A NOTE ON CHARGE QUANTIZATION
THROUGH ANOMALY CANCELLATION

M. Nowakowski\textsuperscript{(a)},\textsuperscript{1} and A. Pilaftsis\textsuperscript{(b)},\textsuperscript{2}

\textsuperscript{(a)}Inst. für Theoretische Teilchenphysik, Universität Karlsruhe, 7500 Karlsruhe, FRG
\textsuperscript{(b)}Inst. für Physik, Johannes-Gutenberg Universität, 6500 Mainz, FRG

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ABSTRACT

In a minimal extension of the Standard Model, in which new neutral fermions have been introduced, we show that the requirement of vanishing anomalies fixes the hypercharges of all fermions uniquely. This naturally leads to electric charge quantization in this minimal scenario which has features similar to the Standard Model: invariance under the gauge group $SU(2)_L \otimes U(1)_Y$, conservation of the total lepton number and masslessness for the ordinary neutrinos. Such minimal models might arise as low-energy realizations of some heterotic superstring models or $SO(10)$ grand unified theories.

\textsuperscript{1} Present address: Physical Research Laboratory, Navrangpura, Ahmedabad 380 009, INDIA
\textsuperscript{2} Address after 1, Oct. 1993, Rutherford Appleton Laboratory, Chilton, Didcot, Oxon, ENGLAND. E-mail address: pilaftsis@vipmza.physik.uni-mainz.de
Recently it has been shown [1-6] that the charge quantization emerges naturally in the one-generation Standard Model (SM) and some of its extensions from the requirement of vanishing anomalies. This is quite a remarkable discovery if we recall that one of the successes of gauge theories is their property that they are renormalizable if all anomalies present in the theory vanish identically. Thus, in principle, for the mechanism of charge quantization we do not impose new constraints on the theory. However, most of the papers which discuss this subject restrict themselves to the one generation case. Then their salient conclusion is that the charge quantization works in SM with one generation only if the right handed neutrino is absent i.e. the Dirac-neutrino is massless. This might give the impression that such considerations of one generation case tacitly assume that the conclusion holds also for arbitrary number of generations ($N_G$). The assumption would be that the generations are replicas of each other as far as charge quantization is concerned. Whether this is indeed the case is an important question in view of the Solar Neutrino Problem [7], the Simpson Neutrino [8] or even the suggested resolution of the $\tau$-puzzle [9] by a heavy $4th$-generation neutrino [10]. In fact, Refs. [5-6] which take into account more than one generation arrive at the conclusion that the charge is de-quantized in SM with massless Dirac-neutrinos. In [5] and the subsequent publications [6] the proof of this statement relies on general arguments which have to do with global, gaugeable, symmetries of the SM Lagrangian (i.e. the abelian lepton and baryon number conservation on classical level).

In this note we wish to show that models which extend the SM by adding neutral
left-handed and right-handed singlets can naturally account for charge quantization in nature. Such models could be realized in the low-energy limit of the heterotic superstring [11-12] or in $SO(10)$ grand unified theories (GUT’s) [13]. It is also important to notice that such low-energy realizations possess attractive features similar to that of the minimal SM. i.e., apart from their invariance under the standard gauge group $SU(2)_L \otimes U(1)_Y$, they conserve total lepton number and provide massless neutrinos to any order of perturbation theory.

Next, we give a short description of the basic low-energy structure of the model. The relevant part of the Yukawa sector containing the masses of the leptons is given by

$$-\mathcal{L}_Y^{\text{leptons}} = \frac{\phi}{v} \bar{f}_{L_i} m_{l_{ij}} l_{R_j} + \frac{\bar{\phi}}{v} \bar{f}_{L_i} m_{D_{ij}} \nu_{R_j} + \bar{S}_{L_i} M_{ij} \nu_{R_j} + \text{h.c.} \quad (1)$$

