NEW IMPROVED OPERATORS: A CONVENIENT REDEFINITION

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ABSTRACT

In this paper we consider a new definition of improved fermion operators, which is very convenient for Monte Carlo simulations and we give the new values of the corresponding renormalization constants. This redefinition includes terms that are formally of order a^2 and do not affect the conclusions about order a improvement brought by the “clover” action.

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1 Introduction

In this paper we consider a redefinition of improved operators, which is very convenient for numerical QCD simulations with the nearest-neighbour improved action [0], also known as the “clover” action, and we present the one-loop values of the corresponding new renormalization constants.

Following the approach presented in refs. [0, 0], it is possible to reduce the systematic error due to the finiteness of the lattice spacing a , from terms of $O(a)$ to terms of $O(g_0^2 a)$. One must work with the “clover” action and, at the same time, transform quark fields in fermion operators according to

$$\begin{aligned}\psi &\rightarrow \left[1 - a\frac{r}{2}(z\overrightarrow{\not{D}} - (1-z)m_0)\right]\psi, \\ \bar{\psi} &\rightarrow \bar{\psi}\left[1 + a\frac{r}{2}(z\overleftarrow{\not{D}} + (1-z)m_0)\right],\end{aligned}\tag{1}$$

where r is the Wilson parameter, m_0 is the bare mass which appears in the lattice action, and the lattice covariant derivatives $\overrightarrow{D}_\mu$ and $\overleftarrow{D}_\mu$ are defined as in ref. [0].

For any real value of the real parameter z , hadronic matrix elements of improved operators do not have terms of the form $(g_0^2)^n a \log^n a$, i.e. terms that are effectively of $O(a)$ in the scaling limit. Note that it is not necessary to “rotate” the operators that create on-shell external states from the vacuum, since different operators, with the same quantum numbers, create states that differ only by an irrelevant normalization constant.

The computation of meson masses and decay constants involves the evaluation of two-point correlation functions of bilinear operators of the kind

$$O_\Gamma^{LOC}(x) = \bar{\psi}(x)\Gamma\psi(x),\tag{2}$$

where Γ is one of the Dirac matrices and the superscript *LOC* stands for ‘local’. In this kind of computations, the definitions of the improved operator that have been used in practice [0, 0] are those with the values $z = 1$ and $z = 0$, in which $O(a^2)$ terms coming from (1) have been dropped, since they are unnecessary for $O(a)$ improvement, that is

$$O_\Gamma^{LOC} \xrightarrow{z=1} O_\Gamma^{LOC}(x) + a\frac{r}{2}\bar{\psi}(x)(\overleftarrow{\not{D}}\Gamma - \Gamma\overrightarrow{\not{D}})\psi(x)\tag{3}$$

and

$$O_\Gamma \xrightarrow{z=0} (1 + arm_0)O_\Gamma^{LOC}(x).\tag{4}$$

For comparison with experimental values, lattice operators must be related to the corresponding renormalized continuum ones. The one-loop computation of the renormalization constants of bilinear improved operators, such as (3) and (4), has been presented in ref. [0]. The perturbative renormalization of analogously improved four-fermion operators has been presented in ref. [0].

The improvement procedure can also be extended to hadronic matrix elements involving heavy quarks [0, 0], treated in the static approximation [0]. The one-loop computation of the renormalization constant of improved heavy-light operators, relevant for B decay and $B\bar{B}$ mixing, can be found in ref. [0].

We note that the renormalization constant of the improved operators, corresponding to the choice $z = 0$ in eq. (1), has a contribution from the $1/a$ divergent terms coming

from the mass renormalization [0, 0]. In this case, at least for the theory with the Wilson action, the one-loop perturbative calculation is known to be inaccurate, and higher orders seem to have a significant numerical effect. In order to take into account higher order terms, in ref. [0] the $1/a$ divergences in the bare mass were resummed to all orders [0], by substituting m_0 in eq. (4) with the “bare” lattice quark mass m_0^q defined as

$$am_0 \longrightarrow am_0^q = \frac{1}{2} \left(\frac{1}{K} - \frac{1}{K_c} \right), \quad (5)$$

where the critical value of the hopping parameter, K_c , is determined non-perturbatively. The choice $z = 1$ is, in any case, highly preferable, since it does not depend on the bare quark mass.

In the case of three-point correlation functions, as for example in the evaluation of B -parameters in mixing amplitudes, the choice $z = 1$ is, however, technically impractical. The problem is that it would involve the covariant derivative acting on the quark propagators at the origin. This requires the prohibitive computation of propagators from several nearest-neighbouring points around the origin.

