**A SUM RULE FOR THE
POLARISED PHOTON STRUCTURE FUNCTION g_1^γ ***

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Abstract

A sum rule relating the first moment of the photon structure function g_1^γ to the three-current AVV correlation functions involving the strong and electromagnetic $U(1)$ anomalies is presented. The relation with radiative pseudoscalar decays is described. The sum rule distinguishes clearly between the expected Nambu-Goldstone realisation of the AVV anomaly and the realisation in terms of massless fermions implicit in perturbation theory. For an on-shell target photon, the first moment of g_1^γ vanishes, while for large values ($\kappa^2 \gg m_\rho^2$) of the off-shell target photon momentum it is determined by the fourth power of the quark charges. The contribution of the singlet axial current differs logarithmically from this canonical value due to the strong anomaly. g_1^γ may be measured in two-photon processes, and some estimates are given for the relevant cross-sections in polarised e^+e^- colliders.

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1. Introduction

Deep inelastic, polarised μp scattering data from the EMC collaboration[1] (see also ref.[2]) have raised a number of interesting theoretical questions bearing on fundamental issues in perturbative and non-perturbative QCD. In particular, the flavour singlet part of the first moment of the polarised proton structure function $g_1^P(x, Q^2)$ was found to be anomalously small (for a review, see e.g. ref.[3]). This is related via the OPE to the forward proton matrix element of the axial $U(1)$ current $J_{\mu 5}^0$, the so-called “spin” of the proton*. Its small value reflects a strong violation of the OZI approximation in the singlet axial-vector channel[4] and the proton “spin” problem has been extensively discussed from this point of view using the $U(1)$ generalisation of the Goldberger-Treiman (GT) relation[5-7]**

Unfortunately, in the case of a proton target, it is not so easy to disentangle target-independent dynamical features of QCD, such as the realisation of the anomaly, with specific properties of the proton itself. Perhaps the cleanest separation is provided by the $U(1)$ GT relation in the form[6,7] in which the matrix element $\langle N | J_{\mu 5}^0 | N \rangle$ is expressed up to kinematical factors as $\chi_t'(0)^{\frac{1}{2}} g_{\eta_0 NN}$, where $\chi_t(p^2)$ is the QCD topological susceptibility $\int dx e^{ip \cdot x} i \langle 0 | T^* Q(x) Q(0) | 0 \rangle$ and $g_{\eta_0 NN}$ is the coupling constant of the nucleon to the OZI Nambu-Goldstone boson η_0 [5-7]. Combining renormalisation group and OZI ideas, we proposed in ref.[7] that the observed suppression of the proton “spin” may be due to an anomalously small value of $\chi_t'(0)$, i.e. a property of the $U(1)$ channel in QCD, independent of the proton structure.

The work presented here was motivated by the idea that by replacing the proton target by a photon, the essential features of the QCD dynamics could be studied independently of the non-perturbative bound-state physics of the target. Deep-inelastic scattering from a photon target is realised experimentally by the two-photon process which at high energies dominates the reaction $e^+e^- \rightarrow e^+e^-X$ in electron-positron colliders.

The photon structure function $F_2^\gamma(y, Q^2)$ has been known since the work of ref.[9] to have a calculable leading contribution, allowing stringent tests of perturbative QCD. (In this paper, we denote by y the Bjorken variable appropriate to the photon, keeping x for the corresponding variable referred to the target electron.) The same analysis[10] also applies to the polarised structure function $g_1^\gamma(y, Q^2)$ measurable in polarised e^+e^- scattering, with the important exception that for the first moment (in the above terminology, the “spin” of the photon) the exactly-calculable contribution of the type identified by Witten[9] vanishes and the leading contribution is again related to the matrix element of $J_{\mu 5}$.

The exceptional character of the first moment of g_1^γ has been recognised independently by Efremov and Teryaev[11]*** and by Bass[12]. In particular, in ref.[12] the sum rule $\int_0^1 dy g_1^\gamma(y, Q^2) = 0$ for real photons was derived. While this result is literally true, the actual experiment to determine the first moment of g_1^γ is sensitive to the off-shell structure

* However, as emphasised in refs.[4-7] and elsewhere, the identification of $\langle N | J_{\mu 5}^0 | N \rangle$ with the nucleon spin is valid only for a free Dirac fermion and in the absence of the $U(1)$ anomaly. The identification of $\langle \gamma | J_{\mu 5}^0 | \gamma \rangle$ as the spin of the photon fails for the same reasons.

** For other work on the $U(1)$ GT relation and the proton “spin”, see ref.[8].

*** The sum rule proposed in ref.[11] on the basis of the simple triangle diagram, equates the first moment of $g_1^\gamma(y, Q^2)$ to the electromagnetic $U(1)$ anomaly coefficient. The reasons why this is not correct, even in the exact chiral limit, will become clear from our analysis. See also ref.[24].

function $g_1^\gamma(y, Q^2; \kappa^2)$, and in fact measures a form factor characterising the fundamental AVV 3-current correlation function $\langle 0 | J_{\mu 5} J_\lambda J_\rho | 0 \rangle$, providing important information on how the $U(1)$ anomaly is realised non-perturbatively in QCD.

We begin our paper with a discussion of the kinematics of the two-photon process in polarised deep-inelastic e^+e^- scattering, and show how the n th y -moment of the off-shell structure function $g_1^\gamma(y, Q^2; \kappa^2)$ is related to the $(n+1)$ st x -moment of the differential cross-section $d^3\Delta\sigma/dQ^2 dx d\kappa^2$. Then, in section 3, we discuss the OPE and show why the $n=1$ moment is exceptional and how it is related to the 3-current correlation function.

The sum rule for $\int_0^1 dy g_1^\gamma(y, Q^2; \kappa^2)$ in terms of the form factors $f^{(r)}(\kappa^2)$ characterising the AVV correlation function is presented in section 4. It is shown that, as also observed in ref.[12], the conservation of the electromagnetic current implies that for small κ^2 the form factors $f^{(r)}(\kappa^2)$ are of $O(\kappa^2)$ and thus vanish for on-shell photons. However, the relation described above between the differential cross section for e^+e^- scattering and the first moment of g_1^γ involves a $1/\kappa^2$ pole originating from the soft photon propagator, so the experiment is indeed sensitive to the off-shell κ^2 dependence of the form factors $f^{(r)}(\kappa^2)$. The form factors are then related using the (anomalous) chiral Ward identities to the off-shell two-photon decay amplitudes of the pseudoscalar mesons π^0 , η and η' (in fact, the direct relation in the flavour singlet case is again with the OZI NG boson η_0) and a careful description of their RG scaling dependence is given. This makes contact with our previous work on radiative pseudoscalar decays[13] and the $U(1)$ GT relation[6,7].

In section 5, we present some further, necessarily more qualitative, considerations on the non-perturbative dynamical features of the form factors. The mass scale governing the κ^2 dependence of the form factors is predicted to be of $O(m_\rho^2)$, and it is explained why this is a consequence of the non-perturbative realisation of the anomalous AVV correlation function in the spontaneously broken chiral symmetry phase of QCD. This is to be contrasted with the perturbative prediction that the appropriate scale is a light quark mass. These results also emphasise the dangers of a perturbative treatment of the analogous Green functions in the case of the proton structure function, with gluons replacing the photons. The “asymptotic” form of the sum rule for $m_\rho^2 \ll \kappa^2 \ll Q^2$ is presented, and it is shown that in this limit the first moment of g_1^γ is proportional to the sum of the fourth power of the quark charges. The renormalisation group and the strong $U(1)$ anomaly imply a logarithmic deviation in κ^2 from the canonical value for the flavour singlet contribution, as well as the usual QCD logarithmic corrections in Q^2 .

The most exciting application of these ideas would be, of course, in the context of two-photon physics at polarised LEP, and in section 6 (see also appendix A) we present some estimates of the relevant cross sections. Our conclusion is that a measurement of g_1^γ may be feasible, though it would be at the limits of the potential of LEP200. An ideal machine for this purpose would combine high luminosity with moderate centre of mass energy, and in this respect PEP is the best suited of existing machines. Unfortunately, there seems to be no possibility of studying the sum rule in detail and determining the shape of the form factors $f^{(r)}(\kappa^2)$ with the luminosity available in current e^+e^- colliders, although the situation is much more promising with the proposed new generation of ultra-high luminosity “factories”.

We summarise our main results in section 7, emphasising the physics encoded in the form factors.

2. Structure Functions and Moment Sum Rules

At high energies, the reaction $e^+e^- \rightarrow e^+e^-X$ is dominated by the two-photon process shown in Fig. 1. We are interested in the deep-inelastic limit characterised by $Q^2, \nu, \tilde{\nu} \rightarrow \infty$ with $x = Q^2/2\nu, y = Q^2/2\tilde{\nu}$ fixed, and with the target photon relatively soft, i.e. $\kappa^2/Q^2 \ll 1$. Here, $Q^2 = -q^2, \kappa^2 = -k^2, \nu = p_2 \cdot q$ and $\tilde{\nu} = k \cdot q$ (see Fig. 1).

We wish to study cross sections for both polarised and unpolarised scattering which are differential w.r.t. Q^2, κ^2 , and x . Q^2 and x are determined from the energy E'_1 and scattering angle θ_1 of the hard-scattered electron by

$$Q^2 = 4EE'_1 \sin^2 \frac{\theta_1}{2} \quad x = \frac{E'_1 \sin^2 \frac{\theta_1}{2}}{E - E'_1 \cos^2 \frac{\theta_1}{2}} \quad (2.1)$$

where E is the incident electron energy. The Bjorken variable y appropriate to the target photon is given in terms of Q^2 and the invariant hadronic mass W by

$$y = \frac{Q^2}{Q^2 + W^2}. \quad (2.2)$$

Measurement of κ^2 is more difficult, since in this kinematical regime the “target” electron is scattered only through a very small angle θ_2 . If this nearly-forward electron is tagged, then κ^2 is found straightforwardly from

$$\kappa^2 \simeq EE'_2 \theta_2^2. \quad (2.3)$$

Otherwise, information on κ^2 must be inferred indirectly from a measurement of the total hadronic energy.

The total cross section is given by

$$\sigma_{\lambda_1 \lambda_2} = 2\pi \alpha^2 \frac{1}{s^2} \int_0^\infty \frac{dQ^2}{Q^2} \int_0^1 \frac{dx}{x^2} L^{\mu\nu}(p_1, \lambda_1; q) W_{\mu\nu}(p_2, \lambda_2; q), \quad (2.4)$$

where we define

$$W_{\mu\nu}(p_2, \lambda_2; q) = \frac{1}{2\pi} \frac{\alpha}{4\pi} x \int_0^\infty d\kappa^2 \int_x^1 \frac{dy}{y^2} L^{\lambda\rho}(p_2, \lambda_2; k) \text{Im} T_{\mu\nu\lambda\rho}(q, k), \quad (2.5)$$

where $L^{\mu\nu}$ is the leptonic tensor for a (massless) electron of helicity λ , i.e.,

$$L_{\mu\nu}(p_1, \lambda_1 = \pm; q) = 4p_{1\mu}p_{1\nu} - 2(p_{1\mu}q_\nu + p_{1\nu}q_\mu) + q^2 g_{\mu\nu} \pm 2i\epsilon_{\mu\nu\alpha\beta} p_1^\alpha q^\beta \quad (2.6)$$

and

$$T_{\mu\nu\lambda\rho}(q, k) = \langle 0 | J_\mu(q) J_\nu(-q) A_\lambda(k) A_\rho(-k) | 0 \rangle. \quad (2.7)$$

Here, A_λ is the photon field and $J_\mu = \sum_{f=1\dots N_F} \hat{e}_f \bar{\psi}_f \gamma_\mu \psi_f$ (where $\hat{e}_f$ are the quark charges in units of e) is the electromagnetic current for QCD with N_F flavours. In these expressions,

and those which follow, we include only the leading order terms in the deep-inelastic limit and neglect the electron mass.

