**Sphalerons and Normal Modes
in the 1+1-dimensional Abelian Higgs Model on the Circle**

Yves Brihaye

CERN, Geneva, Switzerland

Stefan Giller, Piotr Kosinski

Department of Theoretical Physics, University of Lodz

Pomorska 149/153, 90236 Lodz, Poland

Jutta Kunz

Instituut voor Theoretische Fysica, Rijksuniversiteit te Utrecht

NL-3508 TA Utrecht, The Netherlands

and

FB Physik, Universität Oldenburg, Postfach 2503

D-2900 Oldenburg, Germany

Abstract

The Abelian Higgs model is considered on the circle. The periodic sphaleron solutions are constructed explicitly. The equations for the normal modes about these solutions resemble Lamé equations. For special values of the Higgs field mass a number of modes are obtained analytically, including in particular the negative modes.

INTRODUCTION

In the standard model baryon and lepton number are not strictly conserved [1]. In fact, the non-conservation of baryon and lepton number is a non-perturbative phenomenon. Baryon number violating processes involving instantons may be relevant at very high energies [2,3], while sphaleron [4] mediated processes are relevant at high temperatures in the early universe [5,6]. In particular, the baryon asymmetry of the universe may possibly be obtained at the electroweak phase transition.

The sphaleron of the Weinberg-Salam theory is an unstable solution of the classical field equations, describing the top of the energy barrier between topologically inequivalent vacua. It is known only approximately via the numerical solution of a set of coupled non-linear differential equations [4,7]. Likewise, the normal modes of oscillation about the sphaleron are known only numerically. They enter into the baryon decay rate, along with the classical energy of the sphaleron. Attempts to evaluate the baryon decay rate at finite temperature are confronted with many technical problems, aggravated by the lack of a closed form for the classical solution and for the normal modes [8]. This is a motivation to study simpler models. In this respect the Abelian Higgs model in $1 + 1$ dimensions is a particularly useful model [9-12].

Manton and Samols [13] performed a very interesting study of sphalerons on the circle for the simple $\lambda\phi^4$ model. Considering the space variable on the circle they constructed the sphalerons analytically, treating the radius of the circle as a parameter. Only recently was it realized [14] that the mode equation investigated in ref. [13] is in fact a Lamé equation, whose lowest solutions can be expressed in terms of Lamé polynomials and thus be given analytically, too.

In this paper, we consider the Abelian Higgs model on the circle, and we analyze the classical solutions and the normal modes in the same spirit as Manton and Samols [13]. The periodic sphaleron solutions are given analytically in terms of the Jacobi function $\text{sn}(z)$. The equations for the normal modes are technically more complicated than those of refs. [13,14], because one channel of the mode equations consists of a system of coupled, but Lamé-like, Schrödinger equations which depend on two external parameters: the radius of the circle and the ratio of the masses of the gauge and Higgs bosons. In spite of these complications we have obtained analytical solutions for special values of the mass ratio. The solutions are then given in terms of the Jacobi functions $\text{sn}(z)$, $\text{cn}(z)$ and $\text{dn}(z)$.

CLASSICAL SOLUTIONS

We start from the Lagrangian density in 1 + 1-dimensions

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\phi)(D_\mu\phi)^* - V(\phi) , \quad (1)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

$$D_\mu\phi = (\partial_\mu - igA_\mu)\phi ,$$

$$V(\phi) = \lambda(|\phi|^2 - \frac{v^2}{2})^2 ,$$

where, along with refs. [9,13,14], we assume that the space coordinate takes value on a circle of length $\tilde{L}$. For later convenience we define $L \equiv M_H\tilde{L}/2$. The Higgs potential V spontaneously breaks the U(1) symmetry, yielding a gauge boson of mass $M_W = gv$ and a scalar Higgs particle of mass $M_H = \sqrt{2\lambda}v$.

We now consider the classical equations of motion of the Lagrangian (1). Besides the vacua, a trivial static finite energy solution exists for all values of L . This constant sphaleron solution corresponds to the maximum of the Higgs potential. Non-trivial solutions of the equations of motion exist for sufficiently large values of L , in fact for

$$L > L_1 , \quad L_1 = \frac{\pi}{\sqrt{2}} . \quad (2)$$

This critical value of L_1 is to be contrasted with the one obtained in the framework of the non-gauge model: $L_1 = \sqrt{2}\pi$ (ref. [13]).

