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Statistical issues in the 17-keV neutrino experiments

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Abstract

The 17-keV neutrino experiments are reviewed, and the crucial question of observability in the presence of unknown distortions is addressed. Criteria are listed to narrow the field of acceptable experiments.

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1 Introduction.

Evidence for a massive neutrino was first reported in 1985 [1], then in 1989 [2, 3], and has been repeatedly claimed in the last year [4]. Negative results have also been reported by several groups [5], and plenty of theoretical models have appeared [6]. These models must accomodate a lepton sector where mass eigenstates are different from lepton number eigenstates. They must also tackle the somewhat unexpected value of the mass, 17 keV .

In a minimal extension of the Standard Model, the 17-keV neutrino is a Dirac neutrino, because of limits on double β^- decay, and is a mixture dominated by the τ -neutrino, because of limits on μ -neutrino oscillations [7]. This, in turn, will introduce a Cabibbo-Kobayashi-Maskawa lepton matrix where $V_{13} \gg V_{12}, V_{23}$. The CKM quark matrix, on the contrary, is strictly hierarchical. This may mean that the μ -neutrino mass is much larger than 17 keV and is close to the present experimental limit. Non minimal extensions involve introduction of extra particles [8].

With an established 17-keV neutrino, cosmology would go through a major overhaul. 17-keV is near the geometric mean of the excluded neutrino mass range. The universe, filled with $\approx 10^2$ τ -neutrinos per cm^3 would then have $10^1 - 10^2$ too much mass to agree with current estimates of the mass of the universe. Most of the solutions proposed so far invoke new physics. A problem arises also with the 17-keV neutrino lifetime, which would be short, affecting nucleosynthesis in the early universe.

So far the experiments focused on β^- -decay, with one of the experiments using electron capture [9]. Most urgent in this discussion is why experiments where the β^- energy is measured calorimetrically tend to see the effect, and those which use spectrometers do not. My analysis in Section 5 and 6 shows that large continuous distortions in the spectrum can indeed mask or fake a discontinuous kink. In the process I point to some deep inconsistencies in all the spectrometer experiments considered here.

This analysis was spurred by the clash of two arguments.

Supporters of the heavy neutrino say that they measure the electron response function of their experiments in one or two points across the spectrum, while the spectrometers do not and fit them out with arbitrary functions. They add that the arbitrary fitting of a distortion wastes the statistical precision of the experiment. This argument is very qualitative and not true in fact. The statistical error on the heavy neutrino amplitude increases by roughly a factor of two with the addition of one non-physical parameter, and this factor is in general correctly computed. The systematic error, however, which is never quoted, could be of order 0.8% in all of the spectrometer experiments studied here.

Opposers of the heavy neutrino will point to unspecified effects in the detector material as the source of the kink. They will add that a continuous distortion will never mask a kink. This argument sounds right, but in fact is not true, not even mathematically. The kink will be systematically suppressed or amplified, in a continuous way and possibly without impact on the χ^2 , and this effect is related to the presence of zeros in the first two or three derivatives, to the number of unphysical parameters used, and to the magnitude of the distortion. For example, if the distortion is 72 times larger than the kink, the kink can be anywhere between 0 and 2%.

The issue is still very open. The repeated observation of the same kink at the same mass with different nuclei is puzzling. The fact that several spectrometric experiments, despite their larger systematics, consistently agree to $\sin^2 \theta \approx 0$ is equally puzzling. At the end of the paper, I felt that a discussion of future experimental activities was in order.

Section 2 serves as an introduction to the phenomenology of these experiments. Some important theoretical aspects will be only briefly summarized. Section 3 is dedicated to the positive evidence, focusing on the most recent published experiment[11]. Section 4 is dedicated to the negative evidence, focusing on the 1987 experiment of Hetherington *et al.*[12], by general agreement the most complete spectrometer experiment so far. Section 5 and 6 are original work which deals with the observability of a kink in a spectrum when the experimental distortions have to be evaluated with the data themselves. Section 7 contains the conclusions.

2 β decay experiments.

The β^- decay of a nucleus of charge Z is

$$Z \rightarrow (Z + 1) e^- \nu. \quad (1)$$

The source is chosen so that competing processes (electron capture, other β decays, or nuclear transition in the daughter nucleus) are absent or strongly suppressed. The electron kinetic energy E is continuously distributed between 0 and the end-point energy Q . E is the only quantity recorded in the experiment. Relevant sources for this discussion are ^3H , ^{35}S and ^{63}Ni , with end-point energies of 18.7, 167 and 66 keV respectively.

Suppose the final state contains a mixture of two neutrinos, one dominant and massless, the other massive and appearing with small probability $\sin^2 \theta$. The electron kinetic energy spectrum is proportional to (m_e is the electron mass)

$$\begin{aligned} \frac{dN}{dE} &\propto p(E + m_e)(\cos^2 \theta A(0, E) + \sin^2 \theta A(M, E)), \\ A(M, E) &= \text{Re}(\sqrt{(Q - E)^2 - M^2})(Q - E). \end{aligned} \quad (2)$$

The Kurie plot $K(E)$, obtained by dividing out the phase space factor $p(E + m_e)$ and then taking the square root, will look qualitatively like Fig. 1a, showing a kink at $Q - M$. Attempts to fit it with a one-component plot will produce a residual distribution like in Fig. 1b.

Before a spectrum can be fitted it must be deconvoluted from the raw spectrum, parametrized as follows ($\otimes$ means convolution)

$$\begin{aligned} \frac{dN}{dE} &\propto \quad (\text{Kurie plot}) \\ &\otimes (\text{Coulomb factor}) \\ &\otimes (\text{screening effects}) \\ &\otimes (\text{lattice effects}) \\ &\otimes (\text{electron response function}) \\ &\otimes (\text{shape factor}). \end{aligned} \quad (3)$$

The Coulomb factor is due to the e^- being created in a region of strong positive potential. This factor is particularly large for low energy electrons and difficult to calculate precisely. In 1985, the ^3H experiment of Ref.[1] found $\sin^2 \theta \approx 3\%$, later to be reduced to $\approx .5\%$ when more precise calculations of the Coulomb factor became available [13]. For this reason, as well as the proximity of the kink to detector noise, it is safe to ignore altogether ^3H experiments. In this paper the tables from Ref.[14] are used to implement theoretical corrections.

Screening corrections are due to the electron crossing the cloud of electrons surrounding the nucleus. Lattice effects are relevant only for those experiments where the source has been implanted into the detector and ions are subject to the lattice potential. These effects are small or missing in the case of ^{35}S .

The electron response function (ERF) is the convolution of all the effects which have an effect on the acceptance. A major effect, backscattering, is due to the near-brownian motion of $50 - 500\text{keV } e^-$ in matter [15]. Electrons entering a piece of silicon are backscattered with a probability $\approx 13\%$, slowly dependent on energy. A good experiment must control and understand transport of radiation from source to counter to at least two collisions with any part of the apparatus. Predominantly backscattering happens in the source support, degrading the electron energy, and in the counter itself, depositing only part of the energy. This effect disappears with an internal source which is implanted deeply enough into the counter to have full containment.