An alternative procedure has been proposed in ref. [0]. We can use, in all correlation functions, the “effective” propagator

$$S^{eff}(x, 0) = \overrightarrow{R}(x) S^I(x, 0) \overleftarrow{R}(y = 0), \quad (6)$$

where we have defined

$$\overrightarrow{R}(x) \equiv \left(1 - a \frac{r}{2} \overrightarrow{D}(x) \right) \quad , \quad \overleftarrow{R}(y) \equiv \left(1 + a \frac{r}{2} \overleftarrow{D}(y) \right) \quad (7)$$

and $S^I(x, 0)$ is the propagator of the (improved) theory, defined by

$$\Delta_F^I(z, x) S^I(x, y) = \delta(z - y) \quad (8)$$

(Δ_F^I is the improved discretized Dirac operator).

The insertion of S^{eff} in the calculation of n -point correlation functions improves automatically, according to the $z = 1$ prescription, all the operators, including those that create the external states¹⁾. The reason is that the rotation (1), after the fermion functional integration, will end up acting on the fermion propagators; hence, all correlation functions can be expressed in terms of the effective propagator (6). The advantage is that S^{eff} can be easily obtained, by first solving numerically an equation of similar complexity as eq. (8), the difference being that the δ -function in eq. (8) is replaced by a “rotated” δ -function. In fact, by solving the equation

$$\Delta_F^I(z, x) \tilde{S}(x, 0) = \delta(z - y) \overleftarrow{R}(y)|_{y=0} \quad (9)$$

one gets the intermediate quantity

$$\tilde{S}(x, 0) \equiv S^I(x, 0) \overleftarrow{R}(y = 0) \quad (10)$$

from which, without any additional computational effort, by applying $\overrightarrow{R}$, one obtains

$$S^{eff}(x, 0) = \overrightarrow{R}(x) \tilde{S}(x, 0). \quad (11)$$

1) As already noticed, the definition of the operators that create the external states is arbitrary.

It must be noticed that using S^{eff} corresponds to keeping all terms of order a and higher, which arise from the rotations (7). For example, in the case of two-fermion operators (2), this method corresponds to using the new definition of the improved operator:

$$O_\Gamma^I = O_\Gamma^{LOC}(x) + a \frac{r}{2} \bar{\psi}(x) (\overleftarrow{D}\Gamma - \Gamma \overrightarrow{D}) \psi(x) - a^2 \frac{r^2}{4} \bar{\psi}(x) (\overleftarrow{D}\Gamma \overrightarrow{D}) \psi(x), \quad (12)$$

which differs from (3) by the presence of order a^2 terms. These terms are irrelevant for an order a improvement, but they do contribute to the renormalization constants of the operators.

In section 2 we give the new values of the renormalization constants of the improved two-fermion operators defined in eq. (12). Analogous considerations hold in the case of four-fermion improved operators, and we present in section 3 the new values of the corresponding renormalization constants and mixing coefficient, including the contributions of higher orders in a . Finally in section 4 we give the results for the renormalization of the new improved heavy-light operator of $B\bar{B}$ mixing.

2 Renormalization of New Improved Two-Fermion Operators

The numerical results for the one-loop renormalization constants of the new improved bilinear operators O_Γ^I defined in eq. (12) are presented in table 1. These constants differ from those presented in ref. [0] only by the contribution coming from the order a^2 terms.

We use the same notations as in ref. [0], defining

$$Z_\Gamma = \lim_{a \rightarrow 0} \left[1 + \frac{g_0^2}{16\pi^2} C_F (\Delta_\Gamma + \Delta_{\Sigma_1}) \right], \quad (13)$$

where $C_F = (N^2 - 1)/2N$, with N the number of colours. In eq. (13) we can write

$$\Delta_\Gamma = \Delta_\Gamma^{LOC} + \Delta_\Gamma^\otimes, \quad (14)$$

where Δ_Γ^{LOC} is the contribution due to the insertion of O_Γ^{LOC} , $\Delta_\Gamma^\otimes$ is due to the rotated part of the improved operator, including the $O(a^2)$ term, and Δ_{Σ_1} is the contribution of self-energy diagrams. The terms Δ_Γ^{LOC} and Δ_{Σ_1} have been calculated in [0], using the continuum $\overline{\text{MS}}$ renormalization scheme. For completeness we report their numerical values in table 2, for several values of the Wilson parameter r and for the value $F_{0001} \sim 1.31$ of the constant F_{0001} defined in ref. [0]. The new values of $\Delta_\Gamma^\otimes$ are given in table 3, and their analytical form is

