PRECISE AND SIMPLE BEHAVIOR
OF MACROSCOPIC SYSTEMS:
THE CASE OF SUPERCONDUCTIVITY*

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ABSTRACT

We emphasize the theoretical simplicity and experimental precision of superconductivity. Spontaneously broken gauge symmetry and macroscopic occupation of quantum levels lie at the basis of these features. The coherent states formalism proves to be well-suited in clarifying this essential link.

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I. INTRODUCTION

The deepest level in the understanding of Nature is achieved when it becomes possible to derive from general theoretical principles simple physical laws which are experimentally verified to a high degree of precision.

A very successful trend in physics is indeed to focus on *simple* microscopic systems, ranging from atoms to elementary particles, and to explore whether their behavior can be described in the above-mentioned conceptual framework. The wonderful experimental agreement of the theoretical predictions with the hydrogen atom spectrum, with the magnetic moments of the electron and of the muon and, more recently, the outstanding achievement of electroweak unification are examples forcefully indicating that we are on the right track in carrying out this program. This situation lends strong support to the reductionist philosophy, which can be summarized in the sentence: *small is beautiful*.

However, it came somewhat as a surprise at least to some of us, that simple physical laws in wonderful agreement with the experimental data are not an exclusive feature of microscopic systems. Although this recognition was already made, at least partially, many years ago in the field of superfluidity and superconductivity, yet the number of examples of macroscopic systems exhibiting surprisingly simple behavior appears to be rapidly increasing.

The instance of amazing agreement between simple theory and experiment were first met, as mentioned above, in the domain of superconductivity, *e.g.* in the Meißner effect, in the quantization of the magnetic flux and in the Josephson effect. In superfluidity as well, the quantization of the circulation of the superfluid current around a vortex line provided a similar example.

More recently, the magic (10^{-8}) experimental agreement of the theoretical prediction ¹

$$\sigma = n \frac{e^2}{h} (n \text{ integer}) \quad (1.1)$$

for the electrical conductivity in the quantum (integer) Hall effect, together with the discovery of the fractional Hall effect represent admirable and still somewhat surprising (at least in the eyes of the non-experts) analogous cases. Finally, the new challenge for both theorists and experimentalists appears to be the unravelling of possible simple physical laws in the field of high T_c superconductivity. One is therefore naturally led to seek the basic reasons lying at the core of this startling behavior of Nature.

This search, of course, has been intensively carried out in the past. In particular, the spontaneous breaking of gauge symmetry and the macroscopic occupation of quantum states have long been recognized as the key factors for understanding superconductivity.

The aim of this paper is to offer a unified treatment of the role played by these two elements, showing how both of them are needed to explain the extraordinary accuracy of the basic physical laws ruling the behavior of real superconductors.

For this purpose the coherent states formalism proves to be quite convenient. These states are in fact most helpful in disentangling the "classical" from the quantum aspect of superconductivity, thus allowing for a precise definition of the

order parameter and, in particular, of its phase, the cornerstone of the theoretical framework of superconductivity.

We organize the present work as follows. In Section II we briefly revisit the breaking of gauge invariance following the approach of Weinberg.² In Section III we show how our problem can be cast in the language of the coherent states, and in Section IV we utilize a simple field theoretical model to illustrate the main laws of an infinite superconductor and the corrections to these stemming from the presence of the boundaries.

Finally, in the last section we summarize our results and discuss other applications to different sectors of condensed matter physics.

II. SPONTANEOUSLY BROKEN GAUGE SYMMETRY

In this section we briefly revisit Weinberg's analysis of the basic laws of superconductivity like the Meibner effect, the magnetic flux quantization and the voltage-frequency proportionality. In Weinberg's approach these features are shown to be completely model-independent and to follow only from the spontaneous breaking of the electromagnetic gauge invariance: they indeed become exact in the zero momentum-energy limit.

Consider in fact a system (superconductor) made up of radiation and matter. Radiation is described classically by the electromagnetic potential $(\vec{A}, V)$.