with

$$f_{L_i} = \begin{pmatrix} \nu_{L_i} \\ l_{L_i} \end{pmatrix} \quad (2)$$

In Eq. (1) we assume the presence of two singlet neutral fermions $\nu_R$ and $S_L$ for each chiral family. The dimensional $N_G \times N_G$ matrix $M$ can always be brought in a diagonal form by appropriate unitary rotation of the fields $S_L$ and $\nu_R$. Similarly, the charged lepton-mass matrix $m_l$ can also be taken to be diagonal by a basis transformation of the fields $f_{L_i}$ and $l_{R_i}$. However, in this certain weak basis, $m_D$ is in general a non-diagonal $N_G \times N_G$ matrix, since there is not freedom anymore to rotate the fields $\nu_{L_i}$ and $\nu_{R_i}$. The remaining field content for the quarks is completely standard. The Dirac mass matrix $m_D$ is proportional to the vacuum expectation value (VEV) $v$ of the Higgs doublet $\phi$, while $M_{ij}$ can, for simplicity, be considered to be bare masses.
Therefore, no extra Higgs fields are, in principle, required beyond the standard one. The neutrino mass matrix of the model mentioned above provides the existence of \( N_G \) massless neutrinos, whereas the other \( 2N_G \) Weyl fermions pair up into \( N_G \) heavy \textit{Dirac} neutrinos due to the total lepton-number conservation [12]. A surprising feature of this model is the violation of the individual leptonic flavours in spite of the fact that the ordinary known neutrinos are massless [12]. Since the Dirac mass matrix \( m_D \) as predicted by the model under discussion, is generally non-diagonal, its inter-generational mixing structure can naturally explain possible lepton-number violating and \( CP \)-violating signals [14] at the \( Z^0 \) peak.

In an \( SU(N_C) \otimes SU(2)_L \otimes U(1)_Y \) gauge theory the conditions for the anomalies to vanish read [15]

\[
\text{Tr} \left[ U(1)_Y SU(N_C)^2 \right] = \sum_{\text{l.h.p.,quarks}} Y_L - \sum_{\text{r.h.p.,quarks}} Y_R = 0 \quad (3)
\]

\[
\text{Tr} \left[ U(1)_Y SU(2)^2_L \right] = \sum_{\text{all doublets}} Y_L = 0 \quad (4)
\]

\[
\text{Tr} \left[ U(1)_Y^3 \right] = \sum_{\text{l.h.p.}} Y_L^3 - \sum_{\text{r.h.p.}} Y_R^3 = 0 \quad (5)
\]

\[
\text{Tr} \left[ (\text{graviton})^2 U(1)_Y \right] \propto \sum_{\text{l.h.p.}} Y_L - \sum_{\text{r.h.p.}} Y_R = 0 \quad (6)
\]

where l.h.p. (r.h.p.) is a shorthand notation for left handed parts (right handed parts) of the fields assigned according to the chirality operator. For reasons explained in [3] (large \( N_C \) expansion) we will also keep the number of colours \( N_C \) as a free parameter. Eq. (4) is the gravitational anomaly condition [16] which does not play any crucial role here. We will only occasionally mention it. Finally, it should be emphasized that
anomaly conditions (3)-(6) are trace expressions and should be therefore invariant under global basis rotations in the weak space.

In a selfexplanatory way, we introduce the hypercharge assignments

\[ Y_\phi, \ Y_{\phi_i}^L, \ Y_{u_i}^R, \ Y_{d_i}^R, \ Y_{f_i}^L, \ Y_{\nu_i}^R, \ Y_{S_i} \]  

(7)

for the Higgs doublet, the left handed quark doublets and the u- and d-type quarks singlets as well as for the leptons and the additional neutral singlets (the index \( i \) runs from 1 to \( N_G \)). Then, Equations (3)–(6) can be explicitly written as

\[ \sum_{i=1}^{N_G} [2Y_{q_i}^L - Y_{u_i}^R - Y_{d_i}^R] = 0 \]  

(8)

\[ \sum_{i=1}^{N_G} [N_C Y_{\phi_i}^L + Y_{f_i}^L] = 0 \]  

(9)

\[ \sum_{i=1}^{N_G} [2N_C(Y_{q_i}^L)^3 - N_C(Y_{u_i}^R)^3 - N_C(Y_{d_i}^R)^3 + 2(Y_{f_i}^L)^3 + \left( Y_{S_i}\right)^3 - (Y_{\nu_i}^R)^3] = 0 \]  

(10)

\[ \sum_{i=1}^{N_G} [2N_C Y_{q_i}^L - N_C Y_{u_i}^R - N_C Y_{d_i}^R + 2Y_{f_i}^L + Y_{S_i} - Y_{\nu_i}^R - Y_{\nu_i}^R] = 0 \]  