In the $A_0 = 0$ gauge, when the residual gauge symmetry under space-dependent transformations is exploited to eliminate the periodic part in the phase of the Higgs field, the non-trivial static solutions read

$$\begin{aligned} \phi(x) &= \frac{v}{\sqrt{2}} \exp\left(\frac{2i\pi x q}{\tilde{L}}\right) \Phi\left(\frac{M_H}{2}x\right) , \\ A_0 &= 0 , \\ A_1 &= \frac{2\pi q}{g\tilde{L}} . \end{aligned} \quad (3)$$

Here x represents the space coordinate and q represents the Chern-Simons charge of the solution. As shown below, periodicity requires $2q$ to be an integer.

The function $\Phi(z)$ obeys the well-known kink equation

$$\frac{d^2}{dz^2}\Phi = 2\Phi(\Phi^2 - 1) , \quad (4)$$

whose periodic solutions are given in terms of the Jacobi elliptic function $\text{sn}(z)$:

$$\Phi(z) = kb(k)\text{sn}(b(k)z, k) , \quad b^2(k) = \frac{2}{1+k^2} . \quad (5)$$

Here k , ($0 \leq k \leq 1$), is a parameter which fixes the period of the function $\text{sn}(z)$. In fact, the period of the function $\text{sn}(z, k)$ in its first argument is $4K(k)$, where $K(k)$ denotes the complete elliptic function of the first kind.

In order for ϕ , as given in eq. (3), to be a periodic function on the interval $[0, \tilde{L}]$, the length of the circle L and the parameter k should be related as follows

$$L = \frac{2mK(k)}{b(k)} , \quad (6a)$$

and q should be

$$q = \frac{1}{2} + n \quad \text{if } m \text{ is odd} , \quad q = n \quad \text{if } m \text{ is even} . \quad (6b)$$

It is understood that m and n are integers and $m > 0$.

The solution obtained for $m = 1$ has the lowest energy, and is therefore referred to as *the* sphaleron of the model. (We show below that it has a single direction of instability.) Defining the dimensionless variable z as

$$z_\mu = \frac{M_H}{2} x_\mu , \quad (7)$$

the energy of the sphaleron takes the form

$$E_{sp} = \sqrt{2}\lambda v^3 E(k) , \quad E(k) = \int_0^{2K(k)} \left[\left(\frac{d\Phi}{dz} \right)^2 + (\Phi^2 - 1)^2 \right] dz . \quad (8)$$

$E(k)$ is finite and monotonically increasing from $k = 0$ ($L = L_1$) to $k = 1$ ($L = \infty$):

$$E(0) = \frac{\pi}{4\sqrt{2}} , \quad E(1) = \frac{4}{3} . \quad (9)$$

For $L \geq mL_1$ new branches of solutions appear, which can be considered successions of m sphalerons. Therefore we will refer to them as m -sphalerons in the following. Their energies branch off from the energy branch of the constant solution $\phi = 0$ like in Fig. 1 of ref. [13].

NORMAL MODES

To study the normal modes of oscillation about the solutions (3), we consider general fluctuations around these configurations, which are considered as a classical background. We parametrize the fields using the notation of ref. [12]:

$$\begin{aligned}\phi &= \frac{1}{\sqrt{2}} \exp\left(\frac{2i\pi qx}{\tilde{L}}\right) i \left(v\Phi(z_1) + \eta_1(z_0, z_1) + i\eta_2(z_0, z_1) \right) , \\ A_0 &= a_0(z_0, z_1) , \\ A_1 &= \frac{2\pi q}{g\tilde{L}} + a_1(z_0, z_1) .\end{aligned}\tag{10}$$

The functions a_0, a_1 are periodic (in z_1) on $[0, L]$, while the functions η_1, η_2 are periodic (resp. anti-periodic) if $2q$ is even (resp. odd), which is due to the phase factor in front of ϕ in eq. (10).

