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Equations (46) and (51) in section 2.3 should read

$$K = 4(k_{\text{NS}}, k_{\text{S}}, k_{\text{G}}) \quad (46)$$

$$(k_{\text{NS}}^0, k_{\text{S}}^0, k_{\text{G}}^0) = (c_{\text{NS}}, c_{\text{S}}, 0) a(n) , \quad (51)$$

respectively, and the two eigenvalues $\gamma_{\pm}$ in (54) should be

$$\gamma_{\pm} = \frac{1}{2} \left\{ \gamma_{\text{qq}}^0 + \gamma_{\text{GG}}^0 \pm \sqrt{(\gamma_{\text{qq}}^0 + \gamma_{\text{GG}}^0)^2 - 4(\gamma_{\text{qq}}^0 \gamma_{\text{GG}}^0 - 2N_f \gamma_{\text{qG}}^0 \gamma_{\text{Gq}}^0)} \right\} . \quad (54)$$

Real and Virtual Photon Contributions to Inelastic ep Scattering

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Abstract

We investigate direct and resolved photon contributions to hard ep scattering at high energies. Some terms of the direct contribution are already included in the resolved photon one, described in terms of the photon structure functions. A suitable subtraction mechanism has to be found in order to avoid double counting. Inelastic ep scattering requires the knowledge of photon structure functions over a wide range in photon virtuality, in particular from approximately zero up to almost 10^5 GeV^2 at HERA. We analyse the real and virtual photon structure function and we comment on the structure of the photon in the intermediate region of low virtuality. We finally describe how ep cross sections can be obtained as functions of the virtuality of the exchanged photon.

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1. Introduction

HERA is an ideal machine to study the structure of the photon. An enormous range in photon virtuality will be probed, ranging from approximately $s = 10^5 \text{ GeV}^2$ down to almost zero. In this work we denote the photon virtuality by P^2 while we reserve the symbol Q^2 for the typical large scale in the ep processes object of this discussion. This scale coincides with P^2 only in the case of deep inelastic inclusive $ep \rightarrow eX$ scattering.

The class of ep processes we consider here is the rather large class of processes dominated by the exchange of a single photon. Examples are the production of massive gauge bosons, of heavy quarks or more generically of jets, the large scale being in these cases the mass of the gauge boson, the high transverse mass of the heavy quark or the dijet invariant mass, respectively. These reactions share the feature of having the photon contribution to their cross sections split in the two non-independent parts, the so-called **direct** and **resolved** photon contributions. A correct handling of these two parts is required in order to avoid double countings in the evaluation of the differential and total cross sections.

In this and in the following sections we maintain the discussion as general as possible, but refer to the W^+ boson production as an example. The problem of matching the direct and resolved photon contributions arises in the case of gauge boson production already at the leading order in α_s , in contrast for example to the case of jets production, where the matching is required in the next-to-leading order. The matching problem is in fact a general one and the strategy we shall propose in this paper applies to all the above mentioned ep reactions, irrespective of the order in perturbation theory at which it arises.

For definiteness, we indicate these processes as

$$e(l) + p(p_p) \rightarrow e(l') + q(k) + X(p_X) \quad (1)$$

and we show them schematically in Fig. 1. In the case of W^+ production, for example, X

Figure 1: Typical diagram describing inelastic ep processes.

is the W^+ boson plus the proton remnant. The different contributions to the subprocess

$$\gamma^*(p) + p(p_p) \rightarrow q(k) + X(p_X) \quad (2)$$

are shown in Fig. 2 in this specific case. In what follows we use s to denote the center-of-mass

Figure 2: Photon-exchange diagrams contributing to W^+ production. On the left, are shown the diagrams for the direct process, on the right, the resolved process. The wavy, dashed and continuous lines indicate respectively photons, W bosons and quarks.

(CM) energy squared, $s = (l + p_p)^2$ and W^2 for the γp CM energy squared, $W^2 = (p + p_p)^2$.

For massless quarks, the cross sections for the processes (1) are dominated by two large logarithmic integrations. One logarithm originates from the propagator of the exchanged photon, $P^2 = -p^2 = -(l - l')^2$. The other one stems from the propagator of the light quark in the u channel, $u = (p_p - p_X)^2$. The differential cross sections for the reactions (1) have in general the form:

$$\frac{d\sigma_{ep}[\text{direct}]}{dW^2} = \frac{1}{(W^2)^2} \int_{P_{\min}^2}^{P_{\max}^2} \frac{dP^2}{P^2} \int_{|u|_{\min}}^{|u|_{\max}} \frac{d|u|}{|u|} \left\{ A + B \frac{W^2}{W^2 + P^2} + C \frac{|u|}{W^2} + \dots \right\}, \quad (3)$$

where A , B , C , etc., are functions independent of P^2 and u . We have expressed the cross section (3) in such a way as to show the dominant behaviour at small P^2 and $|u|$. At large values of the photon virtuality P^2 and/or of the variable $|u|$ the non-leading terms proportional to B , C , etc., in (3) become important and may even exceed the logarithmic terms. In particular, for the variable P^2 , this shows the existence of a dynamical cut-off in the dP^2/P^2 integration, $P^2 \leq \mathcal{O}(Q^2)$.

In the interesting case $m_p^2 \ll W^2 \ll s$, the P^2 integration limits are given by:

$$m_e^2 \left(\frac{W^2}{s} \right)^2 \leq P^2 \leq s - W^2. \quad (4)$$

Typically, it is $P^2 \ll W^2$ and the integration limits for $|u|$ are then:

$$P^2 \sim P^2 \frac{Q^2}{W^2} \leq |u| \leq W^2 - Q^2 + P^2 \sim Q^2. \quad (5)$$

The scale for the cross section is set by Q^2 , since it is typically $W^2 \sim Q^2$. A closer inspection of (3) yields the following observations:

- The u integration $\int du/u$ yields a potentially large term: $\log(Q^2/P^2)$. For values of P^2 small enough, the lower integration limit $|u|_{\min}$ can become much smaller than the QCD scale parameter Λ . This indicates the limit of applicability of a purely perturbative calculation.
- The large term $\log(Q^2/P^2)$ gets contributions from any order in the strong coupling α_s . Multiple gluon radiation, in fact, brings in a series of extra quark propagators. Each of them yields such a logarithmic term compensating the additional powers of α_s due to the new gluon-quark-antiquark vertex ($\alpha_s(Q^2) \log(Q^2) = \mathcal{O}(1)$).

These features are reminiscent of the peculiar behaviour of the photon at zero or moderate virtuality.

At sufficiently high virtuality the photon is essentially a point-like object, coupling directly to the quarks within the proton, while a real photon behaves rather like a meson. The total γp cross section and the low energy e^+e^- annihilation into hadrons are indeed two well-known examples where the photon can be viewed as a hadron, as successfully described in the context of vector-meson dominance (VMD). At low virtuality one also expects contributions from the partonic constituents of the photons. These two contributions form together the so-called resolved photon contribution, to be distinguished from the direct one, given in (3), where the photon is treated as an elementary particle. This decomposes any ep cross section mediated by photon exchange into two pieces:

$$\sigma_{ep} = \sigma_{ep}[\text{direct}] + \sigma_{ep}[\text{resolved}] . \quad (6)$$

The question arising immediately at this point is how this sum has to be performed. In the specific case of W^+ production it has been addressed in refs. [1–3].

The resolved photon contribution is given in terms of the **photon structure functions (PHSFs)**[4]. Working in a physical gauge, the PHSFs can be shown to be the sum of the so-called ladder diagrams, starting with the point-like γqq vertex or a hadronic input, as shown in Fig. 3[†]. On the other hand, the direct contribution to the cross section is obtained from all the Feynman diagrams of the order in α_s at which one is working. Thus, certain terms of the direct contribution are already included in the resolved one. We denote them as $\sigma_{ep}[\text{dir-sing}]$ since they are the terms responsible for the $|u|$ pole in (3), while we indicate the remaining ones by $\sigma_{ep}[\text{dir-fin}]$. In order to avoid double counting, $\sigma_{ep}[\text{dir-sing}]$ must be subtracted from the simple sum of $\sigma_{ep}[\text{direct}]$ and $\sigma_{ep}[\text{resolved}]$. This roughly amounts to dropping the term A in (3). We say “roughly” since the mismatch in the kinematics of the direct and resolved contributions leaves a small remainder, which we call $\sigma_{ep}[\text{rest}]$.

The total cross section is then given by the sum

$$\sigma_{ep}[\text{total}] = \sigma_{ep}[\text{dir-fin}] + \sigma_{ep}[\text{resolved}] + \sigma_{ep}[\text{rest}] . \quad (7)$$

[†]The role of the photon exchanged in the ep processes considered here is played in these diagrams by the target photon with virtuality P^2 .

Figure 3: Ladder diagrams describing the Q^2 evolution of the distribution of the quark in the photon. A point-like coupling of the target photon to quarks is assumed in a), a hadron-like one in b).

This procedure can be applied to photoproduction ($P^2 = 0$) cross sections or to low- P^2 ep cross sections, i.e. to reactions where P^2 is either measured or integrated over a range well below Λ^2 . At HERA, for example, low- P^2 events can be tagged through the luminosity detector by measuring the energy and the angle of the scattered electron. In these cases, existing parametrizations for the PHSFs can be used to calculate $\sigma_{ep}[\text{resolved}]$. The large terms $\log(Q^2/P^2)$ contributing to the cross section at every order in α_S , are then resummed by the PHSFs. Moreover the use of these structure functions spares us, in principle, from having to apply a not unique lower cut to the integrations du/u or $dp_\perp^2/p_\perp^2$.

The procedure (7), however, cannot be extended to the calculation of P^2 -integrated ep cross sections, since the PHSFs are not known as functions of P^2 over the range that is needed at HERA, i.e. from $P^2 \ll \Lambda^2$ up to $P^2 \sim 10^5 \text{ GeV}^2$. Theoretically only the region of the real PHSFs [5] with $P^2 \ll \Lambda^2 \ll Q^2$, and the region of the virtual PHSFs [6] with $\Lambda^2 \ll P^2 \ll Q^2$ are known. For the intermediate region of $P^2 \sim \Lambda^2$, progress may come from making use of the analytic behaviour of the PHSFs in the P^2 plane. It is also intuitive that for high enough values of P^2 the direct contribution may be sufficient to calculate total and differential cross sections. In order to decide at which value of P^2 this approximation can be used and how the intermediate P^2 region can be dealt with, a review of the current knowledge of the PHSFs is needed.