(11)

The anomaly Eqs. (8)–(11) are not the only source of information on hypercharges in the present model. The gauge invariance of the Yukawa terms

\[ \mathcal{L}_{Yukawa} = \mathcal{L}_{\phi} + \mathcal{L}_{\phi} \]

\[ = -\frac{1}{v} \sum_{\alpha,\beta}^{N_G} \left\{ \overline{q}_{L\alpha} \phi M_{\alpha\beta} d_{R\beta} + \overline{q}_{L\alpha} \phi M_{\alpha\beta} u_{R\beta} \right\} + h.c. \]  

(12)

dictates a relation between \( Y_\phi \) and and the quark hypercharge quantum numbers.

From the first term in (12) which couples to \( \phi \) it follows that \( U(1)_Y \) gauge invariance is verified if

\[ Y_{q_i}^L - Y_{\phi} - Y_{d_j}^R = 0 \]  

(13)
where \( i, j = 1, \ldots, N_G \) which includes \( i = j \) as well as \( i \neq j \). From the second term with the dual field \( \tilde{\phi} \) one gets

\[
Y^L_{q_i} + Y_\phi - Y^R_{u_j} = 0
\]  

(14)

Both equations imply that

\[
Y^L_{q_1} = Y^L_{q_2} = Y^L_{q_3} = \ldots \equiv Y^L_q
\]

\[
Y^R_{u_1} = Y^R_{u_2} = Y^R_{u_3} = \ldots \equiv Y^R_u = Y^L_q + Y_\phi
\]

\[
Y^R_{d_1} = Y^R_{d_2} = Y^R_{d_3} = \ldots \equiv Y^R_d = Y^L_q - Y_\phi
\]

(15)

Eq. (15) is also valid when not all elements of \( M^D \) and \( M^U \) are different from zero. For example, Fritzsch-type mass matrices [17] satisfy the above requirement. Other viable scenarios in GUT models are mass matrices of democratic type [18], where all elements of \( M^U \) and \( M^D \) are nonzero. This also corresponds to the fact that mass-matrix elements should generally not vanish without invoking any additional horizontal symmetry beyond the gauge-group symmetry of the model under consideration. In order to naturally require some matrix elements to vanish, one has to introduce extra Higgs fields obeying certain discrete symmetries. This extension, however, is beyond the minimal realization for a wide class of models considered in this work.

The situation for leptons is a little bit more involved. Consider the Lagrangian of Eq. (1) written in the form

\[
\mathcal{L}^{\text{leptons}}_Y = \mathcal{L}^{\text{leptons}}_\phi + \mathcal{L}^{\text{leptons}}_{\tilde{\phi}} + \mathcal{L}^{\text{leptons}}_{\text{Bare}}
\]  

(16)

Then, all the terms \( \mathcal{L}^{\text{leptons}}_\phi \), \( \mathcal{L}^{\text{leptons}}_{\tilde{\phi}} \) and \( \mathcal{L}^{\text{leptons}}_{\text{Bare}} \) must be separately gauge invariant.
under the $U(1)_Y$ gauge transformations

\[
\begin{align*}
  f_{L\alpha} & \to e^{-iY^L_{f\alpha}\beta(x)} f_{L\alpha} \\
  l_{R\alpha} & \to e^{-iY^R_{l\alpha}\beta(x)} l_{R\alpha} \\
  \nu_{R\alpha} & \to e^{-iY^R_{\nu\alpha}\beta(x)} \nu_{R\alpha} \\
  S_{L\alpha} & \to e^{-iY^{S\alpha}\beta(x)} S_{L\alpha} \\
  \phi & \to e^{-iY_{\phi}\beta(x)} \phi \\
  \tilde{\phi} & \to e^{iY_{\phi}\beta(x)} \tilde{\phi}
\end{align*}
\]

Thus, from $\mathcal{L}^{\text{leptons}}$ we obtain

\[
Y^L_{f_i} - Y_{\phi} - Y^R_{\nu_{i}} = 0 \quad (18)
\]

