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Quantum Symmetry in Quantum Theory

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Abstract

In quantum theory, internal symmetries more general than groups are possible. The generalization of the algebra $\mathcal{G}$ of functions on the group needs to be neither commutative nor associative, and it may involve truncation. With truncation, the dimensions of tensor products of representations of the dual algebra $\mathcal{G}^*$ may be smaller than the product of the dimensions of the original representations.

1 Introduction: Symmetry in Quantum mechanics

Soon after the discovery of quantum mechanics, the exploitation of symmetry in atomic physics played an important role in working out the predictions of quantum mechanics. After that it was taken for granted that symmetries in quantum theory should be groups, as they are in classical physics. However, it need not be so. Supersymmetry was a first exception. Here we will discuss the most general algebraic structures which admit a physical interpretation as symmetries. We call them quasi quantum groups for short. There is some parallelism between the formulation of the theory and of its possible symmetries. In quantum mechanics, the points of phase space go away. But the "algebra of functions" on phase space remains, although it becomes noncommutative. Group elements may be regarded as points g in a space G . In classical mechanics, g can act as a map on points in phase space. When these points are no longer, the group elements are not needed either. It suffices to have an algebra $\mathcal{G}$ of "functions on the group G ". This algebra may be noncommutative. It turns out that it need not be associative either. Quasiasociativity is enough.

To stay as close as possible to the traditional formulation of a symmetry we consider the dual algebra $\mathcal{G}^*$ of the algebra $\mathcal{G}$ of functions on a group. This is a generalization of the group algebra. We will assume that $\mathcal{G}^*$ has a unit element e which acts as the trivial transformation in the Hilbert space of physical states. $\mathcal{G}^*$ is called *cocommutative* if $\mathcal{G}$ is commutative, and *coassociative* when $\mathcal{G}$ is associative. The product in $\mathcal{G}$ yields a coproduct in $\mathcal{G}^*$ which is a homomorphism of algebras,

$$\Delta : \mathcal{G}^* \rightarrow \mathcal{G}^* \otimes \mathcal{G}^* . \quad (1.1)$$

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The various stages of generalization of the notion of symmetry can be characterized by the properties of the coproduct.

cocommutative	coassociative	$\Delta(e) = e \otimes e$	name
yes	yes	yes	group algebra
no	yes	yes	Hopf algebra
no	no	yes	quasi Hopf algebra
no	no	no	weak quasi Hopf algebra

There exists an extensive literature about quantum groups (i.e. quasitriangular Hopf algebras) [3]. Quasi-Hopf algebras were introduced by Drinfeld [4]. The present authors proposed to build in truncation, which leads to what we call weak quasi Hopf algebras [1].

The coproduct is used to define a tensor product $\tau \otimes \tau'$ of representations τ and τ' in vector spaces V and V' of $\mathcal{G}^*$.

$$(\tau \otimes \tau')(\xi) = (\tau \otimes \tau')(\Delta(\xi)) . \quad (1.2)$$

The right hand side involves the outer tensor product $\tau \otimes \tau'$ of representations, which is by definition a representation of $\mathcal{G}^* \otimes \mathcal{G}^*$. If $\Delta(e) \neq e \otimes e$ then there is *truncation*. This means that the "true" representation space of $\tau \otimes \tau'$ is in general a proper subspace of $V \otimes V'$. In this way the truncated tensor products of quantum group representations which appear in conformal field theory can be accommodated.

Among the representations of $\mathcal{G}^*$ there should occur a 1-dimensional representation ϵ . It furnishes a homomorphism of algebras

$$\epsilon : \mathcal{G}^* \rightarrow \mathbb{C} . \quad (1.3)$$

ϵ defines the substitute for the trivial 1-dimensional representation of a group (and its group algebra).

We adopt the framework of second quantized quantum mechanics, so that there are field operators $\Psi(r, t)$ which create particles or excitations. In order that $\mathcal{G}^*$ is a symmetry of a quantum mechanical system, its Hilbert space of physical states $\mathcal{H}$ should carry a unitary representation $\mathcal{U}$ of $\mathcal{G}^*$ with $\mathcal{U}(e) = 1$. Unitarity means that

$$\mathcal{U}(\xi^*) = \mathcal{U}(\xi)^* . \quad (1.4)$$

The Hamiltonian H and the ground state $|0\rangle$ should be invariant,

$$\mathcal{U}(\xi)|0\rangle = |0\rangle \quad , \quad [\mathcal{U}(\xi), H] = 0 \quad \text{for all } \xi \in \mathcal{G}^* . \quad (1.5)$$

In addition, the tensor product is invoked to postulate a transformation law of field operators $\Psi_m(r, t)$ [9]. We will use superscripts $I, J, K, \dots$ to distinguish between field multiplets which transform according to some irreducible representations, and subscripts $i, j, k, \dots$ to distinguish the members of a multiplet. Ψ^j is supposed to transform according to the finite dimensional representation τ^j of $\mathcal{G}^*$. In compact notation this says that

$$\mathcal{U}(\xi)\Psi_k^j(r, t) = \Psi_k^j(r, t)(\tau_k^j \otimes \mathcal{U})(\xi) . \quad (1.6)$$

More explicitly, if

$$\Delta(\xi) = \sum_{\sigma} \xi_{\sigma}^1 \otimes \xi_{\sigma}^2, \quad (1.7)$$

then

$$\mathcal{U}(\xi)\Psi_k^j(r, t) = \sum_{\sigma} \Psi_k^j(r, t) \tau_{ik}^j(\xi_{\sigma}^1) \mathcal{U}(\xi_{\sigma}^2). \quad (1.8)$$

This can be used to move elements of $\mathcal{G}^*$ through field operators from left to right. In the special case where ξ is element of a symmetry group, $\Delta(\xi) = \xi \otimes \xi$, and the field transformation law reduces to the usual one.

From the transformation law of field operators one deduces that many particle states transform according to tensor products of representations.

Summing up, one needs a 1-dimensional representation ϵ in order to formulate the notion of invariance, a coproduct Δ in order to define a tensor product of representations and to formulate a transformation law of field operators, and a $*$ -operation on $\mathcal{G}^*$ in order to state unitarity.

The desire to formulate $\mathcal{G}^*$ -covariant local commutation relations for field operators will lead us to impose additional requirement. Neither commutativity nor associativity of $\mathcal{G}$ (i.e. coassociativity and cocommutativity of $\mathcal{G}^*$) should be given up completely, but they can be weakened to quasi-associativity and braid-commutativity.

When coassociativity is weakened to quasi-coassociativity, the tensor product of representations is not associative, but there is still equivalence of representations

$$((\tau^I \otimes \tau^J) \otimes \tau^K) \cong (\tau^I \otimes (\tau^J \otimes \tau^K)). \quad (1.9)$$

Similarly, braid-commutativity will imply that

$$(\tau^I \otimes \tau^J) \cong (\tau^J \otimes \tau^I). \quad (1.10)$$

The prime indicates interchange of indices ("the indices go with the matrices").

The formulation of the equivalence (1. 9) of triple tensor products of arbitrary representations involves a "reassociator" $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ with certain properties. Similarly, the formulation of braid-commutativity involves an element $R \in \mathcal{G}^* \otimes \mathcal{G}^*$. Relations involving ϵ , Δ and φ and R arise from the requirement that tensoring with the "trivial" 1-dimensional representation ϵ amounts to doing nothing. The existence of φ will enable us to define a $\mathcal{G}^*$ -covariant product of field operators, so that one can write down $\mathcal{G}^*$ -covariant operator product expansions etc.

In addition one requires the existence of an antipode S . It is a (linear) antihomomorphism of algebras

$$S: \mathcal{G}^* \rightarrow \mathcal{G}^*. \quad (1.11)$$

The antihomomorphism property means that $S(\xi\eta) = S(\eta)S(\xi)$. The inverse S^{-1} of the antipode is used to define the contragredient representation $\bar{\tau}$ to a representation τ by

$$\bar{\tau}(\xi) = {}^t\tau(S^{-1}(\xi)). \quad (1.12)$$

for $\xi \in \mathcal{G}^*$. t stands for the transpose of a matrix. When a suitable additional condition on S is imposed, the field transformation law (1. 8) can be inverted as follows

$$\Psi_k^j(r, t) \mathcal{U}(\xi) = \sum_{\sigma} \mathcal{U}(\xi_{\sigma}^2) \bar{\tau}_{ik}^j(\xi_{\sigma}^1) \Psi_k^j(r, t) \equiv (\bar{\tau}_{ik}^j \otimes \mathcal{U})(\Delta(\xi)) \Psi_k^j(r, t). \quad (1.13)$$

This tells us how to move elements of $\mathcal{G}^*$ through field operators from right to left. This relation serves as a starting point for the discussion of the transformation law of adjoints of field operators.

In quantum mechanics, it is very important that one can take the adjoint of operators. In other words, one has a $*$ -operation. This is used to define the notion of a unitary representation through the requirement (1. 4). To give meaning to the left hand side of eq.(1. 4), a $*$ -operation must be defined on $\mathcal{G}^*$. It should be an antilinear antihomomorphism of algebras $\mathcal{G}^* \rightarrow \mathcal{G}^*$. In order that the $*$ -operations preserve the algebraic structures, Δ and ϵ should be $*$ -homomorphisms, i.e.

$$\Delta(\xi^*) = \Delta(\xi)^*, \quad (1. 14)$$

$$\epsilon(\xi^*) = \overline{\epsilon(\xi)}. \quad (1. 15)$$

But $\Delta(\xi) \in \mathcal{G}^* \otimes \mathcal{G}^*$. Therefore we must first say how the $*$ -operation acts on $\mathcal{G}^* \otimes \mathcal{G}^*$. There are two possibilities

$$(\xi \otimes \eta)^* = \xi^* \otimes \eta^*, \text{ or} \quad (1. 16)$$

$$(\xi \otimes \eta)^* = \eta^* \otimes \xi^*. \quad (1. 17)$$

The best known examples of quantum group algebras $\mathcal{G}^*$ are the deformations $U_q(sl_2)$ of the universal enveloping algebra (or the group algebra) of $SU(2)$. They are coassociative but not cocommutative. They admit a $*$ -operation, which is the deformation of the $*$ -operation $\xi^* = \xi^{-1}$ in $SU(2)$, in two cases:

• q real, choice (1. 16) of $*$ -operation,

• $|q| = 1$, choice (1. 17) of $*$ -operation.

