Cross-section estimates for multi-Higgs production at high energies

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Abstract

We present lower limits on the cross-section for the process $q\bar{q} \rightarrow H^* \rightarrow nH$ as predicted by tree-level perturbation theory in the Minimal Standard Model. Emphasis is on high multiplicity n of the final state. Both the bound on the matrix element and the phase-space are treated essentially exactly, which makes our limit on the unitarity-respecting behaviour of this cross-section rigorous to leading order in the quark-Higgs Yukawa coupling.

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1 Introduction

The behaviour of perturbation theory in the production of final states with very high multiplicity has currently attracted renewed interest, because of its possible relevance to studies of instanton-induced $(B + L)$ -violating processes at high energies [1]. Estimates for the probability amplitude of the transition of one off-shell scalar into a large number of scalars have been presented by various authors [2]. It is our aim to refine these calculations, for the more realistic case of the Minimal Standard Model, to a point where the results have a phenomenological meaning.

In this paper we derive estimates for the cross-section for the production of n Higgs bosons in quark-antiquark annihilation:

$$\bar{q}(k_1)q(k_2) \rightarrow H(p_1)H(p_2)\cdots H(p_n) \quad , \quad (1)$$

where the quark-antiquark pair annihilate into a single off-shell Higgs, which subsequently decays into n on-shell Higgses. Throughout, we shall only consider tree-level perturbation theory. In the context of the Minimal Standard Model, this is about the simplest high-multiplicity process, since only Higgs bosons can occur as internal particles. It should be noted, however, that we neglect contributions from annihilation into more than just one off-shell Higgs; at the end of our discussion we shall briefly return to this point.

In a previous publication [3] we have already derived upper and lower bounds for the transition amplitude for $H^* \rightarrow nH$, and we shall use these results as a basis for our cross-section calculation, after having recapitulated them in section 2. We shall then proceed, in section 3, to derive the necessary phase-space volumes, which can conveniently be done by generalizing the old method of Lurçat and Mazur [4]. This treatment is statistical, and our result for the phase-space volume therefore has formal correction terms of order $\mathcal{O}(1/n)$, but we shall use only very high values of n . In section 4 we present numerical results for a rigorous lower limit on the cross-section. In particular we study the values of the multiplicity n and the total energy $\sqrt{s}$ at which the cross-section begins to increase with increasing n (the ‘turnover point’), and those for which the cross-section violates s -wave unitarity (the ‘violation point’). Clearly, at the violation point the tree-level perturbative prediction breaks down completely, but it may be argued that a ‘turnover point’ by itself already indicates this breakdown, even if the corresponding total cross-section does not violate unitarity. We also compare our results with recent work of Voloshin [5], who has performed a similar estimate.

2 The matrix element

For the process (1) there are $2^n - n - 1$ different invariant-mass combinations that enter in the exact calculation of the complete set of s -channel Feynman diagrams. Clearly a brute-force calculation is not feasible at present, and therefore we settle for a lower bound on the amplitude. The matrix element can be written as

$$\mathcal{M} = g_Y \bar{v}(k_1)u(k_2) \frac{1}{s - m^2} a(n) \quad , \quad (2)$$

where $a(n)$ stands for the sum of all diagrams from the first Higgs self-interaction vertex onwards, and m is the Higgs mass. The quark-Higgs Yukawa coupling is denoted by g_Y ,

and we shall denote by λ_3 and λ_4 the triple and quartic Higgs self-coupling constants. In the Minimal Standard Model these read

$$\begin{aligned}\lambda_4 &= 3\sqrt{2}m^2G_F , \\ \lambda_3 &= m\sqrt{3\lambda_4} , \\ g_Y &= m_q\sqrt{G_F\sqrt{2}} ,\end{aligned}\tag{3}$$

where m_q is the quark mass and G_F the Fermi constant. In what follows we have taken $m_q = 5$ GeV. For the $a(n)$, which was the subject of our paper [3], we have the following bounds. Consider only those kinematical configurations of the momenta $p_1, \dots, p_n$ for which

$$p_i \cdot p_j < m^2(1 + \Delta) \text{ for all } i, j ,\tag{4}$$

for some given value of Δ . Depending on the value of Δ we include either the whole phase-space or only part of it, which is all right if we are interested in a lower bound on the cross-section. Clearly, if $\Delta = 0$, all momenta p_i must be collinear and the phase-space volume vanishes. In [3] we derived the following result, with $\omega = 1 + \Delta$ and $\mu = \lambda_3/(m\sqrt{\lambda_4})$:

$$\begin{aligned}|a(n)| &\geq n!(n^2 - 1)m^2\omega b(n) \left(\frac{\lambda_4}{m^2\omega}\right)^{(n-1)/2} , \\ b(n) &= \frac{1}{12^{-1/2+n}} \left[\left(\sqrt{3} + \frac{\mu}{\sqrt{\omega}}\right)^n - \left(-\sqrt{3} + \frac{\mu}{\sqrt{\omega}}\right)^n \right] ;\end{aligned}\tag{5}$$

the special case $\lambda_3 = 0$ was established earlier by Voloshin [1]. For $\Delta = 0$, the inequality for $a(n)$ is actually an equality, and $\Delta = 0$ serves as an *upper* bound on the amplitude, while for $\Delta > 0$ we have a *lower* bound rigorously valid in the part of phase-space considered. For the matrix element squared, summed and averaged over spins and colours, and including a factor $1/n!$ due to the Bose symmetry of the final state, we then find

$$\langle |\mathcal{M}|^2 \rangle = \frac{g_Y^2}{18n!} \frac{s}{(s - m^2)^2} |a(n)|^2 ,\tag{6}$$

having neglected m_q where possible.

3 The phase-space

In order to arrive at a reliable cross-section prediction, the multi-Higgs phase-space has to be evaluated in as exact a manner as possible. In our case, we must implement the restriction (4). However, the inequality (4) embodies a total of $n(n - 1)/2$ different constraints on the phase space, while the number of phase-space variables is only $3n - 4$. Such a hugely overconstrained system is not easily handled, and we therefore impose the even tighter, but simpler, condition

$$p_i^0 \leq m\delta \text{ for all } i , \quad \delta = \sqrt{1 + \Delta/2} .\tag{7}$$

It is easily seen that Eq.(7) implies Eq.(4), so our lower bounds remain rigorous.

We now derive the phase-space volume under the condition (7) using the method of Ref.[4]. Consider the following distribution for the momentum of a single particle:

$$\begin{aligned}\psi(p) d^4p &= \frac{\phi(p) d^4p}{\int \phi(p) d^4p} , \\ \phi(p) &= \delta(p^2 - m^2)\theta(p^0 - m)\theta(m\delta - p^0) \exp(-\beta p^0) ,\end{aligned}\quad (8)$$

where β is a parameter to be determined later. The probability density for the sum, P^μ , of n such momenta, each distributed according to the above single-particle distribution (spd), is then given by

$$\Psi(P) = \int \prod_{i=1}^n (d^4p_i \psi(p_i)) \delta^4 \left(P - \sum_{j=1}^n p_j \right) . \quad (9)$$

The trick is now to obtain a closed-form approximation for $\Psi(P)$ and then to put P^μ to its desired value $P^0 = \sqrt{s}$, $\vec{P} = 0$. The expectation values and variances of the components of p_i^μ and P^μ are given by

$$\begin{aligned}\text{Exp}(p_i^0) &= m \frac{\phi_1}{\phi_0} , \\ \text{Var}(p_i^0) &= m^2 \left(\frac{\phi_2}{\phi_0} - \frac{\phi_1^2}{\phi_0^2} \right) , \\ \text{Exp}(p_i^k) &= 0 , \quad k = x, y, z , \\ \text{Var}(p_i^k) &= \frac{m^2}{3} \left(\frac{\phi_2}{\phi_0} - 1 \right) \quad k = x, y, z , \\ \text{Exp}(P^\mu) &= n \text{Exp}(p_i^\mu) , \\ \text{Var}(P^\mu) &= n \text{Var}(p_i^\mu) , \\ \phi_m &\equiv \int_0^\delta dt \, t^m \sqrt{t^2 - 1} \exp(-\beta m t) .\end{aligned}\quad (10)$$

The central-limit theorem establishes that for large n the distribution $\Psi(P)$ will be approximately a multiple Gaussian:

$$\begin{aligned}\Psi(P) &= \Psi_0(P) \left(1 + \frac{C_1}{\sqrt{n}} + \frac{C_2}{n} + \dots \right) , \\ \Psi_0(P) &= \prod_\mu \frac{1}{\sqrt{2\pi \text{Var}(P^\mu)}} \exp \left(-\frac{(P^\mu - \text{Exp}(P^\mu))^2}{2\text{Var}(P^\mu)} \right) .\end{aligned}\quad (11)$$