Let us now consider the Euclidean version of the action given above (eq. 1), which is suitable for finite temperature. We fix the gauge degrees of freedom by choosing the background gauge condition

$$G(a, \eta) = -\frac{\partial a_0}{\partial z_0} - \frac{\partial a_1}{\partial z_1} + \frac{i}{2} \theta (\eta^* \Phi - \eta \Phi^*) , \quad \theta = \frac{2M_H}{M_W} .\tag{11}$$

The gauge fixed action density then reads

$$\mathcal{S}_{gf} = \frac{1}{4} (F_{\mu\nu})^2 + |D_\mu \phi|^2 + V(\phi) + \frac{1}{2} G^2 .\tag{12}$$

Expanding the action (12) in powers of the fluctuations, the normal modes about the classical solutions obey the following set of Schrödinger equations (obtained from the quadratic form in the fluctuations)

$$(-\partial_1 \partial_1 + 6\Phi^2 - 2)\eta_1 = \omega^2 \eta_1 ,\tag{13}$$

$$(-\partial_1 \partial_1 + \theta^2 \Phi^2) a_0 = \omega^2 a_0 ,\tag{14}$$

$$\begin{pmatrix} -\partial_1 \partial_1 + \theta^2 \Phi^2 & 2\theta \Phi' \\ 2\theta \Phi' & -\partial_1 \partial_1 + (\theta^2 + 2)\Phi^2 - 2 \end{pmatrix} \begin{pmatrix} a_1 \\ \eta_2 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ \eta_2 \end{pmatrix} .\tag{15}$$

The equations for the fluctuations a_0 and η_1 decouple, while the equations for a_1 and η_2 remain coupled in general. However in the case $L = L_1$ (i.e. $k = 0$, $\Phi = 0$) and in the case $\theta = 0$ the system (15) decouples obviously. It can also be decoupled by a

constant transformation in the case $L = \infty$ (i.e. $k=1$). This can be seen by using the identity

$$\frac{d\Phi}{dz} = 1 - \Phi^2 , \quad (16)$$

fulfilled by the classical solution $\Phi = \tanh(z_1)$, obtained in the limit $L \rightarrow \infty$.

Eq. (13) is the central equation in ref. [13]. Recently, it was revisited in ref. [14] in the light of the Lamé equation and its lowest solutions were presented in terms of the Lamé polynomials. (For completeness we give some of these polynomials in Appendix A.) Therefore, we refrain from presenting more details about the analytical solutions of eq. (13). Just note that for the sphaleron solution (i.e. $m = 1$) due to the anti-periodic conditions for the function η_1 , the first relevant eigenvalue for the gauge model is a zero mode given (in the notations of Appendix A) by

$$N = 2 , \quad \eta_1 = cdE_2^0 , \quad \omega^2 = b^2(k)h_2^{(cd)0} - 2 \equiv 0 .$$

This zero mode is associated with the translational invariance of the model.

Equation (14) is also a Lamé equation. For $\theta^2 = N(N+1)$, N integer, $2N+1$ of the eigenvectors are given by the Lamé polynomials (given in Appendix A for $N = 0, 1, 2, 3$). This sector, describing the a_0 fluctuations, contains only positive modes.

We now turn to the coupled system (15). First note that the equations about the m -sphaleron can be easily solved in the limit $L = mL_1$, since then $k = 0$ and $\Phi = 0$, and the system decouples into two harmonic equations.

For even values of m and $L = mL_1$ the eigenvalues organize as follows

$$\omega^2(m, j) = -2 + 2\left(\frac{2j}{m}\right)^2 , \quad \omega^2(m, j) = 2\left(\frac{2j}{m}\right)^2 , \quad (17a)$$

j being an integer which labels the states. For odd values of m one needs to take into account the anti-periodic boundary conditions satisfied by the η fluctuations. Here it is convenient to double the length of the circle, but to retain only those modes that are compatible with the periodicity conditions. These are given by the subset

$$\omega^2(m, j) = -2 + 2\left(\frac{2j+1}{m}\right)^2 , \quad \omega^2(m, j) = 2\left(\frac{2j}{m}\right)^2 , \quad (17b)$$

of the set (17a).