If q is a root of unity, the algebra $U_q(sl_2)$ is not semisimple. As a result, tensor products of irreducible representations are in general not fully reducible. But there is a weak quasi Hopf algebra which is obtained from $U_q(sl_2)$ by a process of truncation. It is semisimple (i.e. isomorphic to a sum of full matrix algebras). The truncation destroys not only $\Delta(e) = e \otimes e$, but coassociativity is weakened to quasi-coassociativity in addition.

Finally we come to a discussion of *statistics and locality*. In second quantized quantum theory, Bose and Fermi statistics are implemented through local commutation or anticommutation relations of field operators which create particles,

$$\Psi_i^j(r, t) \Psi_{i'}^{j'}(r', t) = \pm \Psi_{i'}^{j'}(r', t) \Psi_i^j(r, t) \text{ for } r' \neq r. \quad (1.18)$$

In a relativistic theory this must be true in every Lorentz frame. Consistency of a symmetry with Bose/Fermi statistics requires that this relation should be preserved by a symmetry transformation. This is indeed true for internal symmetry groups, for supersymmetry, and also for conformal transformations of space time [16].

In two and less space dimensions, Bose/Fermi statistics is not the most general possibility, but braid group statistics can also occur. This was explained in detail in the memorable lectures which Jürg Fröhlich presented at the last Cargèse school in

1987 [5]. He proposed that local commutation relations of fields should be replaced by local braid relations as follows.

$$\Psi_i^j(\mathbf{r}, t) \Psi_j^i(\mathbf{r}', t) = \omega \Psi_i^j(\mathbf{r}', t) \Psi_j^i(\mathbf{r}, t) \hat{\mathcal{R}}_{ij}^{ij} \quad (1.19)$$

if $\mathbf{r}' > \mathbf{r}$. We chose to pull out a phase factor ω which will not be determined by the symmetry. A similar relation is valid if $\mathbf{r}' < \mathbf{r}$. It involves the inverse of $\hat{\mathcal{R}}^{ij}$. In one space dimension, $\mathbf{r}$ is a real number, and the condition $\mathbf{r}' > \mathbf{r}$ has the obvious meaning. In 2 space time dimensions, braid group statistics appears in situations where the wave function is multivalued, and the definition of $>$ involves the positions of cuts. We will not go into an explanation of this here.

We take it as a defining feature of a quantum symmetry that it is consistent with local braid relations of this type. It will be seen, however, that braid matrices $\hat{\mathcal{R}}$ with entries in $\mathcal{G}^*$ should be admitted. (Originally it was proposed that they should be numbers.) We showed in [2] that the chiral Ising model - i.e. the conformal field theory with central charge $c = \frac{1}{2}$ - provides an example, with the truncated quantum group algebra $U_q(sl_2)$ with $q = \pm i$ as a symmetry. To the best of our knowledge, this was the first time that the consistency of nonabelian local braid relations (1. 19) has been demonstrated through the construction of a model. Originally it had been proposed that minimal conformal models have quasitriangular Hopf algebras ("quantum groups") as symmetries [7,8], but this identification is not quite satisfactory, because the local braid relations, which should come with the symmetry, are not satisfied [10]. A hint where to go came from the work of Dijkgraaf, Pasquier and Roche [6]. They considered correlation functions of an orbifold model with a quasi Hopf symmetry.

The $\hat{\mathcal{R}}$ -matrix in the local braid relations is determined as part of the specification of the quantum symmetry. It comes from an element $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ which has certain properties. A ((weak) quasi) Hopf algebra equipped with such an R is called quasitriangular. Given the above mentioned "reassociator" $\varphi = \sum \varphi_i^1 \otimes \varphi_i^2 \otimes \varphi_i^3$ one introduces $\varphi_{213} = \sum \varphi_i^2 \otimes \varphi_i^1 \otimes \varphi_i^3$. In this notation, the braid matrices are given by

$$\hat{\mathcal{R}}_{kl}^{ij} = (\tau_k^i \otimes \tau_l^j \otimes U)(\varphi_{213}(R \otimes e)\varphi^{-1}) \in U(\mathcal{G}^*) \quad \text{for all } ijkl. \quad (1.20)$$

The most important property of R is as follows. Given the coproduct Δ there exists another one, Δ' . If $\Delta(\xi) = \sum \xi_i^1 \otimes \xi_i^2$ then $\Delta'(\xi) = \sum \xi_i^2 \otimes \xi_i^1$. R is an intertwiner between these coproducts, that is

$$\Delta'(\xi)R = R\Delta(\xi) \quad \text{for all } \xi \in \mathcal{G}^*. \quad (1.21)$$

The transformation law of adjoint field operators Ψ_j^{j*} can be read of from eq.(1. 13) (assuming the properties of the antipode which are necessary for its validity). The result is of the form of a field transformation law (1. 8), if Δ is a *-homomorphism of algebras with the choice (1. 16) of the *-operation on $\mathcal{G}^* \otimes \mathcal{G}^*$. But with the choice (1. 17) of the *-operation this is not true. Instead the role of ξ^2 and ξ^1 in (1. 8) get interchanged. That is, Δ' appears in place of Δ . But one can use R to define a "covariant adjoint" $\tilde{\Psi}_j^{j*}$ which transforms like Ψ_j^{j*} , except that the representation τ^j gets replaced by its contragredient $\tilde{\tau}^j$. The covariant adjoint is defined as follows.

$$\tilde{\Psi}_j^{j*} = \Psi_k^{k*}(\tilde{\tau}_k^j \otimes U)(R). \quad (1.22)$$

This is a complex linear combination of field components Ψ_j^{j*} times representation operators $\mathcal{U}(\xi)$ of elements $\xi \in \mathcal{G}^*$. It is similar to the Dirac adjoint $\bar{\psi} = \psi^* \gamma^0$ of a Dirac spinor.

2 Group algebras

(Quasi) Quantum group algebras are generalizations of group algebras rather than of the groups themselves. Therefore we start with a review of the properties of traditional symmetries in quantum mechanics, which are groups. All the properties will be reformulated in such a way that they make no reference to the group elements themselves, but only to the group algebra.

If G is a finite group, the elements of its group algebra $\mathcal{G}^*$ are formal sums of group elements a with complex coefficients $\Xi(a)$,

$$\xi = \sum_{a \in G} \Xi(a)a. \quad (2.1)$$

Multiplication is performed with the help of the product in G ,

$$\xi\phi = \sum_{a \in G} \sum_{b \in G} \Xi(a)\Xi(b)ab \quad (2.2)$$

$$= \sum_{c \in G} (\Xi * \Phi)(c)c; \quad (2.3)$$

$$(\Xi * \Phi)(c) = \sum_{a \in G} \Xi(a)\Phi(a^{-1}c). \quad (2.4)$$

This generalizes to Lie groups G in the obvious fashion. If da is left invariant Haar measure then

$$\xi = \int_{a \in G} da \Xi(a)a, \quad (2.5)$$

where $\Xi(\cdot)$ is a generalized function on G [11]. $\mathcal{G}^*$ may be regarded as the dual of the space $\mathcal{G}$ of "good" functions f on G . The corresponding map

$$\xi : \mathcal{G} \rightarrow \mathbb{C} \quad (2.6)$$

is defined by $\xi[f] = \int da \Xi(a)f(a)$. We assume that the constant function $1 \in \mathcal{G}$, and that $\xi \in \mathcal{G}^*$ only if $\xi[f]$ is defined for all $f \in \mathcal{G}$, including $f = 1$. In the Lie group case, the group algebra contains both the group and the universal enveloping algebra of the Lie algebra of G . The corresponding generalized functions Ξ are δ -functions and their derivatives.

We work in the context of second quantized quantum mechanics. There will be a Hilbert space of states $\mathcal{H}$, a Hamiltonian H which is a self adjoint operator on $\mathcal{H}$, a ground state $|0\rangle \in \mathcal{H}$, and there will be field operators $\Psi(\mathbf{r}, t)$ which create particles from the ground state $|0\rangle$.

Let us now suppose that a group G with unit element e is an internal symmetry of such a quantum mechanical system. This means the following:

- The Hilbert space $\mathcal{H}$ carries a unitary representation of G by operators $\mathcal{U}(a)$, $a \in G$, with $\mathcal{U}(e) = 1$. Unitarity means that

$$\mathcal{U}(a^{-1}) = \mathcal{U}(a)^*.$$

• The ground state and the Hamiltonian are invariant,

$$\mathcal{U}(a)|0\rangle = |0\rangle, \quad [\mathcal{U}(a), H] = 0 \quad \text{for all } a \in G. \quad (2.7)$$

• The fields transform covariantly. This means the following. The fields come in multiplets (Ψ_j^I) where J distinguishes different multiplets, and j distinguishes the members of one multiplets. There are finite dimensional representations τ^J of G such that

$$\mathcal{U}(a)\Psi_j^I(\mathbf{r}, t) = \Psi_k^J(\mathbf{r}, t)\tau_{kj}^I(a)\mathcal{U}(a). \quad (2.8)$$

Internal symmetries do not act on space time coordinates of fields. Summation over repeated indices is understood here and throughout.

From the properties stated above, one may work out the transformation properties of many particle states,

$$|I_N i_N, \dots, I_1 i_1\rangle = \Psi_{i_N}^{I_N}[f_N] \dots \Psi_{i_1}^{I_1}[f_1]|0\rangle, \quad (2.9)$$

where $\Psi_i^I[f] = \int d\mathbf{r} \Psi_i^I(\mathbf{r}, t)f(\mathbf{r})$. The time t is thought to be fixed.

To compute $\mathcal{U}(a)|I_N i_N, \dots, I_1 i_1\rangle$ one shifts $\mathcal{U}(a)$ through field operators from left to right, using the field transformation law (2. 8), until it acts on the ground state. Then one uses invariance of the ground state, eq.(2. 7). As a result one finds that the many particle states transform according to the tensor product of the representations $\tau_N^I \dots \tau_1^I$. The tensor product $\otimes$ of two group representations τ and τ' is defined by

$$(\tau \otimes \tau')_{ik,jl}(a) = \tau_{ij}(a)\tau'_{kl}(a). \quad (2.10)$$

The tensor product of group representations is associative. Therefore multiple tensor products are uniquely defined. The transformation law of states comes out as

$$\begin{aligned} \mathcal{U}(a)|I, i\rangle &= |I, k\rangle \tau_k^I(a) \\ \mathcal{U}(a)|I_N i_N, \dots, I_1 i_1\rangle &= |I_N k_N, \dots, I_1 k_1\rangle \otimes (\tau_N^{I_N} \otimes \dots \otimes \tau_1^{I_1})_{k_N \dots k_1, i_N \dots i_1}(a). \end{aligned} \quad (2.11)$$

Summation over $k_N \dots k_1$ is understood.