$$\begin{aligned} \Delta_1^\otimes &= 4\pi^2 \left(-4I_6 + I_7 + I_8 + I_9 - 3I_{10} + I_{11} \right), \\ \Delta_{\gamma_5}^\otimes &= 4\pi^2 \left(-I_8 + I_9 + I_{10} \right), \\ \Delta_{\gamma_\mu}^\otimes &= 4\pi^2 \left(-2I_5 + I_6 + \frac{I_7}{4} - \frac{I_8}{2} - \frac{I_{11}}{4} + I_{12} \right), \\ \Delta_{\gamma_5 \gamma_\mu}^\otimes &= 4\pi^2 \left(-2I_5 - I_6 + \frac{3}{4}I_7 + \frac{I_8}{2} - 2I_{10} + \frac{I_{11}}{4} + I_{12} \right), \\ \Delta_{\sigma_{\mu\nu}}^\otimes &= 4\pi^2 \left(-\frac{8}{3}I_5 + \frac{2}{3}I_6 + \frac{I_7}{2} - \frac{I_9}{3} - I_{10} - \frac{I_{11}}{6} + \frac{4}{3}I_{12} \right). \end{aligned} \quad (15)$$

The integrals $I_{5,7}$ in eq. (15) have been defined in [0] and their analytical form is reported in the appendix, together with the definition of the new integrals $I_{8,12}$.

3 Renormalization of New Improved Four-Fermion Operators

In this section we consider the four-fermion operators

$$O_{\pm} = \frac{1}{2}[(L_{12} \times L_{34}) \pm (2 \leftrightarrow 4)], \quad (16)$$

$$O_{1,2} = [c_{1,2}(L_{12} \times R_{34}) + (T^a L_{12} \times T^a R_{34})] \quad (17)$$

and

$$O_{\mu\nu}^{\pm} = \left\{ [L_{12}(\mu) \times L_{34}(\nu) + (\mu \leftrightarrow \nu)] - \frac{\delta_{\mu\nu}}{2}(L_{12} \times L_{34}) \mp (2 \leftrightarrow 4) \right\} \quad (18)$$

which are relevant for the evaluation of effective weak Hamiltonian matrix elements. We have defined

$$\begin{aligned} c_1 &= -C_F = -\left(\frac{N^2 - 1}{2N}\right), \\ c_2 &= \frac{1}{2C_A} = \frac{1}{2N} \end{aligned} \quad (19)$$

and

$$\begin{aligned} (L_{12} \times L_{34}) &\equiv \sum_{\mu} L_{12}(\mu) \times L_{34}(\mu), \\ (T^a L_{12} \times T^a L_{34}) &\equiv \sum_{\mu} T^a L_{12}(\mu) \times T^a L_{34}(\mu), \\ L_{12}(\mu) &\equiv \bar{\psi}_1(x) \gamma_{\mu}^L \psi_2(x), \\ T^a L_{12}(\mu) &\equiv \bar{\psi}_1(x) T^a \gamma_{\mu}^L \psi_2(x). \end{aligned} \quad (20)$$

Analogous definitions hold by replacing left with right. By ψ_j ($j = 1, 2, 3, 4$) we denote the quark field of flavour j .

We present the results for the finite renormalization constants and the mixing coefficient of the corresponding new improved operators

$$O_{\pm}^I = \frac{1}{2}[(L_{12} \times L_{34}) + (L_{12}^{\otimes} \times L_{34}) + (L_{12} \times L_{34}^{\otimes}) + (L_{12}^{\otimes} \times L_{34}^{\otimes}) \pm (2 \leftrightarrow 4)], \quad (21)$$

$$\begin{aligned} O_{1,2}^I &= \left\{ c_{1,2}(L_{12} \times R_{34}) + (T^a L_{12} \times T^a R_{34}) \right. \\ &\quad + c_{1,2}[(L_{12} \times R_{34}^{\otimes}) + (L_{12}^{\otimes} \times R_{34}) + (L_{12}^{\otimes} \times R_{34}^{\otimes})] \\ &\quad \left. + (T^a L_{12} \times T^a R_{34}^{\otimes}) + (T^a L_{12}^{\otimes} \times T^a R_{34}) + (T^a L_{12}^{\otimes} \times T^a R_{34}^{\otimes}) \right\} \end{aligned} \quad (22)$$

and

$$\begin{aligned}
O_{\mu\nu}^{I(\pm)} = & \left\{ [L_{12}(\mu) \times L_{34}(\nu) + L_{12}^{\otimes}(\mu) \times L_{34}(\nu) \right. \\
& + L_{12}(\mu) \times L_{34}^{\otimes}(\nu) + L_{12}^{\otimes}(\mu) \times L_{34}^{\otimes}(\nu) + (\mu \leftrightarrow \nu)] \\
& - \frac{\delta_{\mu\nu}}{2} ((L_{12} \times L_{34}) + (L_{12}^{\otimes} \times L_{34}) + (L_{12} \times L_{34}^{\otimes}) + (L_{12}^{\otimes} \times L_{34}^{\otimes})) \\
& \left. \mp (2 \leftrightarrow 4) \right\}. \tag{23}
\end{aligned}$$