This asymptotic series in $1/\sqrt{n}$ is formally good for any P^μ but useful only very close to the peak of the Gaussian. We therefore tune the constant β such as to get $\text{Exp}(P^0) = \sqrt{s}$: the other components of P^μ do not need tuning since $\text{Exp}(P^{x,y,z})$ is already 0 because of the isotropy of the spd. The optimal value of β is that given by the equation

$$\frac{\phi_1}{\phi_0} = f \equiv \frac{\sqrt{s}}{nm} ; \quad (12)$$

for this value the quantity C_1 in Eq.(11) vanishes. We may then identify the right-hand-sides of Eqs.(9) and (11) to arrive at an expression (formally good up to terms of order n^{-1}) for the phase-space volume:

$$\begin{aligned} V(\Delta) &= \int \prod_{i=1}^n (d^4 p_i \delta(p_i^2 - m^2) \theta(p_i^0 - m) \theta(m\delta - p_i^0)) \delta\left(\sum_{j=1}^n p_j^0 - \sqrt{s}\right) \delta^3\left(\sum_{j=1}^n \vec{p}_j\right) \\ &= \frac{\sqrt{27} \exp(\beta\sqrt{s}) (2\pi m^2)^{n-2} \phi_0^n}{n^2 (\phi_2/\phi_0 - \phi_1^2/\phi_0^2)^{1/2} (-1 + \phi_2/\phi_0)^{3/2}} . \end{aligned} \quad (13)$$

In the limit $\delta \rightarrow \infty$ we recover the whole phase-space, and in this case (the one studied in [4]) the functions ϕ_m are related to the modified Bessel functions of the second kind [6], with many known analytic and numerical properties:

$$\begin{aligned} \phi_0 &= \frac{1}{\beta m} K_1(\beta m) , \\ \phi_1 &= \frac{2}{\beta^2 m^2} K_1(\beta m) + \frac{1}{\beta m} K_0(\beta m) , \\ \phi_2 &= \left(\frac{6}{\beta^3 m^3} + \frac{1}{\beta m} \right) K_1(\beta m) + \frac{3}{\beta^2 m^2} K_0(\beta m) . \end{aligned} \quad (14)$$

For finite δ the corresponding incomplete functions are not so well known, but, as it turns out, for values of δ typical of our problem, the integrals ϕ_m can readily be obtained numerically to high accuracy by Gaussian integration.

A note is in order on the applicability of the central limit theorem. The value of β obviously depends on δ as well as on f , and f (the average value of p_i^0/m) cannot exceed δ (the maximum value). If f is close to δ , β can go to zero or even become negative, in contrast with Ref.[4] where $\delta = \infty$ and β is always positive. This indicates that the interpretation of β as an ‘inverse temperature’ is totally unjustified for a severely restricted phase-space of this type; for the rest, nothing is wrong with $\beta < 0$. However, one may worry that, for spd’s that are cut off close to the average value, the central limit theorem may not give a good approximation. This will happen when, for the maximum possible value P_m^0 for P^0 , the quantity

$$D = \frac{(P_m^0 - \text{Exp}(P^0))^2}{\text{Var}(P^0)} \quad (15)$$

is of order 1 or smaller. However, since $\text{Exp}((p_i^0)^2) < m^2 \delta^2$, we trivially have

$$D > n \frac{\delta - f}{\delta + f} , \quad (16)$$

so that, for values of n , δ and f typical of our study, D is always large enough to justify our statistical calculation of the phase-space volume.

Before we discuss our numerical results, we want to point out a difference between our treatment and that of Voloshin. Our phase-space cut is actually a cut on the *maximum* value of p_i^0 . The distribution of this maximum is therefore relevant, and a simple toy

model serves to illustrate our point. Consider n numbers x_i , uniformly distributed with sum 1:

$$\int \prod_{i=1}^n dx_i \delta\left(\sum_{j=1}^n x_j - 1\right) = \frac{1}{(n-1)!} ; \quad (17)$$

the spd for this model is just the distribution $n \exp(-nx_i)$, with expectation value $1/n$. The probability density of the maximum $x = \max_i(x_i)$ is therefore

$$F(x) = n^2 e^{-nx} (1 - e^{-nx})^{n-1} ; \quad (18)$$

elementary considerations [7] show that the *expected* value of the maximum $\text{Exp}(x)$, and its standard deviation $\text{Var}(x)^{1/2}$, are both roughly equal to $\log(n)/n$. We may therefore expect that, also in the four-dimensional phase-space treatment, our energy cut (essentially unimportant for very high values of δ) starts to be effective when δ reaches values of order $\log(n)\text{Exp}(p_i^0)/m = f \log(n)$. This corresponds to values of Δ around $2f^2 \log^2(n)$, which is indeed borne out by numerical experience. For completeness, we have also used the phase-space event generator **MAMBO** [8], and found that values of Δ that give half the full phase-space under the invariant-mass cut of Eq.(4) differ by a more or less constant factor of order 1 from those that give half the full phase-space volume using Eq.(13) with the energy cut of Eq.(7)¹. In Ref.[5] it is assumed that the maximum energy among the p_i^0 is higher than the average energy mf by a constant factor, based on equipartition of the particle energies which is assumed to become more pronounced as n increases. In our opinion, this is mistaken, since the spd is (for fixed f) independent of n , and the value of the maximum must therefore grow with n . Nonetheless, the mild (logarithmic) growth of the maximum individual p_i^0 , while decreasing the cross-section, does not change its behaviour qualitatively.