On the other hand, we shall somewhat boldly assume that matter, in the Weinberg limit, is solely described by a single space-time-dependent phase $\theta(\vec{x}, t)$.

The presence of a macroscopic phase $\theta(\vec{x}, t)$ is a clear signature of spontaneously broken gauge symmetry. Yet gauge invariance requires us to have all the observable quantities expressed in terms of the "Higgs combination"

$$\hat{A}_i = A_i - \frac{\hbar c}{q} \frac{\partial \theta}{\partial x_i} \quad (2.1)$$

$$\hat{V} = V + \frac{\hbar}{q} \frac{\partial \theta}{\partial t} \quad (2.2)$$

with q being the charge of the constituents of our superconductors; accordingly, the gauge invariant electric and magnetic fields will read

$$\vec{B} = \text{rot } \hat{\vec{A}} \quad (2.3)$$

and

$$\vec{E} = -\frac{1}{c} \frac{\partial \hat{\vec{A}}}{\partial t} - \text{grad } \hat{V} \quad (2.4)$$

Worth noticing is how the above formulas display transparently the analogy with the electro-weak theory, where, indeed, the vector boson field W_μ is given by

$$W_\mu = A_\mu + \frac{\hbar c}{q} \frac{\partial \theta}{\partial x^\mu} \quad (2.5)$$

with θ being the celebrated Goldstone boson field. Gell-Mann described this situation by saying that the vector field A_μ "eats up" the Goldstone boson, thus acquiring a mass.

Now in the low energy-momentum limit, where the fields $\vec{E}$ and $\vec{B}$ disappear (they depend, through Eqs. (2.3) and (2.4), upon the derivatives of the electromagnetic potential), one can show² on the basis of general arguments that the equilibrium configuration of a superconductor in the absence of external electromagnetic fields is characterized by

$$\hat{\vec{A}} = 0 \quad (2.6)$$

and the equilibrium condition for a junction between two superconductors kept at a different voltage is

$$\vec{V}_1 = \vec{V}_2 \quad (2.7)$$

In the Appendix, exploiting the periodicity of the phase, which implies that θ and $\theta + 2\pi n$ (n integer) are equivalent, we shall show how the Meibner effect and the quantization of the magnetic flux ϕ through a loop of a supercurrent, i.e.

$$\phi = n \frac{\hbar c}{q}, \quad (n = 0, 1, 2, \dots) \quad (2.8)$$

easily follow from (2.6).

We shall see as well that from (2.7) the Josephson formula

$$\Delta V = \frac{\hbar}{q} \nu, \quad (2.9)$$

ΔV being the voltage difference and ν the frequency of the current flowing between the two superconductors, is deduced without difficulty.

What is important to point out here is that in (2.6) and (2.7), and also in (2.8) and (2.9), only the fundamental constants of Nature $\hbar$, c and q enter: precisely this fact accounts for the extreme precision of the above-reported basic laws of superconductivity.

In conclusion, the fundamental result obtained by Weinberg has been to show, in a model-independent way, that the relations (2.8) and (2.9) are valid in the static case and for infinite superconductors without impurities.

In order to apply these results to the real world, one has to explore the size of the corrections occurring at finite momenta.

In general one can write for a physical observable the expression

$$A = A_W (1 + k^2 \ell_A^2) \quad (2.10)$$

with A_W being the Weinberg value and ℓ_A a constant with the dimensions of a length. Thus, in superconductivity phenomenological constants, like the penetration length λ and the coherence length ξ , naturally appear.

In Section IV we shall show that these lengths are small (of the order of 10^{-4} cm) as a consequence of another quantum effect, namely the macroscopic occupation of a quantum level (Bose-Einstein condensation).

APPENDIX

We wish to derive explicitly the fundamental laws of the Meißner and Josephson effect.

