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Analysis of the Stability Domain for the Hénon Map

M. Giovannozzi

Abstract

To give an analytic estimate of the stability domain of non-linear systems, like large accelerators, has been a subject of intense theoretical studies since many years.

Up to now, even for the simple case of the one-dimensional Hénon map, one could not determine the border of the stability domain using analytic tools. On the other hand this estimate can easily be found by tracking through the map.

A promising new attempt is presented here, to estimate the dynamic aperture for the Hénon map via following the invariant manifolds of its hyperbolic fixed point.

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1 Introduction

The standard approach to find the dynamic aperture of an accelerator with nonlinearities, is to use element by element tracking. The dynamic aperture specifies the maximum amplitude up to which the motion stays bounded. In the one-dimensional case the concept of the dynamic aperture is well defined as the regular orbits of an hamiltonian system form closed curves so that the two-dimensional phase space is divided into parts that are completely separated. The tracking has the main drawback of giving little theoretical insight into the causes for the particle losses. Analytical tools are therefore investigated to understand these losses. However, even the most simple case of a linear lattice plus a single sextupole in the one-dimensional case (i.e. the Hénon map [1, 2]) is not fully understood up to now.

In this paper an analytical method is presented (see also [3],[4]) which allows to determine the stability border of the Hénon map following the invariant manifolds that emanate from the hyperbolic fixed point of the map. The idea that the stability domain can be approximated using the invariant manifolds of the system dates back to Poincaré's observation that there is a relation between the homoclinic intersections and the rising of stochastic motion.

In Section 2 we introduce the Hénon map, giving a brief review of its properties; in Section 3 we define and compute the fixed points introducing the concept of invariant manifolds and explaining how to construct them numerically. The results are presented in section 4.

2 General properties of the Hénon map

The Hénon map [1], first introduced in 1969, is a quadratic area-preserving polynomial map. It can be written in the following form:

$$H : \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = R(\omega) \begin{pmatrix} x_0 \\ x'_0 + x_0^2 \end{pmatrix} \quad (x_0, x'_0) \in \mathbb{R}^2 ; \omega \in [0, \pi], \quad (1)$$

where $R(\omega)$ is a rotation matrix. Owing to the area-preserving property it is possible to define a Hamiltonian whose Poincaré section coincides with the Hénon map. This Hamiltonian has the well known form [2]:

$$\mathcal{H}(x, x'; s) = \omega^2 \frac{(x^2 + x'^2)}{2} - \omega \frac{x^3}{3} \sum_{n=1}^{\infty} \delta(s - n). \quad (2)$$

It is easy to see that the Hénon map coincides with the expression of the transfer map of a simple FODO cell with one sextupolar nonlinearity considered in the kick approximation [2]: the rotation represents the effect of the linear part of the cell (in the Courant-Snyder coordinates), whereas the quadratic part comes from the sextupolar nonlinearity.

One peculiar property of the map is the fact that it is symmetric with respect to a line with a slope that depends on the frequency of the map. This can be understood when H is rewritten in the following form:

$$H(\vec{x}) = \mathcal{I}_1 \circ \mathcal{I}_2(\vec{x}). \quad (3)$$

Where $\mathcal{I}_1$ is defined as:

$$\mathcal{I}_1 : \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} \cos \omega & \sin \omega \\ \sin \omega & -\cos \omega \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \quad (4)$$

and $\mathcal{I}_2$ is:

$$\mathcal{I}_2 : \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} - \begin{pmatrix} 0 \\ x_0^2 \end{pmatrix} \quad (5)$$

The two transformations $\mathcal{I}_1, \mathcal{I}_2$ are called *involutions*, i.e. their square is the identity transformation. In particular $\mathcal{I}_1$ represents a reflection around the line of the equation:

$$x' = x \tan \frac{\omega}{2}, \quad (6)$$

which is the line of symmetry as stated above, whereas $\mathcal{I}_2$ is simply a vertical shift.