The rotated part of the operators (21)-(23) now includes higher orders in a , since we have defined

$$L_{12}^{\otimes}(\mu) \equiv a \frac{r}{2} \bar{\psi}_1(x) (\overleftarrow{\mathcal{D}} \gamma_{\mu}^L - \gamma_{\mu}^L \overrightarrow{\mathcal{D}}) \psi_2(x) - a^2 \frac{r^2}{4} \bar{\psi}_1(x) (\overleftarrow{\mathcal{D}} \gamma_{\mu}^L \overrightarrow{\mathcal{D}}) \psi_2(x) \tag{24}$$

(and the same for left $\leftrightarrow$ right). Products of the form $(L_{12} \times L_{34}^{\otimes})$ and $(L_{12}^{\otimes} \times L_{34}^{\otimes})$ are defined analogously to the definitions (20).

We perform the matching between continuum and lattice theories using the dimensional reduction scheme (DRED) to renormalize the continuum; the results can be written in the same form as in the unimproved case [0], that is

$$\begin{aligned}
(O_{\pm})^{CONT} = & \left[1 + \frac{g_0^2}{16\pi^2} F_{\pm} \right] (O_{\pm}^I)^{LATT} + \frac{g_0^2}{16\pi^2} F^* [(O_{\pm}^{VA})^{LATT} \\
& + (O_{\pm}^{SP})^{LATT} + (O_{\pm}^{STP})^{LATT}], \tag{25}
\end{aligned}$$

$$\begin{aligned}
(O_1)^{CONT} = & \left[1 + \frac{g_0^2}{16\pi^2} F_1 \right] (O_1^I)^{LATT} + \frac{g_0^2}{16\pi^2} F^* [(O_1^{VA})^{LATT} \\
& + (O_1^{SP})^{LATT} + (O_1^{STP})^{LATT}], \tag{26}
\end{aligned}$$

$$\begin{aligned}
(O_2)^{CONT} = & \left[1 + \frac{g_0^2}{16\pi^2} F_2 \right] (O_2^I)^{LATT} + \frac{g_0^2}{16\pi^2} F^* [(O_2^{VA})^{LATT} \\
& + (O_2^{SP})^{LATT} + (O_2^{STP})^{LATT}] \tag{27}
\end{aligned}$$

and

$$(O_{\mu\nu}^{\pm})^{CONT} = \left[1 + \frac{g_0^2}{16\pi^2} F_{\mu\nu}^{\pm} \right] (O_{\mu\nu}^{I(\pm)})^{LATT} + \frac{g_0^2}{16\pi^2} F^* [(O_{\mu\nu}^{VA(\pm)})^{LATT} + (O_{\mu\nu}^{TT(\pm)})^{LATT}]. \tag{28}$$

In eqs.(25)-(27), $O_i^{VA,SP,SPT}$ ($i = +, -, 1, 2$) are the usual local operators defined as in refs.[0, 0], while the local operators $O_{\mu\nu}^{VA(\pm)}$ and $O_{\mu\nu}^{TT(\pm)}$ that appear in eq. (28) are defined as:

$$\begin{aligned}
O_{\mu\nu}^{VA(\pm)} = & - \left(\frac{N^2 - 1 \pm N}{16N} \right) \left\{ [(\bar{\psi}_1 \gamma_{\mu} \psi_2)(\bar{\psi}_3 \gamma_{\nu} \psi_4) - (\bar{\psi}_1 \gamma_{\mu} \gamma_5 \psi_2)(\bar{\psi}_3 \gamma_{\nu} \gamma_5 \psi_4) + (\mu \leftrightarrow \nu)] \right. \\
& \left. - \frac{\delta_{\mu\nu}}{2} \sum_{\beta} [(\bar{\psi}_1 \gamma_{\beta} \psi_2)(\bar{\psi}_3 \gamma_{\beta} \psi_4) - (\bar{\psi}_1 \gamma_{\beta} \gamma_5 \psi_2)(\bar{\psi}_3 \gamma_{\beta} \gamma_5 \psi_4)] \mp (2 \leftrightarrow 4) \right\}, \\
O_{\mu\nu}^{TT(\pm)} = & \left(\frac{1}{16N} \right) \left\{ \left[\sum_{\beta} (\bar{\psi}_1 \sigma_{\mu\beta} \psi_2)(\bar{\psi}_3 \sigma_{\nu\beta} \psi_4) + (\mu \leftrightarrow \nu) \right] \right. \\
& \left. - \frac{\delta_{\mu\nu}}{2} \sum_{\alpha\beta} (\bar{\psi}_1 \sigma_{\alpha\beta} \psi_2)(\bar{\psi}_3 \sigma_{\alpha\beta} \psi_4) \mp (2 \leftrightarrow 4) \right\}. \tag{29}
\end{aligned}$$