A. The Meißner Effect

We start from the expression derived in the static limit for a bulk superconductor

$$\hat{\mathbf{A}} = 0 \quad (\text{A.1})$$

with $\hat{\mathbf{A}}$ given in (2.1). Since

$$\vec{B} = \text{rot } \hat{\mathbf{A}} \quad (\text{A.2})$$

the field $\vec{B}$ will vanish deep inside the superconductor. Flux quantization comes about by considering a torus shaped superconductor. Let us in fact perform the integral

$$\oint_{\gamma} \hat{\mathbf{A}} \cdot d\vec{\ell} = 0 \quad (\text{A.3})$$

where γ is a closed path going around inside the torus where (A.1) holds valid. Then the integral (A.3) splits into two pieces

$$\oint \vec{A} \cdot d\vec{\ell} - \frac{hc}{q} (\theta_0 - \theta_0') = 0 \quad (\text{A.4})$$

Since the phase is defined modulo $2\pi n$ we have*

$$\theta_0 - \theta_0' = 2\pi n, \quad (n = 1, 2, \dots) \quad (\text{A.5})$$

from where, using Gauss' theorem, it follows

$$\phi = n \frac{hc}{q}, \quad (\text{A.6})$$

namely the flux quantization.

* Of course one should not confuse the phase θ of the order parameter with the polar angle along the trajectory γ .

B. The Josephson Effect

Let us consider a Josephson's junction and recall (2.2). At the two end of the junction we have

$$V_1 = V_2 \quad (\text{A.7})$$

which can be recast into the form

$$\Delta V = \frac{\hbar}{q} \frac{\partial \theta}{\partial t} \quad (\text{A.8})$$

where

$$\Delta V = V_2 - V_1 \quad (\text{A.9a})$$

and

$$\theta = \theta_1 - \theta_2. \quad (\text{A.9b})$$

Let us consider a fixed potential difference ΔV . From Eq. (A.8) we then get

$$\theta = \frac{q\Delta V}{\hbar} t + \text{constant} \quad (\text{A.10})$$

Now the Josephson current should be invariant under the transformation

$$\theta \rightarrow \theta + 2\pi n \quad (n \text{ integer}) \quad (\text{A.11})$$

and should of course be odd under exchange $1 \leftrightarrow 2$. We can thus write

$$J(t) = S \left(\frac{q\Delta V}{\hbar} t \right) + \text{constant} \quad (\text{A.12})$$

where S should satisfy*

$$S(\theta) = -S(-\theta) \quad (\text{A.13a})$$

$$S(\theta) = S(\theta + 2\pi). \quad (\text{A.13b})$$

Then Eq. (A.12) tells us that

$$J(t) = J \left(t + \frac{2\pi\hbar}{q\Delta V} \right), \quad (\text{A.14})$$

i.e. that the period of the current is

$$T = \frac{\hbar}{q\Delta V} \quad (\text{A.15})$$

and the corresponding frequency is

$$\nu = \frac{q}{\hbar} \Delta V \quad (\text{A.16})$$

which is the fundamental law of the Josephson's effect.

* An elementary example is $S(\theta) = A \sin \theta$, which entails the approximate expression $J(t) = A \sin \left(\frac{q}{\hbar} t + \text{constant} \right)$.

V. CONCLUSIONS

The aim of this paper is the understanding of the simplicity of the theoretical predictions for some fundamental effects in superconductivity, which are experimentally verified with high accuracy.

We have seen that these benchmarks of superconductivity follow from two general principles: spontaneously broken gauge invariance and macroscopic occupation of a single level.

As pointed out by Weinberg, the first of the two principles ensures an exact low energy-momentum limit.

Concerning the importance of the macroscopic occupation of quantum levels we have shown here that it is responsible for the tiny size of the corrections to the Weinberg limit in a realistic superconductor.

Although in the present paper we have used a specific bosonic model, we believe our results to be in fact quite general. Indeed coherent states, which minimize the quantum indeterminacy between conjugate variables, play the key role in our treatment and their use can be extended to deal with a variety of physical problems.