Now we apply this strategy to the system (15), choosing $m = 2$. The eigenmodes of the sphaleron then appear automatically by selecting only the subset (17b). When

$\theta^2 = N(N + 1)$, and N a non-negative integer, the first few (lowest) modes of eq. (15) can be obtained analytically by expanding each component as a linear combination of the Lamé polynomials of degree less or equal to $N + 1$. (Only polynomials with the same parity enter in the combinations.) In the case $L = 2L_1$ the eigenvalues have the following order

$$-2 (1) \quad , \quad 0 (3) \quad , \quad 2 (2) \quad , \quad 6 (2) \quad , \quad 8 (2) \quad , \dots , \quad 2j^2 - 2 (2) \quad , \quad 2j^2 (2) \quad ,$$

with multiplicities indicated in parenthesis. The above degeneracies are lifted for $k \neq 0$. In the following we denote the branches appearing in the above list by

$$a \ ; \ b, c, d \ ; \ e, f \ ; \ g, h \ ; \ i, j \ ; \dots \ ,$$

respectively. Also, we define

$$a_1 = F \quad , \quad \eta_2 = \theta G \quad . \quad (18)$$

The eigenvalues ω^2 and the eigenfunctions F and G , that are obtained from the Lamé polynomials, then read for $N = 0, 1$ and 2

N=0

$$\begin{aligned} \omega_a^2 &= -b^2, & F &= 0, & G &= \text{dn}, \\ \omega_b^2 &= -k^2 b^2, & F &= 0, & G &= \text{cn}, \\ \omega_c^2 &= 0, & F &= 1, & G &= 0, \\ \omega_d^2 &= 0, & F &= 0, & G &= \Phi, \end{aligned} \quad (19)$$

N=1

$$\begin{aligned} \omega_a^2 &= -2, & F &= \Phi', & G &= \Phi^2 - 1, \\ \omega_b^2 &= -2kb^2, & F &= \Phi^2 + \frac{\omega_b^2}{2}, & G &= \Phi', \\ \omega_c^2 &= k^2 b^2, & F &= \Phi \text{cn}, & G &= -\Phi \frac{\text{dn}}{k}, \\ \omega_d^2 &= 2kb^2, & F &= \Phi^2 + \frac{\omega_d^2}{2}, & G &= \Phi', \\ \omega_e^2 &= b^2, & F &= \Phi \text{dn}, & G &= -k\Phi \text{cn}, \\ \omega_f^2 &= 2, & F &= \Phi', & G &= \Phi^2, \end{aligned} \quad (20)$$

N=2

$$\begin{aligned}
\omega_a^2 &= b^2(1 - 2\sqrt{1 + 3k^2}), \\
\omega_b^2 &= k^2 b^2(1 - \frac{2}{k}\sqrt{k^2 + 3}), \\
\omega_c^2 &= 4 - 2b^2\sqrt{1 - k^2 + k^4}, \\
\omega_d^2 &= k^2 b^2(1 + \frac{2}{k}\sqrt{k^2 + 3}), \\
\omega_e^2 &= 2, \\
\omega_f^2 &= b^2(1 + 4k^2), \\
\omega_g^2 &= b^2(1 + 2\sqrt{1 + 3k^2}), \\
\omega_h^2 &= 6, \\
\omega_i^2 &= 4 + 2b^2\sqrt{1 - k^2 + k^4}, \\
\omega_j^2 &= b^2(4 + k^2),
\end{aligned} \tag{21}$$

where we presented only the eigenvalues. The corresponding eigenfunctions F and G for $N = 2$ are given in Appendix B. When possible, we expressed the functions F and G in terms of Φ (see eq. (5)) and of $\Phi' = kb^2 \text{cn dn}$. The elliptic functions (sn, cn, dn) have arguments $b(k)z_1, k$. In general there are $2(2N + 1)$ modes that are polynomials in sn, cn, dn [16].

All modes constructed in this way are normal modes about the 2-sphaleron solution. The branches labelled ω_a^2 and ω_b^2 correspond to negative modes of eq. (15). For the sphaleron solution only the negative modes ω_b^2 should be retained and the positive modes labelled c, d, i, j .

