An Electroweak Monte Carlo for Four Fermion Production

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Abstract

An electroweak event generator for the four-fermion production process $e^+e^- \rightarrow f\bar{f}F\bar{F}$ is described. This generator, **FERMISV**, includes all the lowest-order diagrams involving Z and photon propagators. It uses multichannel phase space methods to map the very many peaks of the differential cross section. A leading-log approach to initial- and final-state radiation has been included to make **FERMISV** useful for phenomenology in the range $20 \text{ GeV} < \sqrt{s} < 200 \text{ GeV}$, where $\sqrt{s}$ is the center-of-mass energy of the initial beam particles. The methodology employed to write this generator allows facile adaptation and generalization.

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1 Introduction

This paper describes a new Monte Carlo whose development has been spurred by the needs of experimenters at LEP, an electron-positron collider, where $\sqrt{s} \approx m_Z$ (LEP 1), and in the future, $\sqrt{s} \approx 200$ GeV (LEP 2). The four-fermion final states are interesting for several reasons. They are a potential background to Higgs searches since, *e.g.*, $e^+e^- \rightarrow q\bar{q}\mu^+\mu^-$ looks like $e^+e^- \rightarrow Z \rightarrow H\mu^+\mu^-$ when the invariant mass of the $q\bar{q}$ system is near that of the Higgs. As signal, many four-fermion final states have a very striking topology and thus are easy to tag experimentally. Thus many such analyses have been conducted at PEP, PETRA and TRISTAN [1], and now as well at LEP [2], where an apparent excess in the $\tau^+\tau^-X$ channel was observed in ALEPH's 1989-1990 data. This excess is not observed by other collaborations. It must be pointed out that while it is interesting to check the Standard Model as many ways as possible, the four-fermion analyses are at best "medium precision" electroweak physics as classified by ref. [3], *i.e.*, accurate to several percent only. To obtain this level of accuracy in the Monte Carlo, initial-state radiation was included in a leading-log approximation. Final-state radiation was also treated in a similar way to give a closer simulation of experimental reality. Electroweak radiative corrections were only included through the usage of the improved Born approximation rather than through explicit calculation of the many additional Feynman diagrams. With the incorporation of initial-state radiation and the improved Born approximation, this Monte Carlo meets the goal of a medium-precision electroweak Monte Carlo.

There have been efforts from various groups to produce a four-fermion generator. The scope of all these generators is more limited for usage at LEP than the possibilities offered by **FERMISV**. We mention here the ones known by us to have been used by experimenters. The earliest effort was that of van Neerven and Vermaseren [4]. Their program included only multiperipheral and bremsstrahlung diagrams (see section 4 for the definitions). The DIAG36 program of Berends, Daverveldt, and Kleiss [5] included the two other diagram groups as well (conversion and annihilation), but did not include any Z propagators. This was remedied by Daverveldt in the generator usually known as **ELW2TAG** [6]. This generator, while rather complete, is not well-suited for LEP energies because it wasn't designed to run efficiently at $\sqrt{s} \approx m_Z$. Consequently, it runs rather slowly. It also has no treatment of either initial-state radiation or the improved Born approximation. Finally, it is not supported by the authors. Nevertheless, it has been used in several of the analyses mentioned above. At TRISTAN there has been use of a generator by Kuroda [7], which is somewhat less complete than **ELW2TAG** since it also lacks a complete treatment of all interferences.

At LEP two additional independent four-fermion generators have emerged: the **EGF4** program [8], used by L3, and **FOURFERM**, a forerunner of **FERMISV**, used by some LEP experimenters [9]. Like **ELW2TAG**, **EGF4** includes all diagrams and their interferences, but it is only optimized for s -channel-dominated processes. **FOURFERM** treats all fermions as being massless. This approximation, which holds for some analyses,

greatly simplifies the calculation of the matrix element. The other relevant feature of **FOURFERM** is that it uses **RAMBO**[11], a phase space generator which produces events with even probability in all regions of phase space. Unfortunately, this method of sampling becomes impractical when a four-fermion Monte Carlo is allowed to go near one of the many peaks in the matrix element. The way this deficiency is addressed in **FERMISV** is via a multi-channel Monte Carlo method.

The rest of the paper is organized as follows. In section 2 we give a general description of the multichannel Monte Carlo integration technique. Section 3 summarizes the properties of the phase space generators in **FERMISV**. The next section describes the generic matrix element in the four-fermion problem, and sets out under which conditions certain matrix elements are negligible. The implementation for leading-log initial- and final-state radiative corrections to the generic process $e^+e^- \rightarrow X$ is then given. Some results from running the generator are presented, followed by our conclusions.

2 Multichannel Monte Carlo

In this section we describe the Monte Carlo strategy underlying the event generator. Although this has already been discussed in the literature, notably ref. [10], we feel that it is important for the understanding of the program layout to review the idea behind multichannel Monte Carlo and the unitary algorithm formalism.

Although an event generator can produce any desired differential cross section, the procedure is most simply described if we concentrate on the total cross section σ for some process:

$$\sigma = \int f(\vec{\phi}) d\vec{\phi} , \quad (1)$$

where $f(\vec{\phi})$ denotes the matrix element squared, suitably summed and averaged over spins, colors, etc., and $\vec{\phi}$ is the phase space variable. In the present case, where we have the process

$$e^+(p_1) e^-(p_2) \rightarrow f_1(q_1) \bar{f}_1(q_2) f_2(q_3) \bar{f}_2(q_4) , \quad (2)$$

$\vec{\phi}$ is the (8-dimensional) momentum configuration

$$\vec{\phi} = \{q_1^\mu, q_2^\mu, q_3^\mu, q_4^\mu\} , \quad (3)$$

(in the following we shall change the notation for the outgoing momenta slightly when need arises, although this should always be clear from the context), and the phase space integration element reads

$$d\vec{\phi} = \frac{1}{(2\pi)^{12}} \left(\prod_{i=1}^4 d^4 q_i \delta(q_i^2 - m_i^2) \right) \delta^4(p_1 + p_2 - q_1 - q_2 - q_3 - q_4) , \quad (4)$$

where m_i is the mass of the particle with momentum q_i . Note that under this convention, Eq.(1) holds provided we include in the definition of $f(\vec{\phi})$ a factor $(2\pi)^4/(2s)$. Finally, any desired experimental cut is of course implemented simply by putting $f(\vec{\phi}) = 0$ in the unwanted region of phase space.

Our Monte Carlo description of the computation of σ according to Eq.(1) is most elegantly described using the unitary algorithm formalism. It starts by considering the identity

$$1 = \int_0^1 dr_1 \int_0^1 dr_2 \cdots \int_0^1 dr_n \equiv \int d\vec{r} \ , \quad (5)$$

where $n \geq 8$ in our case. On the one hand this is a mathematical triviality; on the other hand it is interpreted as the Monte Carlo statement: “there is an algorithm for generating n independent, uniformly distributed random variables $r_i (i = 1, \dots, 8)$ between 0 and 1”. In practice these are given by some pseudo-random number generator. Now consider some transformation (mapping) h from the r_i into phase space variables $\vec{\phi}$:

$$\vec{\phi}(\vec{r}) = h(\vec{r}) \ . \quad (6)$$

In unitary algorithm language we write this as

$$\begin{aligned} 1 &= \int d\vec{r} = \int d\vec{r} d\vec{\phi} \delta(\vec{\phi} - h(\vec{r})) \\ &= \int d\vec{\phi} g(\vec{\phi}) \ , \end{aligned} \quad (7)$$

where $g(\vec{\phi})$ is simply the Jacobian of the transformation:

$$\frac{1}{g(\vec{\phi})} = \left| \frac{\partial h(\vec{r})}{\partial \vec{r}} \right|_{\vec{r}=\vec{r}(\vec{\phi})} \ . \quad (8)$$

Actually, if $n > 8$ the Jacobian does not formally exist (such as when two numbers r_j are used to produce a single phase space coordinate ϕ_i) but in practice the form of $g(\vec{\phi})$ can always be obtained by integrating out over the superfluous degrees of freedom (we shall see an example of this later on). In Monte Carlo language, Eq.(7) means “there is an algorithm that generates variables $\vec{\phi}$ with a probability density given by $g(\vec{\phi})$.” The combination of integration element plus delta function (actually just a complicated way of writing the number “1” is the direct analog of a **FORTRAN** assignment statement: usually, expressions cast in unitary algorithm terms can be straightaway translated into source code for event generators. The phase space point $\vec{\phi}$ is usually referred to as an *event*. Thus we have an algorithm for generating one event.

The *local density* $g(\vec{\phi})$ can always be worked out in closed form for any “self-respecting” mapping h (in the sense that we are only interested in these sorts of

mappings). This means we can evaluate it for *any* event, even if it was generated by some other means than the mapping h . This is used later on. Note that the mapping h may be such that using it we will not cover all of phase space: in that case, $g(\vec{\phi})$ must of course be 0 outside the accessible region. We would like to stress that Monte Carlo programs that do not, at least implicitly, state the form of $g(\vec{\phi})$, *do not really know what they are doing*.

By comparing Eqs.(1) and (7) we readily arrive at the general Monte Carlo recipe for calculating the cross section. From

$$\sigma = \int \left(\frac{f(\vec{\phi})}{g(\vec{\phi})} \right) g(\vec{\phi}) d\vec{\phi} , \quad (9)$$

we see that generating events $\vec{\phi}$ from random numbers $\vec{r}$ with the mapping h and computing the *event weight*

$$w(\vec{\phi}) = \frac{f(\vec{\phi})}{g(\vec{\phi})} , \quad (10)$$

we obtain the Monte Carlo prediction for σ as the average value of $w(\vec{\phi})$ with an estimated error given in the standard statistical manner. The technique of *importance sampling* is just the refinement which aims to make (by hook or by crook) the variance of $w(\vec{\phi})$ much smaller than that of $f(\vec{\phi})$ ⁴.

The *multichannel* case arises when several different mappings are available – different mappings for the same phase space, mappings for different pieces of phase space, or whatever. Let us denote these N different mappings by $h_i(\vec{r})$, each with its own Jacobian function $g_i(\vec{\phi})$. The h_i may actually each employ a different n , but that makes no difference for our discussion. Since in the generation of an event we have, at some point, to make up our minds about which of the alternative h_i we shall use, we can do this by a random choice. Take some partition of unity α_i , with $\sum_{i=1}^N \alpha_i = 1$, and define the cumulative numbers $\beta_i = \alpha_1 + \dots + \alpha_i$. Taking a random number z , uniformly distributed between 0 and 1, we then take mapping h_i if $\beta_{i-1} < z < \beta_i$. This procedure amounts to defining a single, more complicated mapping h :

$$\begin{aligned} 1 = \int d\vec{r} d\vec{\phi} \int_0^1 dz \big\{ & \theta(0 < z < \beta_1) \delta(\vec{\phi} - h_1(\vec{r})) + \\ & \theta(\beta_1 < z < \beta_2) \delta(\vec{\phi} - h_2(\vec{r})) + \dots \\ & \theta(\beta_{N-1} < z < 1) \delta(\vec{\phi} - h_N(\vec{r})) \big\} , \end{aligned} \quad (11)$$

where the function θ of an inequality equals 1 if the inequality holds, zero otherwise. After eliminating the $\vec{r}$ and integrating over z (an example of the case $n > 8$ discussed

⁴When the variance of $w(\vec{\phi})$ has been reduced to zero, we have $g(\vec{\phi}) \propto f(\vec{\phi})$, i.e. we have solved the cross section integral analytically. Real life, however, is rarely so accommodating.

before) we have

$$\begin{aligned} 1 &= \int d\vec{\phi} g(\vec{\phi}) , \\ g(\vec{\phi}) &= \alpha_1 g_1(\vec{\phi}) + \cdots \alpha_N g_N(\vec{\phi}) . \end{aligned} \quad (12)$$

We see that, *whatever h_i was used to generate the event*, the local density to be used in the computation of the event weight must *always* be the $g(\vec{\phi})$ of Eq.(12).