Now we wish to restate these properties and their consequences in terms of the group algebra.

Representations $\mathcal{U}$, τ^J etc. may be extended to the group algebra in an obvious way,

$$\mathcal{U}(\xi) = \int_{a \in G} da \Xi(a) \mathcal{U}(a) \text{ etc.} \quad (2.12)$$

In particular, the trivial 1-dimensional representation is extended to a representation ϵ of $\mathcal{G}^*$,

$$\epsilon(\xi) = \int_{a \in G} da \Xi(a) = \xi[1]. \quad (2.13)$$

Being a representation, ϵ is a homomorphism of algebras $\epsilon: \mathcal{G}^* \mapsto \mathbb{C}$.

Invariance of the Hamiltonian and invariance of the vacuum, eqs.(2. 7) take the form

$$\mathcal{U}(\xi)|0\rangle = |0\rangle \epsilon(\xi), \quad [\mathcal{U}(\xi), H] = 0 \quad \text{for all } \xi \in \mathcal{G}^*. \quad (2.14)$$

To state the transformation law of field operators, we introduce

$$\Delta(a) = a \otimes a \quad \text{for } a \in G. \quad (2.15)$$

This extends to a homomorphism of algebras

$$\Delta: \mathcal{G}^* \mapsto \mathcal{G}^* \otimes \mathcal{G}^*, \quad (2.16)$$

$$\Delta(\xi) = \int_{a \in G} da \Xi(a) a \otimes a. \quad (2.17)$$

If τ and τ' are representations of an algebra $\mathcal{G}^*$ then the outer tensor product $\tau \otimes \tau'$ is a representation of $\mathcal{G}^* \otimes \mathcal{G}^*$. It is defined by

$$(\tau \otimes \tau')(\xi \otimes \eta) = \tau(\xi) \otimes \tau'(\eta), \quad (2.18)$$

etc., where the right hand side involves the standard tensor product of matrices (or linear operators). Writing all or some of the matrix indices we have

$$(\tau \otimes \tau')_{ik,jl}(\xi \otimes \eta) = \tau_{ij}(\xi) \tau'_{kl}(\eta). \quad (2.19)$$

$$(\tau_j \otimes \mathcal{U})(\xi \otimes \eta) = \tau_j(\xi) \mathcal{U}(\eta) \text{ etc.} \quad (2.20)$$

In this notation, the transformation law (2. 8) of field operators under the group extends to the group algebra as follows

$$\begin{aligned} \mathcal{U}(\xi) \Psi_j^I(\mathbf{r}, t) &= \Psi_k^J(\mathbf{r}, t) \int_{a \in G} da \tau_{kj}^J(a) \mathcal{U}(\Xi(a) a) \\ &= \Psi_k^J(\mathbf{r}, t) (\tau_j^J \otimes \mathcal{U})(\Delta(\xi)). \end{aligned} \quad (2.21)$$

$$(2.22)$$

The field transformation law (2. 21) permits to push representation operators $\mathcal{U}(\cdot)$ of elements of $\mathcal{G}^*$ through field operators Ψ from left to right.

Using this fact and the invariance of the ground state (2. 14), one can work out the transformation law of many particle states under the group algebra in the same manner as before in the group situation. The result can again be stated by saying that the multiparticle states transform according to tensor products of representations. Let us explain this in more detail.

Using the definition (2. 15) of the coproduct Δ for the group G the tensor product (2. 10) of group representations can be written as

$$(\tau \otimes \tau')(\xi) = (\tau \otimes \tau')(\Delta(\xi)), \quad (2.23)$$

for group elements $\xi \in G$. The right hand side involves the outer tensor product of representations as described before. Like every group representation, this representation can be extended to a representation of the group algebra $\mathcal{G}^*$. It is given by the same formula, where Δ is the extension (2. 17) of the coproduct (2. 15) to the group algebra. As a result, the field transformation law (2. 21) reduces to (1. 6).

The transformation law of many particle states comes out as

$$\begin{aligned} \mathcal{U}(\xi)|I, i\rangle &= |I, k\rangle \tau_k^I(\xi) \\ \mathcal{U}(\xi)|I_N i_N, \dots, I_1 i_1\rangle &= |I_N k_N, \dots, I_1 k_1\rangle \otimes (\tau_N^{I_N} \otimes \dots \otimes \tau_1^{I_1})_{k_N \dots k_1, i_N \dots i_1}(\xi) \end{aligned} \quad (2.24)$$

for all $\xi \in \mathcal{G}^*$. The brackets in the tensor product are actually unnecessary, because the tensor product of group algebra representations is associative.

Consider the tensor product of some representation τ with the trivial 1-dimensional representation ϵ . This should reproduce τ . From the definitions and $\mathbb{C}$ -linearity of the representations we deduce

$$(\tau \otimes \epsilon)(\xi) = (\tau \otimes \epsilon)(\Delta(\xi)) = \tau((id \otimes \epsilon)\Delta(\xi)), \quad (2.25)$$

and similarly for $\epsilon \otimes \tau$. id is the identity map. The result equals $\tau(\xi)$ if

$$(id \otimes \epsilon)\Delta = id = (\epsilon \otimes id)\Delta. \quad (2.26)$$

Since $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$ for group elements a , eq.(2.26) is true when applied to group elements a . By linearity it is therefore also true on group algebra elements ξ .

Consider next triple tensor products of representations. From the definition (2.23) we deduce

$$(\tau \otimes (\tau' \otimes \tau''))(\xi) = (\tau \otimes \tau' \otimes \tau'')((id \otimes \Delta)\Delta(\xi)), \quad (2.27)$$

$$((\tau \otimes \tau') \otimes \tau'')(\xi) = (\tau \otimes \tau' \otimes \tau'')((\Delta \otimes id)\Delta(\xi)). \quad (2.28)$$

Both sides are equal, i.e. the tensor product of representations is associative. This is true because

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta. \quad (2.29)$$

Both sides of this equation give $a \otimes a \otimes a$ when applied to a group element a . Because group elements span $\mathcal{G}^*$, the equality holds on $\mathcal{G}^*$. A coproduct with the property (2.29) is said to be *cocommutative*.

Let P act on $\mathcal{G}^* \otimes \mathcal{G}^*$ by interchange of factors, $P(a \otimes b) = b \otimes a$. Given one coproduct Δ , there exists another one,

$$\Delta'(\xi) = P\Delta(\xi). \quad (2.30)$$

For group elements $\Delta(a) = a \otimes a = \Delta'(a)$. Therefore

$$\Delta = \Delta' \text{ for group algebras.} \quad (2.31)$$

A coproduct with this property is called *cocommutative*. Cocommutativity can be expressed in a fancier way,

$$\Delta'(\xi)R = R\Delta(\xi) \text{ for all } \xi \in \mathcal{G}^*, \quad (2.32)$$

with $R = e \otimes e$.

3 Weak quasitriangular quasi Hopf algebras and their interpretation as a symmetry

A weak quasitriangular quasi Hopf algebra $\mathcal{G}^*$ is a generalization of a group algebra. It is an algebra equipped with a counit ϵ , a coproduct Δ , a reassociator φ , an R -element $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ and an antipode S with properties as follows.

$\mathcal{G}^*$ is an algebra with unit e . The counit

$$\epsilon : \mathcal{G}^* \rightarrow \mathbb{C}, \quad (3.1)$$

and the coproduct

$$\Delta : \mathcal{G}^* \rightarrow \mathcal{G}^* \otimes \mathcal{G}^* \quad (3.2)$$

are both homomorphisms of algebras, but Δ need not be unit preserving. The homomorphism property requires that $\Delta(\xi\eta) = \Delta(\xi)\Delta(\eta)$. Given the product Δ , there exists another one called Δ' . If $\Delta(\xi) = \sum \xi_p^1 \otimes \xi_p^2$, then Δ' is defined by

$$\Delta'(\xi) = \sum \xi_p^2 \otimes \xi_p^1. \quad (3.3)$$

It is demanded that

$$(id \otimes \epsilon)\Delta = (\epsilon \otimes id)\Delta = id, \quad (3.4)$$

($id =$ identity map). Furthermore one demands that an element $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ is given which implements (weak) quasi-coassociativity of the coproduct, eq.(3.10) below. We admit the possibility that $\Delta(e) \neq e \otimes e$, and we do not demand invertibility of φ , but only existence of a quasiinverse, still denoted by φ^{-1} , such that

$$\varphi\varphi^{-1} = (id \otimes \Delta)\Delta(e), \quad \varphi^{-1}\varphi = (\Delta \otimes id)\Delta(e), \quad (3.5)$$

$$(id \otimes id \otimes \epsilon)(\varphi) = \Delta(e),$$

$$(id \otimes \epsilon \otimes id)(\varphi) = \Delta(e),$$

$$(\epsilon \otimes id \otimes id)(\varphi) = \Delta(e). \quad (3.6)$$

The statement that φ^{-1} is a quasiinverse of φ means that $\varphi\varphi^{-1}\varphi = \varphi$. Finally there should exist $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ such that

$$\Delta'(\eta)R = R\Delta(\eta) \text{ for all } \eta \in \mathcal{G}^*. \quad (3.7)$$

We do not demand that R be invertible, instead it should have a quasiinverse R^{-1} such that

$$RR^{-1} = \Delta'(e), \quad R^{-1}R = \Delta(e). \quad (3.8)$$

It is natural to impose also the following requirement.

$$(id \otimes \epsilon)(R) = e, \quad (\epsilon \otimes id)(R) = e. \quad (3.9)$$

By definition, weak quasi-coassociativity demands that

$$\varphi(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi)\varphi \text{ for all } \xi \in \mathcal{G}^*. \quad (3.10)$$

Following Drinfeld the following relations between Δ , R , and φ are postulated.