A typical ERF is shown in Fig.2. The width of the main peak is what one usually calls the energy resolution. The structure to the left produces a broad reflection of the energy spectrum at scaled-down energies. Such an effect, coupled to a fit which does not account for distortions, will produce non zero values of $\sin^2 \theta$. The exact shape of the broad peak depends strongly on the experimental acceptance [15]. If the tail has no maximum, it will still artificially overweight the lower energies and destroy the fit. The ERF tail, by populating the low energy region, is equivalent to a non linear instrument, with the second derivative of the non linearity being negative.

The shape factor (SF) is an *ad-hoc* extra parameter in the fit, linear and/or quadratic in E , used only by the negative experiments. Originally it was used to implement theoretical corrections. However, both ^{35}S and ^{63}Ni are allowed transitions for which the theoretical SF is expected to be negligible, so in this case the SF is used to take into account parts of the ERF which are not corrected. These missing distortions are, in the experiments considered below, larger than 0.8%. Without SF a fit to Fermi theory would produce a large χ^2 . Even in the best case [12], the shape factor is dominated by an ERF which could not be measured precisely enough with other methods.

3 Evidence for a heavy neutrino.

Evidence for a heavy neutrino has only been reported by experiments where E was measured using an absorption device, either a *Si* or *Ge* semiconductor counter. The source was either implanted with the help of an accelerator [3], or was outside facing the counter [11], or had been dissolved in the counter itself [16], using elements from the same column of the Mendeleiev Table. This technique seems promising, eliminating all possible backscattering effects. At the time this paper was written, however, the experiment was not advanced enough to deliver final results.

Ref. [11] is picked for the comparison, because it is a recent result, with high statistics and improvements over Ref. [2]. In the following two negative results, Refs.[10] and [19], are not studied, because it is not clear whether shape factors were used or not, and because their shape factor seems significantly not flat between the kink and the end point. The follow up of Ref.[10], Ref.[20], contains a quadratic shape factor so the analysis of Sections 5-6 applies.

A planar, thin source of ^{35}S ($2\mu\text{g}/\text{cm}^2$) was deposited on a thin Mylar support ($200\mu\text{g}/\text{cm}^2$) and collimated so that only e^- incident at near normal angles would be accepted by the counter (Fig.3). Data were taken in two one-week runs, with slightly different geometries. Before and

after data taking line sources were used for calibration purposes. The fit is performed with four parameters: a normalization constant C , Q , $\sin^2 \theta$ and M . These parameters are called hereafter physical, or proper to the pure β spectrum. The results, shown in Fig. 4, clearly show a kink at 150 keV, excluding the one-component spectrum with 8σ statistical significance. The results of Runs 1 and 2 were consistent.

More interesting are perhaps Figs. 5 and 6, which show the linearity, resolution and ERF as measured with line sources. Several comments are in order:

- Fig. 5a, being the plot of the peak positions of several sources, shows that the counter is linear, not that the experiment is linear.
- Fig. 5b shows the energy resolution of the counter. Resolution is not important here, as long as it is much smaller than 17 keV. However, Fig. 5b has been obtained with photons only, which fly farther into the counter before depositing energy, and is not necessarily a measurement of the electron resolution.
- Fig. 6a shows the ERF for an e^- line at 129 keV. Fig. 6b, however, is a combination of data from at least two different line sources [17], and not a continuation of Fig. 6a.

Fig. 6b is evidence that the ERF is partially measured in this experiment, but it is not definitive evidence because it has been measured in one point only. If one could use three or more line sources, distributed across the fitted spectrum, one could effectively measure the energy evolution of normalization and shape of the ERF. The claim is that the fraction of β^- backscattered off silicon has been measured so well over the decades, and is so slowly varying, that such a measurement would be redundant.

The low energy tail must be controlled because, again, the kink can be fabricated by degraded particles. This is why backscattering has emerged as a crucial aspect of these experiments. Experimenters are urged to publish as much evidence as possible about their ERF measurements.

4 Evidence against a 17-keV neutrino.

All spectrometer experiments have given only negative results so far. Several of the spectrometer experiments were done with a $\pi\sqrt{2}$ iron-free spectrometer as shown schematically in Fig. 7. In the following, the results of Ref.[18] are not considered because it is unclear whether shape factors were used or not. Other results which became available while this paper was being written are also omitted.

The experiment chosen for the comparison was performed at Chalk River [12], using a ^{63}Ni source. e^- emitted in a narrow angular region at a given momentum were accepted by the spectrometer and transported to a low-threshold counter situated behind a thin window ($168\mu\text{g}/\text{cm}^2$). The source support was orders of magnitude thicker than the one of Ref.[11].

The spectrum was scanned by changing the magnetic field randomly at fixed time intervals, using a source with a long lifetime. The magnetic system was kept stable to one part in 10^5 , and two scans were performed, one between 20 and 67 keV, with 8×10^7 counts (wide scan) and one between 46 and 54 keV, with 10^7 counts (narrow scan), to which the 0.7×10^7 events from the same region of the wide scan were added. The 17-keV kink would have been located at 50 keV.

In this experiment, there are no direct measurements of the ERF. Both the backscattering probability and the window transmission are parametrizations of Monte Carlo simulations, each with one free parameter. Line sources are used to determine the energy resolution of the experiment, which is nearly an order of magnitude better than in calorimetric experiments, but the thick low energy tail (Fig. 8) is explained as shake-up/off effects and not used to correct the data.

Besides the four physical parameters, the fit used:

- a backscattered fraction F_B . Only the normalization was left as a free parameter. This effect changes the 26 keV point by +7.5%.
- a window transmission T . A single parameter, proportional to the window thickness, was varied in the fit. This effect changes the 26 keV point by -10.9%.
- To get a sensible fit, one arbitrary shape factor, chosen to be linear, had to be introduced. This was enough to lower the reduced χ^2 to about 1. One possible explanation is that one important source of systematics had not been corrected. Another possibility is incorrect energy dependence of the corrections applied. The shape factor was about 2.5%, which is .6 times the overall correction from F_B and T .

All of the main corrections are shown in Fig.9[22]. The wide spectrum final result, Fig. 10a, shows no kink. If the second neutrino contribution is held fixed in the fit, a characteristic residual distribution appears, (Fig. 10b) mirroring the one of Fig. 1b. The most restrictive limit, using only the data between 26 and 67 keV, is $\sin^2 \theta < .28\%$ at the 90% C.L. The narrow scan fit sets a limit of $\sin^2 \theta < .44\%$ at the 90% C.L. These results are summarized in Table 1.

When one adds a SF truncated to first order, like in this case, one approximately takes into account missing first order distortions in F_B and T if the distortion is perturbative (*ie*, the Taylor series converges quickly). If the higher orders are large, or if there are maxima and minima in the missing distortion the total effect could be unpredictable and capable of generating systematics.

The higher order terms may be of order percent, *ie* one or more times larger than $\sin^2 \theta$, depending on whether or not the distortion washed the kink away. The analysis presented in the next two Sections shows that very non linear distortions can be fitted out with a linear shape factor. F_B is parametrized as a combination of algebraic and polynomial functions, T is an exponential of a power function. These functions typically have large higher orders. It is assumed that the missing distortion will have similar characteristics.