The values of the finite quantities $F_{\pm}$, $F_{1,2}$, $F_{\mu\nu}^{\pm}$ and F^* are given in table 4. Following ref. [0], their analytical form can be expressed as

$$\begin{aligned}
F_{\pm} &= \left(\frac{N^2-1}{2N}\right) [\Delta_{VA}^+ + (\Delta_{VA}^+)^{\otimes}] + \left(\frac{\pm N-1}{2N}\right) [\Delta_{VA}^+ + (\delta_{VA}^+)^{\otimes}] \\
&\quad + \left(\frac{1 \mp N}{2N}\right) [\Delta_{SP}^+ + (\delta_{SP}^+)^{\otimes}] + \left(\frac{N^2-1}{N}\right) [\Delta_{\Sigma_1}], \\
F_1 &= \left(\frac{N^2-1}{2N}\right) [\Delta_{VA}^+ + (\Delta_{VA}^+)^{\otimes}] + \frac{1}{2N} [\Delta_{VA}^+ + (\delta_{VA}^+)^{\otimes}] \\
&\quad - \frac{1}{2N} [\Delta_{SP}^+ + (\delta_{SP}^+)^{\otimes}] + \left(\frac{N^2-1}{N}\right) [\Delta_{\Sigma_1}], \\
F_2 &= \left(\frac{N^2-1}{2N}\right) [(\Delta_{VA}^+)^{\otimes} - (\delta_{VA}^+)^{\otimes}] + \left(\frac{N^2-1}{2N}\right) [\Delta_{SP}^+ + (\delta_{SP}^+)^{\otimes}] + \left(\frac{N^2-1}{N}\right) [\Delta_{\Sigma_1}], \\
F_{\mu\nu}^{\pm} &= \left(\frac{N^2-1}{2N}\right) [\Delta_{VA}^+ + (\Delta_{VA}^+)^{\otimes}] + \left(\frac{\pm N-1}{2N}\right) [\Delta_{VA}^+ + (\delta_{VA}^+)^{\otimes}] \\
&\quad + \left(\frac{1 \mp N}{N}\right) [\Delta_{\sigma\mu\nu}^{LOC} + \delta_{\sigma\mu\nu}^{\otimes}] + \left(\frac{N^2-1}{N}\right) [\Delta_{\Sigma_1}], \\
F^* &= (-2) [\Delta_{VA}^- + (\Delta_{VA}^-)^{\otimes}] = [\Delta_{SP}^- + (\Delta_{SP}^-)^{\otimes}].
\end{aligned} \tag{30}$$

In eq. (30) we have defined

$$\begin{aligned}
\Delta_{VA}^{\pm} &= \Delta_{\gamma\mu}^{LOC} \pm \Delta_{\gamma_5\gamma\mu}^{LOC}, & \Delta_{SP}^{\pm} &= \Delta_1^{LOC} \pm \Delta_{\gamma_5}^{LOC}; \\
(\Delta_{VA}^{\pm})^{\otimes} &= \Delta_{\gamma\mu}^{\otimes} \pm \Delta_{\gamma_5\gamma\mu}^{\otimes}, & (\Delta_{SP}^{\pm})^{\otimes} &= \Delta_1^{\otimes} \pm \Delta_{\gamma_5}^{\otimes}; \\
(\delta_{VA}^+)^{\otimes} &= \delta_{\gamma\mu}^{\otimes} + \delta_{\gamma_5\gamma\mu}^{\otimes}, & (\delta_{SP}^+)^{\otimes} &= \delta_1^{\otimes} + \delta_{\gamma_5}^{\otimes}.
\end{aligned} \tag{31}$$