On the other hand, from $\mathcal{L}^{\text{leptons}}$ we get

\[
Y^L_{f_i} + Y_{\phi} - Y^R_{\nu_{i}} = 0 \quad (19)
\]

Similar to the case of quarks, both Eqs. (18) and (19) lead to

\[
\begin{align*}
  Y^L_{f_1} = Y^L_{f_2} = Y^L_{f_3} = \ldots & \equiv Y^L_f \\
  Y^R_{e_1} = Y^R_{e_2} = Y^R_{e_3} = \ldots & \equiv Y^R_e = Y^L_f - Y_{\phi} \\
  Y^R_{\nu_1} = Y^R_{\nu_2} = Y^R_{\nu_3} = \ldots & \equiv Y^R_{\nu} = Y^L_f + Y_{\phi}
\end{align*}
\]

Assuming the strong connection between quark-mass matrices and lepton-mass matrices as dictated by $SO(10)$ models (i.e. they are proportional to each other), it is then obvious that hypercharge for each lepton family is uniquely determined, i.e. $Y^L_f$.

Finally, $U(1)_Y$ invariance of $\mathcal{L}^{\text{leptons}}_{\text{Bare}}$ impose the restriction

\[
Y^R_{\nu_i} = Y^R_{\nu} = Y^L_f + Y_{\phi} \quad (21)
\]
It is now an easy task for us to discuss the solution of (8)-(10). On account of Eq. (15), condition (8) is trivially fulfilled since
\[
\sum_{i=1}^{N_G} [2Y_{q_i}^L - Y_{u_i}^R - Y_{d_i}^R] = N_G [2y_q^L - y_u^R - y_d^R] = N_G [2y_q^L - y_q^L - y_\phi - y_q^L + y_\phi] = 0
\]
(22)
whereas Eq. (9) yields
\[
\sum_{i=1}^{N_G} [N_CY_{q_i}^L + y_{f_i}^L] = N_G [N_CY_q^L + y_f^L] = 0
\]
\[\Rightarrow y_f^L = -NCY_q^L\]  
(23)
Condition (10) can now be written down as follows:
\[
2NC(y_q^L)^3 - NC(y_u^R)^3 - NC(y_d^R)^3 + 2(y_f^L)^3 + (y_S)^3 - (y_e^R)^3 - (y_\nu^R)^3 = 0
\]
(24)
By taking into account Eqs. (15), (20) and (21), we arrive at the following expression:
\[
(y_f^L)^3 + 3y_\phi(y_f^L)^2 + 3y_\phi^2y_f^L + y_\phi^3 = 0
\]
(25)
which is independent of \(N_C\). This relation is similar with that which has been originally derived by Geng and Marshak for the one-generation SM [1]. It is now straightforward to see that Eq. (25) has a unique solution, namely
\[
y_f^L = -y_\phi
\]
(26)
Here, we must remark that regardless condition (8) gravitational anomaly (6) alone leads to the same result. One then gets the correct value for the fermion charges by
\[
Q = I_3 + \frac{1}{2Y_\phi}Y
\]
(27)
together with $N_C = 3$ and the Eqs. (15), (20). We complete our proof by noting that
the presence of $S_{L_i}$ in addition to $\nu_{R_i}$ is generally necessary in such an extension of
the SM. Indeed, if all singlets $S_{L_i}$ are absent and the number of $\nu_{R_i}$ equals the number
of $\nu_{L_i}$, the Lagrangian has a gaugable $B - L$ (i.e. baryon–lepton number) symmetry
and charge will be dequantized [5,6]. If all $S_{L_i}$ and $\nu_{R_i}$ are absent, we recover the
prediction for the minimal SM. Analytically, Eq. (25) is also valid when one considers a
general non-diagonal charged lepton-mass matrix (e.g. the mass matrix of democratic
type mentioned above), but this is not sufficient to restore charge quantization, since
our analysis would then be weak-basis dependent. As mentioned above, the conditions
for vanishing anomalies (3)–(6) are trace expressions and are invariant under global
chiral rotations. This observation allows us to show that the three-generation SM
possesses hidden gaugeable symmetries, i.e. $L_e - L_\mu$ or $L_e - L_\tau$ or $L_\mu - L_\tau$, and
the conclusion of charge de-quantization in the SM follows [5]. This indicates that
the presence of $S_{L_i}$ is absolutely necessary to obtain charge quantization, since these
neutral singlets fix the weak space and protect it from the above chiral rotations which
cannot take place without affecting the $U(1)_Y$ symmetry of the model.