Using a linear shape factor of 2.5% when looking for a 3% kink is certainly acceptable, whether or not there are uncorrected maxima and minima. However, when the limits are applied to a 0.8% kink, more study is needed.

Finally, it is assumed in the following that the missing distortion grows from zero to 2.3-2.7% between 26 and 67 keV, *ie* that the relative change of the SF is always fitted correctly although its functional form is not. This assumption was explicitly proven using the Monte Carlo described in the next two Sections.

5 Analysis of the fits.

This Section deals with statistical and analysis issues and studies the observability of a kink when the experimental corrections have to be evaluated with the data themselves. This analysis

is valid for all cases where an unknown source of systematics is fitted with an arbitrary SF. To the best of my knowledge, this study has never been undertaken [23]. What one ultimately wants from this study is an assessment of the systematic errors of the experiments, which are almost never quoted but that must be present if arbitrary shape factors are used.

A kink δ in a curve t is defined as a discontinuity in the first derivative, or the difference of the derivatives taken from the left and the right

$$\delta = \left(\frac{dt}{dx}\right)_- - \left(\frac{dt}{dx}\right)_+. \quad (4)$$

For the sake of clarity the scaled variable $x = E/Q$ is used in most of the following analysis.

Consider the following pedagogical example. Take a straight line with a small kink δ . The interval over which the spectrum is considered can always be mapped into $[0, 1]$ with no loss of generality, the kink being located at x_k . This case is equivalent to a superposition of two Kurie plots, one dominant and one small, the small one having a lower Q , and both neutrinos being massless.

The true spectrum has the form

$$t(x) = \begin{cases} \theta + \mu x, & x < x_k \\ (\theta + x_k \delta) + (\mu - \delta)x, & x > x_k \end{cases} \quad (5)$$

The kink is equal to δ and all higher order derivatives are zero. Suppose one fits Eq.(5) with a straight line

$$f(x) = (\theta - q) + (\mu - m)x. \quad (6)$$

Here greek characters refer to true quantities, roman characters are differences between true and fitted quantities. All variables but θ and μ are of $O(1\%)$ and can be treated perturbatively.

The $\chi^2/dof - 1$ function corresponds to the $\chi^2/dof - 1$ of an experiment with large statistics and frequent sampling ¹

$$\chi^2/dof - 1 = \int_0^{1-\epsilon} dx \frac{(t(x) - f(x))^2}{f(x)}. \quad (7)$$

In the case under study, and to leading order in the $O(1\%)$ parameters, the $\chi^2/dof - 1$ is

$$\chi^2/dof - 1 = \int_0^{1-\epsilon} dx \frac{(q + mx)^2}{\theta + \mu x} + \delta \int_{x_k}^{1-\epsilon} dx \frac{(x_k - x)(\delta(x_k - x) - 2(q + mx))}{\theta + \mu x} \quad (8)$$

Minimization of the χ^2 gives the following compact solution for q and m

$$\begin{aligned} q &= \delta(x_k g_{20}(x_k) - g_{21}(x_k))/g_{20}(0), \\ m &= \delta(x_k g_{10}(x_k) - g_{11}(x_k))/g_{20}(0), \end{aligned} \quad (9)$$

with

$$g_{ij}(u) = h_i(0)h_j(u) - h_{i-1}(0)h_{j+1}(u) \quad (10)$$

and

$$h_0(u) = \frac{1}{\mu} \log\left(\frac{\theta + \mu}{\theta + \mu u}\right), \quad h_n(u) = \frac{1}{n\mu}(1 - u^n) - \theta h_{n-1}(u). \quad (11)$$

¹A small cutoff is needed to avoid the singularity at the end point. In real data, background fluctuations eliminate the singularity. The effect of this cutoff is negligible.

Addition of a linear shape factor gives

$$t(x) \rightarrow t(x)(1 + \alpha x), \quad f(x) \rightarrow f(x)(1 + (\alpha - a)x) \quad (12)$$

and the $\chi^2/dof - 1$ function becomes to leading order

$$\chi^2/dof - 1 \rightarrow \chi^2/dof - 1 + \int_0^{1-\epsilon} dx a^2 x^2 (\theta + \mu x) + 2ax(q + mx) - 2\delta \int_{x_k}^{1-\epsilon} dx (x_k - x)ax. \quad (13)$$

Notice that the $\chi^2/dof - 1$ does not depend on α . The solution for a is

$$a = \frac{\delta [(x_k h_2(x_k) - h_3(x_k))g_{20}(0) - (x_k h_1(x_k) - h_2(x_k))g_{30}(0) + (x_k h_0(x_k) - h_1(x_k))g_{31}(0)]}{\mu (h_4(0)g_{20}(0) - h_3(0)g_{30}(0) + h_2(0)g_{31}(0))} \quad (14)$$

and m and q become

$$\begin{aligned} q &\rightarrow q + a\mu g_{31}(0)/g_{20}(0), \\ m &\rightarrow m - a\theta - a\mu g_{30}(0)/g_{20}(0). \end{aligned} \quad (15)$$

These solutions are used to evaluate numerically the $\chi^2/dof - 1$ in the interval $.1 < x_k < .9$, Fig. 11. When the linear shape factor is added, the $\chi^2/dof - 1$ decreases about one order of magnitude (up to a factor of 19), depending on x_k . An experiment which would get a reduced chi square χ^2/dof of 2.7, at $x_k = 0.65$, would have it reduced to 1.1 with the addition of a shape factor. 0.65 is a typical value for ^{63}Ni experiments.

To understand what is going on look at Fig. 12. Consider the region near the kink. In Fig. 12a), the typical distance between the fitted and true curve is of order $\delta/2$ with two crossings. In Fig. 12b) the true curve only rotates slightly, because the correction is small, and the fit has a way to double the number of crossings and reduce the typical distance to $\delta/4$.

The above example is not applicable to the 17-keV neutrino. First, the heavy neutrino component is not linear and contains all powers of E (or equivalently x). Second, the kink is strictly speaking infinite and is washed down to a finite quantity depending on energy resolution and binning of the experiment.

The best way to study this case is via Monte Carlo, although in principle all calculations could be done analytically. Let $(dN/dx)_0$ be the spectrum with no kink, and $(dN/dx)_1$ the spectrum with the kink. The conditions for masking the kink are that the $\chi^2/dof - 1$ function in the presence of an unknown distortion D

$$\chi^2/dof - 1 = \int_{min}^1 ((\frac{dN}{dx})_1(1 + (\alpha - a)D(x) - (\frac{dN}{dx})_0(1 + \alpha x))^2 dx$$

be small and that the minimum χ^2 be obtained around zero.

Before proceeding with the Monte Carlo analysis some comments are in order. A multiparameter fit will be characterized by the χ^2/dof , the output vectors of the fitted parameters and their estimated errors, called hereafter y_i and σ_i . One can also calculate the correlation coefficients between two parameters, called hereafter V_{ij} as well as the global correlation coefficient for each parameter, called ρ_i . Distortions will affect all these quantities.

When using only physical parameters, the χ^2/dof is a sensitive measurement of the quality of the fit. The statistics considered here (few 10^7 - few 10^8 counts per experiment) produce χ^2/dof distributions with natural widths of about 15%, if the fitting function was the correct

one. With the addition of a one-parameter SF there is a loss of roughly one order of magnitude in $\chi^2/dof - 1$. A large set of distorting functions (those that would have produced χ^2/dof less or of order 2.5) will be fitted by a linear correction with no impact on the χ^2 .