The paper is organized as follows. We devote section 2 to an analysis of the real and virtual PHSFs. In particular, in this and the next section, we restrict ourselves to the case of the transverse PHSF. In section 3 we discuss the P^2 dependence of this PHSF and we argue on the type of approximations one can perform in the range of P^2 where this structure function is not known. These two sections can be skipped by the reader interested only in the discussion of the method to adopt in the calculation of ep cross sections. The strategy we propose in this respect is outlined in section 4. Finally, we summarize our results in section 5.

2. Real and virtual photon structure functions

The PHSFs were introduced and studied experimentally in $\gamma\gamma$ collisions at high-energy e^+e^- colliders [4]. The main interest has been devoted so far to two kinematical regions in the momenta p and q of the two exchanged photons*. In both regions the so-called “probing photon” is very far off shell, $|q^2| = Q^2 \rightarrow \infty$. On the contrary, the “target photon” is in one region very close to the mass shell, $|p^2| = P^2 \approx 0$, but has a large virtuality P^2 in the second one, although still smaller than Q^2 . We shall concentrate in the following on $F_2^\gamma(x, Q^2, P^2)$, where $x = Q^2/(2p \cdot q)$,[†] since this is the dominant PHSF in the regime $P^2 \approx 0$ as well as in the case of an off-shell target photon, provided P^2 is not too close to Q^2 . We shall also refer to $F_2^\gamma(x, Q^2, P^2)$ by the generic term PHSF, unless otherwise stated.

Theoretically, several approaches for the treatment of the PHSF exist. One uses the operator-product expansion (OPE) plus renormalization group (RG) considerations [8]. Another approach is based on evolution equations of the type used for the hadronic structure functions, i.e. the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equations [9]. A third method is based on the formulation of the perturbative QCD evolution as a sum over the so-called ladder diagrams [10]. The last two methods allow perhaps more physical insight but do not provide any information on the boundary conditions to be applied on the PHSF solutions.

In what follows we shall briefly review these three methods with the aim of setting our conventions and of clarifying some important aspect of the PHSF. In particular, we shall treat the two kinematical regions $P^2 \ll \Lambda^2 \ll Q^2$ and $\Lambda^2 \ll P^2 \ll Q^2$ in a way as unified as possible, pointing out the different features arising from the two different configurations of the target photon. In section 2.1 we shall give the QPM contribution to the PHSF. In section 2.2 we shall discuss the **non-singlet** distribution function $\Delta^\gamma(x, Q^2, P^2)$ as obtained in the evolution equation approach and in the diagrammatic one. The non-singlet, **singlet** and **gluon** distribution functions are then derived in the OPE approach in section 2.3 and the boundary conditions obtained in this formalism are also discussed.

Before proceeding, we shall set some definitions. The singlet and non-singlet distribution functions can be expressed in terms of the quark and antiquark distribution as:

$$\begin{aligned}\Sigma^\gamma(x, Q^2, P^2) &= \sum_i \left[q_i^\gamma(x, Q^2, P^2) + \bar{q}_i^\gamma(x, Q^2, P^2) \right] \\ \Delta^\gamma(x, Q^2, P^2) &= c_{\text{NS}} \left[q_u^\gamma(x, Q^2, P^2) - q_d^\gamma(x, Q^2, P^2) \right].\end{aligned}\quad (8)$$

The constant c_{NS} is given by

$$c_{\text{NS}} = 6N_f \left(\langle e^4 \rangle - \langle e^2 \rangle^2 \right) \quad (9)$$

where $\langle e^k \rangle = (\sum_f e_f^k)/N_f$ and N_f is the number of (massless) flavours. In the approximation of all up-like and down-like flavours separately degenerate, and quarks and antiquarks also

*An experimental investigation of the intermediate region of P^2 was performed by the PLUTO Collaboration [7]. The events used for this analysis, with an average Q^2 and P^2 of 5 and .35 GeV² respectively, have allowed the study of the P^2 dependence of the PHSF up to 0.8 GeV².

[†]Throughout this paper, power-like corrections of the type $(P^2/Q^2)^k$ are neglected, since P^2 is always assumed $\ll Q^2$.

degenerate, $F_2^\gamma(x, Q^2, P^2)$ can be decomposed as follows[†]:

$$\begin{aligned} \frac{1}{x} F_2^\gamma(x, Q^2, P^2) &= \chi_S(x, Q^2) * \Sigma^\gamma(x, Q^2, P^2) + \chi_{NS}(x, Q^2) * \Delta^\gamma(x, Q^2, P^2) \\ &+ \chi_G(x, Q^2) * G^\gamma(x, Q^2, P^2) + \chi_\Gamma(x), \end{aligned} \quad (10)$$

where G^γ is the **gluon** distribution function and the symbol $*$ denotes the convolution defined as

$$f(x) * g(x) = \int_x^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y). \quad (11)$$

The first three coefficients χ_i in (10) have a perturbative expansion in α_S . At the leading order, they are

$$\chi_{NS}^0 = 1, \quad \chi_S^0 = \langle e^2 \rangle, \quad \chi_G^0 = 0. \quad (12)$$

The last coefficient χ_Γ will be discussed in section 2.4. If we neglect it, F_2^γ has in leading order the conventional form

$$F_2^\gamma(x, Q^2, P^2) = x \sum_i e_i^2 \left[q_i^\gamma(x, Q^2, P^2) + \bar{q}_i^\gamma(x, Q^2, P^2) \right] \quad (13)$$

and the quark distribution can be expressed as

$$q_i^\gamma(x, Q^2, P^2) = \frac{1}{2N_f} \left\{ \Sigma^\gamma(x, Q^2, P^2) + \frac{e_{q_i}^2 - \langle e^2 \rangle}{\langle e^4 \rangle - \langle e^2 \rangle^2} \Delta^\gamma(x, Q^2, P^2) \right\}. \quad (14)$$

2.1. Parton model results

In the QPM, i.e. at the zeroth order in α_S , or Born approximation, the PHSF is calculated from $\gamma\gamma$ fusion into a quark-antiquark pair. In the limit $m_i^2 \ll P^2 \ll Q^2$, where m_i are the quark masses, one obtains [11]

$$q_i^\gamma(x, Q^2, P^2)|_{\text{QPM}} = 3e_{q_i}^2 \frac{\alpha}{2\pi} \left\{ a(x) \log \left(\frac{Q^2}{x^2 P^2} \right) - 2 + 6x(1-x) \right\}, \quad (15)$$

with $a(x)$ given by

$$a(x) = [x^2 + (1-x)^2]. \quad (16)$$

The leading part of $q_i^\gamma(x, Q^2, P^2)|_{\text{QPM}}$ is linear in $\log(Q^2)$,

$$q_i^\gamma(x, Q^2, P^2)|_{\text{QPM}} = 3e_{q_i}^2 \frac{\alpha}{2\pi} a(x) \log \left(\frac{Q^2}{P^2} \right) \quad (17)$$

and arises from the integration

$$\int_{P^2}^{Q^2} \frac{dp_\perp^2}{p_\perp^2}. \quad (18)$$

The case of a real photon, $P^2 \approx 0$, is only known in the limit $P^2 \ll m_i^2 \ll Q^2$ [11]:

$$q_i^\gamma(x, Q^2, P^2)|_{\text{QPM}} = 3e_{q_i}^2 \frac{\alpha}{2\pi} \left\{ a(x) \log \left(\frac{Q^2}{m_i^2} \frac{1-x}{x} \right) - 1 + 8x(1-x) \right\}. \quad (19)$$

[†]In (10), a particular scheme chosen to carry out mass factorization and coupling-constant renormalization is assumed.

Equation (19) is suited for the contribution of heavy quarks, but becomes ill-defined in the limit of massless quarks. On the other hand, the lower integration limit in (18) cannot be pushed down to $P^2 = 0$, but the $p_\perp^2$ integration has to be cut-off at a scale of order 1 GeV^2 .

We define here also

$$\Delta^\gamma(x, Q^2, P^2)|_{\text{QPM}} = \frac{\alpha}{2\pi} c_{\text{NS}} a(x) \log\left(\frac{Q^2}{P^2}\right) \quad (20)$$

$$\Sigma^\gamma(x, Q^2, P^2)|_{\text{QPM}} = \frac{\alpha}{2\pi} c_S a(x) \log\left(\frac{Q^2}{P^2}\right), \quad (21)$$

where the coefficient c_S is

$$c_S = 6N_f \langle e^2 \rangle. \quad (22)$$

2.2. The evolution equation and diagrammatic approach

Within perturbative QCD, the PHSF can be obtained from the solution of the DGLAP equations. If we restrict ourselves to the leading order in α_S , the evolution equation for the non-singlet distribution function is given in moment space by

$$\frac{\partial}{\partial \log Q^2} \Delta^\gamma(n, Q^2, \mu^2) = \frac{\alpha}{2\pi} k_{\text{NS}}^0(n) + \frac{\alpha_S(Q^2)}{2\pi} P_{\text{NS}}^0(n) \Delta^\gamma(n, Q^2, \mu^2) \quad (23)$$

where the inhomogeneous term k_{NS}^0 and the splitting kernel P_{NS}^0 are

$$k_{\text{NS}}^0(n) \equiv c_{\text{NS}} a(n) = c_{\text{NS}} \int_0^1 dx x^{n-1} a(x) = c_{\text{NS}} \frac{n^2 + n + 2}{n(n+1)(n+2)} \quad (24)$$

$$P_{\text{NS}}^0(n) \equiv P_{\text{qq}}^0(n) = \int_0^1 dx x^{n-1} P_{\text{qq}}^0(x) = \frac{4}{3} \left[-\frac{1}{2} + \frac{1}{n(n+1)} - 2 \sum_{j=2}^n \frac{1}{j} \right]. \quad (25)$$

The evolution equation (23) for the PHSF differs from the equation for a hadronic structure function by the presence of the inhomogeneous term k_{NS}^0 proportional to the electromagnetic coupling α . This term arises from the fact that photons, unlike hadrons, can produce partons of large $p_\perp$ through their direct coupling to quark-antiquark pairs. The QPM result (20) is in fact the solution of (23) for $\alpha_S = 0$ and $\mu^2 = P^2$.

The explicit form of the non-singlet distribution function can be obtained by solving the differential equation (23). The solution of this equation requires the specification of a boundary condition at an initial scale μ^2 . The dependence on this scale has been explicitly shown in (23). We evolve $\Delta^\gamma(n, Q^2, \mu^2)$ at most down to the square momentum of the target photon. Therefore, we take the scale μ^2 of $\mathcal{O}(P^2)$ in the case of a virtual target photon, while we would like to push it to Λ^2 in the case of an on-shell target.

