle \simeq \frac{\int_{r_t}^R \left(1 - \frac{L^2 c^2}{r^2 \omega^2}\right)^{-1/2} \Omega(r) \frac{dr}{c}}{\int_{r_t}^R \left(1 - \frac{L^2 c^2}{r^2 \omega^2}\right)^{-1/2} \frac{dr}{c}} \simeq m \frac{\int_{r_t}^R \Omega(r) \frac{dr}{c}}{\int_{r_t}^R \frac{dr}{c}}, \quad (48)$$

where in the last equality I crudely approximated $(1 - L^2 c^2 / r^2 \omega^2)$ by 1. Thus, in this case one obtains the intuitively appealing result that the splitting between adjacent values of m is an average of the angular velocity, weighted by the sound-travel time. It should be noted that Equation (48) is entirely analogous to Equations (25), (26), and (28) for the frequency change arising from a perturbation to the sound speed.

Figure 4 . Kernels K_{nl} for the frequency splitting in p -modes caused by spherically symmetric rotation (*cf.* Equation (43)). In a) is plotted $R K_{nl}(r)$ for a mode with $l = 1$, $n = 22$ and $\nu = 3233 \mu\text{Hz}$. The maximum value of $R K_{nl}(r)$ is 57. In b) is shown the same mode, on an expanded vertical scale, (—) together with the modes $l = 20$, $n = 17$, $\nu = 3367 \mu\text{Hz}$ (-----), and $l = 60$, $n = 10$, $\nu = 3231 \mu\text{Hz}$ (.....). Notice that the kernels almost vanish inside the turning point radius r_t , and that there is an accumulation just outside the turning point.

Figure 4 shows a few examples of kernels for high-order p -modes. They are clearly large near the solar surface, as is also implicit in Equation (48). Beneath the turning point, the kernels get very small but they are enhanced locally just above it. This effect arises from the term in ξ_h in Equation (43); physically, it corresponds to the fact that the waves travel approximately horizontally in this region, and hence spend a relatively long time there.

2.4.2. General rotation laws. In the general case, where Ω depends on both r and θ , the rotational splitting may be computed from Equations (38) – (40) by evaluating the two-dimensional integral in Equation (39). This integral is, in general, m -dependent and the splitting is no longer a linear function of m . To illustrate the properties of the splitting, it is convenient to rewrite Equation (39) for R_{nlm} using integration by parts:

$$R_{nlm} = \int_0^\pi \sin \theta d\theta \int_0^R P_l^m(\cos \theta)^2 \{ [\xi_r^2 + (L^2 - 1)\xi_h^2 - 2\xi_r\xi_h] \sin \theta \Omega(r, \theta) + \xi_h^2 \left(\frac{3}{2} \cos \theta \frac{\partial \Omega}{\partial \theta} + \frac{1}{2} \sin \theta \frac{\partial^2 \Omega}{\partial \theta^2} \right) \} d\theta \quad (49)$$

(Cuypers 1980). I consider again the case of high-order p -modes; here the terms in ξ_r^2 and $L^2\xi_h^2$ dominate, and consequently

$$\delta\omega_{nlm} \simeq m \frac{\int_0^\pi \sin \theta [P_l^m(\cos \theta)]^2 \int_{r_t}^R \Omega(r, \theta) [\xi_r(r)^2 + L^2\xi_h(r)^2] dr d\theta}{\int_0^\pi \sin \theta [P_l^m(\cos \theta)]^2 d\theta \int_{r_t}^R [\xi_r(r)^2 + L^2\xi_h(r)^2] dr}. \quad (50)$$

Hence, the splitting is simply an average of the angular velocity $\Omega(r, \theta)$, weighted by $[\xi_r(r)^2 + L^2\xi_h(r)^2]P_l^m(\cos \theta)^2$. Approximating the eigenfunction as in the derivation of Equation (48) and using, furthermore, an asymptotic approximation to P_l^m , this may be written as

$$\delta\omega_{nlm} \simeq m \frac{\int_{-\cos \Theta}^{\cos \Theta} (\cos^2 \Theta - \cos^2 \theta)^{-1/2} \int_{r_t}^R \left(1 - \frac{L^2 c^2}{r^2 \omega^2} \right)^{-1/2} \Omega(r, \theta) \frac{dr}{c} d(\cos \theta)}{\frac{1}{\pi} \int_{r_t}^R \left(1 - \frac{L^2 c^2}{r^2 \omega^2} \right)^{-1/2} \frac{dr}{c}}, \quad (51)$$

where $\Theta = \sin^{-1}(m/L)$ (Gough and Thompson 1990, 1991; Gough 1991). It should be noted that a given spherical harmonic is confined essentially to the latitude band between $\pm\Theta$. The variation of the extent of the P_l^m with m/L allows resolution of the latitudinal variation of the angular velocity, in much the same way as the variation of the depth of penetration with ω/L allows resolution of the variation with radius. In particular, with increasing l the sectoral modes (with $l = |m|$) get increasingly confined towards the equator. Thus, the rotational splitting of sectoral modes provides a measure of the solar equatorial angular velocity.

To study the splitting without making the asymptotic approximation, it is convenient to consider a parameterized representation of $\Omega(r, \theta)$, of the form

$$\Omega(r, \theta) = \sum_{s=0}^{s_{\max}} \Omega_s(r) \psi_s(\theta), \quad (52)$$

where the ψ_s are suitable expansion functions (*e.g.* Brown *et al.* 1989, Ritzwoller and Lavelly 1991). Then, the integrals over θ can be evaluated and the rotational splitting can be written as

$$\delta\omega_{nlm} = m \sum_{s=0}^{s_{\max}} \int_0^R K_{nlms}(r) \Omega_s(r) dr, \quad (53)$$

where the kernels K_{nlms} obviously depend on the expansion functions. In the common case where the ψ_s are polynomials in $\cos^2 \theta$, the splitting is a polynomial in odd powers of m . As discussed in section 3.3, it is conventional from the point of view of data analysis to write this in terms of Legendre polynomials:

$$\delta\omega_{nlm} = L \sum_{j=0}^{s_{\max}} \Delta^j \omega_{nl} P_{2j+1} \left(\frac{m}{L} \right). \quad (54)$$

Here, the expansion coefficients $\Delta^j \omega_{nl}$ of the splitting are related to the expansion coefficients Ω_s of the angular velocity through expressions of the form

$$\Delta^j \omega_{nl} = \sum_{s=j}^{s_{\max}} \int_0^R K_{nljs}^j(r) \Omega_s(r) dr. \quad (55)$$

3. Observation of Solar Oscillations

The detectability of solar oscillations with velocity amplitudes less than 1 cm sec^{-1} or relative irradiance amplitudes below 10^{-6} may *a priori* seem incredible. One might have expected that such small effects would have been completely masked by the far larger fluctuations in velocity and intensity which occur on the solar surface as a result of granulation and the magnetic activity. That the oscillations can nevertheless be seen is a result of the fact that they maintain phase over long periods of time, unlike the other fluctuations. Observations that average over N independent random fluctuations, each with an *rms* amplitude of A , produce a signal of $N^{-1/2} A$. Hence, the spatial average over 10^6 granules, each with an *rms* velocity of order 10^5 cm sec^{-1} , results in a velocity noise of order 10^2 cm sec^{-1} per measurement. When, in addition, data over several weeks are combined, the solar “noise” is reduced even further, to a level of a few mm sec^{-1} per frequency bin (*cf.* Jiménez *et al.* 1988).

Needless to say, to utilize this potentially very low noise level, the observing techniques must be such that the least possible noise is introduced in the data by instrumental effects, or effects in the Earth’s atmosphere. It is certainly a triumph of observational ingenuity that it has been possible, in some cases, to achieve data that are largely limited by the solar noise. A detailed summary of the observing techniques that have been utilized to observe solar oscillations is beyond the scope of this Chapter. Instead, I sketch the principles in the observations and discuss some of the problems that are encountered in the analysis of the data. More extensive treatments of observations and data analysis can be found, for example, in Brown (1988).

3.1. SOME OBSERVING TECHNIQUES

The oscillations affect a number of properties of the solar atmosphere, such as its velocity and the continuum intensity and line spectrum that it emits. Hence, there is a corresponding range of potential techniques for observing the oscillations. The choice of techniques depends on noise properties, particularly the ratio between the solar oscillation signal and the solar noise and the effect of the Earth's atmosphere, and on technical convenience. It should be noted also that, in general, the oscillation amplitude increases more rapidly with altitude in the solar atmosphere as the frequency increases. Hence, techniques that probe the upper parts of the solar atmosphere are relatively more sensitive to high-frequency modes.

3.1.1. Doppler observations. Broadly speaking, it appears that the ratio of solar signal to noise is highest for observations of the Doppler velocity. This technique has, in fact, been used for the most extensive studies of solar oscillations. The principle is illustrated schematically in Figure 5: intensities I_r and I_b are measured in narrow passbands on the red and the blue sides of a spectral line; if the line shifts towards the red, I_r decreases and I_b increases. Hence the ratio

$$\frac{I_r - I_b}{I_r + I_b} \quad (56)$$

is a measure of the line shift, and hence of the Doppler velocity (the normalization with the sum of intensities eliminates effects of intensity variations).