The single parameter errors σ_i are either calculated by the fitting routine or explicitly calculated by scanning one parameter and fitting with respect to all others. The single parameter error depends only on the variance distribution and not on the central value of each bin. Once the total statistics is known, the single parameter errors can be predicted precisely. Extra sources of variance, like background subtraction or inefficiency corrections, are typically small and may increase but not decrease these errors.

With the χ^2 sensitivity degraded by the extra parameter, the size of errors becomes an important cross check. If a kink was there and had been washed away, or viceversa, two possibilities exist: one in which the kink had already been washed out in the raw data, and one in which some remnant of the kink was in the raw data and was straightened by applied or fitted corrections. In the former case, there must have been one distortion with some resemblance to the mirror image of Fig. 1a). In the latter, $\sin^2 \theta$ will have an anomalous, non-statistical error as correlation between variables is modified.

It is best to assess the loss of information on any one of the fit parameters by means of the global correlation coefficients ρ_i . These are the largest correlation between parameter y_i and a linear combination of all other parameters y_j . $\rho_3 = 1$ ($\sin^2 \theta$ is the third parameter in all fits presented below) means that $\sin^2 \theta$ can be expressed exactly as a combination of the y_j . As shown later in Table 3 or Table 4, the correlation is high which means that $1 - \rho_3$ is a quantity that can vary a lot.

$1 - \rho_3$ depends essentially on the number of non physical y_j in the fit. If there are none, $1 - \rho_3$ will be 10%-11%. In the case of ^{35}S , where the kink is close to the end point, $1 - \rho_3$ could decrease to 7%, because of background fluctuations. The position of the kink in the spectrum may change it by 1%. $1 - \rho_3$ will change depending on whether the experiment was calorimetric or spectrometric, since the observed spectra differ by a factor proportional to the momentum p and the variance distribution changes. At the .1% level $1 - \rho_3$ may depend on the form or size of an extra correction, and on the particular random scan generated by the program. The fact that $1 - \rho_3$ is about the same whether the fitted correction is 1% or 10% means that it is not the whole story, so I will partially address the problem of size and functional form in the next Section.

6 Simulation algorithm and results.

Mock scans were generated, using the same number of points and total statistics as declared by the experiments. Statistical fluctuations were given to each data point, plus background fluctuations to reproduce the original variance distributions for two of the experiments[24]. Input parameters were in order C , Q , $\sin^2 \theta$ and M (physical parameters), plus a linear and quadratic shape factors with coefficients α and β , and for the analysis of Ref.[12] also a window transmission T and backscattering fraction F_B were added.

Once the mock scan was generated, it was fitted [25] with a theoretical prediction using a subset of the input parameters as fitting variables and the rest as fixed parameters. For each fit presented here, a complete description is given in Ref.[26].

To investigate the effect of non perturbative distortions, a library of test $D_i(x)$ was set up.

Those that were used to produce the plots presented here are listed in Table 2. The chosen distortion was applied to the input function only, multiplied by an appropriately chosen small constant. Therefore a typical application of the program would look at the results of the fit when the true and fitted function were distorted by a non-linear and linear SF

$$t(x) \rightarrow t_0(x)(1 + (\alpha + a)D_i(x)), \quad f(x) \rightarrow f_0(x)(1 + \alpha x). \quad (16)$$

The functions $D_i(x)$ were divided in three blocks; the first had first derivative $D' > 0$ and $D'' < 0$, the second $D', D'' > 0$. The bias in the derivatives is because most shape factors are positive, and most corrections have negative second derivative. The last one was used uniquely for Fig. 22.

Two generating and fitting methods were used. In the first a data point was assumed to be the integral over the bin, in the second it was assumed to be the point value. The two methods are the same away from the kink, but differ in the kink region. Because of frequent sampling in both Ref. [11] and Ref. [12], the difference between the two methods was found to be negligible. Several checks were made, and the program reproduced Eqs.(7-15) to 1% or better.

The first series of fits looked at correlation coefficients for several published experiments. Once the statistics and number of parameters are declared, these coefficients are computable very precisely. M was held fixed at 17 keV, and variations of it did not affect the results. Table 3 shows² the correlation matrices of Ref.[11], Ref.[12] and Ref.[27]. Fig. 13 shows a plot of ρ_3 for five experiments. There is a separation between positive and negative experiments. Each extra parameter diminishes the quantity $1 - \rho_3$ by a factor of 2.5 for ^{63}Ni . ^{35}S changes less. Table 4 shows the expected ρ_3 in some cases of interest.

The second series of fits aimed at cross checking the ^{35}S result of Hime and Jelley[11]. This is particularly easy because only the four physical parameters are used. The errors σ_3 for Runs 1 and 2 are calculated to be $0.094\% \pm .003\%$ and $0.106\% \pm .003\%$ respectively, where the errors depend on the quoted M error. The quoted $\sin^2 \theta$ errors are 0.10% and 0.11% respectively. These results are a trivial consequence of the exact knowledge of the variance distribution.

In a further test, each of the test distortions were added with increasingly larger magnitude to both the ^{35}S and the more recent ^{63}Ni obtained with the same apparatus [28]. In the case of ^{35}S , the χ^2/dof would shoot up before $\sin^2 \theta$ could change appreciably (Fig.14), but in the case of ^{63}Ni one could find functions which would simulate the kink (Figs. 15 and 16). A continuous distortion could explain the ^{63}Ni result, although if one considers the distortion as proper to the apparatus (which was the same for Refs.[11] and [28]) then it would be impossible to obtain a good fit for ^{35}S . Fig. 17 shows that a linear shape factor would put the kink back in its proper place. Ref.[11] cross checks but later will suggest some improvements.

The third series of fits aimed at cross checking the result of Hetherington et al. Fig. 18 shows the results of 100 more scans, where the $D_i(x)$ were used in addition to a linear shape factor, thus allowing for one minimum or maximum which was outside the energy interval considered. The linear shape factor here was simply tuned to have an overall 2-3% increase, as specified at the end of Section 4. It was checked that the linear SF plus the D_i would not produce a minimum or maximum in the fitted energy interval. One scan which masked the kink is shown in Fig. 19. Addition of a quadratic term (Fig. 20) would put the kink back in its proper place. Since the authors did perform a quadratic fit, in which β came out very close to zero[29], these data seem guaranteed against all corrections having up to one zero in the first derivative.

²Most of the analysis here uses the scaled variable $x = E/Q$ which contains a dependence on Q . That would have changed all coefficients between Q and any other variable. This was taken into account, and all coefficients presented in the Tables used E as the spectrum variable.

Fig. 21 and Table 5 show the χ^2 minimum when also F_B and T are allowed to vary and Table 6 shows the correlation coefficients. $1 - \rho_3$ decreases by a factor of three. The result of Fig. 21 is in disagreement with the equivalent result of Ref.[12], which shows a 25% narrower minimum. The discrepancy cannot be attributed to a non linear correction, Table 7. The smaller-than-statistical error on $\sin^2 \theta$, σ_3 , is due to something which has changed the correlation between variables.