4 Results and conclusions

We now proceed to discuss some numerical results. We present cross-sections for various combinations of $\sqrt{s}$, n and Δ . It should be noted that, in principle, *any* choice of Δ above the kinematical limit $(f^2 - 1)$ will result in a valid, rigorous lower bound on the cross-section. Very large values of Δ will, however, add little to the phase-space volume, while drastically suppressing the lower bound on the amplitude, whereas too small values of Δ will tend to cut away so much of the phase-space that the amplitude cannot compensate for it. Therefore, one can in principle optimize the cross-section bound for any combination of $\sqrt{s}$ and n by varying Δ . We feel, however, that too much refinement in the choice of Δ is hardly worth the trouble, since our lower bound is, in any case, extremely loose. In all cases we express our cross-section results in the form of the quantity

$$g = \frac{\lambda_4}{192\pi^2} \log(s\sigma) , \quad (19)$$

a normalization that allows us to compare our results with those of Ref.[5]². We compare our results with the s -wave unitarity limit $4\pi/s$ for incoming fermions, which gives 0.0026

¹The energy cut, being tighter than the invariant-mass cut, leads to higher Δ values.

²Note that in Ref.[5] the coupling constant λ corresponds to our $\lambda_4/6$.

for $m = 200$ GeV (we shall use this value throughout).

First of all we present the cross-section bounds for fixed f , which implies fixed β . In fig. 1, g is given as a function of n for $\beta = 0$, and for various choices for Δ . After initially decreasing with n , the cross-section turns over at some n_0 and then rises continually. At about $3n_0$ the unitarity limit is violated. Since, for fixed f , $\sqrt{s}$ rises with n , no suppression is expected to set in for large n values, and therefore the mere existence of a turnover point at n_0 shows that unitarity violation lurks beyond n_0 . For this relatively simple case we have, for large n

$$n_0 \sim \frac{48\pi^2 \exp(-\beta m f)}{\lambda_4 \phi_0} \left(\frac{2\omega}{1 + \sqrt{\omega}} \right)^2. \quad (20)$$

If, under the same conditions, we would plot the *upper* rather than the lower limit on the cross-section, unitarity would be violated almost immediately (i.e. for a few tens of Higgses). We therefore do not consider the upper limit as a very instructive quantity.

It is of some interest to try to minimize n_0 , as this hints at how low we can make the unitarity violation limit. We find that $dn_0/d\beta = 0$ for $\beta = 0$, and this is the reason for choosing $\beta = 0$ in the figure. For this choice, we may subsequently minimize $\sqrt{s} = n_0 f m$ by varying Δ : this leads to $\Delta \sim 3.575$. We see that without too much trouble we can bring the turnover point n_0 to a respectably low value of 3614, at $\sqrt{s} = 1018$ TeV. Although it cannot be proved from just the lower limit we are studying, we believe that the *real* cross-section will have a turnover point quite a bit lower than this. For completeness, we give in fig. 2 the results for fixed f with the same parameters as before, but this time with $\lambda_3 = 0$. In this case we only plot the results for odd values of n since even values always give a zero cross-section. The cross-section bounds are very much lower, showing that the three-Higgs vertices play a considerable rôle, and that, quantitatively if not qualitatively, considerations based only on the quartic theory may not be reliable.

In fig. 3 we have fixed $\sqrt{s}$ and plot g as a function of $1/f$, where we have adopted the choice $\Delta = \gamma f^2$, with $\gamma = 2.05$; therefore, Δ decreases with increasing n . For (relatively) low multiplicities n the cross-section bound decreases as a function of n , as a good cross-section should; for sufficiently high $\sqrt{s}$, however, the cross-section develops a secondary maximum, which eventually rises to exceed the unitarity limit, before going to zero again as the kinematical threshold is reached for $f \sim 1$. These curves are comparable to the ‘differential holy grail function’ of Ref.[5], and we observe (qualitatively) similar features, including the slight rise of the position of the secondary maximum with $\sqrt{s}$. In fig. 4 we have switched off the triple-Higgs coupling, which again results in a cross-section that is drastically lower, but behaves in roughly the same way with energy. Unitarity is violated at higher energies (in fig. 4 we increased the energies of fig. 3 by a factor 3/2).