The different Doppler techniques are distinguished by the ways in which the passbands are determined and the intensities are measured. Figure 5 refers specifically to the *resonant scattering technique*, where the passbands are defined by scattering off the Zeeman-split components of a K or Na vapour in a permanent magnetic field (Grec, Fossat, and Vernin 1976; Brookes, Isaak, and van der Raay 1978). Their locations, which are determined by the field-strength, are extremely stable; on the other hand, it is not possible with this technique to achieve spatial resolution and, therefore, it has been used mainly to observe oscillations in light integrated over the solar surface. Alternatively, the passbands can be defined by a "filter" which is tuned alternately to the red and the blue wing of the line. If an imaging detector (such as a CCD camera) is placed behind the filter, the intensities I_r and I_b , and hence the velocity, can be determined as a function of position on the solar disk. Such filters can be either based on atomic resonance applied in transmission (Cacciani and Fofi 1978), or one can use a birefringent filter (*e.g.* Libbrecht and Zirin 1986) or a Fabry-Perot (*e.g.* Rust and Appourchaux 1988). A somewhat more elaborate technique is used in the Fourier tachometer (Brown 1984) where the wavelength resolution is achieved by means of a Michelson interferometer. Unlike the simple filter techniques, the signal from the Fourier tachometer is essentially linear in the velocity over a substantial range.

3.1.2. Observations in continuum intensity. The solar five-minute oscillations of low degree were detected in solar irradiance by the Active Cavity Radiometer Irradiance Monitor (ACRIM) instrument on the *Solar Maximum Mission* satellite (Woodard and Hudson 1983); the relative amplitude was, at most, a few parts per million. Detailed results on the five-minute oscillations were obtained from the Soviet satellite *Phobos II* (Fröhlich *et al.* 1990, Toutain and Fröhlich 1992). Observations of the intensity oscillations, both integrated and with spatial resolution, have been made also from the ground (Nishikawa *et al.* 1986,

Figure 5. The average line profile for the K D₁ line in light integrated over the solar disk, normalized to unity in the continuum. For convenience, the abscissa is given both in terms of wavelength shift from the line centre and equivalent Doppler velocity. Also shown are the scattering profiles for K vapour in a cell in a magnetic field of 2000 Gauss. The measurements consist of switching between the two scattering profiles, by changing polarization, to measure alternatively the intensities I_r and I_b in the red and the blue wings of the line. From the ratio in Equation (56), the line-of-sight velocity may then be determined.

Jiménez *et al.* 1987), although fluctuations in the Earth's atmosphere make substantial contributions to the noise. More generally, it appears that the ratio of solar noise to oscillation signal is less favourable for the continuum intensity observations than for the case of Doppler velocity (Harvey 1988). On the other hand, continuum intensity observations are attractive for studying solar-like oscillations in other stars. For stellar observations, the limiting source of noise is photon statistics; hence, Doppler velocity measurements, which only utilize a very small part of the light from the star, require large telescopes or very sophisticated techniques, whereas broad-band intensity observations, in principle, can be made with fairly small telescopes. The main difficulty is again the effect of the Earth's atmosphere, particularly scintillation, which essentially excludes terrestrial observations of this nature. However, from space such observations should be possible even with fairly modest instruments (Baglin 1991).

3.1.3. Observations in spectral line intensity. Problems with fluctuations in the Earth's atmosphere can be reduced substantially by making differential intensity observations, where the ratio between the intensity in the core of a spectral line and in the neighbouring continuum is measured. The amplitude of the oscillations increases with height in the atmosphere.

Hence, the intensity amplitude in the line core, which may be formed high in the atmosphere, is larger than the amplitude in the continuum which is formed at the photosphere. On the other hand, the atmospheric effects are largely the same in the line core and the continuum and cancel in the ratio. Extensive observations of this nature have been made from the South Pole (*e.g.* Jefferies *et al.* 1988, Duvall *et al.* 1991), utilizing the core of the Ca II K line, which is formed roughly at the temperature minimum. An interesting feature of these results is that the observations are more sensitive to oscillations at high frequency than are Doppler observations using spectral lines formed lower in the atmosphere. In fact, mode-like structure was detected at frequencies as high as 6.5 mHz, well above the normally assumed acoustical cut-off frequency $\omega_{ac}/2\pi$ of 5.3 mHz (*cf.* section 2.3). The interpretation of this result is still debated (*e.g.* Balmforth and Gough 1990b, Kumar *et al.* 1990, Kumar and Lu 1991). Leisen and Maltby (1990) detected oscillations with an unexpectedly large amplitude in an infrared band around 2.23 μm . Very recent observations (Leisen, private communication) will allow a detailed investigation of the oscillation amplitude and phase as a function of (radiative) wavelength and oscillation frequency. If their solar origin is confirmed observation of such oscillations will, in addition to supplementing other measurements of the oscillation frequencies, enable investigation of the behaviour of the oscillations in the solar atmosphere; this is crucial for understanding the damping and excitation of the modes, and may eventually provide an independent means for probing the structure of the solar atmosphere (Staiger 1987, Deubner *et al.* 1990, Frandsen 1988).

3.2. THE SEPARATION OF INDIVIDUAL MODES

In the Sun, a very large number of modes are excited simultaneously. As a result, observation at a single point on the solar disk shows an interference between many individual harmonic signals, the result of which has an almost chaotic appearance. It was this fact which for a long time led to the five-minute oscillations being regarded as local phenomena, possibly excited in the solar atmosphere by granulation (see, for example, Noyes and Leighton 1963). The true large-scale nature of the oscillations only came to be realized with the introduction of observations which were more extensive in space and time, hence allowing some separation between the modes (Deubner 1975, Rhodes, Ulrich, and Simon 1977). An interpretation in terms of standing waves in the solar interior had previously been proposed by Ulrich (1970) and Leibacher and Stein (1971).

To illustrate the principles in the mode separation, I note that, according to Equation (6), the observed Doppler velocity on the solar surface is of the form

$$V_D(\theta, \phi, t) = \sin \theta \cos \phi \sum_{n,l,m} A_{nlm}(t) c_{lm} P_l^m(\cos \theta) \cos[m\phi - \omega_{nlm}t - \delta_{nlm}(t)]. \quad (57)$$

Here, the axis of the coordinate system was taken to be in the plane of the sky; longitude ϕ is measured from the central meridian. For simplicity, I assumed that the velocity is predominantly in the radial direction, as is the case for five-minute oscillations of low or moderate degree (*cf.* Equation (17)); the factor $\sin \theta \cos \phi$ results from the projection of the velocity vector onto the line of sight. The amplitudes A_{nlm} and phases δ_{nlm} may vary with time, as a result of the excitation and damping of the modes.

As discussed above, it may be assumed that V_D has been observed as a function of position (θ, ϕ) on the solar surface. The first step in the analysis is then to perform a spatial transform, to isolate a limited number of spherical harmonics. This may be thought of as an integration of the observations multiplied by a weight function $W_{l_0 m_0}(\theta, \phi)$ designed to give greatest weight to modes in the vicinity of $l = l_0, m = m_0$. The result is the filtered time string

$$\begin{aligned} V_{l_0 m_0}(t) &= \int_A V_D(\theta, \phi, t) W_{l_0 m_0}(\theta, \phi) dA \\ &= \sum_{n, l, m} S_{l_0 m_0 l m} A_{nlm} \cos[\omega_{nlm} t + \hat{\delta}_{nlm, l_0 m_0}]. \end{aligned} \quad (58)$$

Here, the integral is over area on the solar disk, and $dA = \sin^2 \theta \cos \phi d\theta d\phi$; also, I introduced the *spatial response function* $S_{l_0 m_0 l m}$, defined by

$$(S_{l_0 m_0 l m})^2 = (S_{l_0 m_0 l m}^{(+)})^2 + (S_{l_0 m_0 l m}^{(-)})^2, \quad (59)$$

where

$$S_{l_0 m_0 l m}^{(+)} = c_{lm} \int_A W_{l_0 m_0}(\theta, \phi) P_l^m(\cos \theta) \cos(m\phi) \sin \theta \cos \phi dA, \quad (60)$$

and

$$S_{l_0 m_0 l m}^{(-)} = c_{lm} \int_A W_{l_0 m_0}(\theta, \phi) P_l^m(\cos \theta) \sin(m\phi) \sin \theta \cos \phi dA. \quad (61)$$

The new phases $\hat{\delta}_{nlm, l_0 m_0}$ in Equation (58) depend on the original phases δ_{nlm} and on $S_{l_0 m_0 l m}^{(+)}$ and $S_{l_0 m_0 l m}^{(-)}$.

It is evident that to simplify the subsequent analysis of the time string $V_{l_0 m_0}(t)$, it is desirable that it contain contributions from a limited number of spherical harmonics (l, m) . This is to be accomplished through a suitable choice of the weight function $W_{l_0 m_0}(\theta, \phi)$ such that $S_{l_0 m_0 l m}$ is large for $l = l_0, m = m_0$ and “small” otherwise. Indeed, it follows from the orthogonality of the spherical harmonics that, if $W_{l_0 m_0}$ is taken to be the spherical harmonic $Y_{l_0}^{m_0}$, if the integrals in Equations (60) and (61) are extended to the full sphere, and if, in the integrals, $\sin \theta \cos \phi dA$ is replaced by $\sin \theta d\theta d\phi$, then essentially $S_{l_0 m_0 l m} \propto \delta_{l_0 l} \delta_{m_0 m}$. It is obvious that, with realistic observations restricted to one hemisphere of the Sun, this optimal level of concentration cannot be achieved. However, the result suggests that suitable weights can be obtained from spherical harmonics. Weights of this nature are almost always used in the analysis. The resulting response functions are typically of order unity for $|l - l_0| \lesssim 2, |m - m_0| \lesssim 2$ and relatively small elsewhere. That the timestring contains modes over a range in l and m is analogous to the quantum-mechanical uncertainty principle between localization in space and momentum (here represented by wavenumber). If the area being analyzed is reduced, the spread in l and m is increased; conversely, intensity observations, which do not include the projection factor $\sin \theta \cos \phi$, effectively sample a larger area of the Sun and therefore, in general, lead to somewhat greater concentration in l and m .