In the following we choose to “build up” the solution of (23) by summing the ladder diagrams depicted in Fig. 3. We start our discussion with the **virtual PHSF** and, for simplicity, we choose the scale μ^2 exactly equal to P^2 .

Starting from the QPM point-like interaction $\gamma^* + \gamma^* \rightarrow q + \bar{q}$ one can obtain a solution of (23) summing up the radiated gluons as shown in Fig. 3a. The homogeneous term in

the left-hand side of (23) accounts for this multiple gluon emission, as in the case of the hadronic structure functions. We obtain:

$$\begin{aligned} \Delta^\gamma(n, Q^2, P^2) = & \frac{\alpha}{2\pi} k_{\text{NS}}^0(n) \left[\int_{P^2}^{Q^2} \frac{d\mu_1^2}{\mu_1^2} \frac{\alpha_S(\mu_1^2)}{2\pi} P_{\text{NS}}^0(n) \int_{P^2}^{\mu_1^2} \frac{d\mu_2^2}{\mu_2^2} \frac{\alpha_S(\mu_2^2)}{2\pi} P_{\text{NS}}^0(n) \int_{P^2}^{\mu_2^2} \frac{d\mu_3^2}{\mu_3^2} \frac{\alpha_S(\mu_3^2)}{2\pi} P_{\text{NS}}^0(n) \right. \\ & \left. + \int_{P^2}^{Q^2} \frac{d\mu_1^2}{\mu_1^2} \frac{\alpha_S(\mu_1^2)}{2\pi} P_{\text{NS}}^0(n) \int_{P^2}^{\mu_1^2} \frac{d\mu_2^2}{\mu_2^2} \frac{\alpha_S(\mu_2^2)}{2\pi} P_{\text{NS}}^0(n) \int_{P^2}^{\mu_2^2} \frac{d\mu_3^2}{\mu_3^2} \frac{\alpha_S(\mu_3^2)}{2\pi} P_{\text{NS}}^0(n) + \dots \right]. \quad (26) \end{aligned}$$

The integrals in (26) are of the following type:

$$\int_{P^2}^{Q^2} \frac{d\mu_1^2}{\mu_1^2} \frac{\alpha_S(\mu_1^2)}{2\pi} P_{\text{NS}}^0(n) \int_{P^2}^{\mu_1^2} \frac{d\mu_2^2}{\mu_2^2} \frac{\alpha_S(\mu_2^2)}{2\pi} P_{\text{NS}}^0(n) = d_{\text{NS}}(n) \left\{ \log \left(\frac{Q^2}{\Lambda^2} \right) - \log \left(\frac{P^2}{\Lambda^2} \right) \left[1 + \log \left(\frac{\alpha_S(P^2)}{\alpha_S(Q^2)} \right) \right] \right\}, \quad (27)$$

where the definitions $d_{\text{NS}}(n) \equiv d_{\text{qq}}(n) = (2/\beta_0) P_{\text{NS}}^0(n)$ and $\alpha_S(Q^2)^{-1} = \beta_0 \log(Q^2/\Lambda^2)/4\pi$ have been used. We shall drop here and in the following the n dependence of d_{NS} in order to allow for more compact forms of the formulas displayed. The sum in (26) then yields

$$\Delta^\gamma(n, Q^2, P^2) = \frac{\alpha}{2\pi} k_{\text{NS}}^0(n) \log \left(\frac{Q^2}{\Lambda^2} \right) \frac{1}{1-d_{\text{NS}}} \left\{ 1 - \left(\frac{\alpha_S(Q^2)}{\alpha_S(P^2)} \right)^{1-d_{\text{NS}}} \right\}. \quad (28)$$

Several remarks are in order. To begin with, the solution (28) can be written as the sum of two terms:

$$\Delta^\gamma(n, Q^2, P^2) = \frac{A(n)}{\alpha_S(Q^2)} + \frac{B(n)}{\alpha_S(P^2)} \left(\frac{\alpha_S(Q^2)}{\alpha_S(P^2)} \right)^{-d_{\text{NS}}} \equiv \Delta^\gamma(n, Q^2)|_{\text{pt}} + \Delta^\gamma(n, Q^2, P^2)|_{\text{had}}. \quad (29)$$

The first term has the $\log(Q^2)$ behaviour of the QPM result and is usually called point-like contribution in the literature, while the second term shows the Q^2 dependence typical of the hadron structure function. Strictly speaking, however, both terms arise from the point-like coupling of the photon to quarks.

Secondly, for $Q^2 \rightarrow \infty$ at fixed P^2 , the solution (28) has the asymptotic form

$$\Delta_{\text{as}}^\gamma(n, Q^2) = \frac{\alpha}{2\pi} k_{\text{NS}}^0(n) \log \left(\frac{Q^2}{\Lambda^2} \right) \frac{1}{1-d_{\text{NS}}}, \quad (30)$$

which coincides with the term denoted as $\Delta^\gamma(n, Q^2)|_{\text{pt}}$ in (29).

Finally, for $P^2 \rightarrow Q^2$ the solution (28) reduces to the QPM result (20).

An ordinary solution of (23) can be obtained starting with a hadronic input distribution $\Delta_0^\gamma(n, P^2)$ taken at the scale P^2 , as shown in Fig. 3b. Following the same procedure as that used for the point-like coupling of the photon to quarks, we obtain the expansion

$$\begin{aligned} \Delta^\gamma(n, Q^2, P^2) = & \Delta_0^\gamma(n, P^2) \left[1 + \int_{P^2}^{Q^2} \frac{d\mu_1^2}{\mu_1^2} \frac{\alpha_S(\mu_1^2)}{2\pi} P_{\text{NS}}^0(n) \right. \\ & \left. + \int_{P^2}^{Q^2} \frac{d\mu_1^2}{\mu_1^2} \frac{\alpha_S(\mu_1^2)}{2\pi} P_{\text{NS}}^0(n) \int_{P^2}^{\mu_1^2} \frac{d\mu_2^2}{\mu_2^2} \frac{\alpha_S(\mu_2^2)}{2\pi} P_{\text{NS}}^0(n) + \dots \right], \quad (31) \end{aligned}$$

which can be resummed to give the hadron-like solution

$$\Delta^\gamma(n, Q^2, P^2) = \Delta_0^\gamma(n, P^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(P^2)} \right)^{-d_{NS}}. \quad (32)$$

Once we name back P^2 as μ^2 , the sum of (28) and (32), which account for the point-like and hadron-like behaviour of the virtual target photon, gives:

$$\Delta^\gamma(n, Q^2, \mu^2) = \Delta_{as}^\gamma(n, Q^2) \left\{ 1 - \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{1-d_{NS}} \right\} + \Delta_0^\gamma(n, \mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{-d_{NS}}. \quad (33)$$

This is the general solution of the differential equation (23), provided the hadronic input distribution $\Delta_0^\gamma(n, \mu^2)$ is identified with the boundary condition to be imposed at the scale μ^2 . As for the value of this boundary condition, no information is provided by any of the two treatments of the PHSF so far outlined.

The solution (33) is decomposed in two parts, the asymptotic term rising with $\log(Q^2)$ and the two hadron-like contributions, coming from the point-like and hadronic coupling of the photon, exhibiting the characteristic scaling violation behaviour. The asymptotic term is the dominant one for not too small values of x . Moreover, the particular splitting of the two hadron-like contributions is the most suitable to show the spurious nature of the singularities affecting the asymptotic solution. The factor $1/(1-d_{NS}(n))$ diverges for $n \approx 0.31$ ($d_{NS}(0.31) = 1$)[‡]. However, the presence of the power $1-d_{NS}(n)$ in the first hadron-like term in (33) is exactly what is needed to completely regularize the full solution. In fact, in the limit $d_{NS} \rightarrow 1$ we have:

$$\lim_{d_{NS} \rightarrow 1} \frac{1}{1-d_{NS}} \left\{ 1 - \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{1-d_{NS}} \right\} = -\log \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right). \quad (34)$$

It is obvious that an artificial splitting of the hadronic and point-like terms reintroduces these spurious singularities in the PHSF.

We come now to discuss the **real PHSF**. We observe immediately that the limit $P^2 \rightarrow \Lambda^2$ in (33) is well defined only for the first term on the right-hand side, i.e. for the term coming from the point-like coupling of the photon to quarks. At variance with what happens in the case of the off-shell target photon, every order in α_S in (26) contributes in this case with the same strength, namely with a term linear in $\log(Q^2/\Lambda^2)$, since $\alpha_S(Q^2) \beta_0 \log(Q^2/\Lambda^2)/(4\pi) = 1$. The summation then simply yields

$$\begin{aligned} \Delta^\gamma(n, Q^2, P^2 = 0) \Big|_{pt} &= \frac{\alpha}{2\pi} k_{NS}^0(n) \log \left(\frac{Q^2}{\Lambda^2} \right) [1 + d_{NS}(n) + (d_{NS}(n))^2 + \dots] \\ &= \frac{\alpha}{2\pi} k_{NS}^0(n) \log \left(\frac{Q^2}{\Lambda^2} \right) \frac{1}{1-d_{NS}(n)} = \Delta_{as}^\gamma(n, Q^2), \end{aligned} \quad (35)$$

[‡]We recall that singularities in the moment space and in the x space are simply related according to:

$$f(n) \sim 1/(n - n_0) \implies f(x) \sim 1/x^{n_0}.$$

i.e. we recover the asymptotic solution already obtained in the case of the virtual target photon, but we lose the hadronic term generated by the photon point-like coupling. Moreover, the fact that the limit $\mu^2 \rightarrow \Lambda^2$ cannot be taken in (32) may be considered as a signal that the “hadronic” contributions are uncalculable in perturbative QCD in the case of an on-shell target. In the past, these contributions were estimated in the VMD approach, where photons are described as hadronic systems with the same quantum numbers (ρ, ω, ϕ) . Naively one could try to decompose the PHSF by summing the VMD contribution to the asymptotic solution (35) as follows:

$$\Delta^\gamma(n, Q^2, P^2 = 0) = \Delta_{\text{as}}^\gamma(n, Q^2) + \Delta^\gamma(n, Q^2, P^2 = 0)|_{\text{VMD}} . \quad (36)$$

Such an estimate can account only for some of the hadron-like terms. In particular, the regularization of the asymptotic solution singularities observed in (34) cannot be reproduced by the approximate solution (36). The neglect of the hadron-like terms regularizing these singularities is responsible for the appearance of non-physical, negative results at small x when parametrizations of the asymptotic solution of the PHSF are used [12].