The numbers α_i are called the *a-priori weights*, and we see that their actual numerical values are unimportant as long as they are properly taken into account in the computation of $g(\vec{\phi})$. This even holds if we wish to change the α_i during the running of the Monte Carlo program. This gives us considerable scope for optimization. For example we may reduce the Monte Carlo error estimate by tuning the values of α_i . Later on we shall present some results on the optimization of our four-fermion generator along these lines. There is actually the possibility of automating the adjustment of α_i so as to make a Monte Carlo program “self-optimizing”. But although this is itself an interesting problem in nonlinear optimization we have not pursued it further, and all our optimizations of the a-priori weights are performed by hand.

There are two cautionary remarks to be made here. In the first place, if a certain h_i reaches to a region in phase space that the others do not, putting its α_i to zero will void that region completely of events, and thus a bias develops in the result. In the second place, since the different h_i are intended in practice to describe different peaking structures of $f(\vec{\phi})$, putting some α_i to very small values forces the program to visit some peaks only rarely — but *if* it does, the Monte Carlo weight will rise with a vengeance! Although the Monte Carlo result will still be correct with probability 1 as the number of events goes to infinity, the very large estimated error for finite numbers of events will be prohibitive.

From the above discussion it should be clear that every *channel* (mapping) should, in a **FORTTRAN** implementation, come as a set of two subprograms: one, to generate $\vec{\phi}$ from $\vec{r}$ with the appropriate mapping $\vec{\phi} = h_i(\vec{r})$; and one, to compute, for an arbitrarily generated event $\vec{\phi}$, the local density $g_i(\vec{\phi})$ for that channel. Although the program speed in the part where the event is generated will not depend on the number N (apart from the very fast step where the choice between channels is made), the part where the local density is evaluated slows down as $1/N$, so that extremely large N should be avoided unless we can for any $\vec{\phi}$ immediatly rule out the majority of the channels since they could not possible give this type of event.

This finishes the description of our Monte Carlo strategy. In what follows we shall discuss the individual mappings h_i and their related g_i , which of course depend on the particulars of our process.

3 The Phase Space Generation Channels

The particular mappings that one uses in a Monte Carlo will naturally depend on the physics of the process under consideration. In electroweak Monte Carlos the main problem is the peaking structure of the integrand when massless or light internal lines become close to their mass shell. To visualize this we present in fig.1 the *kinematical diagrams* for the four-fermion production process. These are inspired by the various Feynman diagrams, but should not be interpreted further than that they inform on the various peaking structures to be expected: they indicate which variables are most appropriate to a given channel. In fig. 1^a we have the *annihilation* where a resonating Z decays radiatively into two fermion pairs. For a typical event, we expect a low mass and a low total energy for the pair momentum $k = k_1 + k_2$; in the diagram depicted we also expect a small angle between k and p_1 . All other distributions are expected to be more or less flat (the exact behaviour will of course be accounted for by the inclusion of the correct matrix element in the weight). Figures 1^{b,c} (*bremsstrahlung* channel) show a similar behaviour for k except that now it also likes to be collinear with the incoming momentum, and no Z resonance occurs: instead, the momentum q_2 wants to be emitted at small angle θ_2 . The other figures 1^{d,e,f,g} (*conversion* channel) and 1^h (*multiperipheral* channel) can be read in the same way. Of course, the rôles of q_1 and q_2 can be interchanged, and so on: in practice the number of channels is greater than 8, although for channels differing by just a relabelling of momenta the same **FORT**RAN code can be used with interchanged arguments. We shall now treat one channel in some detail to illustrate how to obtain the unitary algorithm and the expression for the local density.

For the annihilation channel of diagram 1^a we obviously want to use the invariant mass squared $s_k = k^2 = (k_1 + k_2)^2$ as a variable. We expect its behaviour to be roughly $1/s_k$ in the cross section since the virtual photon gives a factor $1/s_k^2$ in the matrix element squared, partly compensated by the fermion lines of k_1 and k_2 that together give a number of the order of s_k . We therefore use a mapping that will give us a general behaviour $1/s_k^\nu$, and assume lower and upper limits s_k^{min} and s_k^{max} , respectively. The lower limit should be the experimental (or kinematical) lower limit since the diagram will be important there, but the upper limit may be lower than the actual experimental one, because for large values of k^2 the diagram 1^a is not expected to be important, and other channels will provide for a correct population of the phase space. The unitary algorithm that generates s_k is a special case of the following, more general one we shall use extensively:

$$\begin{aligned}
 1 &= \int_0^1 dr \\
 &= \int_0^1 dr \int_{t_0}^{t_1} dt \delta \left(t + a - \left[r(a + t_1)^{1-\nu} + (1-r)(a + t_0)^{1-\nu} \right]^{1/(1-\nu)} \right)
 \end{aligned}$$

Figure 1: The phase space generator variables used in section 3, and their relation to the underlying Feynman diagrams.

$$\begin{aligned}
&= \frac{1}{H(a, \nu, t_0, t_1)} \int_{t_0}^{t_1} \frac{dt}{(a+t)^\nu} , \\
H(a, \nu, t_0, t_1) &= \frac{1}{1-\nu} \left((a+t_1)^{1-\nu} - (a+t_0)^{1-\nu} \right) ,
\end{aligned} \tag{13}$$

if $\nu \neq 1$; and for $\nu = 1$ we have

$$\begin{aligned}
1 &= \int_0^1 dr \int dt \delta(t+a - \exp(r \log(a+t_1) + (1-r) \log(a+t_0))) \\
&= \frac{1}{H(a, 1, t_0, t_1)} \int_{t_0}^{t_1} \frac{dt}{a+t} , \\
H(a, 1, t_0, t_1) &= \log(a+t_1) - \log(a+t_0) .
\end{aligned} \tag{14}$$

This is the first piece of the unitary algorithm to generate the channel 1^a . Note that ν is an adjustable parameter: we may change ν so as to optimize the Monte Carlo without invalidating the result. In practice, $\nu = 1.3$ appears to give the best results for this variable. The internal fermion line in this graph tells us that the invariant mass $s_p = (p_1 + k_1 + k_2)^2$ likes to peak in similar manner. It remains to get the right behaviour of the energy $k^0 = k_1^0 + k_2^0$. This is hard to do directly since k^0 is correlated with both s_k and s_p , but we may use the fact that k^0 depends on the angle θ between the direction of $p+k$ in the lab frame and the direction of k in the rest system of $p+k$:

$$k^0 = \frac{\sqrt{s}}{4} \lambda^{1/2}(s_p/s, m_p^2/s) \lambda^{1/2}(s_k/s_p, m_p^2/s_p) (C + \cos \theta) , \tag{15}$$

where

$$C = \frac{(1 + s_p/s - m_p^2/s)(1 + s_k/s_p - m_p^2/s_p)}{\lambda^{1/2}(s_p/s, m_p^2/s) \lambda^{1/2}(s_k/s_p, m_p^2/s_p)} , \tag{16}$$

and

$$\lambda(x, y) = 1 + x^2 + y^2 - 2x - 2y - 2xy . \tag{17}$$

If θ is small, then k^0 will be large, and vice versa. Therefore, instead of directly generating k^0 , we generate the variable $\cos \theta$ between -1 and 1, with a distribution $1/(C + \cos \theta)$, using the same mapping (and hence the same local density formula) as before. The azimuthal angle ϕ adjoint to θ , the solid angle Ω_{p_2} of p_2 in the lab frame, and the decay angle Ω_{k_1} of k_1 in the k rest frame, are not expected to display any particular peaking behaviour, and we generate them trivially:

$$1 = \int \frac{d\phi}{2\pi} \int \frac{d\Omega_{p_2}}{4\pi} \int \frac{d\Omega_{k_1}}{4\pi} . \tag{18}$$

Putting all these ingredients together, we arrive at the cooking recipe for generating the phase space variables. We are, however, not finished: the global phase space

integration element is not $ds_p ds_k d\cos\theta d\phi d\Omega_{p_2} d\Omega_{k_1}$ but the relativistically invariant one of the previous section. We therefore have to include the Jacobian of the transformation between these two integration elements. For a two-body decay of a momentum p with mass m into two momenta r_1 and r_2 with masses m_1 and m_2 we have

$$\int dr_1 \delta(r_1^2 - m_1^2) dr_2 \delta(r_2^2 - m_2^2) \delta^4(p - r_1 - r_2) = \frac{\pi}{2} \lambda^{1/2}(m_1^2/m^2, m_2^2/m^2) , \quad (19)$$

and we can concatenate three such three-body decays, first from the primary Z into p_2 and $p_1 + k$, then from $p + k$ into p and k , and finally from k into k_1 and k_2 . The total unitary algorithm for this channel is hence given by

$$1 = \left[\int_{s_k^{\min}}^{s_k^{\max}} \frac{ds_k}{s_k^{1.3}} \cdot \frac{1}{H(0, 1.3, s_k^{\min}, s_k^{\max})} \right] \left[\int_{s_p^{\min}=(\sqrt{s_k}+m_q)^2}^{s_p^{\max}} \frac{ds_p}{s_p} \cdot \frac{1}{H(0, 1, s_p^{\min}, s_p^{\max})} \right] \\ \left[\int_{-1}^1 \frac{d\cos\theta}{(C + \cos\theta)^{1.5}} \cdot \frac{1}{H(C, 1.5, -1, 1)} \right] \cdot \int \frac{d\phi}{2\pi} \cdot \int \frac{d\Omega_{p_2}}{4\pi} \cdot \int \frac{d\Omega_{k_1}^*}{4\pi} , \quad (20)$$

and the local density, that is, the probability density that this channel will give an event with the particular momenta p_1, p_2, k_1 and k_2 , is

$$g(\vec{\phi}) = 2 \left[\frac{1}{s_k^{1.3}} \cdot \frac{1}{H(0, 1.3, s_k^{\min}, s_k^{\max})} \right] \left[\frac{1}{s_p} \cdot \frac{1}{H(0, 1, s_p^{\min}, s_p^{\max})} \right] \\ \left[\frac{1}{(C + \cos\theta)^{1.5}} \cdot \frac{1}{H(C, 1.5, -1, 1)} \right] \\ \cdot \frac{2/\pi}{\lambda^{1/2}(s_p/s, m_p^2/s)} \cdot \frac{2/\pi}{\lambda^{1/2}(s_k/s_p, m_p^2/s_p)} \cdot \frac{2/\pi}{\lambda^{1/2}(m_k^2/s_k, m_k^2/s_k)} . \quad (21)$$

In addition to this generator, we also have the possibilities that p_2 takes the rôle of p_1 , and that the rôles of the p 's and k 's are interchanged (in fact, for the final state $f^+ f^- f^+ f^-$ we can also have $p_1 \leftrightarrow k_1$ and so on), giving in total up to 8 different channels. All these alternatives are generated and described by the same formulae and **FORT**RAN routines, with the appropriate argument interchanges, a procedure henceforth referred to as *hand-symmetrization*.