$$(id \otimes id \otimes \Delta)(\varphi)(\Delta \otimes id \otimes id)(\varphi) = (e \otimes \varphi)(id \otimes \Delta \otimes id)(\varphi)(\varphi \otimes e), \quad (3.11)$$

$$(id \otimes \Delta)(R) = \varphi_{23}^{-1} R_{13} \varphi_{213} R_{12} \varphi^{-1}, \quad (3.12)$$

$$(\Delta \otimes id)(R) = \varphi_{312} R_{13} \varphi_{132}^{-1} R_{23} \varphi. \quad (3.13)$$

We used the standard notation. If $R = \sum r_a^1 \otimes r_a^2$ then

$$R_{13} = \sum r_a^1 \otimes e \otimes r_a^2, \quad R_{12} = \sum r_a^1 \otimes r_a^2 \otimes e, \quad (3.14)$$

etc. If s is any permutation of 123 and $\varphi = \sum \varphi_s^1 \otimes \varphi_s^2 \otimes \varphi_s^3$ then

$$\varphi_{(1)(2)(3)} = \sum_s \varphi_s^{s^{-1}(1)} \otimes \varphi_s^{s^{-1}(2)} \otimes \varphi_s^{s^{-1}(3)}. \quad (3.15)$$

Eq. (3.12), or (3.13), implies validity of quasi Yang Baxter equations,

$$R_{12}\varphi_{212}R_{13}\varphi_{12}^{-1}R_{23}\varphi = \varphi_{221}R_{23}\varphi_{231}^{-1}R_{13}\varphi_{213}R_{12}, \quad (3.16)$$

and this guarantees that R together with φ determines a representation of the braid group as will be explained below. There should also exist an antipode

$$S: \mathcal{G}^* \rightarrow \mathcal{G}^* \quad (3.17)$$

which is an antihomomorphism of algebras. The coproduct is used to define the *tensor product of representations of $\mathcal{G}^*$* as in eq.(1. 2). The antipode is used to define the *contragredient $\bar{\tau}$* to a representation τ by eq.(1. 12). One may want to impose it as an additional requirement on the antipode that $\bar{\tau} \otimes \tau$ should contain the trivial representation ϵ for any representation τ . Drinfeld proposed conditions of this type for quasi Hopf algebras [4]. We shall not use them here.

In quantum mechanics it is important to have a $*$ -operation. We say that $\mathcal{G}^*$ is a *weak quasitriangular quasi Hopf $*$ -algebra* if it has the following properties in addition to those listed above.

Both ϵ and Δ are $*$ -homomorphisms of algebras, with the $*$ -operation on $\mathcal{G}^*$ given either by eq.(1. 16) or by eq.(1. 17). In addition we require that the antipode satisfies

$$m_*((S^{-1} \otimes id \otimes id)(id \otimes \Delta)(\xi)) = \epsilon \otimes \xi, \quad (3.18)$$

$$m'_*(id \otimes S^{-1} \otimes id)(id \otimes \Delta)(\xi) = \xi \otimes \epsilon, \quad (3.19)$$

for all $\xi \in \mathcal{G}^*$, with multiplication operators defined by

$$m_*(\xi \otimes \eta \otimes \zeta) = \eta \xi \otimes \zeta, \quad m'_*(\xi \otimes \eta \otimes \zeta) = \zeta \otimes \eta \xi.$$

This condition is used to obtain (1. 13) and the transformation law of adjoint field operators.

A weak quasitriangular quasi Hopf $*$ -algebra $\mathcal{G}^*$ is called a *symmetry* of a quantum mechanical system if a unitary representation $\mathcal{U}$ of $\mathcal{G}^*$ in the Hilbert space of physical state is given, the Hamiltonian and the ground state are invariant, the field operators transform covariantly, and $\mathcal{U}(e) = 1$. All this was explained in the introduction.

Let us now discuss the implications of the assumptions. From the transformation law of field operators and invariance of the ground state one deduces the transformation law of 1-particle states and multiparticle states (2. 9), eqs.(2. 24) in the same manner as for group algebras. The first identity (2. 26) is used in this. The brackets in the result are now important in general, because the tensor product need not be associative, unless the coproduct is coassociative.

The identity (3. 4) guarantees that tensoring with the trivial 1-dimensional representation ϵ amounts to doing nothing, as in the case of group algebras.

The existence of the reassociator φ implies that the tensor product of representations is still quasi-associative in the sense that the equivalence (1. 9) of triple tensor

products of representations holds true. If one of the three representations is the 1-dimensional representation ϵ , the reassociation of the triple tensor product is trivial. This is encoded in the relations (3. 6).

All this is a formal use of φ . Its physical use is to enable us to define products of field operators which transform covariantly under $\mathcal{G}^*$, and to write down $\mathcal{G}^*$ -covariant operator product expansions for products of field operators in local quantum field theories. In addition it enters into the $\mathcal{R}$ -matrix (1. 20) in the $\mathcal{G}^*$ -covariant local braid relations (1. 19).

The coproduct is coassociative if φ is trivial, $\varphi = e \otimes e \otimes e$.

If the coproduct is not coassociative, then the ordinary product $\Psi_j^i \Psi_k^j$ of field operators is not $\mathcal{G}^*$ -covariant. But we may define a covariant product $\times$ as follows

$$(\Psi^J \times \Psi^K)_{jk} = \Psi_j^i \Psi_m^k (\tau_j^i \otimes \tau_m^k \otimes \mathcal{U})(\varphi). \quad (3.20)$$

This transforms according to the representation $(\tau^J \otimes \tau^K)$ of $\mathcal{G}^*$ in the sense of (1. 8). Note that $(\Psi^J \times \Psi^K)_{jk}$ is a sum of terms $\Psi_j^i \Psi_m^k$ multiplied from the right with representation operators $\mathcal{U}(\phi)$ of elements of $\mathcal{G}^*$. The proof of covariance uses the intertwining property (3. 10) of φ .

In this covariant notation, the local braid relations (1. 19) of field operators read as follows (with the same phase factor ω)

$$(\Psi^J \times \Psi^J)_{ij} = \omega(\Psi^J \times \Psi^J)_{ik} \mathcal{R}_{ik,jj}^{JJ} \quad \text{with} \quad (3.21)$$

$$\mathcal{R}_{ik,jj}^{JJ} = (\tau_k^i \otimes \tau_j^j)(R). \quad (3.22)$$

This holds true under the same conditions on the arguments of the field operators, which we neglected to write, as in the braid relations (1. 19). It follows from the intertwining property (1. 21) that both sides of equation (3. 21) transform according to the representation $\tau^J \otimes \tau^J$ of $\mathcal{G}^*$. This shows that the local braid relations which we propose are $\mathcal{G}^*$ -covariant. In the coassociative case $\varphi = e \otimes e \otimes e$, the covariant product is equal to the ordinary one, and the braid relations in the form (1. 19) involve a numerical $\mathcal{R}$ -matrix.

The $\times$ -product is not associative (unless φ is trivial). But the relation (3. 11) ensures that it is quasi-associative in the following sense.

$$((\Psi^J \times \Psi^J) \times \Psi^K)_{ijk} = (\Psi^J \times (\Psi^J \times \Psi^K))_{i,jk} (\tau^J \otimes \tau^J \otimes \tau^K)_{j,jk,i}(e). \quad (3.23)$$

This equation tells us that the triple product on the left hand side is a complex linear combination of the triple products of field operators on the right hand side. The inverse relation is also true. It involves φ^{-1} .

If there is truncation, $\Delta(e) \neq e \otimes e$, then our stated requirement that $\mathcal{U}(e) = 1$ has some important consequences. In particular it implies, together with eq.(3. 6), that

$$\begin{aligned} (\Psi^{I_N} \times (\Psi^{I_{N-1}} \times \dots (\Psi^{I_2} \times \Psi^{I_1}) \dots))_{i_N \dots i_1} |0\rangle &= \Psi_{i_N}^{I_N} \Psi_{i_{N-1}}^{I_{N-1}} \dots \Psi_{i_2}^{I_2} \Psi_{i_1}^{I_1} |0\rangle > (3.24) \\ &= |I_N i_N, \dots, I_1 i_1\rangle. \end{aligned}$$

In words: On the ground state, ordinary products of field operators and covariant products performed in a definite order, are the same.

Finally we turn to the implications of the relations which involve R . It was already mentioned that the fundamental intertwining property (1. 21) of R is crucial to ensure the $\mathcal{G}^*$ -covariance of local braid relations (3. 21).

The element $R \in \mathcal{G}^* \otimes \mathcal{G}^*$, which satisfies the quasi Yang Baxter equations, furnishes a representation of the braid group. Let us recall that the braid group B_n on n threads is generated by elements σ_i and σ_i^{-1} ($i = 1 \dots n-1$) which obey the Artin relations

$$\sigma_i \sigma_k = \sigma_k \sigma_i \quad \text{if } |k-i| \geq 2, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_i^{-1} = \epsilon = \sigma_i^{-1} \sigma_i. \quad (3. 25)$$

The unit element of B_n is written as ϵ . We introduce some notations. Write

$$e^n = \epsilon \otimes \dots \otimes \epsilon \quad (n \text{ factors}) \quad (3. 26)$$

and similarly for ϵ^n . In addition we abbreviate $\mathcal{G}^{*\otimes n} = \mathcal{G}^* \otimes \dots \otimes \mathcal{G}^*$ (n factors), and

$$\begin{aligned} \Delta^n &= (id^{n-1} \otimes \Delta) \dots (id \otimes \Delta) \Delta \quad \text{for } n \geq 2, \\ \Delta^1 &= \Delta, \quad \Delta^0 = id, \quad \Delta^{-1} = \epsilon. \end{aligned} \quad (3. 27) \quad (3. 28)$$

For simplicity we assume that $\mathcal{G}^*$ is semisimple. Then the permutation of factors in $\mathcal{G}^* \otimes \mathcal{G}^*$ can be implemented by a multiplication operator $\mathcal{P} \in \mathcal{G}^* \otimes \mathcal{G}^*$ with the properties

$$(i) \quad \mathcal{P} \xi \otimes \eta = \eta \otimes \xi \quad \text{for all } \xi, \eta \in \mathcal{G}^*,$$

$$(ii) \quad \tau \otimes \tau'(\mathcal{P}) = 0 \quad \text{if } \tau, \tau' \text{ are inequivalent irreducible representations of } \mathcal{G}^*.$$

Theorem (Artin relations) Suppose that $\mathcal{G}^*$ is semisimple. Let $\hat{R}^+ = \mathcal{P}R$ and $\hat{R}^- = R^{-1}\mathcal{P}$, and $n = r+k+1$, $r \geq 0$. Define $\sigma_k^{\pm} \in \mathcal{G}^{*\otimes n}$ by

$$\sigma_k^{\pm} = \Delta^{n-1}(\epsilon)(id^{n-k-1} \otimes \Delta^{k-2})(\epsilon^{n-k-1} \otimes \varphi(\hat{R}^{\pm} \otimes \epsilon)\varphi^{-1}). \quad (3. 29)$$

Then the $\sigma_k^{\pm}$ obey Artin relations (3. 25) with $\epsilon = \Delta^{n-1}(\epsilon)$.