The correct approach for the real PHSF is to stop the evolution at a scale μ^2 where $\alpha_s(Q^2)$ is still quite small, i.e. $\mu^2 \gg \Lambda^2$. The solution for the virtual target photon (33) can still be used in this case, provided the boundary condition $\Delta_0^\gamma(n, \mu^2)$ is fixed from data relative to the scattering of a photon of momentum squared μ^2 off a real or quasi-real one. Parametrizations of the real PHSF have been obtained in this way by several groups, among them Drees and Grassie (DG) [13], Abramowicz, Charchula and Levy (LAC1–LAC3) [14], and Gordon and Storrow [15].

The expression in (33) is then the general solution of the evolution equation (23) for virtual and real PHSF. However, the boundary condition is not fixed in the DGLAP approach and in the diagrammatic one in any of the two cases. We shall see in the OPE approach that it is in fact not calculable in the kinematical region $P^2 \ll \Lambda^2 \ll Q^2$, while it can be evaluated in the case of a virtual target photon provided the ratio P^2/Λ^2 is sufficiently large, i.e. in the kinematical region $\Lambda^2 \ll P^2 \ll Q^2$.

2.3. The OPE approach

In this formalism, the moments of the PHSF can be expressed as a linear combination of the matrix elements of all the relevant local operators evaluated between the target photon states:

$$F_2^\gamma(n, Q^2, P^2) = \sum_i C_n^i \left(\frac{Q^2}{\mu^2}, g^2(\mu), \alpha \right) \langle \gamma(P) | O_i^n(\mu) | \gamma(P) \rangle . \quad (37)$$

The sum in (37) extends to the photon operator $O_\gamma^n(\mu)$ and the fermion and gluon operators collectively denoted as $O_{\text{had}}^n(\mu)$. We indicate the gluon operator as $O_G^n(\mu)$ and the fermion ones as $O_S^n(\mu)$ and $O_{\text{NS}}^n(\mu)$, once they are split into a flavour-blind and flavour-dependent component, i.e. the singlet and non-singlet component.

The expansion (37) has the convenient feature of dividing the dependence on the two scales P^2 and Q^2 (with $P^2 \ll Q^2$) in the matrix elements of the local operators O_γ^n , O_{had}^n and the corresponding Wilson coefficients C_n^γ , C_n^{had} . The dynamic of the probing and the target photon is then separated, except for the link coming from the common scale μ . A pictorial representation of this splitting is given in Fig. 4.

Figure 4: Diagrammatic expression of the expansion (37). The dots indicates higher-order corrections.

Quark and gluon operators, if calculable, can be sandwiched between photon states only at $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\alpha_s)$, respectively, while the matrix element of O_γ^n is $\mathcal{O}(1)$. In contrast, the Wilson coefficient relative to O_γ^n is obviously of $\mathcal{O}(\alpha)$, while C_n^{had} is of $\mathcal{O}(1)$ (see the possible combination (2a) and (1b) in Fig. 4). If for simplicity we drop the moment n dependence, the matrix elements and the corresponding coefficients can then be expressed as

$$\mathcal{A}_\gamma\left(\frac{P^2}{\mu^2}\right) \equiv \langle \gamma(P) | O_\gamma(\mu) | \gamma(P) \rangle, \quad C^\gamma\left(\frac{Q^2}{\mu^2}, g^2(\mu), \alpha\right) \quad (38)$$

and

$$\mathcal{A}_{\text{had}}\left(\frac{P^2}{\mu^2}, g^2(\mu), \alpha\right) \equiv \langle \gamma(P) | O_h(\mu) | \gamma(P) \rangle, \quad C^{\text{had}}\left(\frac{Q^2}{\mu^2}, g^2(\mu)\right). \quad (39)$$

$\mathcal{A}_{\text{had}}$ and C^{had} are vectors containing the matrix elements and the coefficients relative to the non-singlet, singlet and gluon operators, respectively. Notice that no g^2 dependence is present in $\mathcal{A}_\gamma$ since higher orders in α_s for this matrix element can be obtained only at higher order in α .

The requirement for the PHSF to be independent of the scale μ immediately enforces the RG equation

$$\left\{ \left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \mathbf{1} - \begin{pmatrix} \gamma_{\text{had}}(g) & n(g) \\ K(g) & q(g) \end{pmatrix} \right\} \begin{pmatrix} C^{\text{had}}\left(\frac{Q^2}{\mu^2}, g^2\right) \\ C^\gamma\left(\frac{Q^2}{\mu^2}, g^2, \alpha\right) \end{pmatrix} = 0. \quad (40)$$

The matrix inside the curly bracket contains the anomalous dimension of the four operators and their mixings, with γ_{had} a 3×3 matrix, K and n a row and column three-vectors, respectively[†]. The entries in this matrix are in general dependent on g and α . The α dependence will be explicitly discussed later on. The running strong coupling constant g in (40) is always understood as depending on the scale μ .

[†]The notation used here follows the one set by E. Witten in the first of refs. [5]. Since then it has been widely accepted even though the change of symbols from γ to K, n and q may be somewhat confusing.

The integration of (40) between the values of the running strong coupling constant at the initial scale Q^2 and at the subtraction scale μ^2 yields:

$$\begin{aligned} \begin{pmatrix} C^{\text{had}}\left(\frac{Q^2}{\mu^2}, g^2(\mu)\right) \\ C^\gamma\left(\frac{Q^2}{\mu^2}, g^2(\mu), \alpha\right) \end{pmatrix} &= \begin{pmatrix} M & y \\ X & z \end{pmatrix} \begin{pmatrix} C^{\text{had}}(1, g^2(Q)) \\ C^\gamma(1, g^2(Q), \alpha) \end{pmatrix} \\ &\equiv \text{T exp} \int_{g(Q)}^{g(\mu)} \frac{dg'}{\beta(g')} \begin{pmatrix} \gamma_{\text{had}}(g') & n(g') \\ K(g') & q(g') \end{pmatrix} \begin{pmatrix} C^{\text{had}}(1, g^2(Q)) \\ C^\gamma(1, g^2(Q), \alpha) \end{pmatrix}. \end{aligned} \quad (41)$$

The T-ordered exponential of the integral of a matrix A is defined as follows:

$$\text{T exp} \int_{g(Q)}^{g(\mu)} dg' A(g') \equiv 1 + \int_{g(Q)}^{g(\mu)} dg' A(g') + \int_{g(Q)}^{g(\mu)} dg' A(g') \int_{g(Q)}^{g'} dg'' A(g'') + \dots \quad (42)$$

The coefficients C^γ and C^{had} , at Q^2 of $\mathcal{O}(\alpha)$ and $\mathcal{O}(1)$, respectively, still have to be evaluated to this same order in the electromagnetic interaction. This implies $y = n = 0$, while it is $z = 1$. We obtain

$$M\left(\frac{Q^2}{\mu^2}, g(\mu), g(Q)\right) = \text{T exp} \int_{g(Q)}^{g(\mu)} dg' \frac{\gamma_{\text{had}}(g')}{\beta(g')} \quad (43)$$

$$X\left(\frac{Q^2}{\mu^2}, g(\mu), g(Q), \alpha\right) = \int_{g(Q)}^{g(\mu)} dg' \frac{K(g', \alpha)}{\beta(g')} \text{T exp} \int_{g(Q)}^{g'} dg'' \frac{\gamma_{\text{had}}(g'')}{\beta(g'')} \quad (44)$$

For simplicity, we shall drop in the following the coupling constants dependence and the scale dependence in the anomalous dimensions, the beta function, M and X .

Since the non-singlet operator does not mix with the remaining hadronic operators, the matrix γ_{had} can be further decomposed as

$$\gamma_{\text{had}} = \begin{pmatrix} \gamma_{\text{NS}} & 0 \\ 0 & \gamma_{\{\text{SG}\}} \end{pmatrix} \quad \text{and} \quad \gamma_{\{\text{SG}\}} = \begin{pmatrix} \gamma_{\text{qq}} & 2N_f \gamma_{\text{qG}} \\ \gamma_{\text{Gq}} & \gamma_{\text{GG}} \end{pmatrix} \quad (45)$$

with $\gamma_{\{\text{SG}\}}$ the 2×2 matrix of anomalous dimensions for the singlet-gluon subsystem. We notice here that $\gamma_{\text{NS}} = \gamma_{\text{qq}}$. By splitting also the vector K in the three components

$$K = (k_{\text{NS}}, k_{\text{S}}, k_{\text{G}}) \quad , \quad (46)$$

the solution (41) gets the form

$$\begin{pmatrix} C^{\text{NS}}\left(\frac{Q^2}{\mu^2}, g^2(\mu)\right) \\ \begin{pmatrix} C^{\text{S}} \\ C^{\text{G}} \end{pmatrix}\left(\frac{Q^2}{\mu^2}, g^2(\mu)\right) \\ C^\gamma\left(\frac{Q^2}{\mu^2}, g^2(\mu), \alpha\right) \end{pmatrix} = \begin{pmatrix} M_{\text{NS}} & (0 \ 0) & 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{\{\text{SG}\}} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ X_{\text{NS}} & (X_{\text{S}}, X_{\text{G}}) & 1 \end{pmatrix} \begin{pmatrix} C^{\text{NS}}(1, g^2(Q)) \\ \begin{pmatrix} C^{\text{S}} \\ C^{\text{G}} \end{pmatrix}(1, g^2(Q)) \\ C^\gamma(1, g^2(Q), \alpha) \end{pmatrix}. \quad (47)$$

The left multiplication of (47) by the vector of matrix elements $(\mathcal{A}_{\text{NS}}, \mathcal{A}_{\text{S}}, \mathcal{A}_{\text{G}}, \mathcal{A}_\gamma)$ formally gives the solution for the PHSF F_2^T . The use of the Mellin-transformed decomposition (10), allows us to obtain the solution for the non-singlet, singlet and gluon distribution from (47):

$$\Delta^\gamma \chi_{\text{NS}} = [\mathcal{A}_{\text{NS}} M_{\text{NS}} + \mathcal{A}_\gamma X_{\text{NS}}] C^{\text{NS}}(1, \bar{g}^2) \quad (48)$$

$$(\Sigma^\gamma, G^\gamma) \begin{pmatrix} \chi_{\text{S}} \\ \chi_{\text{G}} \end{pmatrix} = [(\mathcal{A}_{\text{S}}, \mathcal{A}_{\text{G}}) M_{\{\text{SG}\}} + \mathcal{A}_\gamma (X_{\text{S}}, X_{\text{G}})] \begin{pmatrix} C^{\text{S}}(1, g^2(Q)) \\ C^{\text{G}}(1, g^2(Q)) \end{pmatrix}. \quad (49)$$

A comment on the remaining piece $C^\gamma(1, g^2(Q), \alpha) \mathcal{A}_\gamma$ in (47) will be made at the end of this subsection.