Having given the details for the method by which one arrives at a unitary algorithm in practice for the annihilation channel, we treat the other channels more superficially, merely giving the unitary algorithm and local density.

3.1 Bremsstrahlung Generators

There are two bremsstrahlung group phase space channels provided in **FERMISV** corresponding to figures 1^{b,c}, called bremsstrahlung forward and bremsstrahlung backward, respectively.

The unitary algorithm for the bremsstrahlung forward phase space generator is

$$\begin{aligned}
1 = & \left[\int_{s_k^{\min}}^{0.4^2 s} \frac{ds_k}{s_k^{1.4}} \cdot \frac{1}{H(0, 1.4, s_k^{\min}, 0.4^2 s)} \right] \left[\int_{\sqrt{s_k}}^{0.95 k_{\max}^0} \frac{dk^0}{\sqrt{k^0}} \cdot \frac{1}{H(0, 1/2, \sqrt{s_k}, 0.95 k_{\max}^0)} \right] \\
& \left[\int_{-\cos \theta_e}^{\cos \theta_e} \frac{d \cos \theta}{(C - \cos \theta)^{1.7}} \cdot \frac{1}{-H(C, 1.7, \cos \theta_e, -\cos \theta_e)} \right] \cdot \int \frac{d\phi}{2\pi} \cdot \\
& \left[\int_{-\cos \theta_{\text{cen}}^*}^{\cos \theta_{\text{cen}}^*} \frac{d \cos \theta_2}{(1 - \cos \theta_2)^{1.7}} \cdot \frac{1}{-H(1, 1.7, \cos \theta_{\text{cen}}^*, -\cos \theta_{\text{cen}}^*)} \right] \cdot \int \frac{d\phi_2}{2\pi} \\
& \cdot \int \frac{d\Omega_{k_1}^*}{4\pi} , \tag{22}
\end{aligned}$$

with

$$s_k = (k_1 + k_2)^2, \quad k^0 = k_1^0 + k_2^0, \tag{23}$$

and

$$C = (k^0 \sqrt{s} - s_k) / (|k| \sqrt{s}). \tag{24}$$

We make a brief diversion from our unitary algorithm description to explain how the integration limit $\cos \theta_e$ for the angle θ is set within **FERMISV**. We wish to prevent too many wasted generation calls which produce very forward and undetected particles; at the same time, we must ensure that we cover at least the central detector region, defined as $\theta_{\text{cen}}^* < \theta < \pi - \theta_{\text{cen}}^*$, where θ_{cen}^* is the central detector cutoff angle θ_{cen} after the Lorentz boost which handles particle kinematics in the case of initial-state radiation (see section 5.3). In fig. 2, which shows the decay into two particles of

Figure 2: A 2-body decay in 3-momentum space.

equal mass in the lab frame, $k \rightarrow k_1, k_2$, it is clear that

$$\min(\theta_1, \theta_2) \leq \theta_{\text{symm}}, \tag{25}$$

where θ_{symm} is the maximal symmetric decay angle defined by

$$\begin{aligned} s_k &= 2k_1 \cdot k_2 + 2m_k^2 \\ &= 2\left(\frac{1}{2}k^0\right)^2(1 - \beta^2 \cos 2\theta_{\text{symm}}) + 2m_k^2, \end{aligned} \quad (26)$$

where

$$\beta = \sqrt{1 - \frac{m_k^2}{(k^0/2)^2}}. \quad (27)$$

Explicitly,

$$\theta_{\text{symm}} = \frac{1}{2} \arccos \left(\frac{1}{\beta^2} \left[1 - \frac{2(s_k - 2m_k^2)}{k^{02}} \right] \right). \quad (28)$$

Now if $\theta_k + \theta_{\text{symm}} < \theta_{\text{cen}}$, then it is guaranteed that at least one of the particles k_1, k_2 will be too forward and hence not in the fiducial volume. Since we are not interested in those cases, we define our integration limits as

$$\theta_e = \theta_k^{\min} = \max(\theta_{\text{cen}}^* - \theta_{\text{symm}}, 0), \quad (29)$$

and

$$\theta_k^{\max} = \pi - \theta_k^{\min}. \quad (30)$$

Returning to the bremsstrahlung forward generator, we have a local density of

$$\begin{aligned} g(\vec{\phi}) &= \frac{2}{(2\pi)^2} \left[\frac{\sqrt{s} + |k| \cos \angle(q_2, k) - k^0}{\sqrt{s} q_2^0} \right] \left[\frac{1}{s_k^{1.4}} \cdot \frac{1}{H(0, 1.4, s_k^{\min}, 0.4^2 s)} \right] \\ &\quad \left[\frac{1}{\sqrt{k^0}} \cdot \frac{1}{H(0, 1/2, \sqrt{s_k}, 0.95 k_{\max}^0)} \right] \\ &\quad \left[\frac{1}{(C - \cos \theta)^{1.7}} \cdot \frac{1}{-H(C, 1.7, \cos \theta_e, -\cos \theta_e)} \right] \cdot \left[\frac{1}{(1 - \cos \theta_2)^{1.7}} \right. \\ &\quad \left. \cdot \frac{1}{-H(1, 1.7, \cos \theta_{\text{cen}}^*, -\cos \theta_{\text{cen}}^*)} \right] \cdot \frac{2/\pi}{\lambda^{1/2}(m_k^2/s_k, m_k^2/s_k)}. \end{aligned} \quad (31)$$

Note that the kinematical diagrams for the bremsstrahlung channels are, in **FERMISV**, not interpreted as three consecutive 2-body decays but rather as a 3-body decay ($p_1 + p_2 \rightarrow q_1 + q_2 + k$) followed by a 2-body decay ($k \rightarrow k_1 + k_2$). For massless electrons (here, a very good approximation) the Jacobian can then be worked out using the identity

$$0 = s + s_k - 2\sqrt{s}(q_1^0 + k^0) + 2q_1^0(k^0 - |k| \cos \angle(q_1, k)). \quad (32)$$

The unitary algorithm for the bremsstrahlung backward generator is given by the prescription

$$1 = \left[\int_{s_k^{\min}}^{0.4^2 s} \frac{ds_k}{s_k^{1.4}} \cdot \frac{1}{H(0, 1.4, s_k^{\min}, 0.4^2 s)} \right] \left[\int_{s_q^{\min}=(\sqrt{s_k}+m_q)^2}^{0.7^2 s} \frac{ds_q}{s_q} \cdot \frac{1}{H(0, 1, s_q^{\min}, 0.7^2 s)} \right]$$

$$\left[\int_{-\cos \theta_{\text{cen}}^*}^{\cos \theta_{\text{cen}}^*} \frac{d \cos \theta_2}{(1 - \cos \theta_2)^{1.7}} \cdot \frac{1}{-H(1, 1.7, \cos \theta_{\text{cen}}^*, -\cos \theta_{\text{cen}}^*)} \right] \cdot \int \frac{d\phi_2}{2\pi} \cdot \left[\int_{-1}^1 \frac{d \cos \theta}{(C + \cos \theta)} \cdot \frac{1}{H(C, 1, -1, 1)} \right] \cdot \int \frac{d\phi}{2\pi} \cdot \int \frac{d\Omega_{k_1}^*}{4\pi} , \quad (33)$$

with

$$s_k = (k_1 + k_2)^2 , \quad s_q = (k_1 + k_2 + q_1)^2 , \quad (34)$$

and

$$C = \frac{(s + s_q)(s_q + s_k)}{(s - s_q)(s_q - s_k)} . \quad (35)$$

The density is

$$g(\vec{\phi}) = 4 \left[\frac{1}{s_k^{1.4}} \cdot \frac{1}{H(0, 1.4, s_k^{\min}, 0.4^2 s)} \right] \left[\frac{1}{s_q} \cdot \frac{1}{H(0, 1, s_q^{\min}, 0.7^2 s)} \right] \left[\frac{1}{(1 - \cos \theta_2)^{1.7}} \cdot \frac{1}{-H(1, 1.7, \cos \theta_{\text{cen}}^*, -\cos \theta_{\text{cen}}^*)} \right] \left[\frac{1}{(C + \cos \theta)} \cdot \frac{1}{H(C, 1, -1, 1)} \right] \cdot \frac{2/\pi}{\lambda^{1/2}(s_q/s, m_q^2/s)} \cdot \frac{2/\pi}{\lambda^{1/2}(s_k/s_q, m_q^2/s_q)} \cdot \frac{2/\pi}{\lambda^{1/2}(m_k^2/s_k, m_k^2/s_k)} . \quad (36)$$

As for the annihilation generator, the bremsstrahlung generators as described above are each optimized for one of the bremsstrahlung Feynman diagrams. The other relevant diagrams are accounted for by hand-symmetrization. We have thus accounted for four (sixteen if the final state is $e^+e^-e^+e^-$) of the bremsstrahlung Feynman diagrams with these generators. The bremsstrahlung channels involving a Z exchange have not yet been considered (though their formulation would be straightforward) because they do not dominate the cross section up to $\sqrt{s} = 200$ GeV.