Whole-disk velocity observations, analyzing light integrated over the solar surface, correspond roughly to unit weight, *i.e.*, essentially to $l_0 = 0, m_0 = 0$. The resulting response function is very small except for $l \leq 3$ (*e.g.* Dziembowski 1977, Hill 1978, Christensen-Dalsgaard and Gough 1982); for whole-disk observations in intensity, the sensitivity is

restricted to $l \leq 2$. Hence, such observations are sensitive only to low-degree modes. The same is evidently true of observations of stellar oscillations.

Having isolated a relatively small number of modes through the spatial analysis, the remaining mode identification is carried out through Fourier analysis in the temporal domain by computing

$$\tilde{V}_{l_0 m_0}(\omega) = \int_0^T V_{l_0 m_0}(t) \exp(-i\omega t) dt, \quad (62)$$

and the power spectrum

$$P_{l_0 m_0}(\omega) = |\tilde{V}_{l_0 m_0}(\omega)|^2. \quad (63)$$

Here, I have arbitrarily taken the start of the observations as the zero-point in time, and T is the total duration of the observations. For a single undamped oscillation, $V_D(t) = \cos(\omega_0 t)$, the power spectrum is

$$P(\omega) = \frac{1}{4} T^2 \text{sinc}[\frac{1}{2} T(\omega - \omega_0)]^2, \quad (64)$$

where $\text{sinc } x = (\sin x)/x$ (I have neglected a corresponding contribution on the negative ω -axis). As shown in Figure 6a and 6b, this gives rise to a single dominant peak in the power spectrum, centred at ω_0 and of width approximately $2\pi/T$; hence, the frequency resolution is improved by extending the duration of the observation (another analogy to the uncertainty principle).

Figure 6 also illustrates another important aspect of the observations: the effects of gaps in the data. Except for the observations from the South Pole, data from a single site typically consist of stretches of 8 – 12 hours, separated by night-time gaps. Fourier analysis of such data results in power spectra such as those shown in Figures 6c and 6f. The central peak is surrounded by *sidebands* separated from it by $1/(1 \text{ day}) = 11.56 \mu\text{Hz}$. In reality, the data contain several closely spaced peaks, arising partly from rotational splitting, partly from the presence of several values of l and m in a single time string; the resulting spectrum can get quite complicated (*cf.* Figure 6f). The presence of noise obviously adds to the confusion. Such effects of gaps motivate setting up networks of observing stations to ensure largely uninterrupted data, or observing from space (see section 7).

3.3. THE OBSERVED SPECTRUM OF OSCILLATIONS

Figure 7 shows an example of an observed power spectrum for the five-minute oscillations. This was obtained by means of Doppler velocity measurements in light integrated over the solar disk and hence, according to section 3.2, is dominated by modes of degrees 0 – 3. The data were obtained from two stations widely separated in longitude, to suppress the daily side-bands, and span 53 days. Thus, the intrinsic frequency resolution, as determined by Equation (64), is smaller than the thickness of the lines. It should be noticed that the peaks occur in pairs, corresponding to the expression for the frequencies given in Equation (29). Also, there is a visible increase in the line-width when going from low to high frequency (see also Grec, Fossat, and Pomerantz 1980). The broadening of the peaks at high frequency is probably caused by the damping and excitation processes; thus, the observations indicate that the damping rate increases with increasing frequency. More careful analyses (see, for example, Libbrecht and Zirin 1986; Libbrecht 1988; Elsworth *et al.* 1990b; Jefferies *et al.* 1991) have shown that the mode lifetimes vary from about a day at high frequency

Figure 6. Illustration of power spectrum analysis. Panels a) – c) show spectra of a single oscillation with frequency $3000\ \mu\text{Hz}$, observed for 8 hours (panel a), 32 hours (panel b), and two 8 hour segments, separated by a gap of 16 hours (panel c). Panels d) – f) similarly show spectra of a timeseries consisting of the superposition of two modes; one with frequency $3000\ \mu\text{Hz}$ and relative amplitude 1, the second with frequency $3045\ \mu\text{Hz}$ and relative amplitude 0.8. The power is on an arbitrary scale.

(as directly visible in the width of the peaks in Figure 7) to several months at the lowest frequencies observed. Another striking aspect of Figure 7 is the well-defined distribution of amplitudes, with a maximum around $3000\ \mu\text{Hz}$ and very small values below $2000\ \mu\text{Hz}$ and above $4500\ \mu\text{Hz}$. The maximum velocity amplitude for a single mode is about $15\ \text{cm sec}^{-1}$. Libbrecht *et al.* (1986) analyzed the observed dependence of mode amplitudes on degree, azimuthal order, and frequency. They found that the distribution of amplitude is largely independent of degree, and that the mode amplitudes depend only weakly on degree (see also Christensen-Dalsgaard and Gough 1982).

Figure 7. Power spectrum of solar oscillations, obtained from Doppler observations in light integrated over the disk of the Sun. The ordinate is normalized to show velocity power per frequency bin. The data were obtained from two observing stations, on Hawaii and Tenerife, and span 53 days. (See Claverie *et al.* 1984).

Extensive tables of five-minute oscillation frequencies, compiled from several sources, were given by Duvall *et al.* (1988) and Libbrecht, Woodard, and Kaufman (1990). To illustrate the quality of current frequency determinations, Figure 8 shows observed frequencies at low and moderate degree from the latter compilation, with error bars magnified by a factor 1000 over the usual 1σ error bars. For the most accurate measurements, the relative standard deviation is less than 10^{-5} , thus substantially exceeding the precision with which the solar mass is known. Precise measurements of frequencies and frequency separations for low-degree modes were recently published by Elsworth *et al.* (1990a, 1991) and Toutain and Fröhlich (1992); such measurements are of great diagnostic importance for the properties of the solar core (*cf.* section 4.2).

From spatially resolved observations, individual frequencies ω_{nlm} can in principle be determined. Because of observational errors and the large amount of data resulting from such determination, it has been common to present the results in terms of coefficients in fits to the m -dependence of the frequencies, either averaged over n at given l (Brown and Morrow 1987) or for individual n and l (*e.g.*, Libbrecht 1989). A common form of the fit is in terms of Legendre polynomials,

$$\omega_{nlm} = \omega_{nl0} + L \sum_{j=1} a_j(n, l) P_j \left(\frac{m}{L} \right). \quad (65)$$

As discussed in section 2.4.2 (*cf.* Equations (54) and (55)) the coefficients a_j with odd j arise from rotational splitting; the coefficients with even j are caused by departures from spherical symmetry in solar structure, or from effects of magnetic fields. It was pointed out by Ritzwoller and Lavelly (1991) that a more suitable expansion of the rotational splitting could be obtained in terms of Clebsch-Gordon coefficients; with a proper choice of expansion

functions ψ , for the angular velocity $\Omega(r, \theta)$ (cf. Equation (52)), the relations corresponding to Equation (55) decouple such that each expansion coefficient for the splitting is related to a single expansion function for the angular velocity. It should be noted that, in general, averaging or expansion of the observed frequencies may involve loss of information; for the purpose of inversion it is, in principle, preferable to work directly in terms of the observed frequencies. On the other hand, by suitably combining the frequencies before inversion, the computational effort required may be greatly reduced.

Figure 8. Plot of observed solar p -mode oscillation frequencies, as a function of the degree l . The vertical lines show the 1000σ error bars. Each ridge corresponds to a given value of the radial order n , the lowest ridge having $n = 1$. (From Libbrecht and Woodard 1990).

4. Computed Oscillation Frequencies

The most immediate use of the observed frequencies for investigating the solar interior is to compare the observations with computed frequencies of solar models. In this way, one obtains a test of the models. Furthermore, by analysing frequencies of models computed with different assumptions, one can get some impression of the sensitivity of the frequencies to the physics of the solar interior and the other ingredients of the solar model computations. This illustrates the diagnostic potential of the frequencies and may provide an indication of the changes required to improve the agreement between computations and observations.

In this section, the relations between physics, model, and frequencies are illustrated by considering three different changes to the physics: a minor change to the opacity, a substantial change to the equation of state, and the inclusion of overshoot below the convection zone. The analysis is carried out by considering differences between the models and the corresponding differences between the frequencies; the dependence of the frequency differences on the model differences can be understood qualitatively from the asymptotic behaviour of

the oscillations discussed in section 2.3. Furthermore, I compare observed frequencies with two different sets of computed frequencies.

A particular feature of the comparison relates to the dependence of the depth of penetration of the modes on the degree at fixed frequency: since modes of high degree penetrate less deeply than modes of low degree, they involve less of the solar mass, and hence their frequencies are easier to perturb. To compensate for this, frequency differences are shown after scaling by the ratio

$$Q_{nl} = \frac{E_{nl}}{\overline{E}_0(\omega_{nl})}. \quad (66)$$

Here, E_{nl} is a measure of the energy of the mode, at fixed surface amplitude, and $E_0(\omega_{nl})$ is the corresponding energy for radial modes, interpolated to the frequency ω_{nl} of the mode being considered. This compensates for the effect of the difference in penetration depth; it may be shown that changes in the model that are localized very near the solar surface cause frequency changes $\delta\omega_{nl}$ such that $Q_{nl}\delta\omega_{nl}$ is essentially independent of l at given frequency (*e.g.* Christensen-Dalsgaard and Berthomieu 1991). In particular, this property may be expected to hold true for the errors introduced into the frequencies by the uncertain physics of the model and the oscillations (effects of convection, nonadiabaticity *etc.*) concentrated near the solar surface.