In order to calculate the matrices M_{NS} , $M_{\{\text{SG}\}}$ and the vectors X_{NS} , X_{S} and X_{G} , we expand all the elements of the reduced 2×2 anomalous dimension matrix and the β function in powers of α_s and we restrict ourselves to the leading order:

$$\beta = -\frac{g^3}{16\pi^2}\beta^0, \quad \gamma_i = \frac{g^2}{16\pi^2}\gamma_i^0, \quad k_i = -\frac{\alpha}{4\pi}k_i^0 \quad (i = \text{NS}, \text{S}, \text{G}). \quad (50)$$

In particular

$$(k_{\text{NS}}^0, k_{\text{S}}^0, k_{\text{G}}^0) = 4 (c_{\text{NS}}, c_{\text{S}}, 0) a(n). \quad (51)$$

where c_{NS} , c_{S} and $a(n)$ are already given in (9), (22) and (24) respectively. We then obtain

$$M_{\text{NS}}^0 = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{-d_{\text{NS}}} \quad (52)$$

$$X_{\text{NS}}^0 = \frac{\alpha}{\alpha_s(Q^2)} \frac{2k_{\text{NS}}^0}{\beta^0(1-d_{\text{NS}})} \left\{ 1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{1-d_{\text{NS}}} \right\} \quad (53)$$

where $d_{\text{NS}} = -\gamma_{\text{NS}}^0/(2\beta^0)$. We observe that (52) is exactly the scaling violating factor obtained in (32), while (53) obtained in (28) from the point-like coupling of the photon to quarks is here the coefficient of the matrix element for the photon operator O_γ . The term with hadronic behaviour in (53) is due to the mixing of the operator O_{NS} with O_γ .

The derivation of the singlet and gluon distribution functions follows the same lines and is only slightly more complicated by the fact that the two distributions are coupled. At leading order in α_s , the 2×2 matrix $\gamma_{\{\text{SG}\}}$ can be expressed as $\gamma_{\{\text{SG}\}}^0 = \gamma_+ P_+ + \gamma_- P_-$, where $\gamma_\pm$ are the eigenvalues of this matrix

$$\gamma_\pm = \frac{1}{2} \left\{ \gamma_{\text{qq}}^0 + \gamma_{\text{GG}}^0 \pm \sqrt{(\gamma_{\text{qq}}^0 + \gamma_{\text{GG}}^0)^2 - 4(\gamma_{\text{qG}}^0 \gamma_{\text{Gq}}^0 - \gamma_{\text{qG}}^0 \gamma_{\text{Gq}}^0)} \right\} \quad (54)$$

and $P_\pm$ the relative projection operators. We obtain

$$\begin{aligned} M_{\{\text{SG}\}}^0 &= P_+ \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{-d_+} + P_- \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{-d_-} \\ (X_{\text{S}}^0, X_{\text{G}}^0) &= \frac{\alpha}{\alpha_s(Q^2)} \frac{2(k_{\text{S}}^0, 0)}{\beta^0} \left\{ \frac{P_+}{1-d_+} \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{1-d_+} \right] + \frac{P_-}{1-d_-} \left[1 - \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{1-d_-} \right] \right\} \end{aligned} \quad (55)$$

where, as before, the definitions $d_{ij} = -\gamma_{ij}^0/2\beta_0$, $d_{\text{qG}} = -N_f \gamma_{\text{qG}}^0/\beta_0$ and $d_\pm = -\gamma_\pm/2\beta_0$ have been introduced.

We are left now with the calculation of the coefficients C^{had} and of the matrix elements $\mathcal{A}_{\text{had}}$, $\mathcal{A}_\gamma$. Of the three coefficients $C^{\text{had}}(1, g^2(Q))$, the ones corresponding to the two fermion operators, $C^{\text{NS}}(1, g^2(Q))$, $C^{\text{S}}(1, g^2(Q))$ are $\mathcal{O}(1)$, while $C^{\text{G}}(1, g^2(Q))$ starts only at $\mathcal{O}(\alpha_s)$. They are exactly the Mellin-transformed coefficients χ_{NS} , χ_{S} and χ_{G} in (10) explicitly given in the leading order in α_s in (12)**.

^{††}The anomalous dimensions γ_{ij} are related to splitting functions $P_{ij}(n)$ ($i, j = q, G$) by $P_{ij} = -\gamma_{ij}^0/4$.

**For details on their derivation, see [16]

As far as the calculation of the matrix elements is concerned, we can immediately observe that $\mathcal{A}_\gamma$ is trivially equal to one, independently of the value of P^2 . In contrast, the value of this momentum plays a crucial role for the determination of the hadronic matrix elements. For the case of a **virtual target** photon, they are calculable, of $\mathcal{O}(\alpha)$ in the electromagnetic coupling and $\mathcal{O}(1)$ in the strong coupling. They are given by:

$$\mathcal{A}_i^0 \left(\frac{P^2}{\mu^2}, g^2, \alpha \right) = \frac{\alpha}{4\pi} \left(-\frac{1}{2} k_i^0 \log \left(\frac{P^2}{\mu^2} \right) + c_i \mathcal{A} \right). \quad (56)$$

Their expression simplifies considerably if the renormalization scale μ^2 is chosen to be equal to P^2 . In this case, apart from a factor α they are simply a function of n and α_S . In the leading order in α_S , it holds

$$(\mathcal{A}_{\text{NS}}^0, \mathcal{A}_{\text{S}}^0, \mathcal{A}_{\text{G}}^0) = \frac{\alpha}{4\pi} (c_{\text{NS}}, c_{\text{S}}, 0) \mathcal{A}. \quad (57)$$

The explicit form of $\mathcal{A}$ can be found in the literature [6]. However, a quick inspection of (48) and (49) shows that M_{NS} and $M_{\{\text{SG}\}}$ are of higher order in α_S with respect to X_{NS} , X_{S} and X_{G} . Therefore, since $\mathcal{A}_{\text{NS}}$, $\mathcal{A}_{\text{S}}$, $\mathcal{A}_{\text{G}}$ and $\mathcal{A}_\gamma$ are of $\mathcal{O}(1)$ in α_S (except for $\mathcal{A}_{\text{G}}$ of $\mathcal{O}(\alpha_S)$), the hadronic matrix elements contribute to the singlet, non-singlet and gluon distribution functions only in the next-to-leading order.

For a **real target** photon, the hadronic matrix elements are unknown. Consistently, the continuation of (56) to $P^2 = 0$ yields divergent quantities. Moreover, in this case, the renormalization must be stopped at a value μ^2 big enough for the integrals in (47) to be well defined. All this makes the case of the real target much more complicated than the case of the virtual one.

In spite of these fundamental differences, we can nevertheless write down the distribution functions $\Delta^\gamma(n, Q^2, \mu^2)$, $\Sigma^\gamma(n, Q^2, \mu^2)$ and $G^\gamma(n, Q^2, \mu^2)$ in a manner valid for both cases. Collecting all the elements discussed so far, we have:

$$\Delta^\gamma(n, Q^2, \mu^2) = \Delta_{\text{as}}^\gamma(n, Q^2) \left\{ 1 - \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{1-d_{\text{NS}}} \right\} + \Delta_0^\gamma(n, \mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{-d_{\text{NS}}} \quad (58)$$

$$\begin{aligned} \Sigma^\gamma(n, Q^2, \mu^2) = & \Sigma_{\text{as}}^\gamma(n, Q^2) \left\{ 1 - c_-^\Sigma \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{1-d_-} + c_+^\Sigma \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{1-d_+} \right\} \\ & + \tilde{\Sigma}_{0-}^\gamma(n, \mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{-d_-} - \tilde{\Sigma}_{0+}^\gamma(n, \mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{-d_+} \end{aligned} \quad (59)$$

$$\begin{aligned} G^\gamma(n, Q^2, \mu^2) = & G_{\text{as}}^\gamma(n, Q^2) \left\{ 1 + c_-^G \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{1-d_-} - c_+^G \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{1-d_+} \right\} \\ & + \tilde{G}_{0-}^\gamma(n, \mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{-d_-} - \tilde{G}_{0+}^\gamma(n, \mu^2) \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu^2)} \right)^{-d_+} \end{aligned} \quad (60)$$

where

$$r_\pm = d_\pm - d_{\text{qq}}; \quad \delta = d_+ - d_-; \quad c_\pm^\Sigma = \frac{r_\mp(1-d_\mp)}{\delta(1-d_{\text{GG}})}; \quad c_\pm^G = \frac{(1-d_\mp)}{\delta}.$$

The asymptotic distribution functions $\Delta_{\text{as}}^\gamma(n, Q^2)$, $\Sigma_{\text{as}}^\gamma(n, Q^2)$ and $G_{\text{as}}^\gamma(n, Q^2)$ factorized in (58–60) are given by

$$\begin{aligned}\Delta_{\text{as}}^\gamma(n, Q^2) &= \frac{\alpha}{2\pi} k_{\text{NS}}^0(n) \log\left(\frac{Q^2}{\Lambda^2}\right) \frac{1}{1 - d_{\text{NS}}} \\ \Sigma_{\text{as}}^\gamma(n, Q^2) &= \frac{\alpha}{2\pi} k_{\text{S}}^0(n) \log\left(\frac{Q^2}{\Lambda^2}\right) \frac{1 - d_{\text{GG}}}{(1 - d_-)(1 - d_+)} \\ G_{\text{as}}^\gamma(n, Q^2) &= \frac{\alpha}{2\pi} k_{\text{S}}^0(n) \log\left(\frac{Q^2}{\Lambda^2}\right) \frac{d_{\text{Gq}}}{(1 - d_-)(1 - d_+)} ,\end{aligned}\tag{61}$$

while $\tilde{\Sigma}_{0\pm}^\gamma(n, \mu^2)$ and $\tilde{G}_{0\pm}^\gamma(n, \mu^2)$ can be expressed in terms of the initial conditions $\Sigma_0^\gamma(n, \mu^2)$ and $G_0^\gamma(n, \mu^2)$ in the following way:

$$\tilde{\Sigma}_{0\pm}^\gamma(n, \mu^2) = \frac{d_{\text{qG}}}{\delta} \left(\frac{r_{\mp}}{d_{\text{qG}}} \Sigma_0^\gamma(n, \mu^2) - G_0^\gamma(n, \mu^2) \right) \tag{62}$$

$$\tilde{G}_{0\pm}^\gamma(n, \mu^2) = \frac{1}{\delta} r_{\pm} \left(\frac{r_{\mp}}{d_{\text{qG}}} \Sigma_0^\gamma(n, \mu^2) - G_0^\gamma(n, \mu^2) \right) . \tag{63}$$

The scale μ^2 in (58–60) can be chosen to be P^2 for a virtual target, but is the scale at which the evolution is stopped ($\mu^2 \gg \Lambda^2$) in the case of a real one. Moreover, the initial conditions $\Delta_0^\gamma(n, \mu^2)$, $\Sigma_0^\gamma(n, \mu^2)$ and $G_0^\gamma(n, \mu^2)$ have to be considered as some input distributions obtainable from experimental data taken at μ^2 in the kinematical region $P^2 \ll \Lambda^2 \ll Q^2$, while have to be set to zero when considering the region $\Lambda^2 \ll P^2 \ll Q^2$.