The tensor $W_{\mu\nu}$ may be decomposed in terms of its independent Lorentz structures as follows, defining “electron structure functions” $F_1^e(x, Q^2)$, $F_2^e(x, Q^2)$ and $g_1^e(x, Q^2)$:

$$2W_{\mu\nu}(p_2, \lambda_2 = \pm; q) = F_1^e(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}) + F_2^e \frac{1}{\nu} (p_{2\mu} - \frac{\nu}{q^2} q_\mu)(p_{2\nu} - \frac{\nu}{q^2} q_\nu) \pm i g_1^e \frac{1}{2\nu} \epsilon_{\mu\nu\alpha\beta} p_2^\alpha q^\beta. \quad (2.8)$$

The cross sections for polarised and unpolarised scattering are readily expressed in terms of these structure functions. Defining $\sigma = \frac{1}{2}(\sigma_{++} + \sigma_{+-})$ and $\Delta\sigma = \frac{1}{2}(\sigma_{++} - \sigma_{+-})$, we have

$$\sigma = 2\pi \alpha^2 \frac{1}{s} \int_0^\infty \frac{dQ^2}{Q^2} \int_0^1 \frac{dx}{x^2} \left[F_2^e \left(\frac{xs}{Q^2} - 1 + \frac{1}{2} \frac{Q^2}{xs} \right) - \frac{1}{2} F_L^e \frac{Q^2}{xs} \right], \quad (2.9)$$

with $F_L^e = F_2^e - 2xF_1^e$, and

$$\Delta\sigma = 2\pi \alpha^2 \frac{1}{s} \int_0^\infty \frac{dQ^2}{Q^2} \int_0^1 \frac{dx}{x} g_1^e \left(1 - \frac{1}{2} \frac{Q^2}{xs} \right). \quad (2.10)$$

Under the relevant experimental conditions, the parameter $\frac{Q^2}{xs} \ll 1$, and from now on in our discussion of the cross sections we keep only the leading terms in eqs.(2.9) and (2.10). In this regime, the differential cross section $d\Delta\sigma/dQ^2$ is suppressed by a factor s/Q^2 relative to the unpolarised cross section $d\sigma/dQ^2$, which is determined entirely by F_2^e .

In the deep-inelastic region, these structure functions may be written as convolutions of the conventional photon structure functions $F_1^\gamma(y, Q^2; \kappa^2)$, $F_2^\gamma(y, Q^2; \kappa^2)$, $g_1^\gamma(y, Q^2; \kappa^2)$ with appropriate Altarelli-Parisi splitting functions. Explicit computations*, contracting $L^{\lambda\rho}$ with the appropriate tensor structures in $T_{\mu\nu\lambda\rho}$ (see below), give

$$\begin{aligned} F_L^e(x, Q^2) &= \frac{\alpha}{2\pi} \int_0^\infty \frac{d\kappa^2}{\kappa^2} \int_x^1 \frac{dy}{y} \frac{x}{y} P_{\gamma e} \left(\frac{x}{y} \right) F_L^\gamma(y, Q^2; \kappa^2) \\ F_2^e(x, Q^2) &= \frac{\alpha}{2\pi} \int_0^\infty \frac{d\kappa^2}{\kappa^2} \int_x^1 \frac{dy}{y} \frac{x}{y} P_{\gamma e} \left(\frac{x}{y} \right) F_2^\gamma(y, Q^2; \kappa^2) \\ g_1^e(x, Q^2) &= \frac{\alpha}{2\pi} \int_0^\infty \frac{d\kappa^2}{\kappa^2} \int_x^1 \frac{dy}{y} \Delta P_{\gamma e} \left(\frac{x}{y} \right) g_1^\gamma(y, Q^2; \kappa^2), \end{aligned} \quad (2.11)$$

with $F_L^\gamma = F_2^\gamma - 2yF_1^\gamma$. The Altarelli-Parisi factors (see ref.[16]) are

$$P_{\gamma e}(z) = \frac{1}{z} (1 + (1-z)^2) \quad \Delta P_{\gamma e}(z) = (2-z). \quad (2.12)$$

* Notice that no approximation, beyond taking the Bjorken limit, is used here. In particular, we do not use the “equivalent photon” method (see, e.g. refs.[14,15] and appendix B). The κ^2 dependence of the photon structure functions is kept manifest.

To understand the origin of these expressions, and to calculate the structure functions, we now introduce the operator product expansion. In this section, we restrict ourselves to the kinematical aspects. The discussion of the actual operators involved and the coefficient functions is deferred to section 3.

The operator product expansion for the product of two electromagnetic currents in QCD is as follows*:

$$\begin{aligned}
i J_\mu(q) J_\nu(-q) &\underset{Q^2 \rightarrow \infty}{\sim} \\
&\sum_{\substack{n=2 \\ \text{even}}} \sum_a O_{a,n}^{\mu_1 \dots \mu_n}(0) (-q^2)^{-n-1} 2^n \left(C_1^{a,n}(-q^2) (q_{\mu_1} \dots q_{\mu_n}) (g_{\mu\nu} q^2 - q_\mu q_\nu) \right. \\
&\quad \left. + C_2^{a,n}(-q^2) q_{\mu_3} \dots q_{\mu_n} (g_{\mu\mu_1} q^2 - q_\mu q_{\mu_1}) (g_{\nu\mu_2} q^2 - q_\nu q_{\mu_2}) \right) \\
&+ \sum_{\substack{n=1 \\ \text{odd}}} \sum_a R_{a,n}^{\mu_1 \dots \mu_n}(0) (-q^2)^{-n} 2^n \left(E_1^{a,n}(-q^2) q_{\mu_2} \dots q_{\mu_n} i \epsilon_{\mu\nu\alpha\mu_1} q^\alpha \right). \tag{2.13}
\end{aligned}$$

Here, $O_{a,n}^{\mu_1 \dots \mu_n}$ and $R_{a,n}^{\mu_1 \dots \mu_n}$ are the set of even and odd-parity, twist 2 operators in QCD (including the pure photon operators) and are listed in section 3.

We now substitute this expression into the four-point function $T_{\mu\nu\lambda\rho}^{**}$, and define the form factors $\hat{O}_{a,n}(\kappa^2)$ and $\hat{R}_{a,n}(\kappa^2)$ by

$$\begin{aligned}
&\langle 0 | O_{a,n}^{\mu_1 \dots \mu_n}(0) A_\lambda(k) A_\rho(-k) | 0 \rangle \\
&= \frac{1}{\kappa^4} \hat{O}_{a,n}(\kappa^2) k^{\mu_3} \dots k^{\mu_n} (k^2 g_\lambda^{\mu_1} g_\rho^{\mu_2} - k_\lambda k^{\mu_1} g_\rho^{\mu_2} - k_\rho k^{\mu_2} g_\lambda^{\mu_1} + k^{\mu_1} k^{\mu_2} g_{\lambda\rho}) \quad (n \geq 2, \text{even}) \\
&\tag{2.14}
\end{aligned}$$

and

$$\langle 0 | R_{a,n}^{\mu_1 \dots \mu_n}(0) A_\lambda(k) A_\rho(-k) | 0 \rangle = \frac{1}{\kappa^4} \hat{R}_{a,n}(\kappa^2) k^{\mu_2} \dots k^{\mu_n} i \epsilon_{\lambda\rho}{}^{\mu_1\alpha} k_\alpha \quad (n \geq 1, \text{odd}). \tag{2.15}$$

Inserting the resulting form for $T_{\mu\nu\lambda\rho}$ into eq.(2.5) for $W_{\mu\nu}$, we find the expressions (2.11) for the electron structure functions, with the photon structure functions given in terms of the Wilson coefficients and form factors. It is easy to check that this reproduces the conventional definition of the photon structure functions in terms of the matrix element $\langle \gamma(k, \epsilon^*) | J_\mu(q) J_\nu(-q) | \gamma(k, \epsilon) \rangle$ in the deep-inelastic limit.

* There are in fact two further terms with independent Lorentz structures in the OPE (2.13) for which the leading operators are twist 3. In general, these contribute to the structure functions F_3 and g_2 . However, they do not give independent contributions to the cross sections considered here.

** Notice that it would be incorrect to remove the photon propagators from $T_{\mu\nu\lambda\rho}$ at this stage and insert the OPE (2.13) into the four-current Green function $\langle 0 | J_\mu(q) J_\nu(-q) J_\lambda(k) J_\rho(-k) | 0 \rangle$. This would miss the effective contact interaction which in the OPE formalism arises from the pure photon operators mentioned above.

The sum rules for the moments of the photon structure functions follow immediately. We find (in each case for $n \geq 1$, odd)

$$\begin{aligned}\int_0^1 dy y^{n-1} F_L^\gamma(y, Q^2; \kappa^2) &= \sum_a C_L^{a, n+1}(Q^2) \hat{O}_{a, n+1}(\kappa^2) \\ \int_0^1 dy y^{n-1} F_2^\gamma(y, Q^2; \kappa^2) &= \sum_a C_2^{a, n+1}(Q^2) \hat{O}_{a, n+1}(\kappa^2) \\ \int_0^1 dy y^{n-1} g_1^\gamma(y, Q^2; \kappa^2) &= \sum_a E_1^{a, n}(Q^2) \hat{R}_{a, n}(\kappa^2),\end{aligned}\tag{2.16}$$

where $C_L = C_2 - C_1$. The study of the $n = 1$ moment sum rule for g_1^γ is the main subject of the paper.

To close this section, we show how these y -moments of F_2^γ and g_1^γ are simply related to the x -moments of the differential cross sections. Substituting eqs.(2.11) into the cross section formulae (2.9) and (2.10) and taking moments w.r.t. x , we find

$$\int_0^1 dx x^n \frac{d^3 \sigma}{dQ^2 dx d\kappa^2} = \alpha^3 \frac{1}{Q^4 \kappa^2} \int_0^1 dz z^n P_{\gamma e}(z) \int_0^1 dy y^{n-1} F_2^\gamma(y, Q^2; \kappa^2) \tag{2.17}$$

and

$$\int_0^1 dx x^n \frac{d^3 \Delta \sigma}{dQ^2 dx d\kappa^2} = \alpha^3 \frac{1}{s Q^2 \kappa^2} \int_0^1 dz z^{n-1} \Delta P_{\gamma e}(z) \int_0^1 dy y^{n-1} g_1^\gamma(y, Q^2; \kappa^2). \tag{2.18}$$

The y and $z = x/y$ integrals on the r.h.s. factorise, leaving the result that the n th moments of the structure functions F_2^γ and g_1^γ are proportional to the $(n+1)$ st moments of the differential cross sections $d\sigma/dQ^2 dx d\kappa^2$ and $d\Delta\sigma/dQ^2 dx d\kappa^2$ respectively. Notice also that in each case the z integral is finite for values of n (viz. 1, 3...) corresponding to the moments of the structure functions given by the sum rules (2.16), while for $n = 0$, which gives the total cross section, the z integration is logarithmically divergent (see section 6).

3. Operator Product Expansion and the $n = 1$ Moment of g_1^γ

In this section, we describe the OPE (2.13) in more detail, and recall briefly the essential results on the photon structure functions obtained in refs.[9,10,17]. (For a review of earlier and subsequent work, see, e.g., ref.[15].) Then we consider the special case of the $n = 1$ moment of g_1^γ and derive the initial version of our sum rule.