3.2 Conversion Generators

Four generators optimized for various conversion channels, shown in figures 1^{d,e,f,g}, are implemented in **FERMISV**. The unitary algorithm for fig. 1^d, a single conversion, is given by

$$1 = \left[\int_{s_k^{\min}}^{0.4^2 s} \frac{ds_k}{s_k^{3/2}} \cdot \frac{1}{H(0, 3/2, s_k^{\min}, 0.4^2 s)} \right] \left[\int_{\sqrt{s_k}}^{k_{\max}^0} \frac{dk^0}{k^{0^{3/2}}} \cdot \frac{1}{H(0, 3/2, \sqrt{s_k}, k_{\max}^0)} \right] \left[\int_{-\cos \theta_e}^{\cos \theta_e} \frac{d \cos \theta_k}{A^2 - B^2 \cos^2 \theta_k} \cdot \frac{AB}{\ln \left(\frac{A+B \cos \theta_e}{A-B \cos \theta_e} \right)} \right] \cdot \int \frac{d\phi_k}{2\pi} \cdot \int \frac{d\Omega_{k_1}^*}{4\pi} \cdot \int \frac{d\Omega_{p_1}^*}{4\pi} \quad (37)$$

with

$$s_k = (k_1 + k_2)^2 , \quad k^0 = k_1^0 + k_2^0 , \quad A = s_k - \sqrt{s} k^0 , \quad \text{and} \quad B = \sqrt{s} |k| . \quad (38)$$

The local density is

$$\begin{aligned}
g(\vec{\phi}) = & \frac{2}{2\pi|k|} \left[\frac{1}{s_k^{3/2}} \cdot \frac{1}{H(0, 3/2, s_k^{\min}, 0.4^2 s)} \right] \left[\frac{1}{k^{03/2}} \cdot \frac{1}{H(0, 3/2, \sqrt{s_k}, k_{\max}^0)} \right] \\
& \left[\frac{1}{A^2 - B^2 \cos^2 \theta_k} \cdot \frac{AB}{\ln \left(\frac{A+B \cos \theta_e}{A-B \cos \theta_e} \right)} \right] \\
& \cdot \frac{2/\pi}{\lambda^{1/2}(m_k^2/s_k, m_k^2/s_k)} \cdot \frac{2/\pi}{\lambda^{1/2}(m_q^2/s_q, m_q^2/s_q)} .
\end{aligned} \tag{39}$$

The unitary algorithm for fig. 1^e, a double conversion, is given by

$$\begin{aligned}
1 = & \left[\int_{s_k^{\min}}^{0.4^2 s} \frac{ds_k}{s_k} \cdot \frac{1}{H(0, 1, s_k^{\min}, 0.4^2 s)} \right] \left[\int_{s_p^{\min}}^{0.4^2 s} \frac{ds_p}{s_p} \cdot \frac{1}{H(0, 1, s_p^{\min}, 0.4^2 s)} \right] \\
& \left[\int_{-\cos \theta_e}^{\cos \theta_e} \frac{d \cos \theta_k}{C - \cos \theta_k} \cdot \frac{1}{-H(C, 1, \cos \theta_e, -\cos \theta_e)} \right] \cdot \int \frac{d\phi_k}{2\pi} \\
& \cdot \int \frac{d\Omega_{k_1}^*}{4\pi} \cdot \int \frac{d\Omega_{p_1}^*}{4\pi} ,
\end{aligned} \tag{40}$$

with

$$s_k = (k_1 + k_2)^2 , \quad s_p = (p_1 + p_2)^2 , \tag{41}$$

and

$$C = (\sqrt{s} k^0 - s_k) / (\sqrt{s} |k|) . \tag{42}$$

The local density is

$$\begin{aligned}
g(\vec{\phi}) = & 2 \left[\frac{1}{s_k} \cdot \frac{1}{H(0, 1, s_k^{\min}, 0.4^2 s)} \right] \left[\frac{1}{s_p} \cdot \frac{1}{H(0, 1, s_p^{\min}, 0.4^2 s)} \right] \left[\frac{1}{C - \cos \theta_k} \right. \\
& \cdot \left. \frac{1}{-H(C, 1, \cos \theta_e, -\cos \theta_e)} \right] \cdot \frac{2/\pi}{\lambda^{1/2}(s_k/s, s_p/s)} \cdot \frac{2/\pi}{\lambda^{1/2}(m_k^2/s_k, m_k^2/s_k)} \\
& \cdot \frac{2/\pi}{\lambda^{1/2}(m_p^2/s_p, m_p^2/s_p)} .
\end{aligned} \tag{43}$$

The unitary algorithm for fig. 1^f, a single conversion accompanied by Z resonance in the other pair, is given by

$$\begin{aligned}
1 = & \left[\int_{s_k^{\min}}^{0.4^2 s} \frac{ds_k}{s_k^{3/2}} \cdot \frac{1}{H(0, 3/2, s_k^{\min}, 0.4^2 s)} \right] \\
& \left[\int_{\sqrt{s_k}}^{k_{\max}^0} \frac{dk^0}{(s_p - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \cdot \frac{2\sqrt{s} m_Z \Gamma_Z}{\tan^{-1} y_{\max} - \tan^{-1} y_{\min}} \right] \\
& \left[\int_{-\cos \theta_e}^{\cos \theta_e} \frac{d \cos \theta_k}{A^2 - B^2 \cos^2 \theta_k} \cdot \frac{AB}{\ln \left(\frac{A+B \cos \theta_e}{A-B \cos \theta_e} \right)} \right] \cdot \int \frac{d\phi_k}{2\pi} \cdot \int \frac{d\Omega_{k_1}^*}{4\pi} \cdot \int \frac{d\Omega_{p_1}^*}{4\pi} \tag{44}
\end{aligned}$$

with

$$\begin{aligned} s_p &= (p_1 + p_2)^2 \\ &= s + s_k - 2k^0\sqrt{s} \ , \end{aligned} \quad (45)$$

and

$$s_k = (k_1 + k_2)^2 \ , \quad k^0 = k_1^0 + k_2^0 \ , \quad A = s_k - \sqrt{s}k^0 \ , \quad B = \sqrt{s}|k| \ , \quad y = (s_p - m_Z^2)/(m_Z\Gamma_Z) \ . \quad (46)$$

The local density is

$$\begin{aligned} g(\vec{\phi}) &= \frac{2}{2\pi|k|} \left[\frac{1}{s_k^{3/2}} \cdot \frac{1}{H(0, 3/2, s_k^{\min}, 0.4^2 s)} \right] \left[\frac{1}{(s_p - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \right. \\ &\quad \cdot \left. \frac{2\sqrt{s} m_Z \Gamma_Z}{\tan^{-1} y_{\max} - \tan^{-1} y_{\min}} \right] \left[\frac{1}{A^2 - B^2 \cos^2 \theta_k} \cdot \frac{AB}{\ln \left(\frac{A+B \cos \theta_e}{A-B \cos \theta_e} \right)} \right] \\ &\quad \cdot \frac{2/\pi}{\lambda^{1/2}(m_k^2/s_k, m_k^2/s_k)} \cdot \frac{2/\pi}{\lambda^{1/2}(m_q^2/s_q, m_q^2/s_q)} \ . \end{aligned} \quad (47)$$

The unitary algorithm for fig. 1^g, a double conversion with both pairs' invariant mass near the Z mass, is given by

$$\begin{aligned} 1 &= \left[\int_{s_k^{\min}=(m_Z-10\Gamma_Z)^2}^{s_k^{\max}=(m_Z+10\Gamma_Z)^2} \frac{ds_k}{(s_k - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \cdot \frac{m_Z \Gamma_Z}{\tan^{-1} y_{\max} - \tan^{-1} y_{\min}} \right] \\ &\quad \left[\int_{s_p^{\min}=(m_Z-10\Gamma_Z)^2}^{s_p^{\max}=(m_Z+10\Gamma_Z)^2} \frac{ds_p}{(s_p - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \cdot \frac{m_Z \Gamma_Z}{\tan^{-1} y_{\max} - \tan^{-1} y_{\min}} \right] \\ &\quad \left[\int_{-\cos \theta_e}^{\cos \theta_e} \frac{d \cos \theta_k}{C - \cos \theta_k} \cdot \frac{1}{-H(C, 1, \cos \theta_e, -\cos \theta_e)} \right] \cdot \int \frac{d\phi_k}{2\pi} \\ &\quad \cdot \int \frac{d\Omega_{k_1}^*}{4\pi} \cdot \int \frac{d\Omega_{p_1}^*}{4\pi} \ , \end{aligned} \quad (48)$$

with

$$s_k = (k_1 + k_2)^2 \ , \quad s_p = (p_1 + p_2)^2 \ , \quad C = (\sqrt{s}k^0 - s_k)/(\sqrt{s}|k|) \ , \quad (49)$$

and

$$y = (s_p - m_Z^2)/(m_Z\Gamma_Z) \ . \quad (50)$$

The local density is

$$\begin{aligned} g(\vec{\phi}) &= 2 \left[\frac{1}{(s_k - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \cdot \frac{m_Z \Gamma_Z}{\tan^{-1} y_{\max} - \tan^{-1} y_{\min}} \right] \\ &\quad \left[\frac{1}{(s_p - m_Z^2)^2 + m_Z^2 \Gamma_Z^2} \cdot \frac{m_Z \Gamma_Z}{\tan^{-1} y_{\max} - \tan^{-1} y_{\min}} \right] \end{aligned}$$

$$\left[\frac{1}{C - \cos \theta_k} \cdot \frac{1}{-H(C, 1, \cos \theta_e, -\cos \theta_e)} \right] \cdot \frac{2/\pi}{\lambda^{1/2}(s_k/s, s_p/s)} \cdot \frac{2/\pi}{\lambda^{1/2}(m_k^2/s_k, m_k^2/s_k)} \cdot \frac{2/\pi}{\lambda^{1/2}(m_p^2/s_p, m_p^2/s_p)} . \quad (51)$$

The single conversion generators described above are each symmetrized with respect to θ_k and thus account for two Feynman diagrams consisting of one photon and one Z propagator. The other conversion channels which have one photon and one Z propagator are accounted for by hand-symmetrization. The double conversion generators are each optimized for one channel. Each gives two (four if the final state is $e^+e^-e^+e^-$) diagrams after hand-symmetrization. Thus all eight (sixteen) of the conversion channels are accounted for with these generators.

3.3 Multiperipheral Generator

One generator optimized for the multiperipheral channels, shown in fig. 1^h, is implemented in FERMISV. The unitary algorithm is given by

$$1 = \left[\int_{s_k^{\min}}^{0.85^2 s} \frac{ds_k}{s_k + \kappa} \cdot \frac{1}{H(\kappa, 1, s_k^{\min}, 0.85^2 s)} \right] \left[\int_{-\cos \theta_{\text{cen}}^*}^{\cos \theta_{\text{cen}}^*} \frac{d \cos \theta_1}{(1 - \cos \theta_1)^{3/2}} \cdot \frac{1}{-H(1, 3/2, \cos \theta_{\text{cen}}^*, -\cos \theta_{\text{cen}}^*)} \right] \cdot \int \frac{d\phi_1}{2\pi} \cdot \left[\int_{-\cos \theta_{\text{cen}}^*}^{\cos \theta_{\text{cen}}^*} \frac{d \cos \theta_2}{(1 - \cos \theta_2)^{3/2}} \cdot \frac{1}{-H(1, 3/2, \cos \theta_{\text{cen}}^*, -\cos \theta_{\text{cen}}^*)} \right] \cdot \int \frac{d\phi_2}{2\pi} \cdot \left[\int_0^{E(1-s_k/s)} \frac{dq_1^0}{E - q_1^0 \tau} \cdot \frac{\tau}{-\ln(1 - \beta \tau)} \right] \cdot \int \frac{d\Omega_{k_1}^*}{4\pi} , \quad (52)$$

with

$$s_k = (k_1 + k_2)^2 , \quad E = \sqrt{s}/2 , \quad \tau = [1 - \cos \angle(q_1, q_2)]/2 , \quad \kappa = 50 \text{ GeV}^2 , \quad (53)$$

and

$$\beta = 1 - s_k/s . \quad (54)$$

The local density is

$$g(\vec{\phi}) = \frac{1}{(2\pi)^2 q_1^0 q_2^0} \left[\frac{1}{s_k + \kappa} \cdot \frac{1}{H(\kappa, 1, s_k^{\min}, 0.85^2 s)} \right] \left[\frac{1}{(1 - \cos \theta_1)^{3/2}} \cdot \frac{1}{-H(1, 3/2, \cos \theta_{\text{cen}}^*, -\cos \theta_{\text{cen}}^*)} \right] \left[\frac{1}{(1 - \cos \theta_2)^{3/2}} \cdot \frac{1}{-H(1, 3/2, \cos \theta_{\text{cen}}^*, -\cos \theta_{\text{cen}}^*)} \right] \left[\frac{1}{E - q_1^0 \tau} \cdot \frac{\tau}{-\ln(1 - \beta \tau)} \right] \cdot \frac{2/\pi}{\lambda^{1/2}(m_k^2/s_k, m_k^2/s_k)} . \quad (55)$$