3 Fixed points and invariant manifolds

The fixed points of a generic map F are the solutions of the equation:

$$F(\vec{x}_0) = \vec{x}_0. \quad (7)$$

We can also define cycles of order m : they are simply fixed points of the m^{th} iterate of the map F :

$$F^m(\vec{x}_0) = \vec{x}_0. \quad (8)$$

In the case of one-dimensional systems these points are classified according to the eigenvalues (λ_1, λ_2) of the linearized map. Owing to the area-preserving property they must fulfill the condition $\lambda_1 \lambda_2 = 1$ so that only three different situations are allowed:

$$\begin{array}{lll} \lambda_1, \lambda_2 \in \mathbb{C} & \lambda_1 = \lambda_2^* & \text{elliptic case} \\ \lambda_1, \lambda_2 \in \mathbb{R} & \lambda_1 \neq \lambda_2 & \text{hyperbolic case} \\ \lambda_1, \lambda_2 \in \mathbb{R} & \lambda_1 = \lambda_2 = 1 & \text{parabolic case} \end{array} \quad (9)$$

The Hénon map has only two fixed points. The origin is an elliptic fixed point, whereas the other one $\vec{x}_h$ with the coordinates:

$$x_h = -2 \tan \frac{\omega}{2} \quad x'_h = -2 \tan^2 \frac{\omega}{2} \quad (10)$$

is hyperbolic over the whole interval of frequencies ω . The linearized map around $\vec{x}_h$ has the following expression:

$$H_L : \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} \cos \omega - 2x_h \sin \omega & -\sin \omega \\ \sin \omega + 2x_h \cos \omega & \cos \omega \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}, \quad (11)$$

and its eigenvectors are given by:

$$\vec{e}_1 = \begin{pmatrix} \sin \omega \\ \cos \omega + 4 \tan \frac{\omega}{2} \sin \omega - \lambda_1 \end{pmatrix} \quad (12)$$

$$\vec{e}_2 = \begin{pmatrix} \sin \omega \\ \cos \omega + 4 \tan \frac{\omega}{2} \sin \omega - \lambda_2 \end{pmatrix}. \quad (13)$$

They define two lines $\mathcal{W}_L^u(\vec{x}_h)$ and $\mathcal{W}_L^s(\vec{x}_h)$, along which the motion induced by the map (11) has an expanding (superscript u) or a contracting (superscript s) behaviour. Also for the

global non-linear map we can define two subspaces emanating from the unstable fixed point such that these sets, called $\mathcal{W}^u(\vec{x}_h)$ and $\mathcal{W}^s(\vec{x}_h)$, show the same expanding or contracting behaviour. These manifolds are uniquely defined and have the important feature of being invariant sets for the map:

$$F(\mathcal{W}^{u,s}(\vec{x}_h)) = \mathcal{W}^{u,s}(\vec{x}_h). \quad (14)$$

Obviously, we can also define invariant manifolds for hyperbolic cycles of order m . In this case, all the previous definitions and properties will hold, replacing the map F with its m^{th} iterate F^m .

From the definition it follows that the invariant manifolds have at least one intersection which is the hyperbolic fixed point. An additional intersection, $\vec{x}_{hom}$ is called *homoclinic* or *heteroclinic* depending on whether the two intersecting manifolds emanate from the same hyperbolic fixed point or not.

The existence of one such point implies that there exist others, in fact an indefinite number of them, as this point belongs to several manifolds for each of which the invariant property (14) has to be fulfilled. Unless the two manifolds coincide completely, (integrable case), they will oscillate around each other (Fig 1). When approaching the hyperbolic fixed point the distance between successive intersections decreases as the period of the motion tends to zero. Due to the area-preserving character of the map, the area enclosed between two successive intersections has to stay constant, leading to larger and larger oscillations near the unstable fixed point.

To construct the whole sets $\mathcal{W}^{u,s}(\vec{x}_h)$, it is sufficient to iterate many times an ensemble of initial conditions belonging to a small part of these manifolds around the hyperbolic fixed point. Moreover it turned out that these initial conditions can be chosen on the eigenvalues of the linearized map as long as one stays within a minimum distance to the hyperbolic fixed point.

4 Results

In a previous study [5] the information about unstable cycles of several orders and the corresponding manifolds were used to determine the dynamic aperture of the Hénon map. This approach is motivated by the fact that whenever a resonance is reached the dynamic aperture is decreased. As, however, the dynamic aperture is the result of a complex relation between resonances and detuning, the method was only partially successful.