For the even-parity operators $O_{a, n}^{\mu_1 \dots \mu_n}$, there are four types, viz. the flavour non-singlet and singlet bilinear quark operators and pure gluon and photon operators. We distinguish these by the index $a = \text{NS}, \text{S}, \text{G}, \gamma$ resp. Explicitly

$$\begin{aligned}O_{\text{NS}, n}^{\mu_1 \dots \mu_n} &= i^{n-1} \text{sym} \left[\bar{\psi} \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \lambda^i \psi \right] & i = 1, \dots, N_F^2 - 1 \\ O_{\text{G}, n}^{\mu_1 \dots \mu_n} &= i^{n-1} \text{sym} \left[\text{tr} G^{\mu_1 \alpha} D^{\mu_2} \dots D^{\mu_{n-1}} G^{\mu_n}_{\alpha} \right]\end{aligned}\tag{3.1}$$

with the obvious definitions for $a = S, \gamma$. The relevance of the photon operators O_γ was first noticed in ref.[9]. The odd-parity operators are similar, viz.

$$\begin{aligned} R_{\text{NS},n}^{\mu_1 \dots \mu_n} &= i^{n-1} \text{sym} \left[\bar{\psi} \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} \lambda^i \gamma_5 \psi \right] \quad i = 1, \dots, N_F^2 - 1 \\ R_{\text{G},n}^{\mu_1 \dots \mu_n} &= i^{n-1} \text{sym} \left[\text{tr} \tilde{G}^{\mu_1 \alpha} D^{\mu_2} \dots D^{\mu_{n-1}} G^{\mu_n}{}_\alpha \right] \end{aligned} \quad (3.2)$$

The Wilson coefficients for the hadron ($h = \text{NS}, S, G$) and photon operators mix under the renormalisation group. We outline here the results for the odd-parity coefficients $E_1^{a,n}$ relevant for g_1^γ . The equivalent results for the $C^{a,n}$ are given in ref.[9]. Keeping only terms relevant to lowest order in α , the RGE reduces to

$$\mathcal{D} \begin{pmatrix} E_1^{h,n} & E_1^{\gamma,n} \end{pmatrix} = \begin{pmatrix} E_1^{h,n} & E_1^{\gamma,n} \end{pmatrix} \begin{pmatrix} \gamma_n^{hh} & \gamma_n^{h\gamma} \\ 0 & 0 \end{pmatrix}, \quad (3.3)$$

where the anomalous dimension matrix comprises a 3×3 matrix γ_n^{hh} in the hadron sector and a 3-dim. column vector $\gamma_n^{h\gamma}$ of $O(\alpha)$ reflecting the mixing of the photon and singlet hadron operators. The solution is standard, and substituting the result back into eq.(2.16) for the moments of g_1^γ we have to $O(\alpha)$, in matrix notation,

$$\begin{aligned} \int_0^1 dy \, y^{n-1} \, g_1^\gamma(y, Q^2; \kappa^2) &= E_1^{h,n}(1, \alpha_s(Q^2), 0) \, \Gamma_n^{hh} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), 0 \right) \hat{R}_{h,n}(\kappa^2, \alpha_s(\mu), \alpha) \\ &+ \left[E_1^{h,n}(1, \alpha_s(Q^2), 0) \, \Gamma_n^{h\gamma} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), \alpha \right) + E_1^{\gamma,n}(1, \alpha_s(Q^2), \alpha) \right] \hat{R}_{\gamma,n}(\kappa^2, \alpha_s(\mu), 0) \end{aligned} \quad (3.4)$$

where

$$\Gamma_n^{hh} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), 0 \right) = \mathcal{T} \exp - \int_0^t dt' \, \gamma_n^{hh}(\alpha_s(t')) \quad (3.5)$$

with $t = \frac{1}{2} \log Q^2 / \mu^2$ and $\mathcal{T}$ denoting t ordering, and

$$\begin{aligned} \Gamma_n^{h\gamma} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), \alpha \right) &= - \int_0^t dt' \, \mathcal{T} \exp \left[- \int_{t'}^t dt'' \, \gamma_n^{hh}(\alpha_s(t'')) \right] \gamma_n^{h\gamma}(\alpha_s(t'), \alpha) \\ &= \frac{\alpha}{\alpha_s(Q^2)} \sum_k \frac{P_n^k}{\left(1 + \frac{\xi_n^k}{2\beta_{(0)}} \right)} \frac{\gamma_{n(0)}^{h\gamma}}{2\beta_{(0)}} + \text{h.o.t.} \end{aligned} \quad (3.6)$$

In this latter expression, the anomalous dimension matrix $\gamma_{n(0)}^{hh}$ has been expanded in terms of its left eigenvalues ξ_n^k and corresponding projection operators P_n^k , viz. $\gamma_{n(0)}^{hh} = \sum_k \xi_n^k P_n^k$. $\gamma_{n(0)}^{h\gamma}$ and $\beta_{(0)}$ are the coefficients of the lowest-order terms in $\gamma_n^{h\gamma}$ and β . The explicit expressions for all these anomalous dimensions are given in ref.[10].

The important features of eq.(3.4) are the following. First, all the quoted terms are the same order in α . The form factors $\hat{R}_{h,n}$ for the hadronic operators are $O(\alpha)$ while Γ_n^{hh}

and the coefficient functions $E_1^{h,n}$ are $O(1)$. On the other hand, the photon form factor $\hat{R}_{\gamma,n}$ is just a calculable contact term of $O(1)$, while both $\Gamma_n^{h\gamma}$ and $E_1^{\gamma,n}$ are $O(\alpha)$.

For $n \geq 3$ odd, the moments receive contributions from both $\hat{R}_{h,n}$ and $\hat{R}_{\gamma,n}$. The dominant contribution for large Q^2 is identified as the $E_1^{h,n} \Gamma_n^{h\gamma} \hat{R}_{\gamma,n}$ term, which is of $O(\log Q^2 / \mu^2)$ because of the inverse power of $\alpha_s(Q^2)$ in $\Gamma_n^{h\gamma}$. We therefore have

$$\int_0^1 dy y^{n-1} g_1^\gamma(y, Q^2; \kappa^2) = \frac{\alpha}{4\pi} b_n \log \frac{Q^2}{\Lambda^2} \quad n \geq 3 \text{ odd.} \quad (3.7)$$

The explicit algebraic form and numerical values of the b_n are given in ref.[10].

The lowest moment, $n = 1$, is exceptional. In this case there is no operator $R_{\gamma,1}$, the only twist 2 operator being the axial current $J_{\mu 5}$. The leading contribution for large Q^2 is therefore

$$\int_0^1 dy g_1^\gamma(y, Q^2; \kappa^2) = E_1^{h,1}(1, \alpha_s(Q^2), 0) \Gamma_1^{hh} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu), 0 \right) \hat{R}_{h,1}(\kappa^2, \alpha_s(\mu), \alpha), \quad (3.8)$$

where for later convenience we keep also the first logarithmic corrections, ie.

$$E_1^{h,1}(1, \alpha_s(Q^2), 0) = E_1^{h,1}(1, 0, 0) L(\alpha_s(Q^2)), \quad (3.9)$$

with

$$L(\alpha_s(Q^2)) = 1 - \frac{3(N_C^2 - 1)}{8N_C} \frac{\alpha_s(Q^2)}{\pi}. \quad (3.10)$$

The coefficients $E_1^{h,1}(1, 0, 0)$ are determined, up to an overall normalisation, by the quark charges $\hat{e}_f$. Letting $\hat{e} = \text{diag}(\hat{e}_1, \dots, \hat{e}_{N_F})$ be the quark charge matrix, we decompose $\hat{e}^2 = \sum_r c^{(r)} \lambda^r$, where λ^r are the diagonal generators of $SU(N_F)$ including $\lambda^0 = 1$. (With these normalisations, $c^{(r)} = 2 \text{tr } \hat{e}^2 \lambda^r$ ($r \neq 0$) and $c^{(0)} = N_F^{-1} \text{tr } \hat{e}^2$.) Then, $E_1^{r,1}(1, 0, 0) = c^{(r)}$. For the special case of QCD with $N_F = 3$, the coefficients are $c^{(0)} = 2/9$, $c^{(3)} = 1/3$ and $c^{(8)} = 1/3\sqrt{3}$.

The non-singlet currents $J_{\mu 5}^r$ ($r \neq 0$) are conserved in the chiral limit, so the corresponding anomalous dimensions $\gamma_1^{rr} = 0$. On the other hand, because of the strong $U(1)$ anomaly, the singlet current $J_{\mu 5}^0$ is not conserved and has an associated anomalous dimension $\gamma_1^{00} = -\gamma$. The sum rule (3.8) for the first moment of g_1^γ therefore becomes,

$$\begin{aligned} \int_0^1 dy g_1^\gamma(y, Q^2; \kappa^2) &= L(\alpha_s(Q^2)) \left(\sum_{r \neq 0} c^{(r)} \hat{R}_{r,1}(\kappa^2, \alpha_s(\mu), \alpha) \right. \\ &\quad \left. + c^{(0)} \exp \left[\int_0^t dt' \gamma(\alpha_s(t')) \right] \hat{R}_{0,1}(\kappa^2, \alpha_s(\mu), \alpha) \right). \end{aligned} \quad (3.11)$$

Finally, we note that the situation for the moments of the structure function F_2^γ is identical, with the important exception that here the n th moment is associated with the operators $O_{a,n+1}$. In particular, even for the first moment of F_2^γ there is a contribution

from the photon operator $O_{\gamma,2}$ and, as explained above, this dominates for large Q^2 . We therefore have simply

$$\int_0^1 dy y^{n-1} F_2^\gamma(y, Q^2; \kappa^2) = \frac{\alpha}{4\pi} a_{n+1} \log \frac{Q^2}{\Lambda^2} \quad n \geq 1 \text{ odd.} \quad (3.12)$$

The algebraic expressions for the a_n are given in refs.[9,17] and numerical values are quoted in [17]. We will use eqs.(3.7) and (3.12) later to estimate the relevant cross sections for polarised deep-inelastic e^+e^- scattering.

4. The g_1^γ Sum Rule, the Axial Anomaly, and Radiative Pseudoscalar Decays

In this section, we study the sum rule (3.11) in more detail, and discuss its precise relation with the strong and electromagnetic $U(1)$ anomalies and the decay amplitudes $P \rightarrow \gamma\gamma$, where P denotes the pseudoscalar mesons π^0 , η and η' . We restrict ourselves here to results which can be obtained from electromagnetic current conservation and the (anomalous) chiral Ward identities (together with the usual dynamical assumption on the resolution of the $U(1)$ problem). Further considerations will be presented in section 5.

The analysis centres on the 3-current correlation functions $\langle 0 | J_{\mu 5}^r(p) J_\lambda(k_1) J_\rho(k_2) | 0 \rangle$. To investigate these, and the associated 1PI proper vertices, we use the formalism developed in refs.[6,7,13]*. Denote by $W[V_5^{r\mu}, \Theta, S_5^r, S^r, S_A^\lambda]$ the generating functional of connected Green functions of the relevant set of composite operators, viz. $J_{\mu 5}^r$, $Q = \frac{\alpha_s}{8\pi} \text{tr} \tilde{G}^{\mu\nu} G_{\mu\nu}$, $\phi_5^r = i\bar{\psi}\gamma_5\lambda^r\psi$, $\phi^r = \bar{\psi}\lambda^r\psi$ and the electromagnetic field A_λ . The corresponding Zumino effective action $\Gamma[V_5^{r\mu}, Q, \phi_5^r, \phi^r, A_\lambda]$ is defined by a partial Legendre transform w.r.t. the sources for all the fields except the current $J_{\mu 5}^r$ ** . Γ is then the generating functional for proper vertices which are 1PI w.r.t. the appropriate pseudoscalar (and electromagnetic) fields only, and it is this property which allows a particularly direct relation between these vertices and the physical pseudoscalar couplings.

In this formalism, we write the amputated, 3-current Green function as

$$\langle 0 | J_{\mu 5}^r(p) J_\lambda(k_1) J_\rho(k_2) | 0 \rangle = -\frac{1}{4\pi\alpha} W_{S_A^\lambda S_{A\sigma}}^{-1} W_{V_5^{r\mu} S_A^\sigma S_A^r} W_{S_{A\tau} S_A^\rho}^{-1} \equiv \hat{W}_{V_5^{r\mu} S_A^\lambda S_A^\rho}, \quad (4.1)$$

the relation to the proper vertices being[7,13]

$$4\pi\alpha \hat{W}_{V_5^{r\mu} S_A^\lambda S_A^\rho} = \Gamma_{V_5^{r\mu} A^\lambda A^\rho} + W_{V_5^{r\mu} \Theta} \Gamma_{Q A^\lambda A^\rho} + W_{V_5^{r\mu} S_5^j} \Gamma_{\phi_5^j A^\lambda A^\rho}, \quad (4.2)$$

* See refs.[7,13] for notation and a full description of this formalism. We will quote freely from these papers, especially ref.[13], in what follows.