We want to point out that the part of the algorithm responsible for the generation of the energy q_1^0 in Eq.(52) will give q_1^0 typically close to the beam energy E if the electron momenta $\vec{q}_1$ and $\vec{q}_2$ are back-to-back (as for a ‘truly multiperipheral’ event) but tends to give a flatter distribution if the electrons come out with sizable acollinearity. This behaviour is roughly what one expects; but the real reason for choosing the behaviour $1/(E - q_1^0\tau)$ is that, under this choice, the energy q_2^0 of the other electron will be similarly distributed as $1/(E - q_2^0\tau)$ (in the limit of massless electrons, which is justified here). This is not only aesthetically pleasing but also ensures that the weight distribution will be symmetrical in q_1 and q_2 . It would be perfectly possible to choose a different distribution in the q_1^0 algorithm without invalidating the Monte Carlo, but in that case the weight distribution would typically be asymmetric, and some additional hand-symmetrization might be called for.

The multiperipheral generator described above is optimized for two channels consisting of two photon propagators. For $e^+e^-e^+e^-$ final states an additional two channels are accounted for after hand-symmetrization.

3.4 The background channel: RAMBO

In **FERMISV** we have also incorporated an all-purpose generator which generates events uniformly in phase space for massless particles, known as **RAMBO** [11]. Though it is thoroughly described in ref. [11], we mention here the features relevant for our formalism. The statement that **RAMBO** generates events uniformly for massless particles implies that

$$g(\vec{\phi}) = \text{const} = 1/(\text{phase space volume}) \quad , \quad (56)$$

while for massive particles with $m \ll \sqrt{s}$, this is still approximately true. In detail for the general case of n (possibly massive) final state particles, we have a relatively simple local density

$$g(\vec{\phi})_{\text{RAMBO}} = 1/W_0 W_m \quad , \quad (57)$$

where

$$W_0 = (\pi/2)^{n-1} w^{2n-4} / \Gamma(n) \Gamma(n-1) \quad , \quad (58)$$

is the phase space volume for the case of massless particles, and

$$W_m = \left[\frac{1}{w} \sum_{i=1}^n |k_i| \right]^{2n-3} \left[\prod_{i=1}^n \frac{|k_i|}{k_i^0} \right] \left[\sum_{i=1}^n \frac{|k_i|^2}{k_i^0 w} \right]^{-1} \quad . \quad (59)$$

In the preceeding equations $w = \sqrt{s}$. Thus, given any event $\vec{\phi}$, its density from **RAMBO** can be calculated. **RAMBO** is included as one of the phase space generators as a fail-safe mechanism which enables us to cover the entire phase space with at least one generator, and thus to avoid the sampling bias mentioned at the end of the previous section. In practice the relative weight described in Eq.(12) of **RAMBO**, α_{RAMBO} , can be fixed to small values, $\alpha_{\text{RAMBO}} \approx 3\%$, since the bulk of the cross section comes from peaks

which are well-modelled by the other phase space generators. The unitary algorithm piece of this generator is more complicated and described in ref. [11], so we do not repeat it here.

Having finished our discussion of the multichannel phase space method, we proceed to describe the treatment of the matrix element in the following section.

4 Calculation of the Matrix Elements

In this section we shall discuss the computation of the matrix elements for the four-fermion process. Let us stress that this calculation is not valid for *all* four-fermion processes, but only for those where no charged-current interactions take place: processes with virtual $W^\pm$ exchange always contain diagrams with three-boson vertices which we do not treat here. On the other hand, the layout of the Monte Carlo simulation is quite independent of the matrix elements and, for instance, a process like

$$e^+e^- \rightarrow \mu^+\mu^-\nu_\mu\bar{\nu}_\mu \quad (60)$$

can easily be incorporated in our treatment once the complete matrix element has been implemented. For this process (at LEP 2 energies) the important channel

$$e^+e^- \rightarrow W^+W^- \rightarrow (\mu^+\nu_\mu)(\mu^-\bar{\nu}_\mu) \quad (61)$$

can be obtained by a trivial modification of the channel

$$e^+e^- \rightarrow Z^0Z^0 \rightarrow (\mu^+\mu^-)(\nu_\mu\bar{\nu}_\mu) \quad . \quad (62)$$

In addition, at LEP 1 energies the W 's are not expected to be on-shell (apart for processes like $e^+e^- \rightarrow \mu^-\bar{\nu}_\mu W^+$ that have a very small cross section), and so in many cases our approach will still be a good medium-precision approximation to the real cross section.

Secondly, we also neglect diagrams with Higgs boson exchange. Of course this is only justified if the Higgs is not on-shell: for processes like $e^+e^- \rightarrow Hl^+l^-$ we are therefore calculating the background rather than the signal. A complete calculation including the Higgs exchange exists [13], but the effects of Higgs exchange are very minor. Again, the complete matrix element can easily be incorporated in our treatment without the need for a modification of the Monte Carlo layout.

With these provisions, all Feynman diagrams can be cast in the form depicted in fig. 3, which we call the *generic diagram*. Its amplitude can be written as

$$\begin{aligned} M(p_1, \lambda_1, p_2, \lambda_2; p_3, \lambda_3, p_4, \lambda_4; p_5, \lambda_5, p_6, \lambda_6) = \\ = S \left((b_1 p_1 + b_2 p_2)^2 - m_b^2 + im_b \Gamma_b \right)^{-1} \times \\ \left((b_1 p_1 + b_2 p_2 + b_3 p_3)^2 - m_3^2 \right)^{-1} \left((b_5 p_5 + b_6 p_6)^2 - m_{b'}^2 + im_{b'} \Gamma_{b'} \right)^{-1} \quad , \end{aligned}$$

Figure 3: The generic 4 fermion Feynman diagram.

$$\begin{aligned}
S &= \bar{u}_{\lambda_1}(p_1)(v_1^b + a_1^b \gamma^5) \gamma^\mu u_{\lambda_2}(p_2) \times \\
&\quad \bar{u}_{\lambda_3}(p_3)(v_3^b + a_3^b \gamma^5) \gamma_\mu Q(v_4^{b'} + a_4^{b'} \gamma^5) \gamma_\nu u_{\lambda_4}(p_4) \times \\
&\quad \bar{u}_{\lambda_5}(p_5)(v_5^{b'} + a_5^{b'} \gamma^5) \gamma^\nu u_{\lambda_6}(p_6) \ , \\
Q &= (b_1(\not{p}_1 + m_1) + b_2(\not{p}_2 + m_2) + b_3(\not{p}_3 + m_3)) \ , \tag{63}
\end{aligned}$$

where

- $b_i = +1(-1)$ if i is an outgoing (incoming) particle
- The bosons b, b' are γ or Z . Of course we have $m_\gamma = \Gamma_\gamma = 0$.
- The v_i^b and a_i^b denote the vector- and axial-vector coupling of the fermion i to the boson b . For $b = \gamma$, $a_i^b \equiv 0$.

We denote both fermion and antifermion spinors by u , and let its type be implied by the type of the argument. To this end it is sufficient to let the fermions have a positive mass, and to take the mass of the antifermions to be negative. Note that

	$p_+ = p_1$	$p_+ = p_3$	$p_+ = p_5$
$p_- = p_2$	A	B	M
$p_- = p_4$	B	C	B
$p_- = p_6$	M	B	A

Table 1: Diagram type as a function of the assignment of the incoming fermion momenta. The notation used is ‘A’ for annihilation, ‘C’ for conversion, ‘B’ for bremsstrahlung, ‘M’ for Multiperipheral, ‘ p_- ’ for incoming electron, and ‘ p_+ ’ for incoming positron.

then, in the numerator of the fermion propagator, the m_1 and m_2 will always cancel: this is the justification for writing the generic diagram in the above form.

The complete set of diagrams is obtained by taking all 6 permutations of p_1, p_3 , and p_5 , and likewise for p_2, p_4 , and p_6 , and all 4 different combinations for b, b' . In many cases a number of diagrams will be forbidden by flavor conservation: the maximum total number of 144 diagrams obtains in the case $e^+e^- \rightarrow e^+e^-e^+e^-$. The evaluation of the matrix element is therefore very straightforward: for each of the 64 possible combinations of the polarizations $\lambda_1, \dots, \lambda_6$ we run over all permutations of the p_i and combinations of the b, b' , summing the results, and summing the squares of the amplitudes over all polarizations.

In many studies, it is a question of interest to know what is the contribution from each of the four groups of annihilation, conversion, bremsstrahlung, and multiperipheral diagrams. The corresponding diagrams are easily recognized by the assignment of the momenta of the incoming electron and positron which can be assigned to label 2,4,6, and 1,3,5, respectively. Table 1 shows which diagram types correspond with which assignment.

We are left with the evaluation of the spinorial part of the generic diagram. This is most compactly done using the formalism developed in ref. [6], which we include here for completeness. We define the following four-fermion function:

$$\begin{aligned}
G_b(p_1, \lambda_1, p_2, \lambda_2; p_3, \lambda_3, p_4, \lambda_4) = \\
= \bar{u}_{\lambda_1}(p_1)(v_2^b + a_2^b \gamma^5) \gamma^\mu u_{\lambda_2}(p_2) \bar{u}_{\lambda_3}(p_3)(v_4^b + a_4^b \gamma^5) \gamma^\mu u_{\lambda_4}(p_4) \quad . \quad (64)
\end{aligned}$$

Using the completeness relation $\sum_\lambda u_\lambda(p) \bar{u}_\lambda(p) = \not{p} + m$, the spinorial part S of our generic diagram can then be written as

$$S = \sum_{\lambda=\pm} \sum_{i=1}^3 b_i G_b(p_1, \lambda_1, p_2, \lambda_2; p_3, \lambda_3, p_i, \lambda) G_{b'}(p_i, \lambda, p_4, \lambda_4; p_5, \lambda_5, p_6, \lambda_6) \quad . \quad (65)$$

The evaluation of G_b is most easily performed with the following explicit representa-

tion of the spinors:

$$\begin{aligned} u_+(p) &= \frac{\not{p} + m}{\eta(p)} u_-(k_0) , \\ u_-(p) &= \frac{\not{p} + m}{\eta(p)} \not{k}_1 u_-(k_0) , \end{aligned} \quad (66)$$

where

- k_0 and k_1 are two auxiliary vectors with the properties $k_0^2 = 0$, $k_1^2 = -1$, and $k_0 \cdot k_1 = 0$, and
- $\eta(p)^2 = 2p \cdot k_0$.