We did not succeed in obtaining analytical solutions for the mode equations of the sphaleron (and the 2-sphaleron) for arbitrary values of θ^2 . Numerical analysis, however indicates that the eigenvalues fall smoothly between the analytically obtained values for $\theta^2 = N(N + 1)$, N integer. For small k this is confirmed by a perturbative calculation

of the modes (i.e. for $L \approx L_1$). The results for the first few eigenvalues are

$$\begin{aligned}
\omega_a^2 &= -2 + k^2(2 - \theta^2) + O(k^3), \\
\omega_b^2 &= -2\sqrt{2}\theta k + k^2\left(\frac{\theta^2}{2} - 1\right) + O(k^3), \\
\omega_c^2 &= k^2\theta^2 + O(k^3), \\
\omega_d^2 &= 2\sqrt{2}\theta k + k^2\left(\frac{\theta^2}{2} - 1\right) + O(k^3), \\
\omega_e^2 &= 2 + k^2\left(\frac{\theta^2}{2} - 3\right) + O(k^3), \\
\omega_f^2 &= 2 + k^2\left(\frac{3\theta^2}{2} - 3\right) + O(k^3), \\
\omega_g^2 &\approx \omega_h^2 = 6 + 5k^2\left(\frac{\theta^2}{3} - 2\right) + O(k^3), \\
\omega_i^2 &\approx \omega_j^2 = 8 + k^2(\theta^2 - 12) + O(k^3).
\end{aligned} \tag{22}$$

These expressions are valid for any $\theta^2 > 0$. Note, that the degeneracy of the branches g and h and of i and j is lifted only by terms of order k^3 .

When one investigates the system (15) in the background of higher multi-sphalerons ($m > 2$) all the modes discussed above are still present but additional branches of eigenvalues appear in between. For instance, in the limit $L = 4L_1$, the spectrum for the 4-sphaleron reads

$$-2 \ (1) \ , \ -\frac{3}{2} \ (2) \ , \ 0 \ (3) \ , \ \frac{1}{2} \ (2) \ , \ 2 \ (2) \ , \ \frac{5}{2} \ (2) \ , \ \frac{9}{2} \ (2) \ , \ 6 \ (2) \ , \ \dots \tag{23}$$

Here we find analytical solutions also for non-integer values of N . Following the same lines as previously, we find in the case $N = 1/2$ (i.e. $\theta^2 = 3/4$) four modes analytically which read

$$\begin{aligned}
\omega^2 &= -\frac{3}{2}, \quad F = \sqrt{\text{dn}z \pm \text{cn}z} (\text{dn}z \pm \text{cn}z), \quad G = -\frac{2}{k} \sqrt{\text{dn}z \pm \text{cn}z} (\text{dn}z \pm k^2 \text{cn}z), \\
\omega^2 &= \frac{1}{2}, \quad F = \sqrt{\text{dn}z \pm \text{cn}z} (\text{dn}z \mp k^2 \text{cn}z), \quad G = 2k \sqrt{\text{dn}z \pm \text{cn}z} (\text{dn}z \mp \text{cn}z).
\end{aligned} \tag{24}$$

CONCLUSIONS

The classical solutions of the 1+1-Abelian Higgs model on a circle of length L form a sequence of m solutions, when $L \geq mL_1$, where L_1 is the critical length at which the first non-trivial sphaleron solution appears. They share this remarkable feature with the sphaleron solutions of the $\lambda\phi^4$ theory on the circle. Dependent upon the value of the

Higgs mass, a similar sequence of sphaleron solutions is present in the Weinberg-Salam theory [17,18].

For the $\lambda\phi^4$ theory the analysis of the small fluctuations about the sphaleron solutions leads to a Lamé equation [14]. Thus the lowest solutions of the eigenvalue problem are known analytically and the eigenvectors are just the Lamé polynomials.

For the Abelian Higgs model the analysis of the small fluctuations about the sphaleron solutions is more involved. Here one obtains a system of four Lamé-like equations, of which only two decouple in general. Furthermore, the equations depend on an additional parameter, θ , the ratio of the Higgs and vector meson masses. Nevertheless, for certain values of this parameter, namely for $\theta^2 = N(N+1)$, with N integer, the lowest solutions can be constructed analytically in terms of the Lamé polynomials.