** In fact, to lowest order in α , the field A_λ simply acts as a source for the electromagnetic current J_λ , so, for our purposes here, we could equally well have defined $W[V_5^{r\mu}, \Theta, S_5^r, S^r, A^\lambda]$ to be the generating functional for Green functions of all the currents and found $\Gamma[V_5^{r\mu}, Q, \phi_5^r, \phi^r, A^\lambda]$ as the corresponding partial Legendre transform, maintaining a symmetry in the formalism between the axial and electromagnetic currents.

(the suffix notation denoting functional differentiation). We will use these proper vertices shortly to discuss the relation with $P \rightarrow \gamma\gamma$ decays.

Electromagnetic current conservation is expressed through the Ward identities

$$ik_1^\lambda \hat{W}_{V_5^{r\mu}} s_A^\lambda s_A^\rho = 0 \quad ik_2^\rho \hat{W}_{V_5^{r\mu}} s_A^\lambda s_A^\rho = 0, \quad (4.3)$$

and the (anomalous) chiral Ward identities are [13] (writing the formulae for equal quark masses m for simplicity)

$$ip^\mu \hat{W}_{V_5^{r\mu}} s_A^\lambda s_A^\rho - 2m \hat{W}_{S_5^r} s_A^\lambda s_A^\rho + \frac{N_C}{4\pi^2} a^{(r)} \epsilon_{\lambda\rho\alpha\beta} k_1^\alpha k_2^\beta = 0 \quad (r \neq 0), \quad (4.4)$$

$$ip^\mu \hat{W}_{V_5^{0\mu}} s_A^\lambda s_A^\rho - 2m \hat{W}_{S_5^0} s_A^\lambda s_A^\rho - 2N_F \hat{W}_{\Theta} s_A^\lambda s_A^\rho + \frac{N_C}{4\pi^2} a^{(0)} \epsilon_{\lambda\rho\alpha\beta} k_1^\alpha k_2^\beta = 0. \quad (4.5)$$

Here*, $a^{(r)} = \text{tr } \hat{e}^2 \lambda^r$ are numerical coefficients determined by the quark charges, which give the relative strengths of the electromagnetic $U(1)$ anomalies for the currents $J_{\mu 5}^r$. For $N_F = 3$ QCD, $a^{(0)} = 2/3$, $a^{(3)} = 1/6$ and $a^{(8)} = 1/6\sqrt{3}$.

It will be useful in what follows to refer to the general Lorentz expansion of the 3-current Green function. We have

$$\begin{aligned} -i\langle 0 | J_{\mu 5}^r(p) J_\lambda(k_1) J_\rho(k_2) | 0 \rangle = & A_1^{(r)} \epsilon_{\mu\lambda\rho\alpha} k_1^\alpha + A_2^{(r)} \epsilon_{\mu\lambda\rho\alpha} k_2^\alpha \\ & + A_3^{(r)} \epsilon_{\mu\lambda\alpha\beta} k_1^\alpha k_2^\beta k_{2\rho} + A_4^{(r)} \epsilon_{\mu\rho\alpha\beta} k_1^\alpha k_2^\beta k_{1\lambda} \\ & + A_5^{(r)} \epsilon_{\mu\lambda\alpha\beta} k_1^\alpha k_2^\beta k_{1\rho} + A_6^{(r)} \epsilon_{\mu\rho\alpha\beta} k_1^\alpha k_2^\beta k_{2\lambda}, \end{aligned} \quad (4.6)$$

where $A^{(r)}(p^2, k_1^2, k_2^2)$ are functions of the invariants formed from the momenta.

For the sum rule for the first moment of g_1^γ , the form factor is given by the 3-current Green function with $p = 0$, $k_1 = -k_2 = k$. Comparing the definitions in eqs.(2.15), (3.2) and (4.6), we find

$$\hat{R}_{r,1}(\kappa^2) = 4\pi\alpha (A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2). \quad (4.7)$$

From the chiral Ward identities, it is clear that $(A_1^{(r)} - A_2^{(r)})$ will be proportional to the coefficients $a^{(r)}$, and it is convenient to define the rescaled form factors $f^{(r)}(\kappa^2; \mu^2)$ by

$$(A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2; \mu^2) = \frac{N_C}{4\pi^2} a^{(r)} f^{(r)}(\kappa^2; \mu^2), \quad (4.8)$$

where we have explicitly displayed the renormalisation scale dependence.

The results (4.7) and (4.8) hold away from the chiral limit, i.e. for $m_\pi^2 \neq 0$. In this paper, we shall limit ourselves to this case, which besides being physically relevant is also the easiest to discuss since it avoids taking the delicate limit $p \rightarrow 0$ in the presence of singularities in p^2 . The case of $m_\pi^2 = 0$ is rather subtle and will be discussed in the context

* Notice that we have changed notation slightly from ref.[13], where $a^{(0)}$ was normalised to 1. The momentum labelling is also different.

of a general analysis of the AVV Green function in a forthcoming paper[24]. The conclusion is that physical quantities such as the cross sections, g_1^γ and the form factors $f^{(r)}(\kappa^2)$ are indeed smooth in the chiral limit, but that the coefficients $A_1^{(r)}$ to $A_6^{(r)}$ specifying the Green function are not. Eqs.(4.7) and (4.8) remain valid in the chiral limit provided $(A_1^{(r)} - A_2^{(r)})$ is replaced by $(A_1^{(r)} - A_2^{(r)})_{\text{reg}}$, i.e. with the pion contribution subtracted out. For $m_\pi^2 \neq 0$, the pion does not contribute to $(A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2)$ *.

With the definition (4.8), we can immediately rewrite the sum rule (3.11) in the following concise form:

$$\int_0^1 dy \, g_1^\gamma(y, Q^2; \kappa^2) = L(\alpha_s(Q^2)) N_C \frac{\alpha}{\pi} \left(\sum_{r \neq 0} c^{(r)} a^{(r)} f^{(r)}(\kappa^2) + c^{(0)} a^{(0)} f^{(0)}(\kappa^2; Q^2) \right). \quad (4.9)$$

For $N_F = 3$ QCD, we find**

$$\int_0^1 dy \, g_1^\gamma(y, Q^2; \kappa^2) = \frac{1}{18} \frac{\alpha}{\pi} \left(1 - \frac{\alpha_s(Q^2)}{\pi} \right) \left(3f^{(3)}(\kappa^2) + f^{(8)}(\kappa^2) + 8f^{(0)}(\kappa^2; Q^2) \right). \quad (4.10)$$

In writing these equations, we have used the RG invariance of the form factors $(A_1^{(r)} - A_2^{(r)})$ for $r \neq 0$ to suppress the μ^2 -dependence in the non-singlet $f^{(r)}(\kappa^2)$, and have eliminated the exponential factor in eq.(3.11) by fixing the renormalisation scale $\mu^2 = Q^2$ in $f^{(0)}(\kappa^2; Q^2)$, using the property that the product $\exp \left[\int_0^t dt' \, \gamma(\alpha_s(t')) \right] \hat{R}_{0,1}(\kappa^2; \mu^2)$ is RG invariant. The result is that the singlet form factor $f^{(0)}(\kappa^2; Q^2)$ in the sum rule retains a non-trivial dependence on Q^2 , scaling with anomalous dimension γ .

Eq.(4.9) is the final form of our sum rule. It completes the separation of the appropriate moment of the differential cross section (see eq.(2.18)) into a known perturbative part and the non-perturbative form factors $f^{(r)}$ defined simply in terms of the 3-current correlation functions. In the discussion that follows, we relate these form factors to the decay amplitudes for $P \rightarrow \gamma^* \gamma^*$ with off-shell photons, and estimate the general features of their κ^2 dependence. We show how the scale controlling their variation with κ^2 depends on the way chirality is realised in the AVV Green function, offering a novel way to distinguish experimentally between the Nambu-Goldstone mode and the realisation through massless fermions.

* In fact, the decoupling of the pion from the cross section can be seen directly by noting that the amplitude for the full process $e^+e^- \rightarrow e^+e^-X$ depends on the matrix element $\langle e | J_{\mu 5}^{(r)}(0) | e \rangle$. Just as in the case of a proton target, the structure function g_1^e is determined solely by the pseudovector form factor in the decomposition of this matrix element, whereas the pion contributes only to the pseudoscalar form factor.

** Heavy quark contributions (which in principle could be separated out experimentally from an analysis of the hadronic decay products) can easily be included. They contribute extra form factors $\frac{32}{3} f^{(U)}(\kappa^2, m_U^2)$ and $\frac{2}{3} f^{(D)}(\kappa^2, m_D^2)$ for u and d -type heavy quarks resp. to the bracket in eq.(4.10).

(i) Non-singlet form factors:

First, consider the implications of electromagnetic current conservation. Substituting the general Lorentz structure (4.6) into the Ward identities (4.3), we find

$$A_1^{(r)} = A_3^{(r)} k_2^2 - A_5^{(r)} \frac{1}{2} (k_1^2 + k_2^2 - p^2). \quad (4.11)$$

Assuming a smooth behaviour of the form factors in the limit $p \rightarrow 0$, $k_{1(2)} \rightarrow +(-)k$, we conclude that

$$A_1^{(r)}(0, \kappa^2, \kappa^2) = -\kappa^2 (A_3^{(r)} - A_5^{(r)})(0, \kappa^2, \kappa^2) = O(\kappa^2). \quad (4.12)$$

Similar results hold for $A_2^{(r)}$.

This smoothness assumption is non-trivial. In particular, we must assume the absence of a $1/p^2$ pole in $A_5^{(r)}$ (and $A_6^{(r)}$), which would be produced by a massless pseudoscalar coupling to $J_{\mu 5}^0$. Eq.(4.12) also assumes that there are no $1/\kappa^2$ poles in the amputated Green functions $A_3^{(r)}, A_5^{(r)}$ (and $A_4^{(r)}, A_6^{(r)}$). Both assumptions are of course justified (remember we are considering the amputated Green functions) provided we are not in the chiral limit. This excludes singularities both at $p^2 = 0$ and $\kappa^2 = 0$.

Now consider the chiral Ward identity (4.4). Defining the form factor $D^{(r)}$ through $2m\hat{W}_{S_5^r S_A^\lambda S_A^\rho} = D^{(r)} \epsilon_{\lambda\rho\alpha\beta} k_1^\alpha k_2^\beta$, we find*

$$(A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2) = -D^{(r)}(\kappa^2) + \frac{N_C}{4\pi^2} a^{(r)}. \quad (4.13)$$

Combining with eq.(4.12), we may rewrite this result as an identity for the form factors $f^{(r)}(\kappa^2)$, viz.

$$f^{(r)}(\kappa^2) = 1 - \frac{D^{(r)}(\kappa^2)}{D^{(r)}(0)}. \quad (4.14)$$

At this point, it is useful to rewrite this relation in terms of the proper vertices derived from Γ . This enables us to make contact with the results of ref.[13] on $P \rightarrow \gamma\gamma$ decays. Writing

$$2m\hat{W}_{S_5^r S_A^\lambda S_A^\rho} = \frac{1}{4\pi\alpha} 2mW_{S_5^r S_5^s} \Gamma_{\phi_5^s A^\lambda A^\rho}, \quad (4.15)$$

taking the limit $p \rightarrow 0$ and using the Ward identity[7] $2mW_{S_5^r S_5^s}(0) = -\frac{1}{2}\langle\phi\rangle\delta^{rs}$, we find $D^{(r)}(\kappa^2)$ in terms of $\Gamma_{\phi_5^r A^\lambda A^\rho}$. Re-expressing in terms of the pseudo NG bosons $\pi^{(r)}$ using

* In the chiral limit, the Ward identity simply gives

$$(A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2) = \frac{N_C}{4\pi^2} a^{(r)}.$$

The way in which this result is reconciled with a smooth behaviour of the form factors $f^{(r)}(\kappa^2)$ as $m_\pi^2 \rightarrow 0$ is explained in ref.[24].

the relation $\pi^{(r)} = 2\langle\phi\rangle^{-1}F_\pi\phi_5^r$, and with the identification[13] of the proper vertices with the pseudoscalar decay couplings, viz. $\Gamma_{\pi^{(r)}A^\lambda A^\rho} = -\frac{1}{2}\epsilon_{\lambda\rho\alpha\beta}k_1^\alpha k_2^\beta g_{\pi^{(r)}\gamma^*\gamma^*}$, we then find*

$$D^{(r)}(\kappa^2) = \frac{1}{8\pi\alpha} F_\pi g_{\pi^{(r)}\gamma^*\gamma^*}(\kappa^2), \quad (4.16)$$

up to corrections of $O(m/\Lambda)$, where $\pi^{(3)} = \pi^0$ and $\pi^{(8)} = \eta$ (neglecting mixing).