The new method presented here allows to determine the dynamic aperture in the whole range of frequencies, simply following the invariant manifolds of the hyperbolic fixed point of the map and computing the minimal distance from the origin of these sets. It exploits the fact that the manifolds are connected via heteroclinic intersections to the unstable cycles that actually determine the border of the stability domain. Two examples are shown in Fig 2,3. The invariant manifolds of the hyperbolic fixed point are compared with the stability region as computed by direct tracking through the map for two different values of the frequency. In both cases the agreement is excellent down to the very details, including the chains of stable islands due to the subresonances. Moreover it is apparent that this result is totally independent on the topology of the border itself especially on the presence of islands. A comparison of the dynamic aperture as a function of the frequency between this new method

and tracking is presented in Fig 4.

The drawback is clearly the need in cpu-time: a tracking run for one frequency requires roughly 2.8s on a Cray Y/MP, while this method takes approximately between 8s and 24s. The time needed also depends very much on the frequency: close to the regime of the first order resonance the time necessary is naturally very short ($\omega/2\pi = 0.50$ in Fig 5A); while in the case close to the fourth order resonance the passing to the manifolds of the fourth order cycle leads to an intermediate regime of a slowly decreasing minimal amplitude and therefore to a large time ($\omega/2\pi = 0.255$ in Fig5B). The particles move extremely slow close to the hyperbolic fixed point especially for small frequencies. This explains the initial plateaus and the difference of their lengths in the two parts of Fig 5.

5 Conclusions

The method outlined here allows to connect the stability region of the Hénon map with the invariant manifolds of the hyperbolic fixed point. To the knowledge of the author, it is the first time that an analytic method is capable to determine the dynamic aperture to this high precision. It must be noted, however, that due to the large amount of cpu-time needed, the method is not yet ready for practical applications.

Even if we have studied a particular map, it is to be expected that similar results can be obtained for any polynomial map in the one-dimensional case.

It is well known that in two-dimensional maps the concept of dynamic aperture is not well defined, at least from a theoretical point of view, as in principle particles close to the origin can escape to infinity due to Arnold diffusion [7]. Nevertheless, in practical cases like accelerators with a time scale of 10^9 iterations, such a diffusive phenomenon is too slow to be noticeable. Experience from tracking [8] has shown that one can specify a dynamic aperture for that time scale. The method outlined here is expected to sample the same barrier of stability as the tracking.

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Figure 1: Example of homoclinic intersections for the invariant manifolds of the Hénon map. The invariant manifolds depicted refer to the case $\omega/2\pi = 0.2450$. Nine homoclinic points are shown which are the result of the oscillations of the manifolds due to the area-preserving property of the Hénon map. It is also apparent that near the hyperbolic fixed point the two manifolds resemble the separatrix of an integrable system. It must be pointed out that here and in the following plots the coordinates used are the normalized Courant–Snyder coordinates.

Figure 2: Invariant manifolds of the Hénon map near the one-third resonance

The invariant manifolds for $\omega/2\pi = 0.404$ are shown together with the tracking data (black area). In this case 1.5×10^5 initial conditions are used, iterated for 5×10^2 turns. The border of the stability domain is reproduced to a very high precision by following the invariant manifolds.

Figure 3: Invariant manifolds of the Hénon map near the one-fourth resonance

The invariant manifolds for $\omega/2\pi = 0.255$ are shown together with the tracking data (grey area). In this case 1.5×10^5 initial conditions are used, iterated for 1×10^3 turns. Also in this case following the invariant manifolds allows to recover the complicated structure of phase space close to the border of the stability.

Figure 4: Dynamic aperture for the Hénon map

The dynamic aperture is depicted for 2000 values of ω (solid line) tracked for 1×10^4 iterations. The circles superimposed represent the minimum distance from the origin of the invariant manifolds for 250 values of ω . The agreement is excellent over the whole range of frequencies.

Figure 5: Plot of the minimal distance of $\mathcal{W}^u(\vec{x}_h)$ for $\omega/2\pi = 0.050$ (part A) and $\omega/2\pi = 0.255$ (part B) from the origin as a function of the cpu-time of the Cray Y/MP needed for the calculations.