For the two decoupled equations the eigenvectors are simply the Lamé polynomials of degree 2 and N . For the two coupled equations, the eigenvectors are linear combinations of the Lamé polynomials, which for $\theta^2 = N(N+1)$ consist of the Lamé polynomials of degree $N \pm 1$. There are $2(2N+1)$ such eigenvectors [16]. (The case $N=0$ is special, since here all equations decouple.) It is intriguing, that analytical solutions of this type can be found, since the construction involves $4N+6$ parameters, solving $4N+14$ algebraic equations.

APPENDIX A

In this Appendix we present some details about the Lamé equation and the first few Lamé polynomials. When k is fixed ($0 \leq k < 1$) the Lamé equation

$$\frac{d^2 E}{dz^2} + [h - N(N+1)k^2 \text{sn}^2(z, k)]E = 0 \quad (A1)$$

admits an infinite but discrete spectrum. (The equation is considered on a period of $\text{sn}(z, k)$). For integer values of N the first few eigenvectors of eq. (A1) are the Lamé polynomials. There are $2N+1$ such polynomials, which are given below for $N = 0, 1, 2, 3$. The notations are those of ref. [15] with $k'^2 = 1 - k^2$.

$$N = 0 :$$

$$uE_0^0(z) = 1, \quad h_0^{(u)0} = 0. \quad (A2)$$

$$N = 1 :$$

$$\begin{aligned} sE_1^0(z) &= \text{sn}z, & h_1^{(s)0} &= 1 + k^2, \\ cE_1^0(z) &= \text{cn}z, & h_1^{(c)0} &= 1, \\ dE_1^0(z) &= \text{dn}z, & h_1^{(d)0} &= k^2. \end{aligned} \quad (A3)$$

$$N = 2 :$$

$$\begin{aligned} scE_2^0(z) &= \text{sn}z \text{cn}z, & h_2^{(sc)0} &= 4 + k^2, \\ sdE_2^0(z) &= \text{sn}z \text{dn}z, & h_2^{(sd)0} &= 1 + 4k^2, \\ cdE_2^0(z) &= \text{cn}z \text{dn}z, & h_2^{(cd)0} &= 1 + k^2, \\ uE_2^{0,1}(z) &= \text{sn}^2 z - \frac{1 + k^2 \pm \sqrt{1 - k^2 k'^2}}{3k^2}, & h_2^{(u)0,1} &= 2(1 + k^2) \mp 2\sqrt{1 - k^2 k'^2}. \end{aligned} \quad (A4)$$

$$N = 3 :$$

$$\begin{aligned} scdE_3^0(z) &= \text{sn}z \text{cn}z \text{dn}z, & h_3^{(scd)0} &= 4 + 4k^2, \\ sE_3^{0,1}(z) &= \text{sn}^3 z - \frac{2(1 + k^2) \mp \sqrt{4 - 7k^2 + 4k^4}}{5k^2} \text{sn}z, \\ h_3^{(s)0,1} &= 5(1 + k^2) \pm 2\sqrt{4 - 7k^2 + 4k^4}, \\ cE_3^{0,1}(z) &= \text{cn}z \left(\text{sn}^2 z - \frac{2 + k^2 \mp \sqrt{4 - k^2 k'^2}}{5k^2} \right), \\ h_3^{(c)0,1} &= 5 + 2k^2 \pm 2\sqrt{4 - k^2 k'^2}, \\ dE_3^{0,1}(z) &= \text{dn}z \left(\text{sn}^2 z - \frac{1 + 2k^2 \mp \sqrt{1 - k^2 + 4k^4}}{5k^2} \right), \\ h_3^{(d)0,1} &= 2 + 5k^2 \pm 2\sqrt{1 - k^2 + 4k^4}. \end{aligned} \quad (A5)$$

When N is half an odd integer there are also analytical solutions of the Lamé equation. The eigenvectors then are algebraic functions of $\text{sn}z$, $\text{cn}z$ and $\text{dn}z$. The solutions corresponding to $N = 1/2$ and $N = 3/2$ are

$$N = 1/2 :$$

$$E_{1/2} = \sqrt{\text{dn}z + \text{cn}z}, \quad h_{1/2} = \frac{1 + k^2}{4}. \quad (\text{A6})$$

$$N = 3/2 :$$

$$\begin{aligned} E_{3/2} &= \sqrt{\text{dn}z + \text{cn}z} \left(\text{dn}z + (k^2 - 1 \pm \sqrt{1 - k^2 + k^4})\text{cn}z \right), \\ h_{3/2} &= \frac{5}{4}(1 + k^2) \mp \sqrt{1 - k^2 + k^4}. \end{aligned} \quad (\text{A7})$$

Unlike the Lamé polynomials, these solutions have a period $8K(k)$. As a consequence other modes (with the same eigenvalue h) can be obtained by shifting z in (A6), (A7) by $2K(k)$, which is equivalent to changing the sign in front of $\text{cn}z$ each time it appears.