Finally, rewriting eq.(4.14) in terms of these pseudoscalar decay couplings, we find

$$f^{(r)}(\kappa^2) = 1 - \frac{g_{\pi^{(r)}\gamma^*\gamma^*}(\kappa^2)}{g_{\pi^{(r)}\gamma\gamma}(0)}, \quad (4.17)$$

or equivalently,

$$f^{(r)}(\kappa^2) = \frac{\pi}{2N_C a^{(r)}} \frac{1}{\alpha} F_\pi \left[g_{\pi^{(r)}\gamma\gamma}(0) - g_{\pi^{(r)}\gamma^*\gamma^*}(\kappa^2) \right]. \quad (4.18)$$

This provides an elegant relation between the first moment of the structure function g_1^γ and the ratio of the off- and on-shell $\pi^{(r)} \rightarrow \gamma\gamma$ amplitudes. Notice that the RG behaviour here is trivial, all the terms being individually RG invariant.

(ii) Singlet form factor:

The analysis for the singlet form factor follows the same lines, but with important differences due to the effect of the strong $U(1)$ anomaly.

Electromagnetic current conservation again implies the relation (4.11), and eq.(4.12) holds since, even in the chiral limit, there is no massless pseudoscalar pole in the singlet channel. The assumption that such a pole is absent is the usual dynamical assumption that the $U(1)$ problem is resolved.

Now consider the chiral Ward identity (4.5), with $m = 0$. In this sub-section, we work entirely in the chiral limit, the effects of light quark mass terms being a small correction of $O(m/\Lambda)$. Defining the form factor B by $\hat{W}_{\Theta S_A^\lambda S_A^\rho} = (2N_F)^{-1}B\epsilon_{\lambda\rho\alpha\beta}k_1^\alpha k_2^\beta$, we find in the $p \rightarrow 0$ limit,

$$(A_1^{(0)} - A_2^{(0)})(0, \kappa^2, \kappa^2) = -B(\kappa^2; \mu^2) + \frac{N_C}{4\pi^2} a^{(0)}, \quad (4.19)$$

and, using $(A_1^{(0)} - A_2^{(0)}) = O(\kappa^2)$, we may rewrite this result as an identity for the form factor $f^{(0)}(\kappa^2; \mu^2)$, viz.

$$f^{(0)}(\kappa^2; \mu^2) = 1 - \frac{B(\kappa^2; \mu^2)}{B(0)}. \quad (4.20)$$

* As a consistency check, notice that the identity $D^{(r)}(0) = N_C a^{(r)}/4\pi^2$, which is a consequence of $(A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2) = O(\kappa^2)$, is equivalent to the relation for $F_\pi g_{\pi^{(r)}\gamma\gamma}$ found in ref.[13]. We can deduce eq.(4.16) less formally simply by pseudo NG boson saturation of the 3-point Green function in eq.(4.15).

We discuss the RG consistency of this result shortly.

To re-express this relation in terms of the proper vertices, we first write

$$\hat{W}_{\Theta S_A^\lambda S_A^\rho} = \frac{1}{4\pi\alpha} \left(W_{\Theta\Theta} \Gamma_{QA^\lambda A^\rho} + W_{\Theta S_5^0} \Gamma_{\phi_5^0 A^\lambda A^\rho} \right), \quad (4.21)$$

and use the Ward identities[7] $W_{\Theta\Theta}(0) = 0$ and $W_{\Theta S_5^0}(0) = -2\langle\phi\rangle/2N_F$ in the limit $p \rightarrow 0$ to determine $B(\kappa^2; \mu^2)$ in terms of $\Gamma_{\phi_5^0 A^\lambda A^\rho}$. Now introduce the OZI Nambu-Goldstone boson η_0 of refs.[5-7] through the relation $\eta_0 = (2\langle\phi\rangle)^{-1} F_{\eta_0}(p^2; \mu^2) \phi_5^0$, where the η_0 decay constant $F_{\eta_0}(0; \mu^2)$ is given by the first moment of the QCD topological susceptibility, i.e.

$$\begin{aligned} F_{\eta_0}(0; \mu^2) &= 2N_F (\chi'_t(0))^{\frac{1}{2}} \\ \chi_t(p^2) &= \int dx e^{ip \cdot x} i \langle 0 | T^* Q(x) Q(0) | 0 \rangle. \end{aligned} \quad (4.22)$$

Expressing $\Gamma_{\phi_5^0 A^\lambda A^\rho}$ in terms of $\Gamma_{\eta_0 A^\lambda A^\rho}$, and with the usual identification[13] with the appropriate pseudoscalar decay coupling, we find

$$B(\kappa^2; \mu^2) = \frac{1}{8\pi\alpha} F_{\eta_0}(0; \mu^2) g_{\eta_0 \gamma^* \gamma^*}(\kappa^2; \mu^2). \quad (4.23)$$

For $\kappa^2 = 0$, this reduces, using $B(0) = N_C a^{(0)}/4\pi^2$, to the identity for $F_{\eta_0}(0; \mu^2) g_{\eta_0 \gamma \gamma}(0)$ found in ref.[13].

With this relation, we can rewrite the form factor $f^{(0)}(\kappa^2; \mu^2)$ in terms of the $\eta_0 \rightarrow \gamma^* \gamma^*$ decay amplitude as follows,

$$f^{(0)}(\kappa^2; \mu^2) = 1 - \frac{g_{\eta_0 \gamma^* \gamma^*}(\kappa^2; \mu^2)}{g_{\eta_0 \gamma \gamma}(0; \mu^2)}, \quad (4.24)$$

in analogy to eq.(4.17). Two aspects are worth commenting on. First, we emphasise that the state involved in eq.(4.24) is the OZI NG boson $|\eta_0\rangle$, which is not a physical state in QCD. Its relation with the physical η' is discussed at length in ref.[7]*. Second, and related to this, the RG properties of the couplings in eq.(4.24) are quite non-trivial. We shall simply quote the results here, which may be inferred from ref.[13]. We find

$$\begin{aligned} \mathcal{D} g_{\eta_0 \gamma \gamma}(0; \mu^2) &= -\gamma g_{\eta_0 \gamma \gamma}(0; \mu^2) \\ \mathcal{D} g_{\eta_0 \gamma^* \gamma^*}(\kappa^2; \mu^2) &= -\gamma g_{\eta_0 \gamma \gamma}(0; \mu^2). \end{aligned} \quad (4.25)$$

* We can re-express the form factor $f^{(0)}$ in terms of the η' -photon couplings and the gluonic coupling $g_{G\gamma\gamma}$ introduced in ref.[13] as follows (disregarding η - η' mixing for simplicity),

$$f^{(0)}(\kappa^2; \mu^2) = \frac{\pi}{2N_C a^{(0)}} \frac{1}{\alpha} F \left[g_{\eta' \gamma \gamma}(0) - g_{\eta' \gamma^* \gamma^*}(\kappa^2) + \frac{F m_{\eta'}^2}{2N_F} \left(g_{G\gamma\gamma}(0; \mu^2) - g_{G\gamma^* \gamma^*}(\kappa^2; \mu^2) \right) \right],$$

where F is the RG invariant singlet decay constant[7,13].

Clearly, this ensures the required RG equation for $f^{(0)}$, viz.

$$\mathcal{D} f^{(0)}(\kappa^2; \mu^2) = \gamma f^{(0)}(\kappa^2; \mu^2). \quad (4.26)$$

It is also interesting to factor out the μ^2 -dependence of $f^{(0)}$ in the form of the decay constant $F_{\eta_0}(0; \mu^2)$, which satisfies the homogeneous RGE $\mathcal{D}F_{\eta_0}(0; \mu^2) = \gamma F_{\eta_0}(0; \mu^2)$. We then find

$$f^{(0)}(\kappa^2; \mu^2) = \frac{\pi}{2N_C a^{(0)}} \frac{1}{\alpha} F_{\eta_0}(0; \mu^2) \left[g_{\eta_0 \gamma \gamma}(0; \mu^2) - g_{\eta_0 \gamma^* \gamma^*}(\kappa^2; \mu^2) \right]. \quad (4.27)$$

Here, the entire μ^2 -dependence is carried by $F_{\eta_0}(0; \mu^2)$, or equivalently by the topological susceptibility, the factor in square brackets being RG invariant. Of course, for the sum rule, all this discussion of μ^2 -dependence translates immediately into the Q^2 scaling dependence of $f^{(0)}(\kappa^2; Q^2)$.

5. Momentum Dependence of the Form Factors

The form factors $f^{(r)}(\kappa^2)$, $f^{(0)}(\kappa^2; \mu^2)$ carry information on the non-perturbative dynamics of QCD, in particular on the realisation of the anomaly, and are therefore not precisely calculable. Nevertheless, from an analysis of the small and large κ^2 behaviour we are able to form a reasonable picture of their momentum dependence and so make the g_1^γ sum rule more predictive.

Consider the non-singlet form factors ($r \neq 0$) first. In the limit $\kappa^2 \rightarrow 0$ the form factors go to zero, $f^{(r)}(\kappa^2) \rightarrow 0$, which is of course just the reflection of the property $(A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2) = O(\kappa^2)$ following from electromagnetic current conservation and the absence of a massless pseudoscalar. We can also make a reliable statement in the large κ^2 limit using the renormalisation group (of course, for the deep inelastic kinematics to be valid we are restricted in practice to $\kappa^2 \ll Q^2$).

The amplitudes $A_i^{(r)}(p^2, \kappa^2, \kappa^2)$ defined in eq.(4.6) satisfy the RGE

$$\mathcal{D}A_i^{(r)} = 0, \quad (5.1)$$

where $\mathcal{D} = \mu \partial / \partial \mu + \beta \partial / \partial \alpha_s + \gamma_m m \partial / \partial m$. Solving this together with the dimensional analysis equation allows us to determine the form of the $A_i^{(r)}$ when any subset of the momenta are scaled uniformly to infinity. In particular, scaling the momenta of the electromagnetic currents from a reference value chosen for convenience as μ^2 , we find (for $i = 1, 2$)

$$A_i^{(r)}(p^2, e^{2t}\mu^2, e^{2t}\mu^2; m; \alpha_s(\mu)) = A_i^{(r)}(e^{-2t}p^2, \mu^2, \mu^2; e^{-t}m(t); \alpha_s(t)), \quad (5.2)$$

where $\alpha_s(t)$ and $m(t)$ are the running coupling and mass defined by

$$\frac{d\alpha_s(t)}{dt} = \beta(\alpha_s(t)) \quad \frac{dm(t)}{dt} = \gamma_m(\alpha_s(t)) m(t), \quad (5.3)$$

with boundary conditions $\alpha_s(t=0) = \alpha_s(\mu)$ and $m(t=0) = m$, and $t = \frac{1}{2} \log \kappa^2 / \mu^2$. For large t , both $\alpha_s(t)$ and $e^{-t}m(t)$ tend to zero, so to leading order the r.h.s. of eq.(5.3) can be evaluated in the free field, chiral limit. For the required combination $(A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2)$ we therefore find the leading large κ^2 behaviour,

$$(A_1^{(r)} - A_2^{(r)})(0, \kappa^2, \kappa^2; m; \alpha_s(\mu)) \underset{\kappa^2 \rightarrow \infty}{\sim} \frac{N_C}{4\pi^2} a^{(r)} \left(1 + O(m^2/\kappa^2) \right), \quad (5.4)$$

the corrections for non-zero m arising from the divergence term in the Ward identity (4.13).