4.1. SENSITIVITY TO CHANGES IN THE PHYSICS OF THE MODEL

4.1.1. Change in the opacity. To illustrate the effects of a subtle change in the opacity, I consider two models differing essentially only in the abundance of iron used in the opacity tables, namely Models 2 and 3 in Table 1 of Chapter 2.

The smaller iron abundance in the tables used for Model 3 leads to a general decrease in opacity in the radiative interior of the model. As noted in Chapter 2, this causes a decrease in the value of Y_0 required to calibrate the model. The detailed effects of the change are illustrated in Figure 9 which shows structure and frequency differences between Models 3 and 2. The decrease in the opacity near the base of the convection zone causes the convection zone to be shallower in Model 3 than in Model 2. In the region that is adiabatically stratified in Model 2 and sub-adiabatically stratified in Model 3, temperature increases more slowly with depth in Model 3. This causes the sharp decrease in the sound speed beneath the convection zone in Model 3 relative to Model 2. In the convection zone, the differences are mainly a result of the difference in Y_0 ; in particular, this causes a significant difference in sound speed (barely visible in the present figure) in the ionization zones of hydrogen and helium. The negative sound-speed difference at the base of the convection zone dominates the frequency differences shown in Figure 9b: the frequencies of modes penetrating into or beyond this region of negative δc are somewhat reduced in Model 3 relative to Model 2; for modes of high degree that are entirely trapped within the convection zone, the frequency change is small and dominated by δc in the near-surface ionization zones. It is interesting that this effect is of considerable magnitude compared with the precision of the observed frequencies. This suggests that helioseismic investigations may be sensitive to even quite subtle effects in the opacity. Indeed, analyses of the observed frequencies (*e.g.* Christensen-Dalsgaard *et al.* 1985, Korzennik and Ulrich 1988) have indicated that the opacity obtained from the CT and LAOL tables might be too low in the temperature region corresponding to the outer parts of the radiative interior; this has been confirmed by

recent independent opacity calculations (Iglesias and Rogers 1991; Yan, Seaton, and Mihalas 1992). Dziembowski, Pamyatnykh, and Sienkiewicz (1992) demonstrated that the use of the new Iglesias and Rogers opacities led to a substantial improvement in the agreement between the computed model and the Sun. However, it remains to be seen whether effects of opacity differences of the magnitude considered in Figure 9 can be distinguished among the other potential uncertainties in the model.

Figure 9. Structure differences, at fixed fractional radius r/R (a), and scaled frequency differences (b), between Model 3 computed with the opacity tables based on the meteoritic Fe abundance and Model 2 computed with tables based on the photospheric Fe abundance, in the sense (Model 3) – (Model 2). In panel a) the lines have the following meaning (note that $\ln$ denotes the natural logarithm):

Thin lines: $\delta \ln q$, where $q \equiv m/M$ is the mass fraction (—); $\delta \ln L$, where L is the luminosity at r (-----); δX , where X is the hydrogen abundance (-----).

Heavy lines: $\delta \ln c$ (—); $\delta \ln p$ (-----); $\delta \ln \rho$ (-----); $\delta \ln T$ (-----).

For the frequency differences shown in panel b) points corresponding to a given value of l have been connected, according to the following convention: $l = 0 - 3$ (-----); $l = 4, 5, 10, 20, 30$ (—); $l = 40, 50, 70, 100$ (-----); $l = 150, 200, 300, 400$ (-----); and $l = 500, 600, 700, 800, 900, 1000$ (-----).

4.1.2. Change in the equation of state. The Eggleton, Faulkner, and Flannery (1973) equation of state (in the following EFF; see Chapter 2) which was used in the models considered so far is extremely simple. In particular, it assumes all ions to be in the ground state, neglects the Coulomb interaction between the particles in the gas, and treats “pressure ionization” in a very crude, although thermodynamically consistent, way. Christensen-Dalsgaard, Däppen, and Lebreton (1988) found that agreement between observed and theoretically computed frequencies could be markedly improved by using the so-called MHD

equation of state (Hummer and Mihalas 1988, Mihalas, Däppen, and Hummer 1988, Däppen *et al.* 1988); this is determined from minimization of an approximation to the free energy containing a substantial number of physical effects. The consequences of the change in the equation of state are illustrated in Figure 10, which shows differences between structure and frequencies of models computed with the EFF and MHD equations of state. The EFF model is Model 1 in Table 1 of Chapter 2, whereas the MHD model differs from that model only in the equation of state. There are clearly substantial differences between the EFF and the MHD models; in particular, the sound speed differs by up to two per cent in the hydrogen and helium ionization zones. This leads to frequency differences of up to $10 \mu\text{Hz}$. For most modes, the frequencies are decreased by the predominantly negative δc . However, very near the surface there is a region where δc is positive, leading to positive frequency differences for high-degree modes which are predominantly trapped in that region. The general effect of the frequency change caused by the introduction of the MHD equation of state is to reduce the difference between the computed and the observed frequencies.

Figure 10. Structure differences, at fixed fractional radius r/R , (panel a) and scaled frequency differences (panel b) between Model 1 computed with the EFF equation of state and an otherwise similar model computed with the MHD equation of state, in the sense (EFF model) – (MHD model). See caption to Figure 9 for details.

It was pointed out by Däppen (1990) that the dominant departures of the MHD equation of state from the simple EFF treatment arise from the Coulomb terms. Indeed, Christensen-Dalsgaard (1991) found that by including the Coulomb effects in the EFF formulation most of the differences with MHD were removed, leaving frequency differences of less than $2 \mu\text{Hz}$ relative to the MHD models. The consequences of improvements in the equation of state, such as the inclusion of Coulomb effects, were also considered by Stix and Skaley (1990). Baturin, Kononovich, and Mironova (1991) and Pamyatnykh, Vorontsov, and Däppen (1991)

analysed the effects of various treatments of the equation of state on the frequencies, in particular as expressed in terms of a phase function $\beta(\omega)$ which is closely related to the Duvall phase $\alpha(\omega)$ introduced in Equation (22). It was found again that in general the more complex, and hence presumably more accurate, treatments of the equation of state improved the agreement between theory and observation. A detailed analysis of the effects of the equation of state on the solar oscillation frequencies was given by Christensen-Dalsgaard and Däppen (1992).

4.1.3. Convective overshoot. As discussed in Chapter 4, convective motion is likely to extend beyond the unstable region; the result is a region where the temperature gradient is slightly subadiabatic, followed by a sharp transition to the purely radiative gradient. The detailed dynamics of this overshoot region, and its extent, is quite uncertain, however. Hence it would be of great interest to be able to study it by means of helioseismology.

To investigate the effect of overshoot on the solar model and its oscillation frequencies, I have considered a simplified model simulating the behaviour discussed in Chapter 4: the temperature gradient was forced to be nearly adiabatic for a specified distance beneath the point of transition from convective instability to convective stability, and this was followed by a discontinuous transition to the radiative gradient. As usual, the model was calibrated to have the observed radius and luminosity; in particular, the inclusion of overshoot requires a change of the mixing-length parameter. The changes in structure resulting from overshoot by $0.029R$, corresponding to 0.36 pressure scale heights at the base of the convection zone, are shown in panel a) of Figure 11. Since the temperature gradient is steeper just beneath the convection zone in the overshoot model, there is a sharp increase in the temperature difference at this point. However, at slightly greater depth the temperature gradient is more shallow in the overshoot model, and the temperature difference decreases again. Hence, there is a sharp peak in $\delta \ln T$ localized to a small region just beneath the convection zone, with a corresponding peak in the sound-speed difference. The change in the mixing-length parameter causes changes of pressure and density within the convection zone, whereas in the bulk of the radiative interior the changes are very small.

The effects on the oscillations are seen most clearly by plotting scaled relative frequency differences against ν/L which, according to Equation (20), determines the depth of the lower turning point; this has been done in panel b) of Figure 11. The results may be understood from the asymptotic expressions (25) and (26) for the frequency differences. For $\nu/L \lesssim 60 \mu\text{Hz}$ the modes are trapped entirely within the convection zone where the sound-speed change is small; so, therefore, are the frequency differences. Modes that penetrate beyond the convection zone experience the sharp peak in $\delta \ln c$; this leads to the rapid increase in $\delta \nu$ as ν/L increases beyond $80 \mu\text{Hz}$. The effect is particularly strong for those modes whose turning points are in the vicinity of the peak in $\delta \ln c$. Mathematically this follows from the fact that the integrand in Equation (25) has an integrable singularity at $r = r_i$; physically, the modes propagate almost horizontally at the lower turning point, spending relatively more time in this region and hence being more sensitive to the change in c . Modes that penetrate substantially beyond the peak in $\delta \ln c$ feel the integrated effect of the change in sound speed, and hence the frequency change is roughly constant. Overlying this general trend the frequency changes exhibit rapid oscillations, barely visible on the scale of the figure. These are caused by the shift with changing frequency of the location of the extrema in the eigenfunctions relative to the peak in $\delta \ln c$. Such a behaviour is characteristic of the frequency response to sharp features in the model (see, for example,

Figure 11. The effects of convective overshoot by a distance of $0.029R$, corresponding to about 0.36 pressure scale heights. Panel a) shows structure differences, in the sense (overshoot model) – (normal model), using the line styles of Figure 9. Panel b) shows the corresponding scaled relative frequency differences; they have been plotted against ν/L which determines the location of the lower turning point (*cf.* equation (20)).

Gough 1990); the “wavelength”, measured in frequency units, of the oscillation in the frequency difference is determined by the depth of the sharp feature.

From these results it is clear that convective overshoot has characteristic effects on the structure and frequencies of the model. The magnitude of the effects evidently depends on the extent of the overshoot region; indeed, it is straightforward to show that the changes in structure and frequencies scale roughly as the square of the extent of overshoot. Whether or not such effects can be studied observationally depends on the ability to extract them from the noise in the observed frequencies, and to distinguish them from the consequences of, for example, errors in the opacities. These questions require further study.