To study this, a smooth function with a broad bump was used (D_{11} , Table 2) and the perturbation was moved through the spectrum. Fig. 22 shows that when the bump is in the low energy region and well below the kink region σ_3 is significantly decreased together with a systematics of order .7% in $\sin^2 \theta$. The low energy region is where most of the missing distortion may be. Ref.[12] does not cross check, although with some caution[30]. The quoted σ_3 error in Ref.[12] is incompatible with statistics.

The fourth series of fits studied what happens when either α or β become very large (more than a factor 10 between the extremes of the fitted region) compared to $\sin^2 \theta$. This applies, for example, to Refs.[27] and [31]. In the former case, both α and β are about 20 times larger than $\sin^2 \theta$ and the shape factor has a peculiar peaking within 1 keV of the expected kink. Fig. 23 shows a $\chi^2 - \sin^2 \theta$ scatter plot when such a peaking is observed. There is a clear tendency for the kink to disappear. These kind of shape factors are obviously unacceptable. The kink also moves a lot in the latter case, Fig. 24, where the overall correction over the fitted spectrum was more than 70 times larger than $\sin^2 \theta$.

The fifth series of fits aimed at studying the statistical significance of narrow interval fits. These have been proposed as the only way to defend data against unpredictable distortions, Ref.[2]. Table 8, however, shows the correlation coefficients for the 46-54 keV scan of Ref.[12]. ρ_3 is .991 even without arbitrary shape factors (it becomes .997 with a linear shape factor, > .9995 with a quadratic shape factor). This much correlation makes the fit impossible. One variation to this theme is to move the narrow window across a larger interval, looking for a dip in the χ^2 correlated with a positive $\sin^2 \theta$, a technique which was used in some experiments. Fig. 25 shows one such scan, where statistical noise is picked up by the fit. The authors of Ref.[12] should not have fitted the narrow scan with a shape factor when $\sin^2 \theta$ was more than .99 correlated. The only way to fit this region would be to fix *both* Q and α to their wide scan values, and fit for C and $\sin^2 \theta$ only. This is a detail, since narrow scan data in all cases have low statistical significance.

There is however one point about narrow scans which should be saved. If a kink was there and disappeared, or was not there and appeared, some hint of the error must have remained in the kink region. There is a discontinuity in either the true or fitted spectrum, and smoothness in the other one. This amounts to creating a small fluctuation in the distribution of the residual χ . As seen in Figs. 16 and 19, there will be a dent at the kink in the residual plot if a kink was missed, and a bump if the kink was fabricated. Notice that such dents may be present in Refs. [12] and [27].

I propose to use some kind of simultaneous fit of the wide and narrow scans together as a mean to decrease the probability of producing a result with a large systematic error. As an example, consider the true shape factor

$$S = \frac{\chi}{fit} + 1.$$

Then fit it with a parabola in the 4 keV below the kink, and with a straight line in the 4 keV

above the kink

$$f_- = \theta_- + \mu_-x + \nu_-x^2, \quad f_+ = \theta_+ + \mu_+x$$

and extract the kink value, called here δ_{17} ,

$$\delta_{17} = \mu_- + 2\nu_-(Q - 17)/Q - \mu_+.$$

Fig. 26 shows distributions of the variable δ_{17} , for "good" and "bad" experiments. Assuming the same statistics as in Ref.[12], a cut on this variable could eliminate 80% of the bad experiments while retaining 80% of the good ones. An increase in statistics will improve separation.

The variable δ_{17} is one of many one could use for a simultaneous wide-narrow fit. Fig. 26 shows that as a stand alone test it is not significant, *ie* small dents or bumps taken out of context a signal do not make.

7 Conclusions.

Results of a statistical analysis of experiments on the 17-keV neutrino were presented. The results are reproducible by any other simple simulation program, quite an achievement in a field of irreproducible shape factors.

The Kurie plot experiments on massive neutrinos face the arduous task of extracting a variable, $\sin^2 \theta$, which is in the best case (Tables 3 and 4) more than 90% correlated with the other variables, and when the total χ^2 has a natural width of $\approx 15\%$. In these conditions kinks can be faked or hidden without any impact on the χ^2 (Figs. 16 and 19). There is little if any chance of finding a large systematic error using the data only. Distortion that fake or mask a kink can be quite different from the shape factor in Fig. 1b) or its mirror image, and the chance to obtain a wrong result is much larger than previously thought.

Arbitrary shape factors substantially increase the global correlation coefficient of $\sin^2 \theta$, called ρ_3 here (Table 4, Fig. 13). Ideally one has a polynomial taking out the distortion, while $\sin^2 \theta$ is uncorrelated and measures the kink. In fact, there is large correlation. Shape factors that change across the fitted spectrum by a quantity much larger than $\sin^2 \theta$ may hijack the fit (Figs. 23 and 24).

One of the negative experiments considered, Ref.[12] showed significant deviations from expected fit results. My fake data were good to 5% in comparison to the data of Ref.[12] and then every tenth point of the original spectrum [24] was used to get a 1% agreement in the variance distribution. Barring the exceptions of Ref.[30], it is certain that Ref.[12] fitted out a strongly non linear distortion with a linear shape factor (Fig. 21, Table 7). It is likely that there was one zero in the first or second derivative of that missing distortion (Table 7 and Fig. 22). A 40% increase in σ_3 while $1 - \rho_3$ decreases by a factor of three (Tables 3 and 6) is statistically impossible for the case under study. I conclude that there is a significant chance that Ref.[12] has underestimated the systematic error (quoted to be 0.14%).

In view of coming new experiments, criteria were sought to discriminate amongst apparently incompatible results. Since there will be more groups using shape factors and claiming sensitivity, I wished also to establish a set of rules to make sure that these arbitrary shape factors are similar to the missing distortions. For example, if one experiment uses a linear shape factor, the missing distortion should have fairly constant first derivative, and small higher order derivatives. When the criteria below are applied, to select "good" experiments, it is still possible to have a "mirror" shape factor obscuring the effect, but at a much smaller probability. These criteria are:

- An experiment should have a direct measurement of the ERF in more than one point across the fitted spectrum.
- An experiment should cross check. In this paper a direct calculation of the errors was used to assess this problem, but a simpler and more comprehensive condition is to request that all fitted correlation coefficients be closer than .005 to the expected value. Larger deviations are associated usually (but not always) with non-parabolic minima or non statistical errors (Fig.21), and in general with large higher orders and/or zeros in them. This criterium will eliminate no "good" experiments. Caveat: following Fig. 18 and Table 7, one could have some distortions producing quite regular errors and still masking or faking the kink. Most of these are eliminated by requesting also an higher order fit, see below.
- At most one small linear shape factor is allowed. Although in principle also a small quadratic shape factor is OK, if it is monotonically increasing; in fact such a situation can always be fitted properly with a linear one, because of the high correlation between Q , α and β (Table 3). Shape factors that peak near $Q - 17$ can always be reinterpreted as consistent with a kink, and shape factors orders of magnitude larger than $\sin^2 \theta$ generate very large systematics. A rough and empirical estimate of where distortions start to have a chance of hiding or faking a kink could be around three times $\sin^2 \theta$. The same caveat of the previous point applies here too. No shape factors will always be an advantage, of course.
- No narrow scans. These are, statistically speaking, insignificant measurements, as ρ_3 is always very close to 1 (Table 8). I side with Ref.[12] and insist that a wide scan is necessary (at least two times the neutrino mass). However, because of the likelihood of masking or making small kinks, one must impose *simultaneous* constraints on the wide region and on the narrow region. Therefore,
- a positive experiment shall have a δ_{17} larger than -0.0002 and a negative experiment δ_{17} smaller than .0002 (Fig.26). This cut is the harshest as it throws away roughly 20% of the "good" experiments but uses the best leftover signature of a missing or faked kink. By increasing the statistics the cut could become more efficient. Any other cut which eliminates fits with a dent or a bump and which is proven against fake data will also do.
- An experiment should show also the results of a fit with a shape factor with one order more (Fig. 17, Fig. 20). This is meant to eliminate those distortions which change the kink without changing the error. Caution should however be used when interpreting this extra piece of information as $1 - \rho_3$ decreases by a factor of two or more.