We can improve on this, however, by using the OPE for the product of two electromagnetic currents together with the results of section 4. The use of the OPE in this context is a subtle theoretical issue which deserves some comment. The simple OPE for the product of two composite operators is only valid when embedded in a Green function for which all the other fields are elementary. In general, it is not correct to apply the OPE for two composites to determine the short distance behaviour of Green functions of several composite operators*.

In our particular example, we are interested in the large κ^2 behaviour of the AVV correlation function $\langle 0 | J_{\mu 5}^r(0) J_\lambda(k) J_\rho(-k) | 0 \rangle$. It is clear that applying the OPE naively to the electromagnetic currents we would predict that the form factor is asymptotically of $O(F_\pi^2/\kappa^2)$. This completely misses the leading $O(1)$ contribution due to the anomaly found above.

One simple way to circumvent this problem (see also ref.[18]), is first to re-express the form factor in terms of the ratio of off- to on-shell pseudoscalar decay amplitudes as in eq.(4.17), then apply the OPE to the matrix element $\langle \pi | J_\lambda(k) J_\rho(-k) | 0 \rangle$ involving the *physical* pion state. Although there is no formal proof of the validity of the OPE in such circumstances, we believe it to be reliable. We then have

$$\langle \pi^{(r)} | J_\lambda(k) J_\rho(-k) | 0 \rangle \underset{\kappa^2 \rightarrow \infty}{\sim} -2\epsilon_{\lambda\rho\alpha\mu} k^\alpha \frac{1}{\kappa^2} E_1^{r,1}(\kappa^2) \langle \pi^{(r)} | J_5^\mu(0) | 0 \rangle, \quad (5.5)$$

the leading behaviour of the Wilson coefficient being given in section 3. The off-shell coupling $g_{\pi^{(r)}\gamma^*\gamma^*}$ therefore falls off as $1/\kappa^2$ so, referring to eq.(4.17), we see that asymptotically $f^{(r)}(\kappa^2) \rightarrow 1$, consistent with eq.(5.4). Including the next-to-leading term given by eq.(5.5), we find

$$f^{(r)}(\kappa^2) \underset{\kappa^2 \rightarrow \infty}{\sim} 1 - \frac{16\pi^2}{N_C} F_\pi^2 \frac{1}{\kappa^2} + \dots \quad (5.6)$$

(where $F_\pi = 93.3\text{MeV}$), up to logarithmic corrections from the Wilson coefficient and terms further suppressed by $O(m/\Lambda)$.

Our conclusion is that the form factors $f^{(r)}(\kappa^2)$ interpolate smoothly between 0 and 1 as κ^2 is varied from zero to infinity. If we parametrise this behaviour by the simple form

* We have already encountered an example of the pitfalls of such a procedure in section 2 (see p.5, footnote **), where we would miss the Witten contribution to the structure function.

$f^{(r)}(\kappa^2) = \kappa^2/(\kappa^2 + M_{(r)}^2)$, we can use eq.(5.6) to give a rough estimate of the “turnover” mass scale as $M_{(r)}^2 \simeq (16\pi^2/N_C)F_\pi^2 \simeq m_\rho^2$.

The important point here is that this is a hadronic mass scale, indicative of the non-perturbative realisation of the anomalous AVV correlation function in QCD in the phase of spontaneously broken chiral symmetry. This should be contrasted with perturbation theory[19] (c.f. eq.(5.4)), which predicts a similar behaviour for the form factor but with a turnover scale of $O(m^2)$, where m is a light quark mass. Our result appears reasonable, since the absence of light states in the vector channels (the vector mesons all have masses of $O(\Lambda)$ even in the chiral limit) suggests that the relevant mass scale for the form factors is m_ρ^2 (or m_ϕ^2 for the octet $f^{(8)}$), consistent with the above estimate. Further discussion can be found in ref.[23].

Now consider the singlet form factor $f^{(0)}(\kappa^2; \mu^2)$. As in section 4, we work for simplicity in the chiral limit $m = 0$. In contrast to eq.(5.1), the singlet amplitudes $A_i^{(0)}(p^2, \kappa^2, \mu^2)$ satisfy a RGE with anomalous dimension γ , viz.

$$\mathcal{D}A_i^{(0)} = \gamma A_i^{(0)}, \quad (5.7)$$

with solution (i=1,2)

$$A_i^{(0)}(p^2, e^{2t}\mu^2, e^{2t}\mu^2; \alpha_s(\mu)) = \Gamma\left(\frac{\kappa^2}{\mu^2}, \alpha_s(\mu)\right) A_i^{(0)}(e^{-2t}p^2, \mu^2, \mu^2; \alpha_s(t)), \quad (5.8)$$

where the anomalous dimension factor Γ is defined by

$$\Gamma\left(\frac{\kappa^2}{\mu^2}, \alpha_s(\mu)\right) = \mathcal{T} \exp - \int_0^t dt' \gamma(\alpha_s(t')). \quad (5.9)$$

This factor is readily evaluated for large κ^2 and, recalling that γ is of $O(\alpha_s^2)$, we have*

$$\Gamma\left(\frac{\kappa^2}{\mu^2}, \alpha_s(\mu)\right) = \exp\left[\frac{\gamma_1}{\beta_0} \frac{\alpha_s(\mu)}{4\pi} + O(\alpha_s(\mu)^2)\right] \left(1 - \frac{\gamma_1}{\beta_0} \frac{\alpha_s(t)}{4\pi} + O(\alpha_s(t)^2)\right). \quad (5.10)$$

For the required case of $p^2 = 0$, the anomalous Ward identity gives

$$(A_1^{(0)} - A_2^{(0)})(0, \mu^2, \mu^2; \alpha_s(t)) = \frac{N_C}{4\pi^2} a^{(0)} + B(0, \mu^2, \mu^2; \alpha_s(t)), \quad (5.11)$$

* The anomalous dimension and beta function can be expanded as follows,

$$\begin{aligned} \gamma &= -\gamma_0 \frac{\alpha_s}{4\pi} - \gamma_1 \frac{\alpha_s^2}{(4\pi)^2} + \dots \\ \beta &= -\beta_0 \frac{\alpha_s^2}{4\pi} - \beta_1 \frac{\alpha_s^3}{(4\pi)^2} + \dots \end{aligned}$$

For QCD with N_C colours and N_F flavours, $\gamma_0 = 0$, $\gamma_1 = 6N_F N_C^{-1}(N_C^2 - 1)$ and $\beta_0 = \frac{2}{3}(11N_C - 2N_F)$.

where B is defined from the Green function $\langle 0|Q(p) J_\lambda(k_1) J_\rho(k_2)|0\rangle$ in section 4. Evaluating this perturbatively for small $\alpha_s(t)$, we find $B = O(\alpha_s(t)^2)$, i.e. the same order neglected in eq.(5.10) for Γ .

Finally, putting these results together and setting the renormalisation scale to Q^2 as explained below eq.(4.10), we find the following asymptotic expression for the singlet form factor $f^{(0)}(\kappa^2, Q^2)$,

$$f^{(0)}(\kappa^2, Q^2) \underset{\kappa^2 \rightarrow \infty}{\sim} 1 + 2 \frac{\gamma_1}{\beta_0^2} \left[\frac{1}{\log Q^2/\Lambda^2} - \frac{1}{\log \kappa^2/\Lambda^2} \right] + O(1/\log^2), \quad (5.12)$$

where we have used the usual leading order approximation $\alpha_s(t)/4\pi \simeq 1/\beta_0 t$ for the running couplings.

Unlike the non-singlet case, the $O(1/\kappa^2)$ power corrections to eq.(5.12) cannot be calculated reliably (since the η_0 is *not* a physical state), although if we were to trust the OPE for the two electromagnetic currents to give these sub-leading terms correctly we would find a correction factor analogous to eq.(5.6) with F_π replaced by $F_{\eta_0}(0; Q^2)$. If so, and if the explanation of the EMC “spin” effect does indeed reside in an anomalously small value of $F_{\eta_0}(0) = 2N_F \chi'_t(0)^{\frac{1}{2}}$, then we would expect a similarly suppressed turnover scale for the singlet form factor. In any case, it presumably remains true that the turnover scale for the singlet form factors is also a typical hadronic scale.

To summarise, we find the following results for the moment sum rule (4.9). At $\kappa^2 = 0$,

$$\int_0^1 dy g_1^\gamma(y, Q^2; \kappa^2 = 0) = 0. \quad (5.13)$$

For κ^2 in the “asymptotic” limit $m_\rho^2 \ll \kappa^2 \ll Q^2$, we have

$$\int_0^1 dy g_1^\gamma(y, Q^2; m_\rho^2 \ll \kappa^2 \ll Q^2) = N_C \frac{\alpha}{\pi} \sum_f \hat{e}_f^4 \left(1 + \tilde{L}(Q^2) + c \tilde{\Gamma}(\kappa^2, Q^2) \right), \quad (5.14)$$

where here

$$c = N_F^{-1} \left(\sum_f \hat{e}_f^2 \right)^2 \left(\sum_f \hat{e}_f^4 \right)^{-1}, \quad (5.15)$$

with

$$\tilde{L}(Q^2) = -\frac{9(N_C^2 - 1)}{2N_C(11N_C - 2N_F)} \frac{1}{\log Q^2/\Lambda^2}, \quad (5.16)$$

and

$$\tilde{\Gamma}(\kappa^2, Q^2) = \frac{27N_F(N_C^2 - 1)}{N_C(11N_C - 2N_F)^2} \left(\frac{1}{\log Q^2/\Lambda^2} - \frac{1}{\log \kappa^2/\Lambda^2} \right), \quad (5.17)$$

substituting for $2\gamma_1/\beta_0^2$ in eq.(5.12). The logarithmic modifications of this sum rule from its canonical value of $N_C(\alpha/\pi) \sum_f \hat{e}_f^4$ are due to the influence of the strong anomaly on the singlet axial form factor and the usual QCD corrections to the operator product coefficient functions[20].

In between these two extremes, the first moment of $g_1^\gamma(y, Q^2; \kappa^2)$ is governed by the expression (4.9), with the form factors interpolating smoothly between the limits. The important prediction here is that because the anomaly is realised non-perturbatively in the spontaneously chiral breaking phase of QCD, the turnover mass scale for the form factors is a hadronic mass of $O(m_\rho^2)$. This is in contrast to a realisation by massless fermions with unbroken chiral symmetry, which would give a turnover scale of $O(m^2)$ determined by a light quark mass. The asymptotic limits themselves do not distinguish between these different realisations of chiral symmetry.

6. Cross Sections for Polarised e^+e^- Scattering

We now present some estimates of the total cross section for polarised deep-inelastic e^+e^- scattering under realistic experimental conditions, with particular reference to polarised LEP. We also discuss the possibilities of verifying the sum rule for the first moment of g_1^γ experimentally and of measuring the κ^2 dependence of the form factors $f^{(r)}(\kappa^2)$.

The total cross sections for unpolarised and polarised scattering are found by integration of the differential formulae (2.17) and (2.18) resp. with $n = 0$. The various differential cross sections of interest can also be read off straightforwardly from the formulae in section 2. (See appendix A for further details and a comparison with the standard treatments.) The limits on the integrals are fixed to match the experimental constraints, and to ensure that the deep-inelastic scaling limit is realised.