The quantities Δ_{Σ_1} , Δ_{Γ}^{LOC} and $\Delta_{\Gamma}^{\otimes}$ have been defined in section 2, and their numerical values have been given in tables 2 and 3. The new quantities $\delta_{\Gamma}^{\otimes}$ can be expressed in terms of the integrals $I_{5,7}$ of refs. [0, 0] and of the new integrals $I_{8,12}$ to be defined in the appendix

$$\begin{aligned}
\delta_1^{\otimes} &= 4\pi^2 \left(-4I_6 + \frac{I_7}{2} + I_8 + I_9 - 3I_{10} + I_{11} \right), \\
\delta_{\gamma_5}^{\otimes} &= 4\pi^2 \left(-\frac{I_7}{2} - I_8 + I_9 + I_{10} \right), \\
\delta_{\gamma\mu}^{\otimes} &= 4\pi^2 \left(-2I_5 + I_6 - \frac{I_7}{4} - \frac{I_8}{2} - \frac{I_{11}}{4} + I_{12} \right), \\
\delta_{\gamma_5\gamma\mu}^{\otimes} &= 4\pi^2 \left(-2I_5 - I_6 + \frac{I_7}{4} + \frac{I_8}{2} - 2I_{10} + \frac{I_{11}}{4} + I_{12} \right), \\
\delta_{\sigma\mu\nu}^{\otimes} &= 4\pi^2 \left(-\frac{8}{3}I_5 + \frac{2}{3}I_6 - \frac{I_9}{3} - I_{10} - \frac{I_{11}}{6} + \frac{4}{3}I_{12} \right).
\end{aligned} \tag{32}$$

4 New Improved Heavy-Light Operator for $B\bar{B}$ Mixing

As has been shown in ref. [0], if b quarks are treated in the static approximation [0], in order to get $O(a)$ improved heavy-light matrix elements, it is sufficient to use the improved theory for light-quark only without modifying the heavy sector. As a consequence of the

fact that only light-quark fields are subjected to the rotation (1), no $O(a^2)$ terms can appear in heavy-light bilinear. Thus in this section we only need to consider four-fermion operators. In particular we consider the four-fermion heavy-light operator involved in $B\overline{B}$ mixing

$$O_L = \frac{1}{2} (\overline{b} \gamma_\mu^L q) (\overline{b} \gamma_\mu^L q), \quad (33)$$

where q denotes a generic light-quark field. The corresponding improved lattice operator is defined as

$$O_L^I = \frac{1}{2} \left[\overline{b} \gamma_\mu^L q \left(1 - a \frac{r}{2} \not{p} \right) \right] \left[\overline{b} \gamma_\mu^L q \left(1 - a \frac{r}{2} \not{p} \right) \right], \quad (34)$$

where we keep the $O(a^2)$ term, which was not included in the computation of ref. [0].

Following [0], we will match this operator at one-loop to the one computed in the continuum full theory, in the limit $m_b \rightarrow \infty$, using the $\overline{MS}$ naïve dimensional renormalization scheme, with fully anticommuting γ_5 . The calculation is done in two steps: the first step, which relates in the continuum the full operator to the effective one, gives the result [0]

$$O_L^{FULL} = \left(1 + \frac{g_0^2}{16\pi^2} \left(-6 \log \frac{\mu^2}{m_b^2} + C_L \right) \right) O_L^{EFF} + \frac{g_0^2}{16\pi^2} C_S O_S^{EFF}, \quad (35)$$

where $C_L = -14$, $C_S = -8$ and

$$O_S = (\overline{b} \not{P}_L q) (\overline{b} \not{P}_L q), \quad (36)$$

$$P_{R,L} = \frac{1 \pm \gamma^5}{2}. \quad (37)$$

In eq. (35), μ is the continuum renormalization point and m_b is the b -quark mass. The result of this first step obviously does not depend on the lattice definition of the theory. The second step of the matching is the one between continuum and lattice effective theories. For the new improved operator (34), by choosing $\mu = a^{-1}$ we get the result

$$O_L^{EFF} = \left(1 + \frac{g_0^2}{16\pi^2} F_L \right) (O_L^I)^{LATT} + \frac{g_0^2}{16\pi^2} [F_N O_N^{LATT} + F_R O_R^{LATT}], \quad (38)$$

where

$$O_R = \frac{1}{2} (\overline{b} \gamma_\mu^R q) (\overline{b} \gamma_\mu^R q), \quad (39)$$

$$\begin{aligned} O_N = & \frac{1}{2} [(\overline{b} \gamma_\mu^R q) (\overline{b} \gamma_\mu^L q) + (\overline{b} \gamma_\mu^L q) (\overline{b} \gamma_\mu^R q) \\ & + 2 (\overline{b} \not{P}_R q) (\overline{b} \not{P}_L q) + 2 (\overline{b} \not{P}_L q) (\overline{b} \not{P}_R q)]. \end{aligned} \quad (40)$$

The values of the finite quantities F_L , F_N , and F_R are given in table 5, where, in analogy with the calculation in the unimproved case, we choose for the Wilson parameter r the same values as in ref. [0].