More details on such representations can be found, for instance, in ref. [14]; here we only give the results. Using the elementary functions

$$\begin{aligned} s_+(p_1, p_2) &= (p_1^y + ip_1^z) \left[\frac{p_2^0 - p_2^x}{p_1^0 - p_1^x} \right]^{1/2} - (p_1 \leftrightarrow p_2) , \\ s_-(p_1, p_2) &= -s_+(p_1, p_2)^* , \\ \eta_i &\equiv \eta(p_i) , \\ \mu_i &\equiv \mu(p_i) = \frac{m_i}{\eta_i} , \end{aligned} \quad (67)$$

we can completely express the function G_b , whose explicit form is given in table 2.

Returning to evaluation of the matrix element, an important computational issue is the fact that the bulk of computing time in **FERMISV** is spent calculating the matrix element (rather than, say, the local densities). Thus large savings in time are realized by evaluating only those diagrams which give a significant contribution. In **FERMISV** a diagram is *not* calculated if any of its fermion lines, *e.g.*, (i, j)

- does *not* conserve helicity, *i.e.*, $\lambda_i \neq \lambda_j$
- *and* has negligible mass with respect to the momentum transfer: $m_i^2/|Q_{i,j}^2| < \epsilon$,

where $(i, j) = (1, 2)$, or $(5, 6)$, and the momentum flow $Q_{i,j,\dots,k}$ is defined as

$$Q_{i,j,\dots,k} = (b_i p_i + b_j p_j + \dots + b_k p_k) . \quad (68)$$

ϵ is set as 8×10^{-4} in **FERMISV**. For the $(3, 4)$ fermion line (cf. fig. 3), a more complicated criterion in addition to $\lambda_3 \neq \lambda_4$ is required before a diagram can be discarded in order to preserve gauge invariance:

- $m_3^2/|Q_{1,2,3}^2|$, and $m_3^2/|Q_{5,6,3}^2| < \epsilon$

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	G_b
$(+, +, +, +)$	$2[(v_1 - a_1)(v_3 - a_3)s_+(p_1, p_3)s_-(p_4, p_2) + (v_1 + a_1)(v_3 - a_3)\mu_1\mu_2\eta_3\eta_4 + (v_1 - a_1)(v_3 + a_3)\mu_3\mu_4\eta_1\eta_2]$
$(+, +, -, -)$	$[G_b(+, +, +, +)]_{\substack{p_3 \leftrightarrow p_4 \\ a_3 \leftrightarrow -a_3}}$
$(-, -, +, +)$	$[G_b(+, +, +, +)]_{\substack{p_1 \leftrightarrow p_2 \\ a_1 \leftrightarrow -a_1}}$
$(-, -, -, -)$	$[G_b(+, +, +, +)]_{\substack{p_3 \leftrightarrow p_4 \\ a_3 \leftrightarrow -a_3 \\ p_1 \leftrightarrow p_2 \\ a_1 \leftrightarrow -a_1}}$
$(+, +, +, -)$	$2(v_1 - a_1)[(v_3 + a_3)\mu_3\eta_2s_+(p_1, p_4) - (v_3 - a_3)\mu_4\eta_2s_+(p_1, p_3)]$
$(-, -, +, -)$	$[G_b(+, +, +, -)]_{\substack{p_1 \leftrightarrow p_2 \\ a_1 \leftrightarrow -a_1}}$
$(+, -, +, +)$	$[G_b(+, +, +, -)]_{\substack{p_1 \leftrightarrow p_3 \\ v_1 \leftrightarrow v_3 \\ p_2 \leftrightarrow p_4 \\ a_1 \leftrightarrow a_3}}$
$(+, -, -, -)$	$[G_b(+, +, +, -)]_{\substack{p_1 \leftrightarrow p_4 \\ v_1 \leftrightarrow v_3 \\ p_2 \leftrightarrow p_3 \\ a_1 \leftrightarrow -a_3}}$
$(+, +, -, +)$	$2(v_1 - a_1)[(v_3 - a_3)\mu_3\eta_1s_-(p_2, p_4) - (v_3 + a_3)\mu_4\eta_1s_-(p_2, p_3)]$
$(-, -, -, +)$	$[G_b(+, +, -, +)]_{\substack{p_1 \leftrightarrow p_2 \\ a_1 \leftrightarrow -a_1}}$
$(-, +, +, +)$	$[G_b(+, +, -, +)]_{\substack{p_1 \leftrightarrow p_3 \\ v_1 \leftrightarrow v_3 \\ p_2 \leftrightarrow p_4 \\ a_1 \leftrightarrow a_3}}$
$(-, +, -, -)$	$[G_b(+, +, -, +)]_{\substack{p_1 \leftrightarrow p_4 \\ v_1 \leftrightarrow v_3 \\ p_2 \leftrightarrow p_3 \\ a_1 \leftrightarrow -a_3}}$
$(+, -, +, -)$	0
$(-, +, -, +)$	0
$(+, -, -, +)$	$2\{(v_1 + a_1)[(v_3 - a_3)\mu_1\mu_3\eta_2\eta_4 - (v_3 + a_3)\mu_1\mu_4\eta_2\eta_3] - (v_1 - a_1)[(v_3 - a_3)\mu_2\mu_3\eta_1\eta_4 - (v_3 + a_3)\mu_2\mu_4\eta_1\eta_3]\}$
$(-, +, +, -)$	$[G_b(+, -, -, +)]_{\substack{p_1 \leftrightarrow p_3 \\ v_1 \leftrightarrow v_3 \\ p_2 \leftrightarrow p_4 \\ a_1 \leftrightarrow a_3}}$

Table 2: Explicit form of G_b . The superscript denoting vector boson type, “ b ”, is dropped for the sake of clarity from the axial and vector coupling terms a_i , v_i .

Further shortcuts are made by discarding those diagrams in which a Z propagator's contribution is negligible compared with a γ propagator in the same place. This criterion is

- $|Q_{i,j}^2|/m_Z^2 < \epsilon$,

where $(i,j) = (1,2)$, or $(5,6)$.

Another feature of **FERMISV** is that the same lines of code get evaluated in each event for the matrix element evaluation. Thus it has been possible to vectorize and even parallelize this section of the program with “do loops” over events at the innermost level. To take advantage of selectively evaluating certain diagrams as well as vectorizing the matrix element requires a computer with gather/scatter capabilities such as the CRAY. A simplified version of **FERMISV** has been ported to the CM-2, a massively parallel computer, as described in ref. [15].

5 Radiative Corrections

In contrast to the Born level reaction $e^+e^- \rightarrow f\bar{f}F\bar{F}$, the matrix elements of the reaction $e^+e^- \rightarrow f\bar{f}F\bar{F}\gamma$ have not been calculated. Therefore, the radiative corrections have been implemented in the Monte Carlo generator following an approximate recipe, the aim of which is to provide an estimate of their effects. In the soft or collinear limits, *i.e.*, when a photon is emitted at either low energies or along the direction of one of the initial or final state particles (or both conditions are true), the process $e^+e^- \rightarrow X\gamma$ can be viewed as resulting from two interfering sub-processes:

- Initial state radiation (ISR): $e^+e^- \rightarrow (e^+e^-)^*\gamma \rightarrow X\gamma$
- Final state radiation (FSR): $e^+e^- \rightarrow X^* \rightarrow X\gamma$

For the sake of simplicity the interferences between the initial state radiative corrections and the final state radiative corrections and the virtuality of the particles of the intermediate state - $(e^+e^-)^*$ or X^* - are neglected. The same description of the process is kept for all photons, irrespective of their energies and directions. Within this approximation, since the invariant mass of the colliding electrons is reduced, the inclusion of the ISR implies in general a modification of the total cross section. The final state kinematics being modified by both the ISR and FSR, both affect the cross section within the user-defined acceptance. Whereas the ISR treatment is expected to give a rather precise description of the initial state effect (at the percent level) the FSR treatment is intended as a means to evaluate systematics attached to the acceptance. In particular, it can only provide a rough estimate of the yield of high energy and isolated photons. The two sub-processes are described in turn below.

5.1 Initial State Radiative Corrections

The photon involved in the sub-process $e^+e^- \rightarrow (e^+e^-)^*\gamma$ is generated according to the differential distribution [16]

$$\frac{d^2 N^{\text{ISR}}}{de_0 d\Omega_\gamma} = \frac{\alpha}{(2\pi)^2} \left[\left(1 + \frac{3}{4}\beta\right) e_0^{\beta-1} - 1 + \frac{e_0}{2} \right] \left(\frac{v \sin \theta_\gamma}{1 - v \cos \theta_\gamma} + \frac{v \sin \theta_\gamma}{1 + v \cos \theta_\gamma} \right)^2, \quad (69)$$

where

- $e_0 = \frac{2k_\gamma^0}{\sqrt{s}}$ is the photon energy in units of the beam energy,
- $\beta = \frac{2\alpha}{\pi} \left(\frac{1+v^2}{2v} \ln \left(\frac{1+v}{1-v} \right) - 1 \right)$,
- $v = \sqrt{1 - \frac{4m_e^2}{s}}$ is the velocity of the electrons,
- θ_γ is the polar angle of the γ with respect to the incoming electron.

5.2 Final State Radiative Corrections

In the low energy limit, the differential distribution of the photon in the process $A \rightarrow B + \gamma$ is given by [17]

$$\frac{d^2 N^{\text{FSR}}}{de_0 d\Omega_\gamma} = \frac{\alpha}{(2\pi)^2} e_0^{\beta_{\text{tot}}-1} \left(\sum_{i=1}^{N_c} \frac{\eta_i Q_i v_i \sin \theta_{\gamma,i}}{1 - v_i \cos \theta_{\gamma,i}} \right)^2, \quad (70)$$

where

- N_c is the total number of charged tracks (including the initial state particles),
- $\eta_i = +1(-1)$ for final state (initial state) particles,
- Q_i is the charge of the i -particle (Since the initial state radiation is accounted for separately, the charges of the initial state particles are set to zero),

- $\cos \theta_{\gamma,i} = \frac{\vec{k}_\gamma \cdot \vec{k}_i}{|\vec{k}_i| k_\gamma^0}$,
- k_i is the four momentum of the i -particle, and
- β_{tot} is given by

$$\beta_{\text{tot}} = -\frac{2\alpha}{\pi} \sum_{i=1}^{N_c} \sum_{j>1}^{N_c} \eta_i \eta_j Q_i Q_j \left[\left(\frac{|k_i \cdot k_j|}{R_{ij}} \right) \ln \left(\frac{|k_i \cdot k_j| + R_{ij}}{m_i m_j} \right) - 1 \right], \quad (71)$$

with $R_{ij} = \sqrt{(k_i \cdot k_j)^2 - (m_i m_j)^2}$. Note that the distributions of Eqs. (69) and (70) are normalized to unity if no cut-off is applied on the photon energy:

$$\int_0^1 de_0 \int d\Omega_\gamma \frac{d^2 N^{\text{ISR/FSR}}}{de_0 d\Omega_\gamma} = 1 \quad (72)$$

The FSR photon is generated as follows:

- (1) A charged particle i is selected within the final state X .
- (2) The photon energy is generated following the distribution

$$\frac{dn}{d\epsilon_0^i} \propto (\epsilon_0^i)^{\beta_{\text{tot}}-1} d_0^i \quad , \quad \text{with } d_0^i = [1 - \epsilon_0^i + \frac{1}{2}(\epsilon_0^i)^2] \quad \text{and } \epsilon_0^i = \frac{k_\gamma^0}{k_i^0} \quad .$$

The charges of the final state particles are set to $Q_j \rightarrow Q_j \sqrt{d_0^j/d_0^i}$ if $k_\gamma^0 < k_j^0$ and to zero otherwise.