We recall that the evolution equation (23) and the corresponding ones for the singlet and gluon distribution functions,

$$\frac{\partial}{\partial \log Q^2} \begin{pmatrix} \Sigma^\gamma(n, Q^2, \mu^2) \\ G^\gamma(n, Q^2, \mu^2) \end{pmatrix} = \frac{\alpha}{2\pi} \begin{pmatrix} k_{\text{S}}^0(n) \\ 0 \end{pmatrix} + \frac{\alpha_{\text{S}}(Q^2)}{2\pi} \begin{pmatrix} P_{\text{qq}}^0(n) & 2N_f P_{\text{qG}}^0(n) \\ P_{\text{Gq}}^0(n) & P_{\text{GG}}^0(n) \end{pmatrix} \begin{pmatrix} \Sigma^\gamma(n, Q^2, \mu^2) \\ G^\gamma(n, Q^2, \mu^2) \end{pmatrix} \tag{64}$$

give the solutions (58–60), but do not provide any information on the boundary conditions $\Delta_0^\gamma(n, \mu^2)$, $\Sigma_0^\gamma(n, \mu^2)$ and $G_0^\gamma(n, \mu^2)$ to be imposed at the initial scale μ^2 . Moreover, they do not give information on the coefficients C^{NS} , C^{S} , C^{G} and therefore they do not describe how the distributions $\Delta^\gamma(n, Q^2)$, $\Sigma^\gamma(n, Q^2)$ and $G^\gamma(n, Q^2)$ contribute to $F_2^\gamma(n, Q^2)$.

We end this paragraph with some comments on the last coefficient $\chi_{\text{r}}(x)$ in (10). In moment space it is simply

$$\chi_{\text{r}} = \mathcal{A}_{\text{r}} C^\gamma(1, g^2(Q), \alpha) \equiv C^\gamma(1, g^2(Q), \alpha) . \tag{65}$$

It contributes to the PHSF with a function of x , scale independent. We neglect this term in the same way as we neglect the x terms in the QPM solutions (19).

3. P^2 dependence

We investigate here the dependence of the PHSF on the photon virtuality P^2 with the aim of understanding how to perform the calculation of $\sigma_{\text{ep}}[\text{resolved}]$ in (7). We can distinguish four regions in the range of virtuality of the photon, accessible in e^+e^- collisions and at HERA:

- i) $P^2 \ll \Lambda^2 \ll Q^2$
- ii) $\Lambda^2 \sim P^2 \ll Q^2$
- iii) $\Lambda^2 \ll P^2 \ll Q^2$
- iv) $\Lambda^2 \ll P^2 \sim Q^2$

The behaviour of the transverse PHSF $F_2^\gamma(n, Q^2, P^2)$ in regions i) and iii) has been widely discussed in the previous section. We summarize here the results obtained, as far as the P^2 dependence is concerned, and we point out a few more aspects so far omitted. The second of the two conditions delimiting the region of virtual PHSFs, $P^2 \ll Q^2$, implies that power-like terms of the type $(P^2/Q^2)^k$ can be neglected. Furthermore this condition guarantees that there are only three independent PHSFs, $F_i(x, Q^2, P^2)$, in two-photon collisions since the other five present in the general case are suppressed by powers of P^2/Q^2 ^{††}. The first condition $\Lambda^2 \ll P^2$ allows the complete calculability of the PHSFs x behaviour at any order in perturbative QCD. The hadronic matrix elements between photon states can be evaluated in this limit and turn out to be contributing to the dominant PHSF, $F_2^\gamma(x, Q^2, P^2)$, only in the next to leading order. At the leading order, the singlet, non-singlet and gluon distributions, in which the transverse PHSF is decomposed, are given in moment space by (58–60) when all the input distributions $\Delta_0^\gamma(n)$, $\Sigma_0^\gamma(n)$ and $G_0^\gamma(n)$ are set to zero. Different values of the renormalization scale μ^2 , which can be chosen to be equal to P^2 , give rise to a logarithmic uncertainty in the evaluation of the virtual PHSF.

The continuation over $P^2 \sim \Lambda^2$ down to the region of the real PHSF is non-trivial. Nevertheless, the solutions (58–60) can still provide a parametrization of the real PHSF of region i), if the distribution functions $\Delta_0^\gamma(n)$, $\Sigma_0^\gamma(n)$ and $G_0^\gamma(n)$, corresponding in this case to uncalculable matrix elements, are taken from experimental data. The scale μ^2 , at which the input distributions are extracted from data, should be large enough to guarantee the reliability of the QCD evolution from Q^2 to μ^2 . The parametrization obtained in this way is obviously P^2 -independent.

Power-like terms $\sim P^2/Q^2$ are important in region iv) and higher-twist contributions should not be neglected in it. Moreover, in this region, $F_2^\gamma(n, Q^2, P^2)$ is no longer the dominant PHSF. The virtual PHSFs $F_2^\gamma(n, Q^2, P^2)$ and $F_L^\gamma(n, Q^2, P^2)$ obtained in region iii) reduce to the respective QPM results when extrapolated to $P^2 \sim Q^2$. In particular, the gluon distribution of a virtual photon vanishes faster than the quark distributions (i.e. faster than $\log(P^2/Q^2)$) when $P^2 \rightarrow Q^2$, in agreement with the fact that there is no gluon contribution to the QPM approximation for the PHSF.

The region $\Lambda^2 \sim P^2 \ll Q^2$ cannot be attacked perturbatively. Not even clear is down to which value of P^2 the virtual PHSF solution can be extrapolated to reach (or partially cover) this region. In the past, attempts were made to estimate this threshold value of P^2 by comparing the perturbatively calculated virtual PHSF (58–60) to VMD contributions with a P^2 dependence of the type:

$$\Delta^\gamma|_{\text{VMD}}(n, Q^2, P^2) = \sum_V \frac{4\pi\alpha}{f_V^2} \left(\frac{m_V^2}{m_V^2 + P^2} \right)^2 \Delta^V(n, Q^2), \quad (66)$$

^{††}There are only two independent PHSFs, F_2^γ and F_L^γ , if the spin average for the target photon is taken.

where $4\pi\alpha/f_V^2$ is the coupling of the photon to the vector meson V and Δ^V the non-singlet distribution function for the meson V . Values for P^2 below 1 GeV^2 were found in this way [11].

Progress in the correct treatment of region ii) can be achieved by observing that, for fixed space-like Q^2 , the moments of the PHSFs are analytic in the P^2 plane. A natural way of making use of this property is to express the generic PHSF $F(n, Q^2, P^2)$ in terms of a dispersion integral [17,18]:

$$F(n, Q^2, P^2) = \int_0^\infty d\sigma^2 \frac{\phi(n, Q^2, \sigma^2)}{P^2 + \sigma^2}, \quad (67)$$

which links perturbative and non-perturbative contributions and, moreover, allows the smooth limit $P^2 \rightarrow 0$.

A qualitative discussion of the weight function can be found in ref. [18]. Different choices of this function allow to recover different approximate solutions for the PHSFs. A constant function in the region $\sigma^2 < Q^2$ and zero otherwise reproduces, for example, the QPM result. Similarly, the point-like and hadron-like terms of the perturbatively calculated solutions, as well as the non-perturbative contributions from Regge poles can also be obtained. In particular, these last terms can be recovered by delta-singularities in the weight function.

A quantitative, model-dependent estimate of (67) is performed in ref. [17]. The physical spectrum is represented as a sum of some non-perturbative contributions due to the dominant vector meson, ρ , and the continuum starting at some threshold value P_0^2 :

$$\begin{aligned} \phi(n, Q^2, \sigma^2) = & \delta(\sigma^2 - m_\rho^2) A_\rho(n, Q^2) + \delta'(\sigma^2 - m_\rho^2) B_\rho(n, Q^2) \\ & + \theta(\sigma^2 - P_0^2) \left\{ \alpha(n, Q^2, \sigma^2) + \frac{\beta(n, Q^2, \sigma^2)}{P^2 + \sigma^2} \right\}, \end{aligned} \quad (68)$$

with the prime indicating the derivative with respect to σ^2 . The unknown functions A_ρ, B_ρ, α and β are fixed by comparing (67) and (68) with the value of $F(n, Q^2, P^2)$ at high- P^2 ($m_\rho^2, P_0^2 \ll P^2 \ll Q^2$) obtained from the simple quark-box-diagram plus some higher-twists corrections. The result for $F_2^\gamma(n, Q^2, P^2)$ and $F_L^\gamma(n, Q^2, P^2)$ obtained through this procedure in ref. [17] is, strictly speaking, correct only for intermediate $Q^2 \lesssim 10 \text{ GeV}^2$ since no QCD evolution is taken into account. However, it may still be worth extrapolating it to higher values of Q^2 and comparing it with the results obtained for regions i) and iii). We expect this extrapolation, which suppresses higher-twists terms of the type m_ρ^2/Q^2 , P_0^2/Q^2 , and P^2/Q^2 , not to affect greatly the P^2 dependence of the PHSFs and we hope to gain some insight on the possibility of connecting regions i) and iii). In the particular case of the transverse PHSF $F_2^\gamma(n, Q^2, P^2)$, the limit $Q^2 \rightarrow \infty$ of the result obtained in ref. [17] becomes, after correcting an error in the quark-box-diagram expression,:

$$F_2^\gamma(n, Q^2, P^2) = D_\gamma(n) + a(n) \left\{ \log \left(\frac{Q^2}{P^2 + P_0^2} \right) + \frac{P_0^2}{P^2 + P_0^2} + \frac{P_0^4}{2(P^2 + m_\rho^2)^2} \left[1 - \frac{1}{64(n-2)} \right] \right\}, \quad (69)$$

where the value 1.5 GeV^2 for P_0^2 has been explicitly used and the term $D_\gamma(n)$ is given by:

$$D_\gamma(n) = \frac{6}{n+1} - \frac{2}{n} - \frac{6}{n+2} + \frac{2}{n^2} - \frac{4}{(n+1)^2} + \frac{4}{(n+2)^2}. \quad (70)$$

Figure 5: Singlet distribution as function of P^2 at $Q^2 = 40 \text{ GeV}^2$ for the different moments $n = 2, 4, 6$. The right solid lines denote the virtual singlet solution (59) (with $\mu^2 = P^2$ and $\tilde{\Sigma}_{0\pm}^\gamma(n, \mu^2) = 0$). The straight solid lines are the real singlet distribution function obtained from the LAC1 input. The dotted lines are the QPM result (21) with the logarithmic term continued to $\log(Q^2/(P^2 + \Lambda^2))$ and the short dashed lines are the singlet part of (69).