As a first estimate, we use the formulae (3.7) and (3.12) for the dominant contributions to the moments of the structure functions. Notice that in these leading terms the κ^2 dependence is absent and the y and Q^2 dependence factorises, viz. $F_2^\gamma(y, Q^2; \kappa^2) = (\alpha/4\pi)a(y) \log Q^2/\Lambda^2$ and $g_1^\gamma(y, Q^2; \kappa^2) = (\alpha/4\pi)b(y) \log Q^2/\Lambda^2$. The functions $a(y)$, $b(y)$ are obtained from the moments a_n , b_n by an inverse Mellin transform. This has been performed numerically for the a_n in refs.[9,17], with the result that $a(y)$ is approximately constant, i.e. $a(y) = \bar{a}$, through most of the range of y . Since the moments b_n have a similar behaviour to the a_n , we also take $b(y) = \bar{b} = \text{const.}$ for the polarised case. Substituting these forms into the cross section formulae, we find the dominant contribution to the total cross section

$$\sigma = \frac{\alpha^4}{2\pi} \bar{a} \frac{1}{Q_{\min}^2} \log \frac{Q_{\min}^2}{\Lambda^2} \log \frac{\kappa_{\max}^2}{\kappa_{\min}^2} \log \frac{x_{\max}}{x_{\min}} \log \frac{y_{\max}}{\langle x \rangle}, \quad (6.1)$$

while

$$\Delta\sigma = \frac{\alpha^4}{2\pi} \bar{b} \frac{1}{s} \log \frac{Q_{\max}^2}{Q_{\min}^2} \log \frac{\langle Q^2 \rangle}{\Lambda^2} \log \frac{\kappa_{\max}^2}{\kappa_{\min}^2} \log \frac{x_{\max}}{x_{\min}} \log \frac{y_{\max}}{\langle x \rangle}, \quad (6.2)$$

where $\langle x \rangle$ denotes the geometric mean of $x_{\max}$ and $x_{\min}$ (similarly for $\langle Q^2 \rangle$) and we have neglected terms of $O(Q_{\min}^2/Q_{\max}^2)$ in eq.(6.1). In deriving these expressions, we have reconverted the decoupled y , z integrals in eq.(2.17) into an integration over y and x ,

since we wish to make cuts on these variables. The corresponding polarisation asymmetry, i.e. the ratio of $\Delta\sigma$ to σ , is

$$\frac{\Delta\sigma}{\sigma} = \frac{1}{2} \frac{\bar{b}}{\bar{a}} \frac{Q_{\min}^2}{s} \log \frac{Q_{\max}^2}{Q_{\min}^2} \left[1 + \log \frac{Q_{\max}^2}{\Lambda^2} \left(\log \frac{Q_{\min}^2}{\Lambda^2} \right)^{-1} \right]. \quad (6.3)$$

Clearly these estimates can be improved by inserting the exact results of the inverse Mellin transform for $a(y)$ and $b(y)$, and by including next-to-leading order[17], and non-perturbative model-dependent (see e.g. ref.[15]), corrections to the structure functions. However, they are sufficient to get a good idea of the experimental situation.

For the structure function F_2^γ , it is shown in ref.[17] that for QCD with four flavours, $1.2 \leq a(y) \leq 1.6$ in the range $0.3 \leq y \leq 0.9$, where Λ is taken to be 0.5 GeV. We therefore approximate $\bar{a} = 1.5$. Higher-order corrections tend to reduce the structure functions by a factor of order 2. The values of b_n which can be extracted from ref.[10] are approximately equal to the corresponding a_{n+1} , so we set $\bar{b} = \bar{a}$ for the estimates below.

The experimental cuts we shall consider are on Q^2 , κ^2 , ν and $\tilde{\nu}$. Initially, $Q_{\min}^2$ is taken simply to ensure that the deep-inelastic limit is realised, although we shall shortly consider the effect of being more selective with the cuts on Q^2 . $Q_{\max}^2$ is limited by detector efficiency. Kinematically, $\nu \geq \tilde{\nu}$ while $\tilde{\nu} = \frac{1}{2}(Q^2 + W^2)$. Since $W_{\min}^2$ is small compared to $Q_{\min}^2$ for the situations we consider, we take $\nu_{\min} = \tilde{\nu}_{\min} \simeq \frac{1}{2}Q_{\min}^2$, while for the upper limit, $\tilde{\nu}_{\max} = \nu_{\max} = \frac{1}{2}s$. Finally, the integration on κ^2 is taken between $\kappa_{\max}^2$, which we shall be interested in varying (we take $\kappa_{\max}^2 = 1\text{GeV}^2$ for the total cross section estimates below), and a lower cut-off set by the electron mass, $\kappa_{\min}^2 = m_e^2$.

Inserting these limits in eq.(6.1) we find*

$$\sigma = \frac{1}{4} \frac{\alpha^4}{\pi} \bar{a} \frac{1}{Q_{\min}^2} \log \frac{Q_{\min}^2}{\Lambda^2} \log \frac{\kappa_{\max}^2}{m_e^2} \left(\log \frac{s}{Q_{\min}^2} \right)^2 \quad (6.4)$$

$$\simeq 0.5 \times 10^{-8} \frac{1}{Q_{\min}^2} \log \frac{Q_{\min}^2}{\Lambda^2} \left(\log \frac{s}{Q_{\min}^2} \right)^2. \quad (6.5)$$

In order to measure the structure function g_1^γ , we need a sufficiently large cross section $\Delta\sigma$. For given s and taking $Q_{\max}^2 \simeq s/4$, we find from eq.(6.2) that $\Delta\sigma/\sigma$ increases with $Q_{\min}^2$ up to a maximum value attained with $Q_{\min}^2$ roughly of order $0.1s$ before falling again. However, for such large values of $Q_{\min}^2$, the total cross section σ becomes small and the event rate will be low. In fact, in order to obtain a statistically significant polarisation asymmetry, we should require $\Delta\sigma/\sigma \geq 1/\sqrt{N}$, where the number of events N is equal to $L\sigma$ with L being the integrated luminosity. This condition turns out to be quite difficult to satisfy in practice. If we take initially a small value for $Q_{\min}^2$, then the event rate $L\sigma$ is high but $\Delta\sigma/\sigma$ is small. Increasing $Q_{\min}^2$, we eventually reach the maximum of $\Delta\sigma/\sigma$ but then the event rate is low. The optimum choice of $Q_{\min}^2$ to satisfy the relation $\Delta\sigma/\sigma \geq (L\sigma)^{-\frac{1}{2}}$ lies somewhere between these points.

* Estimates for the Q^2 distribution of event rates at LEP1 and LEP200 have been given in ref.[21], which discusses the measurement of F_2^γ at high Q^2 . These estimates are in agreement with those inferred from eqs.(6.4),(6.5) within the accuracy of our approximations.

To see what this means in practice, we now consider the parameters appropriate for polarised LEP. For LEP1, $s = 10^4 \text{GeV}^2$ and we consider the design annual integrated luminosity of $L = 100 \text{pb}^{-1}$, although in actual running this value has not yet been achieved. For $Q_{\min}^2 = 4 \text{GeV}^2$, we find $\Delta\sigma/\sigma = 0.55 \times 10^{-2}$ with $\sigma = 85 \text{pb}$ corresponding to an annual event rate of 8.5×10^3 . The maximum value for the polarisation asymmetry, $\Delta\sigma/\sigma = 9.6 \times 10^{-2}$, is found for $Q_{\min}^2 \simeq 10^3 \text{GeV}^2$ but then $\sigma = 0.1 \text{pb}$ only. The optimum choice of $Q_{\min}^2$ to try to satisfy the relation $\Delta\sigma/\sigma \geq (L\sigma)^{-\frac{1}{2}}$ is $Q_{\min}^2 \simeq 50 \text{GeV}^2$ which gives $\Delta\sigma/\sigma = 2.7 \times 10^{-2}$, but with $\sigma = 6 \text{pb}$ the event rate is still too low.

The situation is better at LEP200, mainly due to the higher luminosity. Here, $s = 4 \times 10^4 \text{GeV}^2$ and with a value[21] for the annual integrated luminosity of $L = 500 \text{pb}^{-1}$ we find we can arrange the Q^2 cut to give a reasonable polarisation asymmetry with a sufficient event sample. For a small value $Q_{\min}^2 = 4 \text{GeV}^2$, we find $\Delta\sigma/\sigma = 0.18 \times 10^{-2}$ and $\sigma = 120 \text{pb}$ corresponding to an event rate of 6×10^4 . However, $(L\sigma)^{\frac{1}{2}} \Delta\sigma/\sigma$ is only about 0.4. To maximise $\Delta\sigma/\sigma$, we would take $Q_{\min}^2 \simeq 4 \times 10^3 \text{GeV}^2$ which gives $\Delta\sigma/\sigma = 9.6 \times 10^{-2}$, but the total cross section is then only $\sigma = 0.02 \text{pb}$. The optimum choice is to impose a cut at $Q_{\min}^2 \simeq 100 \text{GeV}^2$. Then, $\Delta\sigma/\sigma = 1.6 \times 10^{-2}$ with $\sigma = 4.3 \text{pb}$ and an event rate of 2.2×10^3 . In this case we have a sufficient number of events to measure g_1^γ , with (bearing in mind the accuracy of our estimates) a reasonable polarisation asymmetry, viz. $(L\sigma)^{\frac{1}{2}} \Delta\sigma/\sigma = 0.8$.

The conclusion of this analysis is that a measurement of g_1^γ at polarised LEP200 may be feasible provided an appropriate cut is made on $Q_{\min}^2$ to maximise the statistical significance of the polarisation asymmetry. In practice, however, the situation is likely to be much more difficult, since the value $L = 500 \text{pb}^{-1}$ may be optimistic and we have not included the effect of polarisation efficiency.

It is clear from this analysis that the optimum conditions for measuring g_1^γ are a combination of high luminosity and moderate s . In this respect, the best existing e^+e^- machine would be PEP, which has (without polarisation) already achieved a luminosity equivalent to 600pb^{-1} per year with proposals[22] for a further increase by a factor of 3. Here, $s = 840 \text{GeV}^2$ and for $Q_{\min}^2 = 4 \text{GeV}^2$ we find $\Delta\sigma/\sigma = 3.2 \times 10^{-2}$ with $\sigma = 40 \text{pb}$. The total event number for $L = 1 \text{fb}^{-1}$ is therefore 4×10^4 with $(L\sigma)^{\frac{1}{2}} \Delta\sigma/\sigma = 6$.

We now discuss the possibility of studying the sum rule for the first moment of g_1^γ experimentally. The first moment of g_1^γ is related to the second x -moment of the differential cross section through eq.(2.18), and substituting the expression (4.10) for the sum rule, we find for $N_F = 3$,

$$\begin{aligned} & \int_0^1 dx \, x \, \frac{d^2 \Delta\sigma}{dQ^2 dx} \\ &= \frac{1}{12} \frac{\alpha^4}{\pi} \frac{1}{s} \frac{1}{Q^2} \left(1 - \frac{\alpha_s(Q^2)}{\pi} \right) \int_0^\infty \frac{d\kappa^2}{\kappa^2} \left(3f^{(3)}(\kappa^2) + f^{(8)}(\kappa^2) + 8f^{(0)}(\kappa^2; Q^2) \right) \end{aligned} \quad (6.6)$$

Compared to the differential cross section $d\Delta\sigma/dQ^2$ itself, we lose a factor of $\log Q^2/\Lambda^2$ because the $n = 1$ moment is non-leading due to the absence of the pure photon operators in the OPE. The logs from the y and x integrals are also absent from the moment, as already noted in section 2.

The most interesting theoretical feature of eq.(6.6), however, is that the integration over κ^2 now yields (using the parametrisation of section 5) $\log(1 + \kappa_{\text{max}}^2/M_{(r)}^2)$, the electron mass being replaced by the turnover mass scale for the form factors $f^{(r)}$. If, as we have suggested, $M_{(r)}^2$ is a hadronic mass scale of $O(m_\rho^2)$, then this replacement suppresses the moment relative to the cross section by a factor of order 15. This is to be contrasted with the prediction of perturbation theory that $M_{(r)}^2$ is a light quark mass squared, which would not give a large suppression. If this suppression could be identified experimentally, it would provide direct evidence for the non-perturbative realisation of the anomalous AVV correlation function in QCD.