- (3) The photon direction is generated according to

$$\frac{d^2n}{d\phi d\cos\theta_{\gamma,i}} \propto \frac{1}{1 - v_i \cos\theta_{\gamma,i}} \quad ,$$

where ϕ is the azimuthal angle of the photon with respect to $\vec{k}_i$.

- (4) The final state kinematics is modified to guarantee $k_{e^+} + k_{e^-} = k_X + k_{\gamma_{\text{FSR}}}$.
- (5) The expression of Eq.(70) is used to produce un-weighted events.

In order to evaluate the possible systematics arising from the fact that only one final state photon is produced, the generation allows for the production of $N_\gamma > 1$ final state photons. In such a case, the above procedure is repeated N_γ times with a β_{tot} parameter reduced by a factor $1/N_\gamma$ to maintain a more or less correct distribution of the total energy flow of the photons.

5.3 The Inclusion of the Radiative Corrections in the Main Generator

The inclusion of the radiative corrections in FERMISV is performed following the procedure described below.

- (1) A photon γ_{ISR} is generated according to Eq.(69) by the sub-generator INITIG. The direction of the photon is forced to be along the beam axis, an approximation which is consistent with the accuracy of the treatment described.

A constant weight is returned by INITIG for low energy photons. The high energy tail of the distribution is enhanced at the generation level to avoid the occurrence of too large weights in step (3).

- (2) The invariant mass squared of the colliding $(e^+e^-)^*$ pair is reduced to $s^* = s(1 - \epsilon_0)$; and the incoming particles' momenta are maintained along the beam axis with the appropriate scaling.
- (3) A phase space channel from the main generator is called to provide a four fermion final state X , in the $(e^+e^-)^*$ center of mass.
- (4) A second photon γ_{FSR} is generated in the $(e^+e^-)^*$ center of mass according to Eq.(70) by the second sub-generator FINALG. Following the procedure of section 5.2, the final state kinematics is modified to ensure $(k_{e^+} + k_{e^-})^* = (k_X + k_{\gamma_{\text{FSR}}})^*$. Step (4) is repeated up to 4 times to give up to 4 final-state photons.
- (5) The final state $(X + \text{FSR photons})^*$ is boosted back to the initial frame to ensure $k_{e^+} + k_{e^-} = k_X + k_{\gamma_{\text{ISR}}} + k_{\gamma_{\text{FSR}}}$.

- (6) The kinematical acceptance routine is called by the main generator. If the final state satisfies the requirements, step (7) is performed, otherwise the process is restarted at step (1).
- (7) The complete matrix element is calculated using the kinematics generated at step (3), *i.e.*, after the initial-state radiation but before final-state radiation. (This is the most cpu-time consuming part of the procedure, hence delayed to the end in **FERMISV**).
- (8) Finally, the total weight of the event is determined, and the algorithm is repeated from step (1) to produce additional events.

Before finishing this discussion on the implementation of radiative effects we want to stress once more that our approach to them is crude. To wit, for hard emitted photons the kinematics of the fermions are distorted with possibly non-negligible but generally unknown effects. In particular in the bremsstrahlung and multipheripheral channels where angular distributions tend to be very peaked in the forward direction, a recoil of the system against a photon's transverse momentum may change the electron angles with a sizable effect on the cross section. However, we hold that our current procedure must be considered as leading to a more-or-less useful estimate of the global radiative effects until (if ever) the complete higher-order calculation is finished.

6 Results

In this section we give some results obtained from running **FERMISV**. Since this generator has many variable parameters, these results are examples of the more interesting numbers we have obtained, but they are not an exhaustive compilation of all the processes accessible with this generator. We first present results obtained from runs with $\sqrt{s} = m_Z = 91.2$ GeV, which are relevant to present LEP experiments, followed by results from runs at $\sqrt{s} = 200$ GeV, which will be relevant for these experiments in a few year's time.

At current LEP experiments there is an ongoing search for the Standard Model Higgs boson. A combined limit of 57 GeV has been reported [18]. Since the production mechanism $e^+e^- \rightarrow H f \bar{f} \rightarrow F \bar{F} f \bar{f}$ leaves four fermions in the final state, it is important to understand the potential backgrounds coming from the diagrams considered in this paper. To this end we have plotted the cross sections relevant to the Higgs search in the $H\mu^+\mu^-$ channel in fig. 4. This channel contributes only about one sixth as much as the $H\nu\bar{\nu}$ channel in setting the limit, but it has a larger background from direct four fermion processes. Naturally, the $H e^+e^-$ signal will also have this type of background. Some minimal experimental cuts have been applied:

- The Higgs and the muons are more than 15 degrees from the beam axis.
- $m_{\mu^+\mu^-} > 5$ GeV.

Figure 4: Cross sections relevant to the Higgs search via $e^+e^- \rightarrow H\mu^+\mu^-$ at $\sqrt{s} = m_Z$. The effects of initial-state radiation have been included, but no QCD corrections have been made to the four fermion cross sections. 26

At these masses the Higgs particle decays into quark pairs, mostly $b\bar{b}$, so the comparable direct 4-fermion production is the reaction $e^+e^- \rightarrow q\bar{q}\mu^+\mu^-$. For these events the cuts as spelled out above were applied, except that the two quarks were each required to be in the central detector. In fig. 4 we have also plotted the Higgs cross section divided by a factor of two (dashed line) to fairly compare to the differential cross section obtained in four fermion events, *i.e.*, the Higgs signal will show up in events whose reconstructed invariant mass is smeared out by the resolution, here assumed to be 2 GeV as found in [19]. Thus the ratio between the heights of the dashed curve (effective Higgs signal) and the dotted curve (four fermion background) gives the signal-to-noise ratio. We see that this background is well below the signal except at masses so high that the $Z \rightarrow q\bar{q}$ background will undoubtedly be dominant.

To illustrate the large peaks in the cross section and how the different diagram groups contribute to the overall distributions, we have made several characteristic plots with the following cuts applied to the generator:

- $m_{e^+e^-} > 0.05$ GeV;
- ISR turned on, with the cut-off $E_{\gamma\text{ISR}}/E_{\text{beam}} = 0.8$;
- All fermions must be in the central detector (15 degrees).

These cuts and the Z pole heavily favor the annihilation group, which dominates the total cross section when $\sqrt{s} = m_Z$. The other three groups prefer to throw particles down the beam pipes. Figures 5 and 6 show the differential cross sections as functions of $m_{e^+e^-}$ and $m_{\mu^+\mu^-}$, respectively. Whereas only the annihilation and conversion groups have pronounced peaks in $d\sigma/dm_{e^+e^-}$, virtually all four groups have a peaking structure in $d\sigma/dm_{\mu^+\mu^-}$, with the conversion diagrams being especially sharp. Fig. 7 shows the distribution $d\sigma/d|\cos\theta_{e^+}|$ for the bremsstrahlung, multiperipheral, and all groups combined. In groups with t -channel exchanged bosons (bremsstrahlung and multiperipheral), the beam particles tend to go forward in an attempt to keep the Q^2 of the exchange photon low, producing the pronounced peak in this distribution.

The result of using the initial-state radiator formalism of section 4 is shown in fig. 8, where the correction introduced to the Born order diagram is plotted as a function of the center-of-mass energy for $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ with our standard cuts listed above. The figure separates all the diagram groups, and relegates the annihilation group to its own plot because its correction becomes so large. Fortunately, from the point-of-view of the overall cross section, just as the correction to annihilation is becoming very large, the overall contribution from this group is decreasing. The dependence of the total cross section error on the accuracy of our ISR treatment is therefore weak, and the total radiative correction does not exceed $\pm 23\%$ in the range of energies studied.

Figure 5: Differential cross section $d\sigma(e^+e^- \rightarrow e^+e^-\mu^+\mu^-)/dm_{e^+e^-}$ versus $m_{e^+e^-}$ at $\sqrt{s} = m_Z$. ISR has been included.

Figure 6: Differential cross section $d\sigma(e^+e^- \rightarrow e^+e^-\mu^+\mu^-)/dm_{\mu^+\mu^-}$ versus $m_{\mu^+\mu^-}$ at $\sqrt{s} = m_Z$. ISR has been included.

Figure 7: Differential cross section $d\sigma(e^+e^- \rightarrow e^+e^-\mu^+\mu^-)/d|\cos\theta_{e+}|$ versus $|\cos\theta_{e+}|$ at $\sqrt{s} = m_Z$. ISR has been included.

Figure 8: Correction to the total cross section of $\sigma(e^+e^- \rightarrow e^+e^-\mu^+\mu^-)$ due to initial-state radiation as a function of the center-of-mass energy $\sqrt{s}$. The factor δ relates the Born and the ISR-corrected cross section by the relation $\sigma_{\text{ISR}} = \sigma_{\text{Born}}(1 + \delta)$.

We have also made extensive runs at the Z pole of almost every possibly interesting final state producible with the generator, ranging from the interesting ($ee\mu\mu$) to the very irrelevant, and very rare ($\nu_\mu\nu_\mu\nu_\tau\nu_\tau$)! Tables of cross sections for these runs are in tables 3 (ISR included) and 4 (no ISR). In addition to serving as references for users of the generator, some of these numbers are interesting in their own right. For instance we see that the ratio of Xdd to Xuu cross sections is not always one fourth, indicating that the $Zq\bar{q}$ vertex plays a role in these cross sections and should not be ignored.

The efficiency in the last column is defined as

$$\text{efficiency} = \frac{\langle w \rangle}{w_{\max}} , \quad (73)$$

with w defined in Eq.(10): the fraction of internally generated events which could be written out as unweighted events in the limit of high statistics. This number of course depends on the generator weights α_i (cf. Eq. (12)) which are used. A set of reasonable α_i are provided in **FERMISV** for the more common generator applications of $ee\mu\mu$ and $eeee$ final states generated with the cuts listed below. The other channels use the $ee\mu\mu$ values and could be further optimized. For the tables we applied these settings and cuts:

- $m_{f\bar{f}} > 0.05 \text{ GeV}$, $f \neq q$;
- $m_{q\bar{q}} > 0.27 \text{ GeV}$;
- no QCD corrections have been applied;
- ISR turned on, with the cut-off $E_{\gamma\text{ISR}}/E_{\text{beam}} = 0.8$ (table 3 only);
- all fermions must be in the central detector (15 degrees), except neutrinos;
- $m_Z = 91.2 \text{ GeV}$, $\Gamma_Z = 2.497 \text{ GeV}$, $\sin^2 \theta_W = 0.2291$;
- $\alpha_{\text{EM}}(m_Z) = 1/128$ at each Z boson vertices.