Figure 6: The same singlet distribution functions shown in Fig. 5 are here plotted for $Q^2 = (80 \text{ GeV})^2$. The only curve omitted is the singlet distribution obtained from (69).

It is evident from (69) how this description allows a smooth limit $P^2 \rightarrow 0$.

We show in Figs. 5 and 6 the P^2 dependence of $\Sigma^\gamma(n, Q^2, P^2)$ in the regions listed before for the three moments $n = 2, 4, 6$ at $Q^2 = 40 \text{ GeV}^2$ and $Q^2 = M_W^2$, respectively. The values 200 MeV for the QCD parameter Λ and four for the number of flavours, n_f , have been used to obtain these plots. Both figures show the result from the LAC1 parametrization for the real singlet distribution function (independent of P^2) at low- P^2 . At high- P^2 we show the behaviour of the virtual singlet distribution as obtained from (59). Throughout the full range of P^2 we plot the QPM result (21) with the logarithmic term continued to $\log(Q^2/(P^2 + \Lambda^2))$. For $Q^2 = 40 \text{ GeV}^2$, we show also the solution obtained from (69) without the term $D_\gamma(n)$ and without the last term in the square bracket for the moment $n = 2$.

From Figs. 5 and 6, we see how the virtual PHSF increases as P^2 decreases. Extrapolating the real PHSF towards larger P^2 and the virtual one towards lower values of P^2 we obtain a matching of the two distributions at $P \sim 0.05\text{--}0.5 \text{ GeV}$. The precise value of P , which we call P_{cut} , depends on n and Q^2 . The matching point increases with increasing n while it decreases for increasing values of Q^2 . It also depends on the particular parametrization used for the real PHSF. Values as high as 0.8 for P_{cut} are obtained for large values of n when the DG parametrization is used.

The crossing of the real and virtual solutions at P_{cut}^2 suggests to construct a PHSF covering the whole P^2 range by taking the real PHSF (constant in P^2) for $P^2 < P_{\text{cut}}^2$ and the virtual solution for $P^2 > P_{\text{cut}}^2$. The so-constructed PHSF will be continuous for a particular value of n , but discontinuous for different moments if a fixed value of P_{cut} is used. The optimal choice for P_{cut} depends therefore on the average x (or n) and on the particular scale of the process Q^2 . A reasonable, “average” value of P_{cut} is, for example, the mass of the ρ meson. As long as one integrates over a sufficiently extended P^2 interval, the error introduced in this way in the above approximation should be small.

The justification for this procedure is enforced by the particularly smooth behaviour across the region $P^2 \sim \Lambda^2$ of the QPM result and of the singlet distribution obtained in (69). Both curves show a very slow onset of a P^2 dependence for increasing P^2 and then smoothly approach at large P^2 the lower curve representing the virtual solution. They lay above the virtual PHSF result since this is reduced by the QCD evolution in Q^2 .

While completely similar conclusions can be drawn in the case of the non-singlet distribution function $\Delta^\gamma(n, Q^2, P^2)$, the continuation of the gluon distribution from $P^2 \ll \Lambda^2$ to $P^2 \gg \Lambda^2$ is more problematic. The gluon distribution shown in Fig. 7 vanishes faster than the quark distributions for increasing P^2 , making a matching of the solutions for virtual and real photons more subtle. The choice $P_{\text{cut}} \sim m_\rho$ made for the quark distributions can still be reasonably adopted for the matching point of the real and virtual gluon distributions at large n . For $n = 2$, however, the real photon distribution as given by the LAC1 parametrization is far above the virtual solution at $P^2 = m_\rho^2$ and matches the latter only for values as small as $P^2 = 10^{-3} \text{ GeV}^2$. Two explanations are plausible: either the gluon distribution in a real photon is less singular than predicted by this parametrization, or effects of the photon virtuality set in much earlier than for quark distributions. More work is needed in this direction.

We have repeatedly stressed that the virtual PHSF approaches the QPM structure function for large P^2 . As can be seen in Figs. 5 and 6, the quality of this approximation

Figure 7: The gluon distribution in the virtual photon (60) as a function of the photon virtuality at $Q^2 = 40 \text{ GeV}^2$ ($n_f = 4$ and $\Lambda = 0.2 \text{ GeV}$). The solid, dashed and long-dashed line, refer to $n = 2, 4$ and 6 , respectively. The values of the three moments of the LAC1 parametrization of the gluon distribution in the real photon are 17.1 ($n = 2$), 0.18 ($n = 4$), and 0.045 ($n = 6$).

depends on n and Q^2 . For example, the 25% deviation of the QPM result from the virtual PHSF result for the singlet distribution $\Sigma^\gamma(n=2, Q^2=40 \text{ GeV}^2, P^2)$ at $P^2 = 1 \text{ GeV}^2$ is already reduced to 10% at $P^2 = 3 \text{ GeV}^2$, while for $Q^2 = M_W^2$ the 40% deviation at $P^2 = 1 \text{ GeV}^2$ is reduced to 10% only at $P^2 = 300 \text{ GeV}^2$. Moreover, this approximation is not adequate for the gluon in a virtual photon, in spite of the fact that its distribution quickly approaches zero for increasing P^2 . No gluon, in fact, contributes to the PHSF at the Born level.

Nevertheless, since calculations with the QPM expressions are considerably simpler than those using the full virtual solutions, one may still want to approximate the virtual PHSFs by their Born expressions in the region $P^2 > P_{\text{cut}}^2$. In particular, in the case of the transverse PHSF $F_2^\gamma(n, Q^2, P^2)$, we obtain the following approximation for the non-singlet, singlet and gluon distribution functions, over the full range of virtuality P^2 :

$$\begin{aligned} \Delta^\gamma(n, Q^2, P^2) &= \Delta^\gamma(n, Q^2)|_{\text{REAL}} \theta(P_{\text{cut}}^2 - P^2) + \Delta^\gamma(n, Q^2, P^2)|_{\text{QPM}} \theta(P^2 - P_{\text{cut}}^2) \\ \Sigma^\gamma(n, Q^2, P^2) &= \Sigma^\gamma(n, Q^2)|_{\text{REAL}} \theta(P_{\text{cut}}^2 - P^2) + \Sigma^\gamma(n, Q^2, P^2)|_{\text{QPM}} \theta(P^2 - P_{\text{cut}}^2) \end{aligned} \quad (71)$$

$$G^\gamma(n, Q^2, P^2) = G^\gamma(n, Q^2)|_{\text{REAL}} \theta(P_{\text{cut}}^2 - P^2) .$$

The distributions $\Delta^\gamma(n, Q^2, P^2)|_{\text{QPM}}$ and $\Sigma^\gamma(n, Q^2, P^2)|_{\text{QPM}}$ are given by (20) and (21), respectively, while $\Delta^\gamma(n, Q^2)$, $\Sigma^\gamma(n, Q^2)$ and $G^\gamma(n, Q^2)$, with the explicit label REAL in (71), can be taken from existing parametrizations of the real PHSF. This approximation may yield sufficiently accurate P^2 -integrated cross sections, depending on the relative size of the low- and high- P^2 regions. Needless to say, the use of the full virtual solution in the region $P^2 > P_{\text{cut}}^2$ is preferable, especially in the case of the gluon distribution.

Before closing this section we briefly comment on the result obtained for the moment $n = 2$. We observe that the agreement between the virtual PHSF and the QPM result at high- P^2 is better for $n = 2$ than for higher moments. Moreover, the real singlet distribution, quite below the QPM result and the one obtained from (69) at $P^2 \sim 0$ for $n = 4, 6$, is in this case at a reduced distance from it. The peculiarity of this value is easily explained if we keep in mind that we are in this case close to the rightmost singularity displayed by the asymptotic part of the singlet distribution (62) at $n_0 = 1.59$ ($d_+(1.59) \sim 1$). The efficiency of the suppression factors $1/(1-d_-)(1-d_+)$ is then reduced, as shown in (34), therefore giving values for the real and virtual singlet functions closer to the ones non-QCD evolved.

4. Calculation of ep cross sections

It is clear from the above discussion that the resolved photon contribution to P^2 -integrated ep cross sections cannot be exactly calculated and that a suitable approximation has to be found. The approximation (71) for quarks and gluon distribution functions suggests the splitting of the P^2 integration into a low- and a high- P^2 region, as follows:

$$\sigma_{ep}[\text{total}] = \int_{P_{\text{min}}^2}^{P_{\text{cut}}^2} dP^2 \frac{d\sigma_{ep}[\text{total}]}{dP^2} + \int_{P_{\text{cut}}^2}^{P_{\text{max}}^2} dP^2 \frac{d\sigma_{ep}[\text{total}]}{dP^2} \equiv \sigma_{ep}[\text{total}]|_{\text{low-}P^2} + \sigma_{ep}[\text{total}]|_{\text{high-}P^2} . \quad (72)$$

The total cross section can then be calculated using different approximations in the two regions, motivated by the following arguments.