4.2. COMPARISON WITH OBSERVED FREQUENCIES

A detailed analysis of the observed frequencies is outside the scope of this Chapter. However, to indicate the typical behaviour found in comparisons of the computed frequencies with observations, Figure 12 shows scaled differences between observed frequencies from the compilation by Libbrecht *et al.* (1990) and the frequencies of two models computed with the MHD equation of state. In panel a) the MHD model considered in Figure 10 was used; this was computed with the CT opacities. The model considered in panel b) was computed with a version of the LAOL tables similar to the one used for Model 2 in Table 1 of Chapter 2, although with a heavy element abundance of 0.02. There are striking differences between the two sets of results: the variation of the differences with frequency is considerably larger in the CT than in the LAOL case, whereas the scatter with the degree of the mode is slightly larger in the LAOL case. These properties are a result of the differences between the two opacities and the resulting differences between the models, which are similar to those between Models 1 and 2. In particular, inversion of the observed frequencies indicates that the depth of the solar convection zone is $0.287 \pm 0.003R$ (Christensen-Dalsgaard, Gough, and Thompson 1991), somewhat larger than the values of $0.279R$ and $0.274R$, respectively, for the CT and the LAOL models. The effect on the frequencies is similar to the differences illustrated in Figure 9, with a transition between modes trapped within the convection zone and modes penetrating beyond it. This leads to the l -dependence of the frequency differences; since the CT model is closer to the Sun in this respect, the variation with degree is slightly smaller. The smaller variation of the differences with frequency for the LAOL model is a result of the substantially larger opacity at low temperature in the LAOL tables; since the resulting changes in the model are concentrated near the surface, they lead to scaled frequency differences that depend predominantly on frequency. These results might suggest that the atmospheric behaviour of the LAOL opacities should be preferred. It is important to remember, however, that the calculation neglects a number of complications near the solar surface, such as nonadiabaticity and dynamical effects of convection; such effects would be expected to lead to scaled frequency shifts relative to the observations that depend on frequency but not on degree, much as is obtained for the CT model, and of a comparable magnitude. Thus, the apparently better agreement between theory and observation in Figure 12b is not immediately significant.

There has been extensive discussion about the observed and computed values of the frequency separation $\delta\nu_{nl}$ (see Equation (32)). As mentioned in section 2.3, the importance of this quantity lies in the fact that it is sensitive to the core structure of the model, and hence may help elucidating those properties of the model which are involved in the discrepancy between the predicted and observed flux of solar neutrinos (*cf.* Chapter 2). Faulkner, Gough, and Vahia (1986) and Däppen, Gilliland, and Christensen-Dalsgaard (1986) noted that $\delta\nu_{nl}$ was decreased by the inclusion in the model of energy transport by “weakly interacting massive particles” (WIMPs) as had been suggested previously as a way of reducing the solar neutrino flux. Indeed, these early results suggested that $\delta\nu_{nl}$ was somewhat higher than observed for normal solar models, so that the agreement between theory and observation could be improved by considering models with WIMPs. In contrast, partial mixing of the solar core, which had also been proposed as a means of reducing the predicted neutrino flux, led to a substantial increase in $\delta\nu_{nl}$. The current situation can be judged from Figure 13, in a form first proposed by Elsworth *et al.* (1990a): this shows

Figure 12. Differences between observed frequencies of solar oscillations, from the compilation by Libbrecht *et al.* (1990), and adiabatic frequencies for two solar models, in the sense (observation) – (model). The models were computed with the MHD equation of state and had a heavy element abundance of 0.02. In panel a) the CT opacities were used, whereas the model in panel b) used LAOL opacities computed with the photospheric iron abundance. The mode sets and line styles are the same as in Figure 9b.

the coefficients $\bar{d}_l$ and s_l of the fit (34), for $l = 0$ and 1. The figure includes results for a substantial number of “normal” solar models using different physics, for models with reduction in the core opacity simulating the effects of WIMPs, and for a model with a partially mixed core. In addition, observed values obtained by Gelly *et al.* (1988) and Elsworth *et al.* (1990a) are indicated. Details about the models were given by Christensen-Dalsgaard (1991, 1992ab). The results for the normal models fall close to the observations, although with a slight but systematic tendency for $\bar{d}_l$ to be larger than observed. In contrast, $\bar{d}_l$ for WIMP-like models with the observed neutrino flux is far lower than observed (see also Elsworth *et al.* 1990a and Cox, Guzik, and Raby 1990) whereas, for the partially mixed model, $\bar{d}_l$ is much larger than observed, even though the neutrino flux is still in excess of the observed value (see, for example, Cox and Kidman 1984, Provost 1984). These results indicate that it will be very difficult to find a model that is consistent with both the observed neutrino flux and the observed oscillation frequencies. As discussed in Chapter 2, this strengthens the case for solutions to the neutrino problem in terms of the properties of the neutrinos.

5. Inverse Analyses

The observed solar oscillation frequencies are integral measures of conditions in the solar interior. A simple example is provided by the rotational splitting, which is a weighted

Figure 13. Observed and computed coefficients $\overline{d}_l$ and s_l in the fit in Equation (34) to the scaled frequency separation d_{nl} . The solid (for $l = 0$) and dashed (for $l = 1$) error boxes show the observed results of Elsworth *et al.* (1990a), whereas the dotted error box gives what are essentially averages of $\overline{d}_0$, $\overline{d}_1$ and s_0 , s_1 , based on observations by Gelly *et al.* (1988). The points clustering close to the error boxes are for normal solar models with a variety of physics. The remaining points are identified in the figure by the symbol type. Those to the left are for different simulations of the effects of WIMPs, including two models (SW2 and EW2) with approximately the observed flux of ^{37}Cl neutrinos. The points to the far right are for a model with a partially mixed core. Adapted from Christensen-Dalsgaard (1992a) where further details about the models are given.

integral of the angular velocity over the region where the mode is trapped. Because of the variation of the turning point radius r_t with degree and frequency, different modes sample different parts of the Sun. Hence, the variation of the splitting with degree, say, provides an indication of the variation of Ω with r . This also suggests that it may be possible to obtain localized information about Ω : roughly speaking, the difference between splittings for modes with different r_t should be a measure of the rotation in the region between the turning points of the modes.

The expression (42) for the splitting caused by spherically symmetric rotation is a particularly simple example of a relation between observable properties of oscillation frequencies and properties of the solar interior. The determination of $\Omega(r)$ from the $\delta\omega_{nlm}$ constitutes an *inverse problem*. Such problems have a vast literature, covering their application in, for example, geophysics and radiation theory (*e.g.* Deepak 1977, Parker 1977, Craig and Brown 1986, and Tarantola 1987). The application to the solar inverse problem was discussed by, for example, Gough (1978, 1985), Thompson (1991), and Gough and Thompson (1991). Christensen-Dalsgaard, Schou, and Thompson (1990) made a systematic comparison of different inversion techniques, as applied to the problem of spherically symmetric rotation.

5.1. BASIC PRINCIPLES OF INVERSE ANALYSIS

As an illustration of some general properties of inverse analyses, it is instructive to consider briefly the technique of *optimally localized averages*, developed by Backus and Gilbert (1970), as applied to inversion for a spherically symmetric angular velocity $\Omega(r)$. The inverse problem may be formulated as

$$\Delta_i = \int_0^R K_i(r) \Omega(r) dr, \quad (67)$$

where, for notational simplicity, I represent the pair (n, l) by the single index i . Δ_i is the scaled rotational splitting $m^{-1} \beta_{nl}^{-1} \delta \omega_{nlm}$, so that the kernels K_i are normalized as in Equation (45). The principle of the method is to construct a linear combination

$$\tilde{\Omega}(r_0) = \sum_i c_i(r_0) \Delta_i = \int_0^R K(r; r_0) \Omega(r) dr \quad (68)$$

of the observed data, where

$$K(r; r_0) \equiv \sum_i c_i(r_0) K_i(r). \quad (69)$$

The goal is to choose coefficients $c_i(r_0)$ such as to make $K(r; r_0)$ approximate as far as possible a delta function $\delta(r - r_0)$ centred on r_0 . Then, $\tilde{\Omega}(r_0)$ provides an approximation to $\Omega(r_0)$. If this can be done for all r_0 , an estimate of $\Omega(r)$ is obtained.

The coefficients $c(r_0)$ are determined by minimizing

$$\cos \eta \int_0^R (r - r_0)^2 K(r; r_0)^2 dr + \sin \eta \sum_{ij} E_{ij} c_i c_j, \quad (70)$$

subject to the constraint

$$\int_0^R K(r; r_0) dr = 1; \quad (71)$$

here E_{ij} is the covariance matrix of the data. The effect of the minimization is most easily understood for $\eta = 0$. Minimizing Equation (70) subject to Equation (71), ensures that $K(r; r_0)$ is large close to r_0 , where the weight function $(r - r_0)^2$ is small, and small elsewhere. This is precisely the required “delta-ness” of the combined kernel. However, with no further constraints, the optimization of the combined kernel may result in numerically large coefficients of opposite sign. Hence, the variance in $\tilde{\Omega}$, which can be estimated as

$$\sigma^2(\tilde{\Omega}) = \sum_{ij} E_{ij} c_i c_j, \quad (72)$$

would be large. The effect of the second term in Equation (70), when $\eta > 0$, is to restrict $\sigma^2(\tilde{\Omega})$. The size of η determines the relative importance of the localization and the size of the variance in the result. Hence, η must be determined to ensure a trade-off between the localization and the error, and η is generally known as *the trade-off parameter*.