Though **FERMISV** has not yet been optimized for energies above m_Z , it can still be used for this regime, if one accepts lower operating efficiencies. With this in mind, we have produced table 5 with the same parameter settings as table 3, except

- $\sqrt{s} = 200 \text{ GeV}$
- $E_{\gamma\text{ISR}} < 0.9E_{\text{beam}}$.

The latter requirement loosens the range of ISR photon energies to allow for radiation down to the Z pole, a dominant process for many kinds of physics at this energy, as shown in Fig. 8. Indeed, it is precisely this phenomenon which lowers the efficiency of generation since the ISR generator of section 4 has not been modified to produce

channel	σ_{M}	σ_{B}	σ_{C}	σ_{A}	σ_{tot}	effic.(%)
$eeee$	$.034 \pm .002$	$.69 \pm .01$	$.152 \pm .003$	$1.02 \pm .01$	$1.93 \pm .02$	3.7
$ee\mu\mu$	$.024 \pm .001$	$.276 \pm .002$	$.193 \pm .001$	$1.413 \pm .004$	$1.93 \pm .01$	6.9
$ee\tau\tau$	$.024 \pm .001$	$.027 \pm .001$	$.150 \pm .002$	$.853 \pm .007$	$1.06 \pm .01$	8.1
$\mu\mu\mu\mu$	—	—	$.042 \pm .001$	$.450 \pm .003$	$.498 \pm .003$	9.8
$\mu\mu\tau\tau$	—	—	$.042 \pm .001$	$.427 \pm .003$	$.475 \pm .003$	10.5
$\tau\tau\tau\tau$	—	—	$1.26 \pm .01$ $\times 10^{-3}$	$.0436 \pm .0003$	$.0454 \pm .0003$	15.9
$eeuu$	$.015 \pm .001$	$.33 \pm .01$	$.55 \pm .01$	$2.05 \pm .02$	$2.97 \pm .02$	6.5
$eedd$	$.0008 \pm .0001$	$.083 \pm .002$	$.65 \pm .01$	$.61 \pm .01$	$1.32 \pm .01$	2.8
$eebb$	$5.8 \pm .2$ $\times 10^{-4}$	$2.46 \pm .04$ $\times 10^{-3}$	$.62 \pm .01$	$.300 \pm .003$	$.89 \pm .01$	2.7
$ee\nu_\mu\nu_\mu$	$2.40 \pm .04$ $\times 10^{-8}$	$1.27 \pm .04$ $\times 10^{-4}$	$.300 \pm .003$	$1.10 \pm .02$ $\times 10^{-4}$	$.299 \pm .003$	0.9
$\mu\mu uu$	—	—	$.184 \pm .003$	$1.22 \pm .01$	$1.42 \pm .01$	8.4
$\mu\mu dd$	—	—	$.169 \pm .003$	$.351 \pm .003$	$.514 \pm .005$	5.5
$\mu\mu bb$	—	—	$.153 \pm .002$	$.153 \pm .001$	$.298 \pm .003$	4.5
$\mu\mu\nu_\tau\nu_\tau$	—	—	$.073 \pm .001$	$1.08 \pm .02$ $\times 10^{-4}$	$.073 \pm .001$	0.9
$\nu_\mu\nu_\mu\nu_\tau\nu_\tau$	—	—	$5.8 \pm .1$ $\times 10^{-8}$	$6.0 \pm .1$ $\times 10^{-5}$	$6.0 \pm .1$ $\times 10^{-5}$	0.3

Table 3: Cross sections (in picobarns) at $\sqrt{s} = m_Z = 91.2$ GeV, $E_{\gamma_{\text{ISR}}} < 0.8E_{\text{beam}}$. The four-neutrino process is described as $\nu_\mu\nu_\mu\nu_\tau\nu_\tau$ because this final state has no lowest-order diagrams involving W 's, unlike the state $\nu_e\nu_e\nu_\mu\nu_\mu$. A similar comment holds, *e.g.*, for the $\mu\mu\nu_\tau\nu_\tau$ process: both $\mu\mu\nu_e\nu_e$ and $\mu\mu\nu_\mu\nu_\mu$ have contributions from diagrams with W 's. In practice, however, at LEP 1 energies the diagrams involving a space-like W propagator will make a small effect by the measure of the medium-precision physics of **FERMISV**. This conclusion is supported by the analysis of contributing diagrams in $e^+e^- \rightarrow \nu\bar{\nu}\gamma$ [21], where the same situation exists modulo internal boson conversion to a fermion pair. At LEP 2 these diagrams are expected to have a somewhat larger effect.

channel	σ_{M}	σ_{B}	σ_{C}	σ_{A}	σ_{tot}	effic.(%)
$eeee$	$.029 \pm .003$	$.68 \pm .01$	$.211 \pm .003$	$1.37 \pm .01$	$2.27 \pm .02$	5.3
$ee\mu\mu$	$.024 \pm .002$	$.28 \pm .01$	$.262 \pm .004$	$1.92 \pm .01$	$2.49 \pm .02$	8.3
$ee\tau\tau$	$.023 \pm .001$	$.027 \pm .001$	$.210 \pm .003$	$1.14 \pm .01$	$1.40 \pm .01$	8.3
$\mu\mu\mu\mu$	—	—	$.056 \pm .001$	$.619 \pm .003$	$.679 \pm .004$	13.5
$\mu\mu\tau\tau$	—	—	$.056 \pm .001$	$.579 \pm .003$	$.640 \pm .004$	13.4
$\tau\tau\tau\tau$	—	—	$.00130 \pm .00001$	$.0597 \pm .0003$	$.0612 \pm .0003$	22.4

Table 4: Selected cross sections (in picobarns) at $\sqrt{s} = m_Z = 91.2$ GeV, no ISR.

channel	σ_{M}	σ_{B}	σ_{C}	σ_{A}	σ_{tot}	effic.(%)
$eeee$	$6.5 \pm .2$	207 ± 1	$9.5 \pm .3$	$4.4 \pm .2$	197 ± 1	1.3
$ee\mu\mu$	$5.4 \pm .1$	93 ± 1	$15.9 \pm .3$	$6.0 \pm .3$	106 ± 1	0.6
$eeuu$	$3.4 \pm .1$	120 ± 1	36 ± 1	$7.1 \pm .3$	149 ± 1	0.3
$eedd$	$.22 \pm .01$	37 ± 1	36 ± 1	$1.8 \pm .1$	68 ± 2	0.1
$\mu\mu uu$	—	—	$30.1 \pm .3$	$4.6 \pm .2$	$34.7 \pm .4$	0.2
$\mu\mu dd$	—	—	$28.9 \pm .4$	$1.09 \pm .04$	$29.6 \pm .4$	0.3
$\mu\mu\nu_\tau\nu_\tau$	—	—	$13.2 \pm .2$	$.030 \pm 0.001$	$13.3 \pm .2$	0.2
$\nu_\mu\nu_\mu\nu_\tau\nu_\tau$	—	—	$9.0 \pm .1$	$.052 \pm 0.001$	$9.1 \pm .1$	1.9
$\nu_\mu\nu_\mu\nu_\mu\nu_\mu$	—	—	$4.50 \pm .03$	$.026 \pm 0.001$	$4.55 \pm .03$	0.01

Table 5: Cross sections (in femtobarns) at $\sqrt{s} = 200$ GeV, $E_{\gamma_{\text{ISR}}} < 0.9E_{\text{beam}}$.

more of these sorts of high-weight events. One interesting difference from the generation at $\sqrt{s} = m_Z$ is that the effects of interference between diagram groups is more pronounced. For instance the bremsstrahlung cross section of the $eeee$ final state alone is greater than the total cross section.

We have also investigated the behaviour of $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ more fully at $\sqrt{s} = 200$ GeV. Fig. 9 shows the differential cross section in the plane of $(m_{e^+e^-}, m_{\mu^+\mu^-})$, while Figures 10 and 11 show how the various diagram groups behave. These figures show very clearly the peaks due to the Z and photon propagators, or the combination of the two (conversion and annihilation diagrams).

The conversion diagrams are of particular interest because the subset of diagrams involving only Z propagators has been extensively studied previously [20]. This subset is responsible for the double Z mass peak in the middle of the top plot in Fig. 11. Although the other diagram groups do not peak in this area, it turns out their contribution is not negligible. In the restricted mass range

- $|m_{e^+e^-} - m_Z|, |m_{\mu^+\mu^-} - m_Z| < 10$ GeV,

we find that the conversion diagrams contribute only 84% of the total cross section of $2.31 \pm .02$ pb (with ISR turned on). The remainder comes from bremsstrahlung (11%) and multiperipheral (5%) diagrams. This implies that future studies in this regime which involving a final-state e^+e^- pair cannot afford to neglect the considerable effect of these extra diagrams. However, for final states without an e^+e^- pair, however, the annihilation diagrams can be safely ignored to an accuracy of 0.2%.

7 Conclusion

We have presented in detail a new 4-fermion electroweak generator, **FERMISV**. We use the method of multichannel phase space integration and standard spinor techniques. These techniques allow careful modelling of the many peaks in the matrix elements, so the net result is an efficient generator. Additionally, a leading-log treatment of initial- and final-state radiation has been included for the first time in a 4-fermion generator of this type.

While **FERMISV** is targeted mainly towards LEP experimenters looking for a fully tagged signature at LEP 1, its methodology is flexible enough to permit facile extension to other scenarios, *e.g.*, inclusion of WZ and WW diagrams, or optimization for the no-tag configurations; thus making this generator very relevant for LEP 2 and future e^+e^- linear collider studies. We have also shown that in order to have the correct phenomenology for the process $e^+e^- \rightarrow e^+e^-f\bar{f}$ at $\sqrt{s} = 200$ GeV, it is important to include the often-overlooked multiperipheral and bremsstrahlung diagram groups.

Figure 9: Differential cross section $d^2\sigma(e^+e^- \rightarrow e^+e^-\mu^+\mu^-)/dm_{e^+e^-}dm_{\mu^+\mu^-}$ at $\sqrt{s} = 200$ GeV. The bins are 10 GeV by 10 GeV.

Figure 10: Differential cross section $d^2\sigma(e^+e^- \rightarrow e^+e^-\mu^+\mu^-)/dm_{e^+e^-}dm_{\mu^+\mu^-}$ at $\sqrt{s} = 200$ GeV, for multiperipheral and bremsstrahlung diagrams.

Figure 11: Differential cross section $d^2\sigma(e^+e^- \rightarrow e^+e^-\mu^+\mu^-)/dm_{e^+e^-}dm_{\mu^+\mu^-}$ at $\sqrt{s} = 200$ GeV, for conversion and annihilation diagrams.

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