The proof is spelled out in [12]. [If semisimplicity is not assumed, there is still a representation of B_n by linear maps $\mathcal{G}^{*\otimes n} \mapsto \mathcal{G}^{*\otimes n}$, but these linear maps are not necessarily implemented by multiplication operators.]

We remark that this representation of the braid group can act on the Hilbert space of physical states $\mathcal{H}$. In the algebraic approach to quantum field theory, this representation of the braid group is determined by the algebra of observables. It is the starting point from which one tries to recover the symmetry algebra - see Fredenhagen's lectures at this school [13]. We will not go into this now.

Drinfeld's relations (3. 12) and (3. 13) imply the quasi Yang Baxter equation. They imply also that covariant products of field operators obey local braid relations, under appropriate conditions on their arguments, if the individual fields do.

A field operator $\Phi(x, t)$ is called $\mathcal{G}^*$ -invariant if it transforms according to the trivial 1-dimensional representation ϵ , or, equivalently, if $\mathcal{U}(\xi)\Phi = \Phi\mathcal{U}(\xi)$ for all $\xi \in \mathcal{G}^*$. It follows from eq.(3. 9) that $\mathcal{G}^*$ -invariant fields Φ , which one may construct as composites from $\mathcal{G}^*$ -covariant fields, for instance, obey ordinary local commutation

relations with all braid-local fields, except for the possible appearance of a phase factor ω_1 which depends on the fields

$$\Phi(x, t)\Psi_j^i(x', t) = \omega_1 \Psi_j^i(x', t)\Phi(x, t) \quad \text{if } i \neq j. \quad (3. 30)$$

In algebraic field theory one demands that the phase factor ω_1 is 1 if Φ is an observable.

4 Truncated quantum group algebras

The best known examples of quasitriangular Hopf algebras are the deformations $U_q(s\mathfrak{g})$ of the universal enveloping algebra (or the group algebra) of $SU(2)$. They are indexed by a complex number q .

The quantum group algebra $U_q(s\mathfrak{g})$ is generated by elements $q^{\pm H/2}$ and $S_{\pm}$ subject to the relations

$$q^{H/2} q^{-H/2} = q^{-H/2} q^{H/2} = 1, \quad (4. 1)$$

$$q^{H/2} S_{\pm} = q^{\pm \frac{1}{2}} S_{\pm} q^{H/2}, \quad (4. 2)$$

$$[S_+, S_-] = \frac{q^H - q^{-H}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \quad (4. 3)$$

Here and in the following we write 1 in place of e for the unit element in $\mathcal{G}^* = U_q(s\mathfrak{g})$.

We denote the comultiplication in $U_q(s\mathfrak{g})$ by Δ_q in order to distinguish it from the comultiplication Δ for the truncated quantum group algebra which we are going to introduce. It is given by

$$\Delta_q(q^{\pm H/2}) = q^{\pm H/2} \otimes q^{\pm H/2} \quad (4. 4)$$

$$\Delta_q(S_{\pm}) = S_{\pm} \otimes q^{H/2} + q^{-H/2} \otimes S_{\pm}. \quad (4. 5)$$

If q is real, a $*$ -operation can be defined by

$$S_{\pm}^* = S_{\pm}, \quad (q^{\pm H/2})^* = q^{\pm H/2}.$$

If $|q| = 1$, a $*$ -operation can be defined by

$$S_{\pm}^* = S_{\mp}, \quad (q^{\pm H/2})^* = q^{\mp H/2}.$$

In both cases this is an automorphism satisfying $\Delta_q(\xi^*) = \Delta_q(\xi)^*$ provided we adopt the appropriate convention for the adjoint on $\mathcal{G}^* \otimes \mathcal{G}^*$, viz. (1. 16) if q is real, and (1. 17) if $|q| = 1$.

The counit is

$$\epsilon(q^{\pm H/2}) = 1, \quad \epsilon(S_{\pm}) = 0,$$

and the antipode S acts as

$$q^{\mp H/2} = S(q^{\pm H/2}), \quad -q^{\pm 1/2} S_{\pm} = S(S_{\pm}). \quad (4. 6)$$

There exists a canonical R -element for $U_q(s\mathfrak{g})$ which we denote by R_q . It is given by

$$R_q = q^{H \otimes H} \sum_{n \geq 0} \frac{(1 - q^{-1})^n}{[n]!} q^{-\frac{1}{2}n(n-1)} q^{nH/2} S_+^n \otimes q^{-nH/2} S_-^n \quad (4. 7)$$

If $|q| = 1$ and if we adopt the choice (1. 17) of the $\ast$ -operation, viz. $(a \otimes b)^\ast = b^\ast \otimes a^\ast$ we get that R_q is unitary,

$$R_q^\ast = R_q^{-1} = (S \otimes id)(R_q).$$

The "quantum dimension" d_τ of a representation τ is defined by

$$d_\tau = \text{tr} \tau(q^H). \quad (4.8)$$

If q is a (primitive p -th) root of unity, then $U_q(sl_2)$ is not semisimple, and tensor products of its irreducible representations are in general not fully reducible. Its irreducible representations τ^j with nonzero quantum dimension are called "physical" representations. They are labelled by $J = 0, \frac{1}{2}, \dots, \frac{1}{2}(p-2)$. They have dimension $2J+1$.

We denote the tensor product of $U_q(sl_2)$ -representations by $\otimes_q$. The tensor product of $\tau^I \otimes_q \tau^J$ of two physical representations decomposes in general into physical representations, plus unphysical representations with quantum dimension 0. If we multiply $(\tau^I \otimes_q \tau^J)(\xi)$ with a projection operator P_{IJ} which cuts away the contribution with zero quantum dimension, one obtains what is known as the truncated tensor product of physical representations of $U_q(sl_2)$.

A weak quasitriangular quasi Hopf algebra $\mathcal{G}^\ast$ is canonically associated with $U_q(sl_2)$ with q a root of unity. As an algebra $\mathcal{G}^\ast = U_q(sl_2)/\mathcal{I}$, where $\mathcal{I}$ is the ideal which is annihilated by all the physical representations $\tau^I, 2I = 0 \dots p-2$, of $U_q(sl_2)$. $\mathcal{G}^\ast$ is semisimple, its representations are fully reducible, and the irreducible ones are precisely the physical representations of $U_q(sl_2)$. Let

$$u(I, J) = \min\{|I+J|, p-2-I-J\} \quad (4.9)$$

and let P_{IJ} be the projector on the physical subrepresentations $K, |I-J| \leq K \leq u(I, J)$ of the tensor product $\tau^I \otimes_q \tau^J$ of $U_q(sl_2)$ representations. There exists $P \in \mathcal{G}^\ast \otimes \mathcal{G}^\ast$ such that $P_{IJ} = (\tau^I \otimes \tau^J)(P)$. The coproduct in $\mathcal{G}^\ast$ is determined in terms of the coproduct Δ_q in $U_q(sl_2)$ as

$$\Delta(\xi) = P \Delta_q(\xi), \quad (4.10)$$

hence $\Delta(e) = P \neq e \otimes e$. This coproduct specifies a tensor product $\otimes$ which is equal to the truncated tensor product of physical $U_q(sl_2)$ representations. Thus

$$\tau^I \otimes \tau^J = \bigoplus_{|I-J| \leq K \leq u(I, J)} \tau^K. \quad (4.11)$$

There exists an element $\varphi \in \mathcal{G}^\ast \otimes \mathcal{G}^\ast$ such that $\varphi_{IK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi)$ implements the well known unitary equivalence of the truncated tensor products $\tau^I \otimes (\tau^J \otimes \tau^K)$ and $(\tau^I \otimes \tau^J) \otimes \tau^K$. A truncated tensor product $\otimes$ is defined also for basis vectors $\hat{e}_i^I$ in the dual representation spaces $\hat{V}^I$ on which $\mathcal{G}^\ast$ acts from the right, viz. $\hat{e}_i^I \otimes \hat{e}_j^J = \hat{e}_i^I \otimes \hat{e}_j^J P_{IJ}$. The map φ_{IJK} can be specified by its action on triple truncated products of basis vectors, together with the condition $\varphi = (id \otimes \Delta)(\Delta(e)\varphi)$, viz.

$$\sum_{i,j,k,p} \hat{e}_i^I \hat{e}_j^J \hat{e}_k^K \hat{e}_p^L \varphi_{IJK} = \sum_{Q, i,j,k,q} P_{PQ} \hat{e}_i^I \hat{e}_j^J \hat{e}_k^K \hat{e}_q^Q \varphi_{IJK} \hat{e}_p^L \hat{e}_q^Q \hat{e}_k^K \hat{e}_i^I \otimes \hat{e}_j^J \otimes \hat{e}_k^K \quad (4.12)$$

with fusion matrices given by 6J-symbols,

$$F_{PQ|KL}^{IJ} = \left\{ \begin{matrix} K & J & P \\ I & L & Q \end{matrix} \right\}_q.$$

[...] are Clebsch Gordan coefficients for $U_q(sl_2)$ (and at the same time for $\mathcal{G}^\ast$).

The R -element of $\mathcal{G}^\ast \otimes \mathcal{G}^\ast$ is given in terms of the R -element R_q for $U_q(sl_2)$ by

$$R = R_q \Delta(e) = \Delta^q(e) R_q, \quad (4.13)$$

while antipode and counit are the same as in $U_q(sl_2)$. It is shown in ref. [1] that the defining properties of a weak quasitriangular quasi Hopf algebra are satisfied.