Of course, the total cross-section for $q\bar{q}$ annihilation into any number of Higgs bosons consists of a sum over the Higgs multiplicity n , and the ‘holy grail function’ is this sum rather than the cross-section for any given n . However, from figures 3 and 4 it is clear that the total cross-section is completely dominated by just a few multiplicities, be they low or high. By the measure of g it is, therefore, quite a good approximation to replace the total cross-section by the maxima of the ‘differential holy grail function’. In fig. 5 we plot these maxima as a function of $\sqrt{s}$, again taking $\Delta = \gamma f^2$ throughout. For ‘low’

values of $\sqrt{s}$ the cross-section bound is dominated by $n = 2$, and is in this case essentially independent of $\sqrt{s}$. This changes when the top of the secondary maximum in fig. 3 starts to exceed the two-Higgs cross-section, after which the unitarity limit is rapidly crossed: g is roughly proportional to $\sqrt{s} \log(\sqrt{s})$. Note that in calculations of the $B + L$ -violating rather than the $B + L$ -conserving cross section, which has no appreciable contribution from low n values, this holy grail function would rise smoothly with energy even below the unitarity violation point.

Before finishing we want to comment briefly on the Higgs-fermion Yukawa coupling. A full calculation should take into account also diagrams where more than one primary off-shell Higgs is radiated off the fermion line; note that each set of graphs with a given power of g_Y also contains of the order of $n!$ Feynman diagrams; there is thus no combinatorial reason to neglect these additional contributions. In fact, diagrams with two primary Higgses are easily seen to interfere *destructively* with the graphs we consider in this paper. However, in the case $\lambda_3 = 0$ these graphs are absent, since any primary Higgs must decay into an odd number of on-shell Higgses, so that in that case the first contribution of subleading order in g_Y comes from diagrams with three primary Higgses, which interfere *constructively*. Assuming that the qualitative behaviour of the theory is the same with λ_3 zero and non-zero we feel justified to neglect the Yukawa-suppressed diagrams. Another point in case is the fact that the Yukawa coupling, being proportional to the fermion mass, is different for different initial fermion states – it is, then, hard to imagine how the enormous cancellations needed to bring the tree-level cross-section below the unitarity limit could be realized in each different case.

In conclusion, we have calculated lower bounds on the multi-Higgs production cross-section, which are rigorous within the context of tree-level perturbation theory and truncation to leading order in the Yukawa coupling of the Higgs to the initial fermions. We find that the cross-section as a function of the Higgs multiplicity n increases after some value n_0 and violates the unitarity limit at roughly $3n_0$. We give an analytical estimate for n_0 for fixed $\sqrt{s}/n$. Our results hold for the case where triple and quartic Higgs self-interactions are present: switching off the triple-Higgs couplings leads to cross-sections that are significantly smaller, but display the same qualitative behaviour.

Our results are meaningful to the extent that we prove the breakdown of tree-level perturbation theory for sufficiently high values of n . Obviously, in those circumstances the theory has to be unitarized either in a nonperturbative way (as conjectured by Voloshin [5]) or by some as yet unknown perturbative mechanism: these considerations, however, are beyond the scope of this paper.

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Figure 1: Lower bounds on the total cross-section as a function of n , with $\beta = 0$, for various values of Δ . The open circles refer to the unitarity limit: the solid line is for $\Delta = 2$, the dashed line for $\Delta = 3.5$, and the triangles refer to $\Delta = 10$.

Figure 2: Lower bounds on the total cross-section as a function of n , with $\beta = 0$, for various values of Δ , for a theory with $\lambda_3 = 0$. The symbols have the same meaning as in fig. 1.

Figure 3: Lower bounds on the total cross-section as a function of $1/f$, for various choices of $\sqrt{s}$. The open circles denote the unitarity limit, and the values of $\sqrt{s}$ are 500 TeV (solid curve), 1500 TeV (dashed curve), 2888 TeV (triangles) and 4000 TeV (stars). For each value of f we used $\Delta = 2.05f^2$.

Figure 4: Lower bounds on the total cross-section as a function of $1/f$, for various choices of $\sqrt{s}$, in a theory with $\lambda_3 = 0$. The symbols refer to the same energies as in fig. 3, multiplied in each case by $3/2$.

Figure 5: The total cross-section as a function of $\sqrt{s}$, as estimated from the maximum of the cross-section as a function of n .