A look at the published data seems to indicate that the statistical criteria listed above would eliminate all the negative experiments considered here, but it is left to the authors to look at their data. The positive experiments do not offer definitive proof of a total ERF measurement, but having found the kink in the same apparatus with two different nuclei seems to eliminate the possibility of one common missing distortion as the source of the kink.

The conclusions of this review differ from those of Ref.[32]. I do not share the same enthusiasm for anything sporting a decent χ^2 , or the same belief in infinitely small systematic errors. Conclusions based on "experiment counting" (one counts experiments and the majority wins) is most definitely not the way to assess this controversy. Figures such as Fig. 27, which compiles the results obtained worldwide so far, do not mean that there is a huge gap between positive and negative experiments. Fig. 15 or 18 show that systematic errors are in general not measurable

using the data only, and the presence of an arbitrary shape factor, while not allowing an unique determination of the systematic error, should be enough to raise the systematic error to a few per thousands at least. The separation in ρ_3 between a minority of positive and a majority of negative results remains puzzling (Fig. 13).

The main limit of the analysis presented here is that missing distortions will perforce have infinite degrees of freedom which cannot be all reproduced with a finite set of simulated experiments. However, the main conclusions of this paper do not depend on these infinite degrees of freedom. The only real drawback is the impossibility to define a probability of faking a kink versus shape factor size. One should also be careful not to try to extract statistical information from the plots using the $D_i(x)$.

We clearly need more experiments. In 1987, $\sin^2 \theta$ used to be 3%. When one experiment with a shape factor smaller than $\sin^2 \theta$ came along, it completely settled the matter[12]. Notice that this remains true even after the analysis presented here.

So, few, ultimate experiments are preferable to a lot of results with arbitrary inputs by the experimenters. We need:

- spectrometer experiments. The superior energy resolution is a minor detail, but it is important to avoid the suspicion that something in the detector material faked the kink. If backscattering is the prime suspect, then also the internal source experiment in Berkeley will prove significant.
- Using anything but 3H . A firm conclusion about whether ^{35}S or ^{63}Ni is the best source, or equivalently if a fitted interval of 100 or 40 keV is best, could not be found. ^{63}Ni has a smaller ρ_3 and a more rapidly decreasing statistical error, but because 17 keV is a larger fraction of the fitted interval a smooth distortion will generate larger systematics. ^{35}S , if there are large distortions, will have the fit hijacked by the low part of the spectrum, which has the most statistics. Besides, ρ_3 is much more sensitive to background fluctuations. In a good experiment, and with the same statistical significance, they may turn out to be an equally good choice.
- with no shape factors, to eliminate arbitrary intervention by the experimenters, and
- surviving the cuts listed above. While this paper was being written, I was informed that one such negative experiment may have been performed[33].

As one last remark, notice that these experiments are close to their limit of observability. Even in the most ideal of cases a few percent corrections are necessary. Below the few per thousands level neutrino oscillation experiments become the best way to study neutrino mixing. I do not think that a ν_e disappearance experiment is feasible at this level of precision, but a $\nu_e - \nu_\tau$ oscillation experiment may be [34]. This is possibly the most urgent, answerable question in a different experimental environment other than nuclear β^- decay.

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[22] The window transmission was

$$P(E) = 1 - T \exp(-.431(E - 7)^{.65})$$

with $T=2.08$ and E expressed in keV, according to Fig. 7 of Ref.[12].

The backscattering probability was given by an integral in momentum according to Fig. 6 of Ref.[12] and formula provided by the authors. With P the momentum corresponding to Q kinetic energy, one has (Ref.[29])

$$N_{exp}(p) = \int_p^P dp' [\delta(p - p') + \frac{3F}{2(1 - .425)^3} (\frac{p}{p'} - (\frac{1 + .425}{2})^2) \frac{dN_{th}}{dp'}].$$

Since this correction depends slightly on $\sin^2 \theta$, the convolution was performed separately on the light and heavy neutrino components, and the results tabulated for use in the generating and fitting routines.

The window correction was applied after (ie, outside) the convolution integral.

- [23] D.W. Hetherington, private communication; J.J. Simpson, private communication.
- [24] To allow a complete cross check of my method, while preserving confidentiality, the authors of Refs. [11] and [12] sent every tenth data point of their scans. This allowed to include small sources of variance and to get an agreement of about 1% on the variance distribution.
- [25] The standard Cern routine Minuit is used here. The routine has the option of robust search for the minimum (SIMPLEX), and a faster and more accurate routine which uses derivatives (MIGRAD). The χ^2 is continuous in all variables but Q and M . Q varies very little (typically .05 keV) while M could create noise by moving across one or more data points during the fit. The latter routine was used when M was held fixed, the former was used otherwise. In all the results presented here, M was held fixed. To control the numerical stability of the routine, the derivatives were either numerically or analytically calculated.
- [26] In each Table or Figure a set of 8 bits is given, each bit corresponding to one parameter being turned on/off in the generation (T) and fit (F). The bits correspond, in order, to C , Q , $\sin^2 \theta$, M , α , β , F_B and T . For example, $T = 11100000$ will mean a purely theoretical spectrum in input, mass to be specified; $F = 11101000$ means that C , Q , $\sin^2 \theta$ and α were varied in the fit. Fixed variables were, if specified, the same in input and in output. If not specified, they were all zero.
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[34] G. Bonvicini, in preparation.

Table Captions.