The minimization problem defined by Equations (70) and (71) leads to a set of linear equations for the c_i . To characterize the properties of the inversion, it is convenient to consider a measure of the width of the averaging kernels $K(r; r_0)$ and the *error magnification*

$$\Lambda(r_0) = \left[\sum_i c_i(r_0)^2 \right]^{1/2} ; \quad (73)$$

Λ is defined such that if the standard error $\sigma(\Delta_i)$ is the same for all the modes the standard error in the result of the inversion is

$$\sigma[\tilde{\Omega}(r_0)] = \Lambda(r_0)\sigma(\Delta_i) . \quad (74)$$

By considering the *trade-off curve*, where Λ is plotted against the width for varying η , one may choose an appropriate value of η . It is evident that this value will depend on the properties of the data, particularly the level of errors, and on the desired properties of the solution.

Figure 14. Averaging kernels $K(r; r_0)$ at selected radii ($r_0/R = 0.1, 0.2, \dots, 1.0$) for inversion by means of optimally localized averages. The kernel at $r_0/R = 0.5$ is shown as a bolder curve. From Christensen-Dalsgaard *et al.* (1990).

An illustration of the use of this method is provided by the results obtained by Christensen-Dalsgaard *et al.* (1990). They considered a set consisting of about 830 modes at selected degrees between 1 and 200, and frequencies between 2000 and 4000 μHz . Examples of averaging kernels $K(r; r_0)$ are shown in Figure 14. The trade-off parameter was chosen such that the error magnification at $r_0 = 0.5R$ was close to 1. It should be realized that the kernels entering into the combination are of the form shown in Figure 4. Thus, a very large degree of cancellation has been achieved of the dominant contribution from near the surface. Nevertheless, it is obvious that the averaging kernels are only approximate realizations of delta functions; structure on a scale smaller than roughly $0.05 R$ is not resolved.

This limitation is inherent in any inversion method. Indeed, it is evident that from a finite set of data one can never completely resolve the function $\Omega(r)$. Thus, the solution must be constrained. The constraint that is invoked in the present method, and in most other inversion methods, is that the solution be smooth. This is ensured by the representation of the solution by the averaging kernels whose shape is determined by the minimization in Equation (70).

A second commonly used technique is the regularized least-squares method (see, for example, Craig and Brown 1986). Here $\Omega(r)$ is parameterized, often as a piecewise constant function, and the parameters are determined through a least-squares fit to the data by minimizing the sum of the squared differences between the observed splittings and the splittings computed from the parameterized representation of Ω . In general, this least-squares solution needs to be regularized to obtain a smooth solution. This is achieved in the minimization by adding to the sum of squared differences a multiple of the average of the square of Ω , or the square of its first or second derivative; the weight given to this term serves as a trade-off parameter, determining the balance between resolution and error for this method. Another inversion technique is spectral expansion, where Ω is approximated as a linear combination of the kernels (see, for example, Backus and Gilbert 1967; Gough 1985). Finally, from the asymptotic properties of p -modes it follows that the inverse problem can be formulated approximately as an integral equation of the Abel type, the solution to which can be written down analytically; this leads to a very simple, although approximate, technique for inversion of data consisting only of such modes (*e.g.* Gough 1984). In these methods, also, there are parameters which determine the trade-off between resolution and error.

These methods are all *linear*, in the sense that the result of the inversion depends linearly on the data. For any such method, there exist coefficients $c_i(r_0)$ such that the solution $\tilde{\Omega}(r_0)$ may be written in terms of the observed data Δ_i as in Equation (68) and hence, by using Equation (67), may be expressed from the original angular velocity $\Omega(r)$ through an averaging kernel $K(r; r_0)$. It should be noticed that, once the parameters of the inversion have been determined, the averaging kernels are independent of the data (however, they obviously depend on the weights given to each data point, and hence on the assumed errors in the data). As discussed extensively by Christensen-Dalsgaard *et al.* (1990), a quantitative comparison of different inversion methods can be carried out in terms of the averaging kernels and the coefficients $c_i(r_0)$.

So far, I have considered inversion for a function that depends on r alone. It is evidently desirable, however, to carry out inversion for more general properties, for example, the angular velocity $\Omega(r, \theta)$, which are functions both of r and θ . This may be carried out by expansion in terms of suitable functions of θ , with coefficients that are functions of r (see section 2.4.2). The inverse problem then reduces to inversions for the expansion functions (see, for example, Korzenik *et al.* 1988, Brown *et al.* 1989, Thompson 1990). Alternatively, one may perform a direct two-dimensional inversion by means of a regularized least-squares technique (*e.g.* Sekii 1990, 1991 and Schou 1991a). Finally, a two-dimensional asymptotic inversion technique may be developed by noting that, in the asymptotic Equation (51), the dependence on θ leads to an integral equation of the Abel type in latitude (Kosovichev and Parchevskii 1988, Gough 1991, Gough and Thompson 1990, 1991).

Figure 15. The inferred solar rotation rate obtained from inversion of rotational splitting observations, given on the form of coefficients in an expansion like Equation (54). Results are displayed at three target co-latitudes: $\theta_0 = 0^\circ$ (the pole), $\theta_0 = 45^\circ$, and $\theta_0 = 90^\circ$ (the equator). Dashed lines indicate 1- σ error bars based on the observers' error estimates. (From Schou *et al.* 1992).

5.2. RESULTS ON SOLAR INTERNAL ROTATION

A substantial number of inversions have been carried out to investigate properties of the solar internal rotation (for a review, see, for example, Christensen-Dalsgaard 1990). Early analyses in general concentrated on the splitting for sectoral modes with $l = m$ which, according to section 2.4, predominantly give information about the angular velocity at the equator. Duvall *et al.* (1984) found that the result was generally at, or possibly somewhat below, the surface equatorial angular velocity, although there were slight indications of substantially faster rotation of a small core. Additional evidence for fast core rotation was obtained by Toutain and Fröhlich (1992); nevertheless, the rotation of the core is still uncertain. More recently, observations of the dependence of the splitting on m has enabled investigation of the latitude-dependence of the internal angular velocity (*e.g.* Brown *et al.* 1989, Christensen-Dalsgaard and Schou 1988, Korzennik *et al.* 1988, Dziembowski, Goode, and Libbrecht 1989, Rhodes *et al.* 1990, and Thompson 1990). Goode *et al.* (1991) recently carried out a comprehensive analysis of several different sets of data.

As an example, I present results obtained by Schou, Christensen-Dalsgaard, and Thompson (1992) from rotational splitting observations by Libbrecht (1989). The observations were given as coefficients $\Delta^j \omega_{nl}$ in the expansion (54), with $s_{\max} = 2$. Consequently, the inversion was carried out by determining expansion functions $\Omega_s(r)$ as in Equation (52). The

inversion for each of the Ω , was performed by means of the technique of optimally localized averages, discussed in section 5.1. The results are presented in Figure 15, in the form of inferred angular velocities at the equator, the latitude 45° , and the pole. It is striking that in much of the convection zone the angular velocity is quite similar to the behaviour on the surface; it should be noted that the inversion does not impose continuity with the surface angular velocity. In the lower part of the convection zone there appears to be a transition such that the angular velocity in the radiative interior is roughly independent of latitude, at a value intermediate between the surface equatorial and polar values, but substantially closer to the former.

In interpreting results such as these, it is important to keep in mind the limited resolution of the inversion. Thus, with the data that were used, it is not possible to distinguish between the gradual transition to latitude-independent rotation in Figure 15 and a discontinuous transition at, for example, the base of the convection zone. Because of the simple three-term representation (54) of the splitting, the resolution in latitude is even poorer. To illustrate the resolution in both r and θ , Schou *et al.* (1992) used generalized averaging kernels $K(r, \theta; r_0, \theta_0)$ defined such that the inferred angular velocity $\tilde{\Omega}(r_0, \theta_0)$ is related to the true angular velocity $\Omega(r, \theta)$ through

$$\tilde{\Omega}(r_0, \theta_0) = \int_0^\pi \int_0^R K(r, \theta; r_0, \theta_0) \Omega(r, \theta) r dr d\theta. \quad (75)$$

Examples of such kernels are shown in Figure 16. It is evident that the kernels have a substantial extent in latitude. Also, the figure shows that what was inferred to be the polar angular velocity in fact corresponds to extrapolation from lower latitudes; indeed, it is obvious that the rotation of the region very near the pole has little effect on the frequency splittings, and hence cannot be determined from the inversion.

5.3. RESULTS ON SOLAR INTERNAL STRUCTURE

It follows from the discussion in section 2.2 that the frequencies of adiabatic oscillation of a solar model can be written as functionals of the dependence on r of density $\rho(r)$ and adiabatic exponent $\Gamma_1(r)$:

$$\omega_{nl} = \mathcal{F}_{nl}^{(\text{ad})}[\rho(r), \Gamma_1(r)]. \quad (76)$$

Here, $\mathcal{F}_{nl}^{(\text{ad})}$ is defined through the way in which $\rho(r)$ and $\Gamma_1(r)$ enter into the coefficients in Equations (9) – (11). Hence, the dependence is quite complicated and non-linear. To get information about solar structure from a given set of observed frequencies $\omega_{nl}^{(\text{obs})}$, one must, therefore, “solve” the non-linear equations

$$\omega_{nl}^{(\text{obs})} = \mathcal{F}_{nl}^{(\text{ad})}[\rho(r), \Gamma_1(r)], \quad (77)$$

to determine $\rho(r)$ and $\Gamma_1(r)$; this, in general, requires iterative techniques. A further difficulty arises from nonadiabaticity and other effects that were neglected in the simple dependence expressed formally in Equation (76). Hence, the inverse problem for solar structure is considerably more complicated than the simple linear problems considered in section 5.1 and 5.2.