APPENDIX B

Here we give the eigenfunctions F and G of eq. (15) which can be expressed in terms of the Lamé polynomials for $\theta^2 = 6$.

- a) $F = 6\Phi^2 \text{cn} + (\omega_a^2 - 3b^2)\text{cn}, \quad G = (2\Phi' \text{cn} - \frac{\Phi^2 \text{dn}}{k}) - \text{dn}(\frac{\omega_a^2 - 3b^2}{6k}),$
- b) $F = 6\Phi^2 \text{dn} + (\omega_b^2 - 3b^2 k^2)\text{dn}, \quad G = (2\Phi' \text{dn} - \Phi^2 k \text{cn}) - \text{cn}(\omega_b^2 - 3b^2 k^2) \frac{k}{6},$
- c) $F = 2\Phi\Phi', \quad G = \Phi^3 - (\frac{8 - \omega_c^2}{6})\Phi,$
- d) F, G see branch b with $\omega_b \rightarrow \omega_d,$
- e) $F = \Phi^3 - \Phi, \quad G = \frac{\Phi\Phi'}{2},$
- f) $F = 2k\Phi^2 \text{cn} - kb^2 \text{cn}, \quad G = -\Phi^2 \text{dn},$
- g) F, G see branch a with $\omega_a \rightarrow \omega_g,$
- h) $F = \Phi^3, \quad G = \frac{\Phi\Phi'}{2},$
- i) F, G see branch c with $\omega_c \rightarrow \omega_i,$
- j) $F = 2\Phi^2 \text{dn} - k^2 b^2 \text{dn}, \quad G = -k\Phi^2 \text{cn}.$

REFERENCES

- [1] G. 't Hooft, Phys. Rev. Lett. 37 (1976) 8.
- [2] A. Ringwald, Nucl. Phys. B330 (1990) 1.
- [3] *Baryon Number Violation at the SSC?*, Santa Fe Workshop, eds. M. Mattis and E. Mottola (World Scientific, Singapore, 1990).
- [4] F. R. Klinkhamer and N. S. Manton, Phys. Rev. D30 (1984) 2212.
- [5] V. A. Kuzmin, V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B155 (1985) 36, B191 (1987) 171.
- [6] M. E. Shaposhnikov, CERN Preprint, CERN-TH.6304/91 and references therein.
- [7] B. Kleihaus, J. Kunz and Y. Brihaye, Phys. Lett. B273 (1991) 100.
- [8] L. Carson, X. Li, L. McLerran and R. Wang, Phys. Rev. D42 (1990) 2127.
- [9] A. I. Bochkaev and M. E. Shaposhnikov, Mod. Phys. Lett. A2 (1987) 991.
- [10] D. Yu. Grigoriev and V. A. Rubakov, Nucl. Phys. B299 (1988) 67.
- [11] D. Yu. Grigoriev and V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B216 (1989) 172, Nucl. Phys. B326 (1989) 737.
- [12] L. Carson, Phys. Rev. D42 (1990) 2853.
- [13] N. S. Manton and T. M. Samols, Phys. Lett. B207 (1988) 179.
- [14] J.-Q. Liang, H. J. W. Müller-Kirsten and D. H. Tchrakian, Phys. Lett. B282 (1992) 105.
- [15] F. M. Arscott, *Periodic differential equations* (Pergamon, Oxford, 1964).
- [16] Y. Brihaye and J. Kunz, in preparation.
- [17] J. Kunz and Y. Brihaye, Phys. Lett. B216 (1989) 353.
- [18] L. G. Yaffe, Phys. Rev. D40 (1989) 3463.