Since $\Delta(e)^\ast = \Delta(e^\ast) = \Delta(e)$, the unitarity of R_q implies that

$$R^\ast = R^{-1}. \quad (4.14)$$

5 A model with quantum symmetry and local braid relations

The chiral Ising model provides an example of a conformal field theory which admits the truncated quantum group algebra $\mathcal{G}^\ast$ canonically associated with $U_q(sl_2)$ as a symmetry ($q = \pm i$). It has field operators which act in a positive definite Hilbert space of physical states, and which satisfy $\mathcal{G}^\ast$ -covariant local braid relations. This was demonstrated in [2].

The chiral Ising model lives on a circle S^1 . Its chief observable field is the stress energy tensor $T(z)$. $z \in S^1$ is regarded as a complex number of modulus 1. The stress tensor furnishes a Virasoro algebra with generators $L_n = L_{-n}$, $n = 0, \pm 1, \pm 2, \dots$, and central charge $c = \frac{1}{2}$.

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \quad (5.1)$$

It can be constructed from a "universal Majorana algebra" $Major$ with generators b_a , $a = 0, \pm \frac{1}{2}, \pm 1, \dots$, $b_a^\ast = b_{-a}$. $Major$ contains a central element $Y = 4b_0^2 - 1$ which has eigenvalues ± 1 . The generators are the Fourier modes of a Fermi field,

$$\psi(z) = \sum_{a \in \frac{1}{2}\mathbb{Z}} b_a z^{-a-\frac{1}{2}}. \quad (5.2)$$

It is a two-valued function on the circle and satisfies anticommutation relations

$$\{\psi(z), \psi(w)\} = \pi i [\delta(z-w) - Y \delta(z - e^{2\pi i} w)], \quad (5.3)$$

or, equivalently

$$\{b_a, b_c\} = \frac{1}{2} [1 + (-1)^{2a} Y] \delta_{a,-c}. \quad (5.4)$$

The stress energy tensor is obtained as

$$T(z) = -\frac{1}{2} \lim_{w \rightarrow z} \left\{ \frac{1}{2} \psi(w) \psi'(z) + \frac{1}{2} \psi(w e^{2\pi i}) \psi'(z e^{2\pi i}) - (w-z)^{-2} \right\}, \quad (5.5)$$

$$\psi'(z) = \frac{\partial}{\partial z} \psi(z).$$

The Virasoro algebra with central charge $c = \frac{1}{2}$ admits 3 inequivalent representations π^J which we label by $J = 0, \frac{1}{2}, 1$. They act in Hilbert spaces $\mathcal{H}^J$ with lowest weight vectors $|\lambda_J > \text{of weight } \lambda_J = 0, \frac{1}{16}, \frac{1}{2}$.

All these irreducible Virasoro representation spaces are actually representation spaces for an algebra of observables $\mathcal{A}$ which is spanned by all those even products of Majorana fields which are invariant under rotations of the circle by 2π . Inequivalent positive energy representations of the observable algebra are known in quantum mechanics as *superselection sectors*. Thus we have 3 superselection sectors, including the vacuum sector $\mathcal{H}^0$ which contains the ground state $|0\rangle$. π^0 is called the vacuum representation of $\mathcal{A}$.

All these inequivalent irreducible positive energy representations of the algebra of observables can be obtained from the vacuum representation π^0 by composition with a suitable $*$ -morphism ρ_J of the algebra of observables $\mathcal{A}$.

$$\pi^J(A) = \pi^0(\rho_J(A)). \quad (5.6)$$

By definition, a $*$ -morphism ρ is a $\mathbb{C}$ -linear map $\mathcal{A} \rightarrow \mathcal{A}$ which obeys

$$\rho(A_1)\rho(A_2) = \rho(A_1 A_2), \quad \rho(A^*) = \rho(A)^*. \quad (5.7)$$

In our model, suitable morphisms are known explicitly. They are given by the following morphisms of the universal Majorana algebra. [10]

$$\rho_0 = id \quad (5.8)$$

$$\rho_{1/2}(b_a) = \begin{cases} ib_{a+1/2} & a \geq \frac{1}{2} \\ \frac{1}{\sqrt{2}}(b_{1/2} - b_{-1/2}) & a = 0 \\ -ib_{a-1/2} & a \leq -\frac{1}{2} \end{cases}, \quad \rho_{1/2}(Y) = -Y \quad (5.9)$$

$$\rho_1(b_a) = \begin{cases} -b_a & a \neq 0 \pm \frac{1}{2} \\ b_{-a} & a = 0, \pm \frac{1}{2} \end{cases}, \quad \rho_1(Y) = Y. \quad (5.10)$$

ρ_1 is an automorphism, i.e. bijective, but $\rho_{1/2}$ is not surjective on $\mathcal{A}$ because $b_r(b_{\frac{1}{2}} + b_{-\frac{1}{2}})$, ($r \in \mathbb{Z} + \frac{1}{2}$) and $b_n b_0$, ($n \in \mathbb{Z}$) are not in its range.

The Hilbert space of physical states of our model will be the sum of $2J+1$ copies $\mathcal{H}_m^J$, $m = -J, \dots, J$ of each of our inequivalent representation spaces $\mathcal{H}^J$ of $\mathcal{A}$.

$$\mathcal{H} = \bigoplus_{J=0, \frac{1}{2}, 1}^J \bigoplus_{m=-J}^J \mathcal{H}_m^J. \quad (5.11)$$

A representation π of $\mathcal{A}$ on all of $\mathcal{H}$ is defined by its restrictions to the subspaces $\mathcal{H}_m^J$,

$$\pi(A) = \pi^J(A) \text{ on } \mathcal{H}_m^J \text{ (} m = -J, \dots, J \text{)}. \quad (5.12)$$

All infinite-dimensional Hilbert spaces of the world are the same. In view of the equivalence of representations (5.6) we may introduce identification maps i_{Jm} which identify the representation spaces $(\pi^J, \mathcal{H}_m^J)$ and $(\pi^0 \circ \rho_J, \mathcal{H}^0)$. Thus,

$$i_{Jm}^J : \mathcal{H}^0 \rightarrow \mathcal{H}_m^J \quad (5.13)$$

is bijective and has the intertwining property

$$\pi^J(A) i_{Jm}^J = i_{Jm}^J \pi^0(\rho_J(A)). \quad (5.14)$$

Now we are ready to describe the action of the truncated quantum group algebra $\mathcal{G}^*$ on the Hilbert space $\mathcal{H}$ of physical states.

$\mathcal{U}(\xi) i_{Jm}^J |\psi\rangle = i_{Jm}^J |\psi\rangle$ for arbitrary $|\psi\rangle \in \mathcal{H}^0$. The symmetry acts as a *gauge symmetry (of first kind)*, i.e. all observables are invariant

$$[U(\xi), A] = 0 \text{ for all } A \in \mathcal{A}, \quad \xi \in \mathcal{G}^*. \quad (5.16)$$

Field operators which make transitions between the sectors $\mathcal{H}_m^J$ with different J are constructed from observables by use of the general bosonization formula of Doplicher Haag and Roberts.

$$\Psi_m^J(w) = \Gamma_m^J A_J(w). \quad (5.17)$$

All the dependence on the space time point w is in the second factor $A_J(w)$ which is an (unbounded) observable, i.e. a limit of elements in the observable algebra. These factors commute therefore with $\mathcal{U}(\xi)$, $\xi \in \mathcal{G}^*$. Only the "constant fields" Γ_m^J transform nontrivially under the symmetry. We will first describe some of their properties. Below we will give an explicit formula which will show that operators with these properties exist.

(i) intertwining property for representations of $\mathcal{A}$,

$$\pi(A) \Gamma_m^J = \Gamma_m^J \pi(\rho_J(A)). \quad (5.18)$$

(ii) $\mathcal{G}^*$ -covariance,

$$\mathcal{U}(\xi) \Gamma_m^J = \Gamma_m^J (\gamma_{km}^J \otimes \mathcal{U})(\Delta(\xi)). \quad (5.19)$$

(iii) Action on lowest weight vectors

$$\Gamma_m^J |0\rangle = |\lambda_J >_m \in \mathcal{H}_m^J \quad (5.20)$$

The subscript m on $|\lambda_J >_m$ distinguishes between lowest weight vectors in the copies $\mathcal{H}_m^J$ of $\mathcal{H}^J$. The $\mathcal{G}^*$ -covariance of the field operators (5.17) follows from $\mathcal{G}^*$ -covariance of Γ_m^J and $\mathcal{G}^*$ -invariance of the observable factor. The intertwining property (i) and the multiplication law in $\mathcal{A}$ permit to push observables $T \in \mathcal{A}$ through the field (5.17). Therefore their commutators can be computed in principle. The factor $A_J(w)$ in the bosonization formula is implicitly determined by the requirement that the field is relatively local to the observables, in particular to the stress tensor.

$$[\Psi_m^J(w), T(z)] = 0 \text{ if } w \neq z. \quad (5.21)$$

This determines A_J up to a Klein transformation, i.e. up to multiplication with an element in the center of the observable algebra. This arbitrariness can be eliminated by suitable conventions, after a "point at infinity" is chosen on the circle. An illustrative calculation is found in ([15]) which shows how A_J is actually computed in a simpler model.

The existence of Γ_m^J needs only a limited amount of dynamical information: It must be possible to put the list of sectors into correspondence with the list of irreducible representations of the symmetry algebra in such a way that the fusion rules of the quantum field theory match with the tensor product decomposition of the symmetry algebra. To obtain local braid relations for the field operators more dynamical information will be needed. This will be discussed later on.

We turn to a discussion of the fusion rules. A product of morphisms is again a morphism. It is defined in the obvious way,

$$(\rho \circ \rho')(A) = \rho(\rho'(A)). \quad (5.22)$$

In our model the Hamiltonian is

$$L_0 = \sum_{a>0} ab_{-a}b_a + \frac{1}{8}b_0^2. \quad (5.23)$$

Summation runs over $a = \frac{1}{2}, 1, \frac{3}{2}, \dots$. It can be verified that $\rho(L_0)$ is a positive operator when ρ is equal to ρ_J or equal to a product of such morphism. Therefore $\pi^0 \circ \rho$ is a positive energy representation for any such ρ . We write $[\rho]$ for the equivalence class of this positive energy representation of $\mathcal{A}$.