Unfortunately, the absence from this moment of the kinematical logarithmic enhancements $(\log s/Q_{\text{min}}^2)^2$, as well as the factor $\log\langle Q^2 \rangle/\Lambda^2$ from the running coupling constant, which occur in the cross section itself is already quite enough to rule out the possibility of measuring the moment at LEP. Experimentally, all that will be observed is the vanishing of $\int_0^1 dx \ x \ L \ d\Delta\sigma/dx$.

The same considerations of course mean that it will not be possible to verify the asymptotic sum rule (5.14) at polarised LEP. In principle, we would take the differential cross section w.r.t. κ^2 , measuring κ^2 in the range $m_\rho^2 \ll \kappa^2 \ll Q^2$ by a direct tag of the target electron. This would be quite feasible for such values of κ^2 since the scattering angle θ_2 could then be greater than the minimum detection angle. The ($N_F = 3$) sum rule predicts

$$\int_0^1 dx \ x \ \frac{d^3 \Delta\sigma}{dQ^2 dx d\kappa^2} \Big|_{m_\rho^2 \ll \kappa^2 \ll Q^2} = \frac{\alpha^4}{\pi} \frac{1}{s \ Q^2 \ \kappa^2} \left(1 + \tilde{L}(Q^2) + \frac{2}{3} \tilde{\Gamma}(\kappa^2, Q^2) \right), \quad (6.7)$$

and it would be interesting to look for the anticipated logarithmic deviation of the singlet contribution from the canonical value.

The conclusion is that the event rates even at LEP200 are simply far too small to permit a study of the sum rule for the first moment of g_1^γ . With the available luminosity, the vanishing of the moment $\int_0^1 dx \ x \ L \ d\Delta\sigma/dx$ is a consequence only of the kinematics of the reaction. Again the situation would be better at a polarised machine with the parameters of PEP, although even here the details of the sum rule would still be out of reach. A good test of the sum rule requires very much higher luminosities, but may in principle be possible in the context of polarised two-photon physics in the proposed ultra-high luminosity B factories.

7. Summary

We have derived the following sum rule for the first moment of the polarised photon structure function $g_1^\gamma(y, Q^2; \kappa^2)$ in terms of form factors $f^{(r)}(\kappa^2)$, $f^{(0)}(\kappa^2; Q^2)$ characterising the three-current AVV correlation function:

$$\int_0^1 dy g_1^\gamma(y, Q^2; \kappa^2) = 2N_C \frac{\alpha}{\pi} \left(\sum_{r \neq 0} a^{(r)2} f^{(r)}(\kappa^2) + \frac{1}{2N_F} a^{(0)2} f^{(0)}(\kappa^2; Q^2) \right), \quad (4.9)$$

where $a^{(r)} = \text{tr } \hat{e}^2 \lambda^r$ are numerical coefficients determined by the quark charge matrix $\hat{e}$. The $\kappa^2 = 0$ limit,

$$\int_0^1 dy g_1^\gamma(y, Q^2; \kappa^2 = 0) = 0, \quad (5.13)$$

is a consequence of electromagnetic current conservation. The asymptotic form for $m_\rho^2 \ll \kappa^2 \ll Q^2$,

$$\int_0^1 dy g_1^\gamma(y, Q^2; m_\rho^2 \ll \kappa^2 \ll Q^2) = N_C \frac{\alpha}{\pi} \sum_f \hat{e}_f^4 \left(1 + \tilde{L}(Q^2) + c \tilde{\Gamma}(\kappa^2, Q^2) \right), \quad (5.14)$$

differs logarithmically in κ^2 from the canonical value $N_C(\alpha/\pi) \sum_f \hat{e}_f^4$ due to the strong anomaly, as well as displaying the usual QCD logarithmic corrections in Q^2 .

In between these limits, even a rough determination of the shape of the form factors as a function of κ^2 would distinguish clearly between the realisation of the anomalous AVV correlation function through massless fermions characteristic of unbroken chiral symmetry (as predicted in perturbation theory) and the expected non-perturbative realisation through Nambu-Goldstone bosons in the spontaneously broken chiral symmetry phase of QCD.

Unfortunately, in view of its theoretical interest, the sum rule for the first moment of g_1^γ cannot be measured with the luminosity available in existing e^+e^- colliders. A measurement of the structure function g_1^γ itself is at the limits of the possibilities of polarised LEP200, but would easily be performed at a high-luminosity, moderate s , polarised collider with the characteristics, for example, of PEP. A thorough test of the moment sum rule requires very much higher luminosity, but may be possible in a programme of polarised two-photon physics in the new generation of $e^+e^- B$ factories.

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Appendix A Cross section formulae

Since we are interested in studying the behaviour of the photon structure functions for off-shell momenta κ^2 , we have not used the standard “equivalent photon” formalism (see, e.g., refs.[14,15,23]). We therefore collect here some formulae for various differential cross sections to illustrate how the standard results are recovered from our exact formulae in the usual approximation of a quasi-real, collinear target photon.

First, recall from section 2 our formula for the differential cross section $d^2\sigma/dQ^2 dx$ for the reaction $e^+e^- \rightarrow e^+e^-X$ (keeping only the F_2 terms),

$$\frac{d^2\sigma}{dQ^2 dx}(ee \rightarrow eeX) = 2\pi\alpha^2 \frac{1}{Q^4} \left(1 - t + \frac{1}{2}t^2\right) \frac{1}{x} F_2^e(x, Q^2), \quad (\text{A.1})$$

where $t = Q^2/xs$, together with the relation

$$F_2^e(x, Q^2) = \frac{\alpha}{2\pi} \int_0^\infty \frac{d\kappa^2}{\kappa^2} \int_x^1 \frac{dy}{y} \frac{x}{y} P_{\gamma e}\left(\frac{x}{y}\right) F_2^\gamma(y, Q^2; \kappa^2). \quad (\text{A.2})$$

In the standard approach, the differential cross section $d^2\sigma/dQ^2 dy$ for electron photon scattering $e^\pm\gamma \rightarrow e^\pm X$ is considered, i.e.

$$\frac{d^2\sigma}{dQ^2 dy}(e\gamma \rightarrow eX) = 2\pi\alpha^2 \frac{1}{Q^4} \left(1 - \hat{t} + \frac{1}{2}\hat{t}^2\right) \frac{1}{y} F_2^\gamma(y, Q^2), \quad (\text{A.3})$$

with $\hat{t} = Q^2/y\hat{s}$, where $\hat{s} = 2k.p_1$ replaces $s = 2p_2.p_1$ and the appropriate Bjorken variable is y .

Defining $\frac{d^2\sigma}{dQ^2 dy}(e\gamma^* \rightarrow eX)$ by replacing $F_2^\gamma(y, Q^2)$ in eq.(A.3) by the off-shell structure function $F_2^\gamma(y, Q^2; \kappa^2)$, we can write

$$\frac{d^2\sigma}{dQ^2 dx}(ee \rightarrow eeX) = \frac{\alpha}{2\pi} \int_0^\infty \frac{d\kappa^2}{\kappa^2} \int_0^1 \frac{dy}{y} P_{\gamma e}\left(\frac{x}{y}\right) \frac{d^2\sigma}{dQ^2 dy}(e\gamma^* \rightarrow eX) \frac{1 - t + \frac{1}{2}t^2}{1 - \hat{t} + \frac{1}{2}\hat{t}^2} \quad (\text{A.4})$$

expressing the $e^+e^- \rightarrow e^+e^-X$ cross section as a convolution of the $e^\pm\gamma^* \rightarrow e^\pm X$ “cross section” with an Altarelli-Parisi factor and integrating over the off-shell photon momentum.

The approximation used in the equivalent photon method is to take the target photon to be quasi-real and collinear. In this approximation, we take $\kappa^2 \simeq 0$, and have $z = x/y = k.q/p_2.q \simeq E_\gamma/E$, where E_γ is the target photon energy.

This gives $y\hat{s} = xs$ and so the extra factors cancel in eq.(A.4). The κ^2 integral decouples, leaving

$$\frac{d^2\sigma}{dQ^2 dx}(ee \rightarrow eeX) \simeq \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} P_{\gamma e}\left(\frac{x}{y}\right) \frac{d^2\sigma}{dQ^2 dy}(e\gamma \rightarrow eX) \int_0^\infty \frac{d\kappa^2}{\kappa^2}. \quad (\text{A.5})$$

In the same way, we can relate the differential cross sections $d\sigma/dQ^2$. With the same approximation,

$$\frac{d\sigma}{dQ^2}(ee \rightarrow eeX) \simeq \frac{\alpha}{2\pi} \frac{d\sigma}{dQ^2}(e\gamma \rightarrow eX) \int_0^1 dz P_{\gamma e}(z) \int_0^\infty \frac{d\kappa^2}{\kappa^2}, \quad (\text{A.6})$$

the integrals over x and y factorising as already illustrated in section 2. Also,

$$\frac{d^2\sigma}{dQ^2 dy}(\epsilon\epsilon \rightarrow \epsilon\epsilon X) \simeq \frac{\alpha}{2\pi} \frac{d^2\sigma}{dQ^2 dy}(\epsilon\gamma \rightarrow \epsilon X) \int_0^1 dz P_{\gamma\epsilon}(z) \int_0^\infty \frac{d\kappa^2}{\kappa^2}. \quad (A.7)$$

In the experimental situation envisaged in, e.g., ref.[23], the target electron is not tagged, being scattered into an angle less than $\theta_{2\max}$. This sets an upper limit on the κ^2 integration of $\kappa_{\max}^2 = E^2\theta_{2\max}^2$ and, as in the text, we take $\kappa_{\min}^2 = m_e^2$. A cut is also placed on the minimum photon energy, $E_\gamma \geq E_{\min}$, which determines $z_{\min} = E_{\min}/E$. The “photon flux” factors in eqs.(A.6), (A.7) then become

$$\frac{\alpha}{2\pi} \int_{z_{\min}}^1 dz P_{\gamma\epsilon}(z) \int_{\kappa_{\min}^2}^{\kappa_{\max}^2} \frac{d\kappa^2}{\kappa^2} = 2 \frac{\alpha}{\pi} \log \frac{E}{E_{\min}} \log \frac{E\theta_{2\max}}{m_e}. \quad (A.8)$$

Finally, as a consistency check, we recover the formulae for the total cross section given in ref.[23] in which the structure function is crudely approximated by a constant, i.e. $F_2^\gamma(y, Q^2; \kappa^2) = \bar{F}_2^\gamma$. In this case, our exact result (c.f. eq.(2.17)), with $\tilde{\nu}_{\max} = s/2$ and cuts on $\tilde{\nu}_{\min}$ and $Q_{\min}^2$ such that the deep-inelastic kinematics is valid, viz.

$$\sigma = \alpha^3 \int_{Q_{\min}^2}^{Q_{\max}^2} \frac{dQ^2}{Q^4} \int_{\kappa_{\min}^2}^{\kappa_{\max}^2} \frac{d\kappa^2}{\kappa^2} \int_{z_{\min}}^1 dz P_{\gamma\epsilon}(z) \int_{y_{\min}}^{y_{\max}} \frac{dy}{y} \bar{F}_2^\gamma \left(1 - t + \frac{1}{2}t^2\right) \quad (A.9)$$

($t = Q^2/zys$), reduces in the approximation of a quasi-real collinear target photon and $Q^2/xs \ll 1$ to

$$\sigma = 4\alpha^3 \frac{1}{Q_{\min}^2} \log \frac{E\theta_{2\max}}{m_e} \log \frac{E}{E_{\min}} \log \frac{2E^2}{\tilde{\nu}_{\min}} \bar{F}_2^\gamma. \quad (A.10)$$

This reproduces the expression derived in ref.[23] using the more complicated equivalent photon procedure.

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Figure Caption

FIG.1 Kinematics of the two-photon reaction $e^+e^- \rightarrow e^+e^-X$.