1. From Ref.[12]. Results of the fit to the data. Indicated are the energy interval considered, the endpoint Q , the shape factor α , the heavy neutrino branching $\sin^2 \theta$ and reduced chi square χ^2/dof .
2. Missing distortions used in the analysis to investigate the systematic error of $\sin^2 \theta$. The first seven have second derivative $D'' < 0$, the next three $D'' > 0$, the last was used only for Fig. 22, and the gaussian was used with a variable peak position. The factors k_i were varied by hand and are listed for each Figure or Table.
3. Global (ρ_i) and partial (V_{ij}) correlation coefficients for the free parameters of several fits. A fixed neutrino mass $M = 17keV$ was considered. These parameters were obtained by assuming that the true spectrum was exactly what the experimenters were fitting. a) Ref.[11]. True spectrum is a β^- decay spectrum. My fit values[26]: $T=11100000$, $F=11100000$, $M = 17keV$. b) Ref.[12]. True spectrum is β^- times a linear shape factor. My fit values: $T=11101000$, $F=11101000$, $M = 17keV$, $\alpha = .025$. c) Ref. [27]. True spectrum is β^- times $(1+\alpha(1-x))(1+\beta(1-x)^2)$. My fit values: $T=11101100$, $F=11101100$, $M = 17keV$, $\alpha = .25$, $\beta = .32$. Notice that there are no background fluctuations in this fit, as opposed to the previous two. Q correlation coefficients are the most affected by background fluctuations.
4. Global correlation coefficients for neutrino mixing probability. Calorimetric and spectrometric spectra differ by a factor proportional to the momentum p . Only statistical variance was considered. Notice from Table 3 that in the case of ^{35}S there is some difference when background fluctuations are included. Shape factors, when present, were of the form $1 + \alpha x + (\beta x^2)$.
5. Comparison of values obtained in Ref.[12], labeled [12], and expected fitted parameters, labeled e . Notice the different fluctuations of α , F_B and T in the two cases. This may or may not be due to the very high V_{ij} between these corrections. My fit values: $T=11101000$, $F=11101011$, $M = 17keV$, $\alpha = .025$, $F_B = .185$, $T = -2.08$, $\sin^2 \theta = 0$.
6. Correlation coefficients corresponding to the fit presented in the previous Table.
7. Statistical error as calculated from the width of the χ^2 minimum. First row: from Ref.[12]. Second: as calculated from Fig. 21. All others: with addition of non linear distortions, using the same scans of Fig. 18.
8. Correlation coefficients for narrow scan, Ref.[12]. The true fit was assumed to be the same used in the experiment. My fit values: $T=11101000$, $F=11101000$, $M = 17keV$, $\alpha = .025$, $F_B = .185$, $T = -2.08$.

Interval(keV)	$Q(keV)$	α	$\sin^2 \theta$	χ^2/dof
50-67	$66.946 \pm .010$	$.023 \pm .007$	-	1.01
26-67	$66.941 \pm .005$	$.026 \pm .001$	$-.0011 \pm .0010$	.86
26-67	$66.946 \pm .005$	$.027 \pm .001$	0	.86
46-54	$66.994 \pm .123$	$.012 \pm .011$	$-.0017 \pm .0027$	1.14
46-54	66.946	$.031 \pm .012$	$-.0011 \pm .0027$	1.15

Table 1.

$D(x)$	i
$k_1 \log[E/(Q - 17)] \exp(-.5(90x)^{.17})$	1
$k_2 \log(3/2 - x)$	2
$k_3 \arcsin x$	3
$k_4 \exp(-.5(90x)^{.17})$	4
$k_5 \arctan x$	5
$k_6 \tanh x$	6
$k_7 \sqrt{1 + x}$	7
$k_8(4 + x)/(2 - x)$	8
$k_9 x^2 \sqrt{1 + x^2}$	9
$k_{10} e^x$	10
$.05\sqrt{x} + .005 \exp(-(x - x_b)^2/.106)$	11

Table 2.

Parameter	ρ_i	V_{i1}	V_{i2}
C	.962	—	—
Q	.984	— .93	—
$\sin^2 \theta$	.943	— .72	.89

a)

Parameter	ρ_i	V_{i1}	V_{i2}	V_{i3}
C	.994	—	—	—
Q	.832	.47	—	—
$\sin^2 \theta$	.958	— .52	.36	—
α	.996	— .97	— .30	.69

b)

Parameter	ρ_i	V_{i1}	V_{i2}	V_{i3}	V_{i4}
C	.9998	—	—	—	—
Q	.992	— .5	—	—	—
$\sin^2 \theta$	.952	.65	.24	—	—
α	.99996	— .98	.36	— .7	—
β	.99986	.97	— .31	— .996	.73

c)

Table 3.

Interval(keV)	Nucleus	Experiment	Shape factor	ρ_3
26-67	^{63}Ni	Spectrom.	α	.954
46-54	^{63}Ni	Calorim.	α	.997
35-67	^{63}Ni	Calorim.	-	.898
35-67	^{63}Ni	Calorim.	α, β	.978
120-167	^{35}S	Calorim.	-	.892
110-167	^{35}S	Spectrom.	α, β	$\approx .97$
110-167	^{35}S	Calorim.	α, β	$\approx .96$

Table 4.

$\sin^2 \theta$	$\chi^2/dof[12]$	χ^2/dof_e	$\alpha[12]$	$F_B[12]$	$T[12]$	α_e	$(F_B)_e$	T_e
-0.0022	120.6	120.6	.034	.184	2.10	.003	.146	1.96
0.	124.	121.9	.048	.192	2.09	.023	.172	2.03
0.0022	128.5	125.9	.056	.191	2.07	.044	.202	2.10

Table 5.

Parameter	ρ_i	V_{i1}	V_{i2}	V_{i3}	V_{i4}	V_{i5}
C	.99998	-	-	-	-	-
Q	.992	.85	-	-	-	-
$\sin^2 \theta$	.989	-.79	-.56	-	-	-
α	.99996	-.998	-.86	.81	-	-
F_B	.9997	.87	.70	-.86	-.50	-
T	.99991	-.96	-.79	.95	.65	-.98

Table 6.

Input data	$\chi^2 \rightarrow \chi^2 + 1$
Ref.[12]	0.00140
Expected	0.00187
$D_4(x)$	0.0016
$D_8(x)$	0.0017
$D_1, D_3, D_{10}(x)$	0.0018
$D_2(x)$	0.0020
$D_7(x)$	0.0024

Table 7.

Parameter	ρ_i	V_{i1}	V_{i2}	V_{i3}
C	.99999	—	—	—
Q	.99994	.96	—	—
$\sin^2 \theta$	.997	-.8	-.6	—
α	.99999	-.998	-.98	.8

Table 8.