Figure 16. Contour plots of two-dimensional averaging kernels $R^{-2}K(r, \theta; r_0, \theta_0)$ (cf. Equation (75)) for the inversion shown in Figure 15, at a target radius $r_0 = 0.65R$ and target co-latitudes $\theta_0 = 0^\circ, 45^\circ$, and 90° . The plots are in the (r, θ) plane, with the polar axis towards the top of the page. Positive contours are indicated by solid lines, negative contours by dashed lines; Δ is the value of the separation between contour levels. From Schou *et al.* (1992).

5.3.1. Asymptotic sound-speed inversion. Simple inversion procedures can be obtained from the asymptotic properties of the oscillations which were discussed in section 2.3. In the asymptotic limit, the frequencies are determined by the sound speed and by the function $\alpha(\omega)$ which describes the behaviour of the modes very near the solar surface. The dependence of frequencies on sound speed is given by Equations (23) and (24). Here, the functions $F(\omega/L)$ and $\alpha(\omega)$ can be determined by fitting the observed frequencies to a relation of the form given in Equation (23). Given F , Equation (24) provides an integral equation for the sound speed c as a function of r . This can be inverted analytically (Gough 1984) to yield

$$r = r\left(\frac{c}{r}\right) = R \exp \left[-\frac{2}{\pi} \int_{c_0/R}^{c/r} \left(w^{-2} - \frac{r^2}{c^2} \right)^{-1/2} \frac{dF}{dw} dw \right], \quad (78)$$

where c_0 is the surface value of c ; hence, $c(r)$ can be determined. This procedure was described in more detail by Gough (1986b). It was applied to observed frequencies by Christensen-Dalsgaard *et al.* (1985) who were able to determine the sound speed in much of the Sun with a precision of considerably better than 1 per cent. Similar inversion techniques based on the asymptotic expression (22) have been developed (*e.g.* Brodsky and Vorontsov 1987, Kosovichev 1988, Sekii and Shibahashi 1989, and Vorontsov and Shibahashi 1990). They are mainly distinguished by the methods of fitting the data to the asymptotic expression, particularly the separation into the parts depending on ω/L and ω . It should be noted that, in these procedures, the uncertain influence of nonadiabaticity and other effects near the solar surface is eliminated through the function $\alpha(\omega)$.

A very attractive feature of these inversion methods is that they are *absolute* : the sound speed $c(r)$ is obtained directly from the data, without any use of a solar model. However, they suffer from systematic errors arising from inaccuracies in the asymptotic Equation (22). It was shown by Christensen-Dalsgaard, Gough, and Thompson (1989) that these errors, to a large extent, cancel if one considers instead differences between frequencies of pairs of models; this suggests that a *differential asymptotic inversion* of the solar data may be more accurate. This can be carried out by fitting Equation (27) to differences between observed solar frequencies and those of a suitable reference model; the resulting function $H_1(\omega/L)$ is linearly related to the sound-speed difference $\delta c/c$ between the Sun and the model, and this relation can be inverted analytically to obtain an estimate of δc . The effects of near-surface uncertainties are eliminated in the fit through the term $H_2(\omega)$ in Equation (27). Christensen-Dalsgaard, Gough, and Thompson (1991) used this method to infer the solar sound speed and hence the depth of the solar convection zone.

5.3.2. Linearized structure inversion. To move beyond the asymptotic treatment, one must consider the general relation (76) for the frequencies, possibly also taking into account departures from the adiabatic approximation. As is common for non-linear equations, Equation (77) is “solved” through linearisation around an initial reference model. Let $(\rho_0(r), \Gamma_{1,0}(r))$ correspond to the reference model, which has oscillation frequencies $\omega_{nl}^{(0)}$. We seek to determine corrections $\delta\rho(r) = \rho(r) - \rho_0(r)$ and $\delta\Gamma_1(r) = \Gamma_1(r) - \Gamma_{1,0}(r)$ to match the differences $\omega_{nl}^{(\text{obs})} - \omega_{nl}^{(0)}$ between the observed frequencies and those of the reference model. By linearizing Equation (77), assuming $\delta\rho$ and $\delta\Gamma_1$ to be small, one obtains

$$\omega_{nl}^{(\text{obs})} - \omega_{nl}^{(0)} = \int_0^R K_{nl}^{(\rho)}(r) \delta\rho(r) dr + \int_0^R K_{nl}^{(\Gamma_1)}(r) \delta\Gamma_1(r) dr, \quad (79)$$

where the kernels $K_{nl}^{(\rho)}$ and $K_{nl}^{(\Gamma_1)}$ are determined from the eigenfunctions in the reference model. An additional constraint is that the mass of the Sun and the reference model be the same; i.e.,

$$\delta M = 4\pi \int_0^R \delta\rho(r) r^2 dr = 0. \quad (80)$$

In Equation (79), a term may be included which takes into account the uncertainties introduced by the surface layers. Indeed, it was argued in section 4 that such uncertainties introduce frequency changes $\delta\omega_{nl}$ such that the scaled frequency difference $Q_{nl}\delta\omega_{nl}$ is a function of frequency alone. This suggests adding to the right-hand side of Equation (79) a term $Q_{nl}^{-1}\mathcal{G}(\omega)$, where the function $\mathcal{G}(\omega)$ is determined as part of the inversion (see Dziembowski, Pamyatnykh, and Sienkiewicz 1990, and Däppen *et al.* 1991). Note that this is closely analogous to the determination of the function $\alpha(\omega)$ or $H_2(\omega)$ in the absolute and differential asymptotic inversions.

After linearization, the inverse problem has been reduced to a form similar to that considered in section 5.1, and the techniques discussed there can be used, with comparatively little modification, to infer the corrections $\delta\rho$ and $\delta\Gamma_1$ to the reference model. The process could then, in principle, be iterated by adding the corrections to the reference model, computing a new reference model by imposing again the constraint of hydrostatic equilibrium, and repeating the inversion. So far, there is little experience with the properties of such iteration, however.

Figure 17. Corrections to a reference solar model, obtained by inverting differences between the observed frequencies and the frequencies of the model. Panel a) shows corrections to $u \equiv p/\rho$, and panel b) shows corrections to ρ . The vertical bars indicate the errors in the results, based on the errors in the observed frequencies, whereas the horizontal bars provide a measure of the resolution in the inversion (from Däppen *et al.* 1991).

As an example of recent inversions, one may consider the analysis by Däppen *et al.* (1991), who carried out the inversion for the corrections to the model by means of the method of optimally localized averages, discussed in some detail in section 5.1. Typical results are illustrated in Figure 17, for the corrections $\delta\rho$ and δu ($u = p/\rho$ being closely related to the sound speed) which should be applied to the model to approximate the Sun. Here, the reference model was Model 13 of Table 2 in Christensen-Dalsgaard *et al.* (1991); this used rather similar physics to the model with MHD equation of state and LAOL opacities considered in Figure 12; however, the opacity was increased somewhat in the vicinity of the base of the convection zone, to increase the convection zone depth to $0.286R$. The figure shows that there are systematic differences between the Sun and the model but that these are generally fairly small. It is probable that the positive values of

$\delta u/u$ around $r = 0.5R$ result from errors in the opacities used to compute the reference model. Further work is required to test whether all the discrepancies between the solar data and the model can be accounted for in terms of such simple changes to the physics used in the model calculation.

6. Frequency Variation with Time

It is of obvious interest to study the temporal variation of the solar oscillation frequencies, to look for effects of possible changes in solar structure associated with the solar cycle. Such effects might help elucidate the mechanisms responsible for the variations of solar magnetic activity.

Evidence for time variation of the frequencies was obtained by Woodard and Noyes (1985), who found an average frequency decrease of about $0.4 \mu\text{Hz}$ from 1980 (near solar maximum) to 1984 (approaching minimum) in low-degree modes. More detailed results, although still restricted to low-degree modes, were obtained by Elsworth *et al.* (1990c), who followed the frequency change through an entire solar cycle; the amplitude of the change was roughly consistent with the variation found by Woodard and Noyes (1985); Elsworth *et al.* (1990c) also presented evidence that the frequency variation was correlated with the smoothed sunspot number.

Figure 18. Frequency shift as a function of frequency, using frequencies from 1986 as a reference. The frequency dependence was obtained by averaging over modes in the range $4 \leq l \leq 140$ in degree. Data from 1988 are denoted by circles, data from 1989 by squares. From Woodard and Libbrecht (1991).

Much more detailed results were obtained by Libbrecht and Woodard (1990); they determined the frequency change between 1986 and 1988 for a large number of modes, and hence were able to study the dependence of the frequency shift on the frequency and degree of the modes. Typical results, which also include data from 1989, are shown in Figure 18. As 1986 was near solar minimum, 1988 was on the ascending branch of the solar cycle, and 1989 near solar maximum, the results again show a close correlation between the magnetic activity and the frequencies. Furthermore, the fact that the frequency change is a rapidly increasing function of frequency, being very small at low frequency, strongly suggests that it is caused by effects localized very near the solar surface. This is confirmed by analysis of the l -dependence of the change. An even closer association with the surface magnetic field was found by Woodard *et al.* (1991), who followed the frequency change on a month-by-month basis. Figure 19 shows the average frequency change relative to 1986. Also shown is a magnetic field index, defined as the average of the absolute value of the magnetic field as obtained from Kitt Peak magnetograms and assumed to correspond to the mean square field (see Chapter 6). The correlation between the frequency change and the field index is striking.

Figure 19. The solid curves show average frequency shifts over 23-day intervals, taking the average frequency for 1986 as a reference. The dashed curves show a properly scaled magnetic field index, obtained as an average of the absolute value of the magnetic field. From Woodard *et al.* (1991).