For completeness we mention the case of Majorana interactions [19,20]. Here
we have to include in the Lagrangian additional mass terms of the form

$$\mathcal{L}_{\text{Majorana}}^\nu = -\frac{1}{2}\bar{\nu}_{R_i}m_{M_{ij}}(\nu_{R_j})^C - \frac{1}{2}\bar{S}_{L_i}\mu_{ij}(S_{L_j})^C + \text{h.c.}$$

(28)

with $C$ being the charge conjugation operator. These models lead to Majorana neutrinos even in the absence of the $m_{M_{ij}}$ term (the latter is then sometimes called
"µ"-model [19]). Of course, such models break in general the total lepton number $L$.

Here, under $U(1)_Y$ gauge transformations $S_{L_i}$, $\nu_{R_i}$ transform according to (17) and the corresponding transformation for their charge conjugate fields are given by

$$
(v_{R_j})^C = e^{i Y_{\nu_j}^R \beta(x)} (v_{R_j})^C
$$

$$
(S_{L_j})^C = e^{i Y_{S_j} \beta(x)} (S_{L_j})^C
$$

Imposing $U(1)_Y$-gauge invariance on $\mathcal{L}_{\text{Majorana}}^\nu$, one then gets $Y_{\nu}^R = Y_S = 0$ and charge is quantized. The phenomenology of the models considering right-handed neutrinos in the SM has been recently investigated in [20,21], which can account, among others, for possible lepton-number violating effects in high energy collider machines.

However, the presence of Majorana interactions as described by $\mathcal{L}_{\text{Majorana}}^\nu$ is not a sufficient condition to restore charge quantization. To give such a counter-example example let us write down a ‘bare-bones-like’ model similar to Ref. [22]. Here we have

$$
m_D = \text{diag}(m_1, m_2, m_3)
$$

$$
m_l = \text{diag}(m_e, m_\mu, m_\tau)
$$

and

$$
m_M = \begin{pmatrix}
0 & a & 0 \\
a & 0 & b \\
0 & b & 0
\end{pmatrix}
$$

For the sake of illustration, we assume that $S_{L_i}$ are absent. From gauge invariance one obtains the following set of equations

$$
Y_{\nu_e}^R = Y_{\nu_e} = -Y_{\nu_\mu}^R \equiv Y^R
$$

$$
Y_{f_\tau}^L = Y_{f_\tau}^L = Y^R - Y_\phi, \quad Y_{f_\mu}^L = -Y^R - Y_\phi
$$

$$
Y_{e}^R = Y_{\tau}^R = Y^R - 2Y_\phi, \quad Y_{\mu}^R = -Y^R - 2Y_\phi
$$
However, the requirement of anomaly cancellations yields now ‘only’

$$9Y^L_q + Y^R - 3Y_\phi = 0$$  \hfill (33)

Clearly $Y^R$ remains undetermined and the charge would be de-quantized. This can be attributed to the fact that the Yukawa sector as described by the special form of Eqs. (30) and (31) possesses the local hidden symmetry $L = L_e - L_\mu + L_\tau$ which dequantizes charge. Due to this hidden symmetry, the Weyl mass-eigenstates are degenerate in pairs and hence form Dirac neutrino fields. Nevertheless, if in this certain weak basis we assume that the matrix $m_M$ contains five nonzero elements, for example, the above symmetry spoils and charge is quantized. Furthermore, one then gets a mass spectrum of six non-degenerate Majorana neutrinos.

In summary, there are generally three minimal extensions of the Standard Model by including new neutral singlets, in which the charge of fermions is quantized:

(i) SM with non-degenerate Majorana neutrinos.

(ii) SM with Dirac neutrinos where, however, the number of right handed neutrino fields ($N^\nu_R$) must not equal the number of left handed ones ($N^\nu_L$) and with the additional restriction $N^\nu_R \neq 0$ (see also [6]).

(iii) the models with additional left- and right-handed neutral singlets, where quark- and lepton-mass matrices are proportional to each other as dictated by a wide class of GUT models. The last scenario has been discussed in the present work.
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