Only two PHSFs, $F_2^\gamma(x, Q^2, P^2)$ and $F_L^\gamma(x, Q^2, P^2)$ contribute to the total ep cross section, if the system X in (1) is integrated out. At sufficiently small P^2 , the transverse PHSF $F_2^\gamma(x, Q^2, P^2)$ is the dominant one. The P^2 dependence of $F_2^\gamma(x, Q^2, P^2)$ is negligible and the approximation of real PHSF can be used up to P_{cut}^2 . The decomposition of $F_2^\gamma(x, Q^2, P^2)$ into quark and gluon distribution functions and the Q^2 evolution of these distributions are well known up to the next-to-leading order accuracy. Existing parametrizations of $q^\gamma(x, Q^2)$ and $G^\gamma(x, Q^2)$ can be used to calculate the resolved photon contribution to the given ep process. The direct photon contribution has to be added to the resolved one as in (7), i.e. the initial-state mass singularity (the large logarithm $\log Q^2/P^2$ together with certain non-log terms) of the direct process has to be absorbed into the quark (or gluon) distribution functions of the photon. Furthermore, the ep cross section can be expressed in terms of

the one for the γp subprocess in the Weizsäcker-Williams approximation [20]. Thus, the ep cross section in the low- P^2 region is given by:

$$\sigma_{ep}[\text{total}]|_{\text{low-}P^2} = \int d\eta \int_{P_{\text{min}}^2}^{P_{\text{cut}}^2} \frac{dP^2}{P^2} f_\gamma(\eta) \left\{ \sigma_{\gamma p}^{\text{REAL}}[\text{resolv}] + \sigma_{\gamma p}^{\text{REAL}}[\text{dir-fin}] + \sigma_{\gamma p}^{\text{REAL}}[\text{rest}] \right\} \quad (73)$$

where the variable $\eta = (W^2 + P^2 - m_p^2)/s \sim W^2/s$ is used to integrate over the γp CM energy W and $f_\gamma(\eta)$ is the familiar function $f_\gamma(\eta) = (\alpha/2\pi)(1+(1-\eta)^2)/\eta$. The superscript REAL in (73) reminds that the approximation of real photon is used in the calculation of the γp cross sections.

In the high- P^2 region, the situation is more complicated by the fact that power-like corrections of the type $(P^2/Q^2)^k$ become important for $P^2 \rightarrow Q^2$. In this limit, the longitudinal PHSF $F_L^\gamma(x, Q^2, P^2)$ is not negligible anymore with respect to $F_2^\gamma(x, Q^2, P^2)$. Given its dominant behaviour $\mathcal{O}(P^2/Q^2)$, the addition of this PHSF at $P^2 = P_{\text{cut}}^2$, does not produce a sudden change in the P^2 dependent cross section for small enough values of P_{cut}^2 and for reasonably large scales Q^2 . In the same limit $P^2 \rightarrow Q^2$, also higher-twist corrections cannot be neglected. These corrections, together with target masses effects, cannot be simply included in the non-corrected virtual PHSFs solutions by suitable redefinitions of the variable x (as was shown in ref. [21] for the case of the quark-box-diagram), but they have to be calculated order by order in perturbation theory. It is also clear that, in order to keep in account all the $(P^2/Q^2)^k$ terms, the full ep cross section should be calculated rather than being obtained in the Weizsäcker-Williams approximation*, from the respective γp cross section. The ep cross section in the high- P^2 region is then

$$\sigma_{ep}[\text{total}]|_{\text{high-}P^2} = \int_{P_{\text{cut}}^2}^{P_{\text{max}}^2} dP^2 \frac{d}{dP^2} \left\{ \sigma_{ep}^{\text{VIRT}}[\text{resolv}] + \sigma_{ep}^{\text{VIRT}}[\text{dir-fin}] + \sigma_{ep}^{\text{VIRT}}[\text{rest}] \right\} . \quad (74)$$

All the contributions from virtual PHSFs enter in $\sigma_{ep}^{\text{VIRT}}[\text{resolv}]$, while the contributions from W and/or Z exchange, relevant at $P^2 \gtrsim M_W^2$, can be included in $\sigma_{ep}^{\text{VIRT}}[\text{dir-fin}]$. At the HERA energy, the region of phase space where P^2 exceeds these values is quite limited, making these contributions rather small.

We can still go one step further in the approximations made in the high- P^2 region. As argued in the previous section, the virtual PHSFs may be replaced by their QPM expressions. In doing so, the calculation of $\sigma_{ep}[\text{total}]|_{\text{high-}P^2}$ simplifies considerably. It is merely given by the direct process:

$$\sigma_{ep}[\text{total}]|_{\text{high-}P^2} = \int_{P_{\text{cut}}^2}^{P_{\text{max}}^2} dP^2 \frac{d\sigma_{ep}^{\text{VIRT}}[\text{direct}]}{dP^2} = \int_{P_{\text{cut}}^2}^{P_{\text{max}}^2} dP^2 \frac{d}{dP^2} \left\{ \sigma_{ep}^{\text{VIRT}}[\text{dir-sing}] + \sigma_{ep}^{\text{VIRT}}[\text{dir-fin}] \right\} . \quad (75)$$

*This approximation, with explicit dP^2/P^2 integration, is still quite good also in the high- P^2 region if the dynamical cut-off at $P^2 \sim Q^2$, is respected. This cut-off arises from the fact that the $1/P^2$ dependence typical of the low- P^2 region, changes to $1/P^4$ for $P^2 \sim Q^2$.

The approximations proposed above for the high- and low- P^2 regions are constructed for the limits $P^2 \rightarrow Q^2$ and $P^2 \rightarrow 0$, respectively. Both approximations become worse as P^2 approaches P_{cut}^2 , and the P^2 distribution is, in general, not continuous at $P^2 = P_{\text{cut}}^2$. The failure of both approximations at $P^2 \sim P_{\text{cut}}^2$ may or may not have a sizeable effect on P^2 integrated cross sections or distributions depending on several, process-dependent factors.

To begin with, the uncertainty in the calculation of ep cross sections is smaller the smaller is the size of the “uncertainty interval” for P_{cut} compared to the total P^2 range. Moreover, as should be clear from the discussion in the previous section, quark-initiated processes can be calculated more reliably than gluon-initiated ones because of the softer P^2 dependence of quarks distribution functions. Non-negligible is also the role of the direct finite terms to the sensitivity of the ep cross section to different possible approximations. The larger is their contribution to the total cross section, the less is the latter affected by the matching procedure of the singular terms. An important parameter in this respect is the value of the typical scale of the process Q^2 . This scale determines the effective upper and lower limits of the P^2 integration, as already mentioned, and the size of the direct finite terms with respect to the resolved photon ones. It determines also the average value of x , $x \propto Q^2/s$, responsible for the type of dependence on P_{cut} . This is sharper for smaller values of x . Finally, for higher values of Q^2 , the approximation of the virtual photon distributions with the ones obtained in the QPM is worse than for smaller scales Q^2 .

To illustrate these concepts we consider the particular case of W^+ production in ep collisions at HERA ($\sqrt{s} = 314$ GeV). In Fig. 8 we plot the P^2 dependence of the direct finite and direct singular contributions to the total cross section as well as the values of the resolved photon contributions, constant in P^2 , given by the LAC1 and the DG parametrization of the real PHSF. The results shown are obtained by using the Weizsäcker-Williams approximation in the low- and high- P^2 regions and by neglecting the exchange of heavy gauge bosons. Moreover, the EHLQ1 parametrization [22] for the quark distribution in the proton is used and both scales appearing in the proton and photon structure functions are set equal to M_W .

We observe that the direct singular contribution shows the $\log(Q^2/P^2)$ behaviour typical of the QPM approximation. It coincides with the resolved photon contribution in the DG parametrization at $P_{\text{cut}}^2 \sim (2m_\rho)^2$, but at quite smaller values of P_{cut}^2 if the LAC1 parametrization of the real PHSF is used. The figure shows clearly that the use of the direct singular term at low- P^2 would overestimate the value of the total cross section, while the use of the resolved photon contribution would lead to an overestimation at high- P^2 . The direct finite term hardly shows a P^2 dependence in the considered P^2 interval, due to the smallness of the contribution of power like terms $(P^2/Q^2)^k$, for such a high value of Q^2 . Moreover, it is similar in size to the resolved and direct singular contributions, diluting therefore the uncertainty of the matching procedure. The main uncertainty left in the calculation of W^+ production is then the factorization scale uncertainty [1] and the one coming from the differences in the existing parametrizations of the PHSF. Results ranging between 0.58 and 0.69 pb are obtained for the total cross section at $\sqrt{s} = 314$ GeV.

5. Conclusions and outlook

In this paper we have analysed the problem of matching the direct and resolved photon contributions to hard, inelastic ep scattering. The range of virtuality P^2 of the exchanged

Figure 8: P^2 dependence of the different contributions to the cross section for W^+ production at HERA. The two solid lines are the resolved photon contributions in the LAC1 (upper curve) and DG (lower curve) parametrization of the real PHSF. The short-dashed and long-dashed lines are the contributions coming from the direct finite and the direct singular term, respectively.

photon is already quite large at the HERA energy. This fact makes, in general, the use of the real PHSF inadequate to calculate the resolved photon contributions to P^2 -integrated ep cross sections. The problem will be more serious for higher CM energies of the ep system.

We have reviewed the known results on the P^2 dependence of the PHSF for $P^2 \ll \Lambda^2 \ll Q^2$ and $\Lambda^2 \ll P^2 \ll Q^2$ and argued on the possibility of analytically continuing the results obtained in the latter region to the values given by the parametrizations used in the former one. More work is needed in this direction. The obvious goal is to obtain a parametrization of the PHSF in the full range of P^2 to be probed by HERA and future colliders.

Lacking such a parametrization, approximations have to be used to calculate the resolved photon contributions to ep cross sections. At low photon virtuality, $P^2 \lesssim \Lambda^2$, a parametrization of parton distributions within a real photon together with the Weizsäcker-Williams approximation can be used for the calculation of the resolved photon contributions. When adding the direct process, initial-state mass singularities have to be absorbed in the photon-parton and proton-parton distributions as in the case of purely hadronic reactions. If this sum of resolved and direct finite contributions is used also at high- P^2 , a P^2 -integrated

cross section free of double counting can be obtained.

The use of P^2 -independent, real photon distribution functions does not reproduce the correct decrease of the cross section with increasing P^2 . In this paper we have proposed to split the integration over the photon virtuality into a low- and a high- P^2 region. In the high- P^2 region, the virtual photon structure functions can be approximated by the direct singular term. This should account for the dominant decrease of the total ep cross section, since the virtual photon distribution functions quickly approach the respective lowest-order expressions. In a more refined calculation one would use the full virtual photon distributions at high- P^2 . This is especially necessary for gluon-initiated processes, since the lowest-order gluon solution vanishes.

With an appropriate choice of the cut applied to divide the P^2 region, P_{cut}^2 , the total cross section can be made continuous in P^2 . This is not necessarily true for differential distributions. In particular, at small x , P_{cut}^2 depends strongly on the precise value of x . Besides producing a discontinuity in P^2 , the uncertainty in P_{cut}^2 can then affect differential distributions, like the rapidity one, since real and virtual (and even more QPM) distribution functions have different small x behaviours. Work on the comparison of the small x behaviour of real and virtual photon distribution functions is in progress.

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