Figure Captions

1. a) Example of Kurie plot with an heavy neutrino. b) Shape factor with an heavy neutrino.
2. Electron response function. Shown are the gaussian resolution peak and, amplified by 10, the tail due to backscattering off the detector.
3. From Refs.[11] and [28]: schematic of the experiment. a) Si(Li) detector, b) source substrate, c) Al detector aperture, d) Cu source aperture, e) Al antiscatter baffle, f) linear motion feed-through, g) liquid nitrogen cryo-panel, h) Teflon centering ring.
4. From Ref.[11]. Shape factors for a) Run 1 and b) Run 2 obtained by dividing the experimental spectra by the least squares fit when no mixing is allowed. The data go off scale above 161 keV and should be read against the right hand scale. c) Shape factor for combined data of Runs 1 and 2 when normalizing to a single component spectrum to the data above 150 keV.
5. From Ref.[11]. a) Deviation of measured pulse heights from a linear calibration curve. Full circles denote photopeaks from X -rays from ^{57}Co and ^{109}Cd . Open circles are from tests using a precision pulser in input to the detector preamplifier. b) Detector resolution deduced from the full width of the photopeaks from ^{57}Co and ^{109}Cd .
6. From Ref.[11]. a) 129.36 keV internal conversion line from ^{57}Co . Points are experimental and the smooth curve models the electron distribution after accounting for ionization losses and finite resolution of the detector. b) Distribution of electrons contained in the low energy tail of the ERF. The histogram and dashed curve represent the component due to backscattering of normally incident electrons on Si. The dotted curve is the energy loss component due to ionization in the Au contact on the front of the detector. The solid curve combines the two components and experimental points are shown. The distribution is normalized using a bin width of 0.04 units of $x = E/Q$.
7. From Ref.[12]. Plan section through the vacuum chamber of the Chalk River iron-free $\pi\sqrt{2}$ beta spectrometer.
8. From Ref.[12]. 53 keV L_1 and L_2 conversion electron lines from ^{169}Yb .
9. Partial and total corrections applied in Ref.[12]. See Ref.[22] for formulae. Dashed: backscattering. Dot-dashed: window transmission. Solid: sum of these main two corrections. Not shown are other smaller corrections applied to the data. Dotted: linear shape factor.
10. From Ref.[12]. Ratio of data to fit for the wide scan spectrum. a) $\sin^2 \theta$ fixed to zero, neutrino mass fixed to 17 keV, $F = 11001000$, see Ref.[26], F_B and T fixed to their Monte Carlo value. For comparison, the dashed line shows the expected shape for $\sin^2 \theta = .03$. b) Best fit for $\sin^2 \theta = .03$. The other fit results were $Q = 67.019$ keV, $\alpha = .32$, $\chi^2/dof = 7.76$. As discussed in Section 7, these plots certainly exclude a 3% mixing.
11. $\chi^2/dof - 1$ function when attempting to fit a kinky line with a straight line. Solid line: normal case. Dashed line: when a linear shape factor $(1 + \alpha x)$ is added to both true and fitted lines.

12. Qualitative explanation of rapid decrease of $\chi^2/dof - 1$. a) Fit with a straight line. b) Fit with a straight line times a linear shape factor.
13. Global correlation coefficient ρ_3 for the results considered in this analysis. Only statistical errors are assumed. The lack of information about background subtraction mostly affects the Q correlation coefficients. Dark: negative results. Shaded: positive results.
14. $\chi^2 - \sin^2 \theta$ scatter plot, ^{35}S . Statistics and energy interval according to Ref.[11]. $T=11100000$, $F=11100000$, input values: $M = 17$ keV, $\sin^2 \theta = 0$. Different markers correspond to the different distortions D_i (Table 3) applied to the input spectrum. The value of the constants k_i is specified in the Figure near the respective marker.
15. Same as previous figure, ^{63}Ni . Statistics and energy interval according to Ref.[28]. Input values, markers meaning and constants as described in Fig. 14. Statistical fluctuations only are assumed.
16. One of the scans plotted in the previous Figure. Fit input were: $Q = 67$ keV, $\sin^2 \theta = 0$, and the distortion was D_4 . Fit output was: $Q = 67.002$ keV, $\sin^2 \theta = 0.0077 \pm .0009$, $\chi^2/dof = .945$.
17. Same as Figure 15, but one linear shape factor was added in the fit. $T=11100000$, $F=11101000$, input values: $M = 17$ keV, $\sin^2 \theta = 0$.
18. $\chi^2 - \sin^2 \theta$ scatter plot, ^{63}Ni . Statistics and energy interval according to Ref.[12]. $T=11101000$, $F=11101000$, Input values: $M = 17$ keV, $\sin^2 \theta = .008$, $F_B = .185$, $T = 2.08$, different markers correspond to different D_i (Table 2) applied to the input function, and for each the chosen value of k_i (first value) and α (second value) are given.
19. One of the scans plotted in the previous Figure. Fit input were: $Q = 67$ keV, $\sin^2 \theta = 0.008$, $F_B = .185$, $T = 2.08$, and the distortion was D_{10} . There was no zero in D' over the interval considered. Fit output was: $Q = 67.030$ keV, $\sin^2 \theta = -.001 \pm .001$, $\chi^2/dof = .99$.
20. Same as Fig. 18, but one quadratic shape factor was added in the fit. $T=11101000$, $F=11101100$, input values: $M = 17$ keV, $\sin^2 \theta = 0.008$, $F_B = .185$, $T = 2.08$, α and k_i factors listed in Figure have same values as Fig. 18.
21. χ^2 minimum used to evaluate the expected single parameter error σ_3 . Statistics and energy interval from Ref.[12]. Parabola from one Monte Carlo scan. $T=11101011$, $F=11101011$, where $\sin^2 \theta$ was scanned. Input values: $M = 17$ keV, $\sin^2 \theta = 0$, $\alpha = 0.025$, $F_B = .185$, $T = 2.08$. Distortions were NOT used here.
22. Same statistics and energy interval as in previous Figure. Dependence of σ_3 and $\sin^2 \theta$ when one "bumpy" distortion was in the true spectrum. Distortion was D_{11} with the median value of the gaussian x_b scanning across the spectrum. Squares: best fit for $\sin^2 \theta$. Circles: fitted single parameter error σ_3 . Triangles: the reduced χ^2/dof is also shown. Input values: $M = 17$ keV, $\sin^2 \theta = 0.008$, $F_B = .185$, $T = 2.08$. Notice that kink would have been located at about .66.
23. $\chi^2 - \sin^2 \theta$ scatter plot, ^{35}S . Statistics and energy interval according to Ref.[31]. $T=11101000$, $F=11101100$, Input values: $M = 17$ keV, $\sin^2 \theta = .008$. Different markers correspond to different D_i (Table 2), including value of factors k_i , first value, and shape factor α , second value. Scans were plotted only when the fitted shape factor peaked within 5 keV of expected kink.

24. $\chi^2 - \sin^2 \theta$ scatter plot, ^{35}S . Statistics and energy interval according to Ref.[27]. T=11101000, F=11101100, Input values: $M = 17$ keV, $\sin^2 \theta = .008$. Different markers correspond to different D_i (Table 2), including value of constants k_i , first value, and shape factor α , second value.
25. $\chi^2 - \sin^2 \theta$ plots versus x , ^{63}Ni . Statistics of 10^8 counts over the 26-67 keV interval. A 10 keV interval was moved across the spectrum in steps of .8 keV, and for each step a fit was made. The fitted mass was held fixed at the center of the fitted interval. The χ^2 and $\sin^2 \theta$ output values are plotted versus the center of the fitted interval. T=11101000, F=11101000, distortion D_1 was used, $k_1 = -.12, \alpha = .15$. Other input values: $M = 17$ keV, $\sin^2 \theta = .008$.
26. δ_{17} distributions, as described in the text. Statistics and energy interval equal to those of Ref.[12] were considered, including both narrow and wide scan, and the two were added together. Grey: scans where the fitted and true $\sin^2 \theta$ values were within .0025 of one another. Dark: scans where a .8% kink was input but the fit found a value to within .0025 of zero. Distortions and fit parameters as in Fig. 18.
27. From Ref.[32]. Negative and positive results on the 17 keV neutrino.

Fig. 1a)

Fig. 1b)

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 15

Fig. 17

Fig. 19

Fig. 21

Fig. 22

x

Fig. 23

Fig. 24

Fig. 25a

Fig. 25b

Fig. 26

Fig. 27