Libbrecht and Woodard (1990) also noticed another close connection between the oscillation frequencies and surface activity. As discussed in section 3.3, departures from spherical symmetry induce even terms in the expansion given in Equation (65) of the frequency splitting. Libbrecht and Woodard found that the even coefficients a_i , and their change with time, displayed a behaviour qualitatively similar to the frequency change shown in Figure 18. Furthermore, the latitude variation inferred for the asphericity responsible for the even coefficients was very similar to the measured variation with latitude of the solar-limb

brightness; as the limb brightness is correlated with solar activity, this gives further support for the direct relation between frequency changes and activity.

The physical cause of the frequency changes has not been definitely established; however, it seems likely that direct magnetic effects dominate. Gough and Thompson (1988) showed that effects of a fibril magnetic field could account for the closely related even component of the frequency splitting. This description was further elaborated by Goldreich *et al.* (1991) and Dziembowski and Goode (1991) who showed that it could account for some, if not all, of the features in the observed frequency change; in particular, the frequency change was found to be proportional to the mean square field, in accordance with the results shown in Figure 19. An alternative explanation, in terms of variations in the chromospheric magnetic field which might change the atmospheric boundary conditions for the oscillations, was proposed by Campbell and Roberts (1989) and Evans and Roberts (1990) (see Chapter 6).

Goode and Dziembowski (1991) and Goode *et al.* (1991) speculated that there might be evidence for change during the solar cycle in the helioseismically inferred equatorial angular velocity near $r \simeq 0.4R$; the statistical significance of such variation was questioned by Schou (1991b), however. It might also be noted that Dziembowski and Goode (1989, 1991) found evidence for a strong (megagauss) field near the base of the solar convection zone, which apparently does not change during the solar cycle. It is evident that such results, if confirmed by improved observations and a more careful analysis, are of great importance for our understanding of the solar cycle.

7. Prospects

Although substantial efforts are under way to increase our theoretical understanding of solar oscillations and to develop new analysis tools, it is probably a fair assessment that the major developments in helioseismology in the coming years will result from improvements in the observations. The principal problems in current data are the presence of gaps, leading to sidebands in the power spectra, and the effects of atmospheric noise. The problem with gaps will be overcome through observations from global networks; nearly continuous observations, which are furthermore free of effects of the Earth's atmosphere, will be obtained from space. The result, ten years from now, should be greatly improved sets of frequencies, extending to high degree, from which one may expect a vast improvement in our knowledge about the solar interior.

7.1. NETWORK OBSERVATIONS OF SOLAR OSCILLATIONS

As shown in Figure 6, gaps in the timeseries introduce sidebands in the spectrum; these add confusion to the mode identification and contribute to the background of noise in the spectra. Largely uninterrupted timeseries of a few days' duration have been obtained from the South Pole (*e.g.* Grec *et al.* 1980 and Duvall *et al.* 1991); however, to utilize fully the phase stability of the modes at relatively low frequency requires continuous observations over far longer periods, and these cannot be obtained from a single terrestrial site.

Nearly continuous observations can be achieved from a network of observing stations, suitably placed around the Earth (*e.g.* Hill and Newkirk 1985). An overview of current network projects was given by Hill (1990). A group from the University of Birmingham has operated such a network for several years, to perform whole-disk observations using the resonant scattering technique (*e.g.* Aindow *et al.* 1988). Among the significant results already obtained from this effort are the frequency separations (Elsworth *et al.* 1990a), which were discussed in section 4.2 and illustrated in Figure 13. A similar network (the IRIS network) is under construction by a group at the University of Nice (Fossat 1991).

An even more ambitious network is being established in the GONG project, organized by the National Solar Observatory of the United States (for an introduction to the project, see Harvey, Kennedy, and Leibacher 1987). This project involves the setting up at carefully selected locations of six identical observing stations. They use the Michelson interferometer technique, to observe solar oscillations of degrees up to around 250. In addition to the design and construction of the observing equipment, a great deal of effort is going into preparing for the merging and analysis of the very large amounts of data expected, and into establishing the necessary theoretical tools. The network is expected to become operational in 1994.

7.2. HELIOSEISMIC INSTRUMENTS ON SOHO

Major efforts are going into the development of helioseismic instruments for the *SOHO* satellite, which has a planned launch in 1995. *SOHO* will be located near the L_1 point between the Earth and the Sun, and hence will be in continuous sunlight. This permits nearly unbroken observations of solar oscillations. A further advantage is the absence of effects from the Earth's atmosphere. These are particularly troublesome for observations of high-degree modes, where seeing is a serious limitation (*e.g.* Hill *et al.* 1991), and for intensity observations of low-degree modes, which suffer from transparency fluctuations.

SOHO will carry three instrument packages for helioseismic observations:

- The GOLF instrument (for Global Oscillations at Low Frequency; see Gabriel *et al.* 1991). This uses the resonant scattering technique in integrated light. Because of the great stability of this technique, it is hoped to measure oscillations at comparatively low frequency, possibly even *g*-modes. Unlike the *p*-modes, which have formed the basis for helioseismology so far, the *g*-modes have their largest amplitude near the solar centre; hence, detection of these modes would greatly aid the study of the structure and rotation of the core. Also, since the lifetime of *p*-modes increases rapidly with decreasing frequency, very great precision is possible for low-frequency *p*-modes.
- The SOI-MDI experiment (for Solar Oscillations Investigation – Michelson Doppler Imager; see Scherrer, Hoeksema, and Bush 1991) will use the Michelson interferometer technique. By observing the entire solar disk with a resolution of 4 arcseconds, and parts of the disk with a resolution of 1.2 arcseconds, it will be possible to measure oscillations of degree as high as a few thousand; furthermore, very precise data should be obtained on modes of degree up to about 1000, including those modes for which ground-based observation is severely limited by seeing. As a result, it will be possible to study the structure and dynamics of the solar convection zone, and of the radiative interior, in great detail.

- The VIRGO experiment (for Variability of solar IRradiance and Gravity Oscillations; see Andersen 1991). This contains radiometers and Sun photometers to measure oscillations in solar irradiance and broad-band intensity. It is hoped that this will allow the detection of g -modes; furthermore, the observations will supplement those obtained in Doppler velocity, particularly with regards to investigating the phase relations for the oscillations in the solar atmosphere.

7.3. ASTEROSEISMOLOGY

It is of obvious interest to extend seismic studies to stars other than the Sun. Here the definition of “seismic studies” is somewhat imprecise, since even inferences based on the periods of “classical” variable stars could be included. However, it is probably more reasonable to restrict the term to those cases where rich spectra of oscillations can be observed, and where, therefore, there is a considerable amount of information about the properties of the star. Very interesting examples are provided by the white dwarfs (for a review, see Winget 1988), where extensive spectra of g -mode frequencies have been obtained from intensity observations. Spectra of high-order p -modes have been observed in the rapidly oscillating Ap-stars (see Kurtz 1990); these spectra show some similarities to the observed spectrum of solar oscillations, although the amplitudes are larger by two to three orders of magnitude.

Detection of solar-like oscillations in other stars has proved more difficult, as a result of their expected very low amplitude. Brown *et al.* (1991) obtained power spectra from Doppler observations of Procyon, which strongly suggested the presence of p -mode oscillations, although no definite frequency determinations were possible because of the effects of sidebands caused by the gaps in the data. Possible evidence for oscillations has also been obtained for α Cen A (Butcher, Christensen-Dalsgaard, and Frandsen 1990; Pottasch, Butcher, and van Hoesel 1992); these included a slight suggestion of equally-spaced frequencies, as might have been anticipated from the asymptotic Equation (29), corresponding to a separation $\Delta\nu$ of about $110\ \mu\text{Hz}$.

As discussed in section 3.1.2, Doppler observations of stellar oscillations suffer from the fact that only a small part of the spectrum, and hence of the available photons, is used; as a result, photon statistics limit the accuracy that can be achieved, restricting the observations to fairly bright stars and demanding the use of large telescopes. No such limitation affects intensity measurements, where a sufficient number of photons can be obtained with a modest telescope; however, fluctuations in the Earth’s atmosphere, particularly scintillation, make the detection of oscillations at the solar amplitude very difficult. This evidently suggests that the observations be made from space. Indeed, observations of solar-like intensity oscillations in a few selected stars, using a 6 cm telescope, will be made in the EVRIS experiment (for Etude de la Variabilité, de la Rotation et des Intérieurs Stellaires) on the Soviet Mars probe *MARS 94*. Substantially more ambitious observations are planned in the *PRISMA* project (for Probing Rotation and Interior of Stars: Microvariability and Activity), which is currently undergoing Phase A studies in ESA (see Lemaire *et al.* 1991). Here, it is expected to observe of order 100 stars with a precision sufficient to detect solar-like oscillations.

7.4. CONCLUDING REMARKS

I hope that this Chapter has given some impression of the exciting results and tremendous possibilities of seismic investigations of the Sun. Twenty years ago, the concept of getting even rough observational information about the properties of the solar interior might have seemed utterly unrealistic. Now, we can imagine studying the Sun to a level of detail exceeding even what is possible for the interior of the Earth. It seems possible that subtle effects in the physics of matter in the Sun can be investigated, potentially yielding new information about the properties of dense plasmas. Finally, observations of solar-like oscillations in other stars now seem to be at the point where observations of solar oscillations were about 15 years ago. Although the data for other stars will never be as detailed as what is available for the Sun, this is partly compensated for by the possibility of studying stars of different masses and in different stages of evolution. By testing the basic assumptions of stellar evolution calculations, the results of such work will be of great importance to large areas of astrophysics.

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