If ρ is not one of the irreducible morphisms ρ_J then $[\rho]$ is in general a reducible representation of $\mathcal{A}$. It can be decomposed into irreducible representations. The result is as follows.

$$[\rho_{1/2}^2] = [\rho_0] + [\rho_1] \quad (5.24)$$

$$[\rho_{1/2}\rho_1] = [\rho_{1/2}\rho_{1/2}] = [\rho_{1/2}] \quad (5.25)$$

$$[\rho_1^2] = [\rho_0] \quad (5.26)$$

and $[\rho_0\rho_J] = [\rho_J]$ for $J = 0, \frac{1}{2}, 1$. We see that this matches with the tensor product decomposition for the irreducible representations of the truncated quantum group algebra for $U_q(st_2)$ with $q = \pm i$. Let us write $[J]$ for the equivalence class of the representation τ^J of $\mathcal{G}^*$. Formula (4. 11) yields

$$[1/2] \otimes [1/2] = [0] + [1] \quad (5.27)$$

$$[1/2] \otimes [1] = [1] \otimes [1/2] = [1/2] \quad (5.28)$$

$$[1] \otimes [1] = [0]. \quad (5.29)$$

Let us briefly explain how the fusion rules are derived in the algebraic frame work. To do this one uses intertwiners T and their associated projectors $\Pi = T^*T$. The intertwiners are elements T of the observable algebra, and $\pi^0(T)$ are bounded operators in $\mathcal{H}^0$. To reduce $\pi^0 \circ \rho_K \rho_J$ into irreducible subrepresentations $\cong \pi_0 \circ \rho_L$, one needs projection operators $\pi^0(\Pi(L_K^J))$ which commute with the representation operators $\pi^0 \circ \rho_K \rho_J(A)$ for all $A \in \mathcal{A}$

$$\rho_K \rho_J(A) \Pi(L_K^J) = \Pi(L_K^J) \rho_K \rho_J(A), \quad (5.30)$$

One needs also intertwining operators $\pi^0(T(L_K^J))$ which establish the unitary equivalence of $\pi_0 \circ \rho_K \rho_J$ restricted to $\pi^0(\Pi_L)\mathcal{H}_0$ with $\pi_0 \circ \rho_L$,

$$\rho_L(A)T(L_K^J) = T(L_K^J)\rho_K \rho_J(A), \quad (5.31)$$

$$T(L_K^J)^*T(L_K^J) = \Pi(L_K^J). \quad (5.32)$$

We say that $T(L_K^J)$ is an intertwiner "from $\rho_K \rho_J$ to ρ_L ".

Given J, K the number of linearly independent intertwining operators $\pi^0(T(L_K^J))$ is equal to the number of times that $[\rho_L]$ appears in $[\rho_K \rho_J]$. In our model this multiplicity is at most one. The projectors satisfy the completeness relation

$$\sum_L \Pi(L_K^J) = 1. \quad (5.33)$$

The intertwiners are known explicitly [10]. For instance

$$T(\frac{1}{2} \frac{1}{2}) = \frac{1}{2}[1 - Y](1 - 2b_{-1/2}b_{1/2}) + \frac{1}{2}[1 + Y]\sqrt{2}b_0(b_{-1} - b_1). \quad (5.34)$$

This particular intertwiner is unitary. It follows that $[\rho_{1/2}\rho_1]$ is irreducible and equivalent to $[\rho_{1/2}]$.

With the knowledge that fusion rules and tensor product decomposition match, we can write down an expression for the "constant fields"

$$\Gamma_m^J = \sum_{K,L} \sum_{i,j} \langle L_K^J | i \rangle_{q^*} \langle j |_{q^*} \pi_0(T(L_K^J)) \delta_{Li} \quad (5.35)$$

$[\cdot]_q$ are Clebsch Gordan coefficients for $U_q(st_2)$ and for $\mathcal{G}^*$ at the same time. One verifies that Γ_m^J has all the properties listed above.

Now we have an action of the symmetry algebra in the Hilbert space $\mathcal{H}$ of physical states, and field operators on this space which are relatively local to the observables and which transform covariantly under $\mathcal{G}^*$. But we have used only part of the dynamical information. More is needed to get also the local braid relations.

The morphisms ρ_J of the algebra of observables and their intertwiners determine the 6J-symbols of the symmetry algebra, and a representation of the braid group for every J , as follows.

Intertwiners from $\rho_I \rho_J \rho_K$ to ρ_L can be constructed from the elementary intertwiners in two different ways, decomposing either $\rho_I \rho_J$ first, or $\rho_I \rho_K$. In either way one obtains a complete set of intertwiners. Therefore there must be a linear relation between these two sets of intertwiners. It is known as the "fusion identity" and reads

$$T(L_I^N) \rho_I T(L_N^J) = \sum_M F_{NM|IK} T(L_M^K) T(L_M^J). \quad (5.36)$$

in the absence of multiplicities. The explicit form of the fusion matrix depends on phase conventions. When suitable phase conventions are imposed then F is a numerical matrix which is given by 6J-symbols.

$$F_{PQ|KL} = \{ \begin{matrix} K & J & P \\ L & L & Q \end{matrix} \}_q.$$

The algebra of observable $\mathcal{A}$ and its morphisms determine a unitary representation of the braid group by elements of $\mathcal{A}$ of the following form.

$$\sigma_1 = \epsilon_J, \quad \sigma_n = \rho_J^{n-1}(\epsilon_J) \quad (n \geq 2) \quad (5.37)$$

This is discussed in some detail in Fredenhagen's lectures at this school. There is a formula for ϵ_J , but it is easier to determine it from its properties. ϵ_J is an intertwiner from ρ_J^2 to ρ_J^2 . In other words, it is in the commutant of $\rho_J^2(\mathcal{A})$. A nontrivial commutant exists only if ρ_J is not an automorphism. Otherwise, ϵ_J is a phase factor. In our model, the interesting case is $J = \frac{1}{2}$. The commutant of $\rho_{1/2}^2$ is spanned by the projection operators

$$\Pi_L \equiv \Pi\left(\frac{1}{L}\right), \quad L = 0, 1.$$

Thus

$$\epsilon_{1/2} = \gamma_0 \Pi_0 + \gamma_1 \Pi_1. \quad (5.38)$$

The phase factors γ_L could a priori be elements of the center of $\mathcal{A}$, i.e. be linear combinations of 1 and Y . They can be determined, up to an overall phase factor, from the requirement that (5.37) is actually a representation of the braid group, i.e. satisfy the Artin relations. The remaining phase factor is determined by Fredenhagen's general spin and statistics theorem. As a result one obtains for the chiral Ising model

$$\gamma_0 = i\gamma, \quad \gamma_1 = -\gamma \quad \text{with} \quad \gamma = -e^{i3\pi/8}. \quad (5.39)$$

Using the explicit definition and homotopy invariance of ϵ_J as discussed in Fredenhagen's lectures, one can show that the local braid relations of field operators (5.17) with the same J are implied by the following property of the "constant fields"

$$(\Gamma^J \times \Gamma^J)_{ij} \epsilon_J = (\Gamma^J \times \Gamma^J)_{ij} \mathcal{R}_{ij}^{JJ}. \quad (5.40)$$

All the quantities in this identity have been defined. It can be verified that the identity is indeed satisfied for $J = \frac{1}{2}$ [2]. The proof uses the hexagon- and pentagon identities of 6J-symbols. Validity of covariant operator product expansions can also be proven. They are valid on all of $\mathcal{H}$, i.e. on all superselection sectors. Validity of local braid relations for arbitrary I, J follows from their validity for $I = J = \frac{1}{2}$, from the operator product expansions for $I = J = \frac{1}{2}$ and from the fact that local braid relations for covariant products of fields follow from those for individual fields. This last fact is a consequence of Drinfeld's relations (3.12), (3.13).

6 An algebraic approach to the construction of Bose fields in two space time dimensions

The central idea of algebraic field theory is to start from an algebra of observable fields and its positive energy representations. The corresponding representation spaces are called superselection sectors. One constructs then new field operators which are relatively local with respect to the observables and which make transitions between superselection sectors. These new fields are constructed by use of the general bosonization formula of Doplicher Haag and Roberts, eq.(5.17).

For the purpose of mathematical physics, any algebra of local Bose fields may be declared to be the algebra of observables to start with. One may choose to start with a smaller algebra of observables than the full algebra of observables which one would like to have. After construction of the new fields one may select among them those that are Bose fields, and obtain in this way a larger algebra of observables which contains the desired one.

A simple example of such a construction was given by Buchholz, Mack and Todorov in [15]. In that example, the $U(1)$ -current algebra was the starting point, and the possible maximal extensions of it to larger algebras of observables were classified.

Here we wish to consider conformal field theory in 2-dimensional space time, with Lorentzian metric, i.e. on a tube $S^1 \times \mathbb{R}$. [17]. In particular, we consider the Wess-Zumino-Novikov-Witten models. This will enable the reader to compare with Ludwig Faddeev's lectures [14]. We wish to explain how the extra variables ("zero modes") which appear in his approach in addition to the chiral currents would be reconstructed by an algebraic field theory approach. The considerations of this section could be made explicit for $SO(N)$ -WZNW-models at level 1 because the morphisms of the chiral algebras are known in this case [18].

In the WZNW-model for group G one wishes to construct a renormalized version of the field operators

$$g : (x, t) \mapsto g(x, t) \in G \quad (6.41)$$

where the space coordinate x serves to fix the position on the circle and t is a time coordinate. We write $\partial_\pm$ for differentiation with respect to $x^\pm = \frac{1}{2}(t \pm x)$.

The ultimate algebra of observables $\mathcal{A}$ is supposed to be generated by the fields $g(x, t)$. A truly smaller algebra is generated by the chiral currents,

$$J_\pm = \partial_\pm g g^{-1}. \quad (6.42)$$

This smaller algebra factors into chiral algebras,

$$\mathcal{A}^+ \times \mathcal{A}^- \quad (6.43)$$

Already at a classical level, the desired fields $g(x, t)$ cannot be reconstructed from the chiral currents alone, because a constant of integration $g(0,0)$ is missing. The problem is to reconstruct the quantum version of this constant of integration.

In the algebraic approach one will first reconstruct chiral fields, using the bosonization formula

$$\psi_m^{I\pm}(x^\pm) = \Gamma_m^{I\pm} A_f^{\pm}(x^\pm) \quad (6.44)$$

This involves "constant fields" $\Gamma^{I\pm}$ and factors A which are constructed from the chiral currents. The formula for A in terms of currents will depend on I . We indicate this dependence. The constant fields will transform covariantly under a quantum symmetry while the chiral observables $A_f^{\pm}(x^\pm)$ are invariants under the quantum symmetry. The chiral fields $\psi^{I\pm}$ will satisfy local braid relations by construction.