In terms of the quantities $D_{L,N,R}$ and $D_{L,N,R}^I$, defined respectively in refs.[0] and [0], we can write

$$\begin{aligned} F_L &= D_L + D_L^I + \delta_L \\ F_N &= D_N + D_N^I \\ F_R &= D_R + D_R^I + \delta_R. \end{aligned} \tag{41}$$

When comparing the numerical values of F_L in table 5 with those of refs.[0, 0], it is important to notice that the value we use for D_L takes into account the fact that the time dependence of Monte Carlo data is usually fitted to correlation functions of the form

$$A e^{-Bt}. \tag{42}$$

When using the fit (42), the heavy-quark self-energy contributes to D_L with a “reduced” value that includes the perturbative contribution of the linearly divergent part to the heavy-wave function renormalization. On this subject, see also refs.[0, 0].

In eq. (41) the new contributions $\delta_{L,R}$ can be expressed in terms of the integrals $I_{8, 11}$ defined in the appendix, and they have the form

$$\begin{aligned} \delta_L &= -\frac{4}{3}\pi^2 \left(I_9 - I_{10} + \frac{1}{2}I_{11} \right) \\ \delta_R &= -\frac{4}{3}\pi^2 \left(I_8 - 2I_{10} + \frac{1}{2}I_{11} \right). \end{aligned} \tag{43}$$

Finally, as an example, we compute the numerical factors which relate the quantities $f_B^2 B_B$ and B_B in the $\overline{MS}$ continuum renormalization scheme at a scale $\mu = 2$ GeV and for $m_b = 5$ GeV, to the same quantities measured on a lattice with parameters: $\beta = 6.0$, i.e. $\alpha^{LATT} = 0.08$, $a^{-1} = \mu = 2$ GeV and $r = 1$. We find that, within our precision, the inclusion of the new contribution of terms of $O(a^2)$ does not modify the previous numerical results presented in ref. [0], that is

$$(f_B B_B)^{CONT} = (0.80)(f_B^2 B_B)^{LATT} \tag{44}$$

and

$$B_B^{CONT} = (1.02)B_B^{LATT}. \tag{45}$$

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Appendix

In this appendix we report the analytical expression of the integrals $I_{5,7}$ defined in ref. [0], together with the definition of the new integrals $I_{8,12}$ which appear in eqs.(15), (32) and (43)

$$\begin{aligned}
I_5 &= \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{4r^2}{\Delta_2^2} \left(\Delta_4 - \frac{\Delta_3^2}{\Delta_1} \right) \right\}, \\
I_6 &= \int \frac{d^4 k}{(2\pi)^4} \left\{ \frac{4r^2}{\Delta_2^2} \left(\Delta_3 \Delta_5 + r^2 (\Delta_4 - \Delta_3 \Delta_5) (\Delta_3 - \Delta_1) \right. \right. \\
&\quad \left. \left. - \frac{\Delta_3}{\Delta_1} (\Delta_3 \Delta_5 + \Delta_3 - \Delta_4) \right) \right\}, \\
I_7 &= \int \frac{d^4 k}{(2\pi)^4} \left\{ -\frac{4r^2}{\Delta_2} \left(-\Delta_5 + \frac{(\Delta_3 - \Delta_4 + \Delta_3 \Delta_5)}{\Delta_1} \right) \right\}
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
I_8 &= \int \frac{d^4 k}{(2\pi)^4} \left\{ -r^2 \left(\frac{4}{\Delta_1 \Delta_2^2} \left(-\Delta_3^2 \Delta_5 + r^4 \Delta_1^2 (\Delta_3 \Delta_4 - \Delta_3^2 \Delta_5 - \Delta_1 \Delta_3) \right) \right. \right. \\
&\quad \left. \left. + 2 \left(\frac{\Delta_3}{\Delta_1 \Delta_2} (\Delta_5 + r^2 \Delta_1 (1 + \Delta_5)) - r^2 \frac{\Delta_4}{\Delta_2} \right) - \frac{\Delta_5}{4\Delta_1} \right) \right\}, \\
I_9 &= \int \frac{d^4 k}{(2\pi)^4} \left\{ -\frac{4r^4 \Delta_3}{\Delta_1 \Delta_2^2} \left(\Delta_5 \Delta_1^2 + \Delta_3 (\Delta_5 \Delta_3 - \Delta_4) \right) \right\}, \\
I_{10} &= \int \frac{d^4 k}{(2\pi)^4} \left\{ -\frac{4r^4 \Delta_3^2}{\Delta_2^2} \right\}, \\
I_{11} &= \int \frac{d^4 k}{(2\pi)^4} \left\{ -16r^4 \frac{\Delta_3}{\Delta_2^2} (\Delta_4 - \Delta_3 \Delta_5) \right\}, \\
I_{12} &= \int \frac{d^4 k}{(2\pi)^4} \left\{ -4 \frac{r^4}{\Delta_2^2} \Delta_1 \Delta_4 \right\}.
\end{aligned} \tag{47}$$