Two dimensional fields of the form

$$\Gamma_i^{I+} A_f^+(x^+) \Gamma_j^{I-} A_f^-(x^-) \quad (6.45)$$

will also satisfy local braid relations. (Left and right moving building blocks commute with each other)

The matrix elements τ_{ij} of any representation of $\mathcal{G}^*$ are linear functions on $\mathcal{G}^*$, and may therefore be regarded as elements of $\mathcal{G}$. The formula (7. 4) for the product implies that

$$\tau_{ij} \cdot \tau'_{kl}(\xi) = (\tau \otimes \tau')_{ik,jl}(\xi) \quad (7.5)$$

In this sense, tensor products of representations of $\mathcal{G}^*$ become ordinary products of elements of $\mathcal{G}$. There is also a coproduct in $\mathcal{G}$ which comes from the product in $\mathcal{G}^*$.

We write $|\psi\rangle > f$ in place of $|\psi\rangle > \otimes f$ for elements of $\mathcal{H} \otimes \mathcal{G}$ and regard them as linear functions on $\mathcal{G}^*$ with values in $\mathcal{H}$, viz.

$$(|\psi\rangle > f)(\xi) = |\psi\rangle > f(\xi).$$

In this notation, the coaction $\hat{U}$ of $\mathcal{G}$ on $\mathcal{H}$ is given by

$$\hat{U}|\psi\rangle > (\xi) = U(\xi)|\psi\rangle. \quad (7.6)$$

Imagine for a moment that a basis $|\phi_\alpha\rangle$ has been introduced in $\mathcal{H}$, and define matrix elements of $\hat{U}$ by

$$\hat{U}|\phi_\alpha\rangle > = |\phi_\beta\rangle > \hat{U}_{\alpha\beta} \quad \text{with } \hat{U}_{\alpha\beta} \in \mathcal{G}.$$

The matrix elements $\hat{U}_{\alpha\beta}(\cdot)$ of $\hat{U}(\cdot)$ are functions on $\mathcal{G}^*$, hence elements of $\mathcal{G}$, and so are $\hat{U}_{\alpha\beta}$. Both are in fact equal, by definition (7. 6),

$$\hat{U}_{\alpha\beta}(\xi) = U_{\alpha\beta}(\xi).$$

$\hat{U}$ and U are therefore "basically the same".

Since $\hat{U}_{\alpha\beta}$ is an element of $\mathcal{G}$, the product $\hat{U}_{\alpha\beta} \cdot \tau'_{ij}$ is defined as a product in $\mathcal{G}$, and equal to the tensor product of representations $(\hat{U} \otimes \tau')_{\alpha\beta,ij}$ by (7. 5). We continue to use this when the reference to a basis in $\mathcal{H}$ is dropped.

The 1-dimensional representation ϵ of $\mathcal{G}^*$ is also a function on $\mathcal{G}^*$, hence an element of $\mathcal{G}$. It is the identity in $\mathcal{G}$, i.e.

$$\epsilon \cdot f = f \quad \text{for all } f \in \mathcal{G}.$$

The defining properties of a symmetry (cp. introduction) can now be rewritten as follows

$$\hat{U}|0\rangle > = |0\rangle > \epsilon, \quad (7. 7)$$

$$\hat{U}H = H\hat{U}, \quad (7. 8)$$

$$\hat{U}\Psi_j^i = \Psi_k^j(\tau_{ij}^i \cdot \hat{U}). \quad (7. 9)$$

On the right hand side of eq.(7. 8), the Hamiltonian H is understood to act on the first factor $\mathcal{H}$ of $\mathcal{H} \otimes \mathcal{G}$; a more careful notation would be to write $H \otimes \epsilon$ in place of H . Similarly, the factor Ψ_k^j on the right hand side of eq. (7. 9) is understood to act on $\mathcal{H}$, and so on.

The module-picture, which is used in the body of this paper, was introduced for quantum groups by Buchholz, Mack and Todorov [9]. The comodule picture was introduced by Reshetikhin and Smirnov [19]. We see that both pictures are equivalent.

It remains to pick out those linear combinations for which the local braid relations reduce to Bose commutation relations. The guiding principle here is that a nontrivial transformation law under a true quantum symmetry is in general not compatible with Bose commutation relations. Considering fixed time, one finds that the Bose fields should be invariant under the diagonal quantum symmetry algebra which acts on the left and right movers alike. One has the freedom of a choice of basis in index space so that $\sum_i \Gamma_i^+ \Gamma_i^-$ are invariants. This gives a list of candidates for Bose fields indexed by I ,

$$\phi^{(I)}(x) = \sum_i \Gamma_i^+ \Gamma_i^- A_I^+(x^+) A_I^-(x^-).$$

Which is g would have to be found out from further requirements. From an algebraic point of view, this question is actually not so interesting. One is happy once one has the algebra of Bose fields. Not all the candidates are actually Bose fields in general, though. Covariantly defined composites of braid local fields are braid local fields, as a consequence of Drinfelds relations. This implies that $\mathcal{G}^*$ -invariant fields obey "anyonic" commutation relations which may differ from Bose commutation relations by phase factors. Therefore one must work out these phase factors. In the case of the conformal Ising model which was considered in the last section, and which is very similar to the WZNW-case, $I = 0, \frac{1}{2}, 1$. For $I = 1$, the new field turns out to be a Fermi-field. The case $I=0$ is always trivial because $\Gamma^{0\pm} = 1$; therefore one gets back the old observables. The only new Bose field comes from $I = \frac{1}{2}$ in the case of the Ising model.

7 The comodule picture

In order to have a conventional picture of a symmetry, we used the algebra $\mathcal{G}^*$. An element ξ of $\mathcal{G}^*$ will be called a "quasi-rotation" in the following text. The Hilbert space of physical states is a representation space (=module) for $\mathcal{G}^*$. Thus there is a map

$$U : \mathcal{G}^* \otimes \mathcal{H} \mapsto \mathcal{H} \quad (7.1)$$

It takes $\xi \otimes |\psi\rangle$ into $U(\xi)|\psi\rangle$. Given any quasirotation ξ , its action $U(\xi)$ on any element $|\psi\rangle \in \mathcal{H}$ is defined and produces another element of $\mathcal{H}$.

One may instead consider the action $\hat{U}$ of an unspecified element of $\mathcal{G}^*$ on $|\psi\rangle \in \mathcal{H}$. It produces an object which becomes an element of $\mathcal{H}$ when a quasirotation ξ is supplied to it. Mathematically

$$\hat{U} : \mathcal{H} \mapsto \mathcal{H} \otimes \mathcal{G}, \quad (7.2)$$

where $\mathcal{G}$ is the dual bialgebra to $\mathcal{G}^*$. Given such a linear map (7. 2), one says that $\mathcal{H}$ is a comodule for $\mathcal{G}$, and $\hat{U}$ defines a coaction of $\mathcal{G}$ on $\mathcal{H}$. The algebra $\mathcal{G}$ is a generalization of the algebra of (good) functions on a group. Its elements $f \in \mathcal{G}$ are linear maps

$$f : \mathcal{G}^* \mapsto \mathbb{C}. \quad (7.3)$$

The product $\cdot$ in $\mathcal{G}$ is furnished by the coproduct in $\mathcal{G}^*$,

$$f_1 \cdot f_2(\xi) = (f_1 \otimes f_2)(\Delta(\xi)). \quad (7.4)$$

8 Supersymmetry

It is instructive to consider supersymmetry as a special case of a quantum symmetry. Supersymmetry is not a purely internal symmetry, since it includes the space time translation generators P_μ among its generators. But this can be accommodated.

We adjoin fermionic parity $(-)^F$ to the generators of the supersymmetry algebra. It becomes a quasitriangular Hopf algebra ("quantum group algebra") by defining coproduct, counit and antipode as described below, and an R-element which reproduces the standard commutation / anticommutation relations. The coproduct is coassociative, therefore $\varphi = e \otimes e \otimes e$ and $\Delta(e) = e \otimes e$.

Apart from generators of groups, the supersymmetry algebra contains fermionic generators Q_α^A which anticommute with $(-)^F$. These fermionic generators obey anticommutation relations with Fermi field operators and commutation relations with Bose fields. Both are reproduced by the coproduct defined below.

$$\Delta(Q_\alpha^A) = Q_\alpha^A \otimes e + (-)^F \otimes Q_\alpha^A \quad (8.1)$$

$$R = \frac{1}{2} [e \otimes e + (-)^F \otimes e + e \otimes (-)^F - (-)^F \otimes (-)^F], \quad (8.2)$$

$$S(Q_\alpha^A) = (-)^F Q_\alpha^A. \quad (8.3)$$

Obviously, the coproduct is not cocommutative. It may be called "graded cocommutative". The counit is the obvious one, $\epsilon(Q_\alpha^A) = 0$ etc. All of Drinfelds relations are satisfied.

Let us also consider the comodule picture. We remember that representation functions τ_{ij}^I of $\mathcal{G}^*$ are elements of $\mathcal{G}$ whose product is given by the tensor product of representations of $\mathcal{G}^*$. Let us choose bases in the representation spaces such that fermionic parity $(-)^F$ is diagonal,

$$\tau_{jk}^I((-)^F) = s_j^I \delta_{jk},$$

and set

$$s_{ji}^{II} = \begin{cases} -1 & \text{if } s_j^I = -1 \\ s_i^I & \text{otherwise} \end{cases} \quad (8.4)$$

$$= +1 \quad \text{otherwise} \quad (8.5)$$

The relation $\Delta'(\eta)R = R\Delta(\eta)$ translates into commutation relations in $\mathcal{G}$,

$$\tau_{ik}^I \cdot \tau_{jl}^I = \pm \tau_{ji}^I \cdot \tau_{lk}^I \quad \text{with} \quad \pm = s_{ij}^{II} s_{kl}^{II}. \quad (8.6)$$

One may express this result by saying that $\mathcal{G}$ is graded commutative.

The considerations of this section illustrate the point that the introduction of quantum symmetries continues a process of generalization which began with the introduction of supersymmetry.

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