As in ref. [0], the symbols Δ_i denote:

$$\begin{aligned}
\Delta_1 &= \sum_{\mu} \sin^2\left(\frac{k_{\mu}}{2}\right), \quad \Delta_2 = \sum_{\mu} \sin^2(k_{\mu}) + \left(2r \sum_{\mu} \sin^2\left(\frac{k_{\mu}}{2}\right)\right)^2, \\
\Delta_3 &= \frac{1}{4} \sum_{\mu} \sin^2(k_{\mu}), \quad \Delta_4 = \sum_{\mu} \sin^2\left(\frac{k_{\mu}}{2}\right) \cos^4\left(\frac{k_{\mu}}{2}\right), \quad \Delta_5 = \sum_{\mu} \cos^2\left(\frac{k_{\mu}}{2}\right).
\end{aligned} \tag{48}$$

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Z_1	Z_{γ_5}	Z_{γ_μ}	$Z_{\gamma_5\gamma_\mu}$	$Z_{\sigma_{\mu\nu}}$
0.94	0.77	0.90	0.98	0.98

Table 1: Numerical values of the renormalization constants of the new operators O_Γ^I of eq. (12) for $r = 1$, $N = 3$ and $g_0^2 = 1$.

r	Δ_1^{LOC}	$\Delta_{\gamma_5}^{\text{LOC}}$	$\Delta_{\gamma_\mu}^{\text{LOC}}$	$\Delta_{\gamma_5\gamma_\mu}^{\text{LOC}}$	$\Delta_{\sigma_{\mu\nu}}^{\text{LOC}}$	Δ_{Σ_1}
0	-33.0	-33.0	-8.8	-8.8	0.7	-6.1
0.2	-21.7	-27.0	-8.3	-5.6	0.2	-6.5
0.4	-14.6	-20.8	-7.4	-4.3	-0.6	-7.4
0.6	-11.9	-17.0	-6.8	-4.3	-1.2	-8.2
0.8	-10.7	-14.6	-6.4	-4.4	-1.7	-8.8
1	-10.0	-13.1	-6.1	-4.6	-1.9	-9.2

Table 2: Numerical values of $\Delta_\Gamma^{\text{LOC}}$ and Δ_{Σ_1} for several values of the Wilson parameter r .

r	$\Delta_1^\otimes$	$\Delta_{\gamma_5}^\otimes$	$\Delta_{\gamma_\mu}^\otimes$	$\Delta_{\gamma_5\gamma_\mu}^\otimes$	$\Delta_{\sigma_{\mu\nu}}^\otimes$
0	0	0	0	0	0
0.2	0.7	-0.1	1.3	1.7	1.9
0.4	3.3	-0.4	2.9	4.8	4.6
0.6	6.3	-1.3	3.7	7.4	6.6
0.8	9.2	-2.6	3.8	9.7	7.9
1	12.0	-4.3	3.5	11.7	8.9

Table 3: Numerical values of the new $\Delta_\Gamma^\otimes$, including the order a^2 contribution.

r	F_+	F_-	F_1	F_2	$F_{\mu\nu}^+$	$F_{\mu\nu}^-$	F^*
0	-22.3	-69.8	-30.2	-101.5	-43.4	-27.6	0
0.2	-18.5	-54.7	-24.5	-78.8	-34.6	-22.5	6.1
0.4	-14.9	-42.2	-19.4	-60.4	-27.0	-17.9	10.0
0.6	-12.8	-35.6	-16.6	-50.9	-23.0	-15.4	12.7
0.8	-11.6	-32.0	-15.0	-45.6	-20.7	-13.9	15.7
1	-10.9	-29.9	-14.1	-42.5	-19.3	-13.0	19.4

Table 4: Numerical values of the finite constants $F_\pm$, $F_{1,2}$ and of the mixing coefficient F^* for several values of the Wilson parameter r and for $N = 3$.

r	F_L	F_N	F_R
0	-31.5	0	0
0.25	-28.3	-8.0	-2.4
0.5	-25.2	-10.5	-3.4
0.75	-23.4	-12.3	-3.9
1	-22.6	-14.0	-4.3

Table 5: Numerical values of the finite constant F_L , and of the mixing coefficients F_N , F_R for several values of the Wilson parameter r and for $N = 3$.