In particular, we are currently studying "fermionic" superconductivity:¹¹ here coherent states are obtained by acting on the "bare" vacuum with an exponentiated operator bilinear in the fermionic fields and appear as an extension of the celebrated Bogoliubov transformation.

At the conclusion of this work several questions come to mind. First of all, one is naturally led to ponder whether further, simple and precise theoretical predictions can still be unravelled in condensed matter physics. In this connection the magic precision of the formula (1.1) for the quantum Hall effect, explained on the basis of topological argument, is tale-telling.

Analogously, we believe it is not unlikely that some of the well-known features of superfluidity (associated with the Goldstone mode of symmetry breaking) might display similar simple and general characteristics.

Another point deserving further consideration is the close analogy with particle physics. In addition to the previously mentioned fundamental Higgs combinations, which play such a key role in the electroweak theory, it is worth reminding that the background expansion (3.15) also appear in the non-linear sigma model for elementary particles and it might also intervene in the quantum theory of gravitation.

In conclusion, it is our belief that further theoretical studies of condensed matter physics, based on methods developed in particle physics, can still be a source of progress in both fields.

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B. The Time-Dependent Case

We now extend the model discussed in the previous subsection allowing for time dependence.

In this case we should consider the equation

$$\left(H - i\hbar \frac{\partial}{\partial t}\right) |G(t)\rangle = 0 \quad (4.22)$$

which follows from the minimization of the action

$$I = \int dt \left\langle G(t) \left| \left(H - i\hbar \frac{\partial}{\partial t} \right) \right| G(t) \right\rangle = \int dt \left\langle 0 \left| U^{-1}(t) \left(H - i\hbar \frac{\partial}{\partial t} \right) U(t) \right| 0 \right\rangle \quad (4.23)$$

where the new coherent state reads

$$|G(t)\rangle = U(t)|0\rangle = \exp \left\{ - \int (F^*(\vec{x}, t) \psi(\vec{x}) - F(\vec{x}, t) \psi^\dagger(\vec{x})) d\vec{x} \right\} |0\rangle \quad (4.24)$$

the order parameter $F(\vec{x}, t)$ being now time-dependent.

Accordingly, we are led to employ time-dependent coherent field operators as well, i.e.

$$\psi_c(\vec{x}, t) = U^{-1}(t) \psi(\vec{x}) U(t) = F(\vec{x}, t) + \psi(\vec{x}) \quad (4.25)$$

Let us now minimize (4.23). In doing so it is clear that the Hamiltonian H will play a role identical to the one played before: in this connection all one has to do is to replace $F(\vec{x})$ with $F(\vec{x}, t)$.

Of course, an additional term $q\psi^\dagger\psi V$ will also appear in H . This, as we shall see later, should be combined with the one arising from the time derivative in (4.23), which reads

$$\begin{aligned} & \left\langle 0 \left| U^{-1}(t) i\hbar \frac{\partial}{\partial t} U(t) \right| 0 \right\rangle \\ &= i\hbar \left\langle 0 \left| U^{-1}(t) \left[\psi^\dagger(\vec{x}) \frac{\partial F(\vec{x}, t)}{\partial t} - \psi(\vec{x}) \frac{\partial F^*(\vec{x}, t)}{\partial t} \right] U(t) \right| 0 \right\rangle \quad (4.26) \\ &= i\hbar \left\langle 0 \left| \left[\psi_c^\dagger(\vec{x}, t) \frac{\partial F(\vec{x}, t)}{\partial t} - \psi_c(\vec{x}, t) \frac{\partial F^*(\vec{x}, t)}{\partial t} \right] \right| 0 \right\rangle. \end{aligned}$$

Keeping then, as we did before, the lowest order in the coherent field operator, namely the time-dependent order parameter $F(\vec{x}, t)$ only, we obtain the following expression for the action

$$I = \int dt \mathcal{L}_{\text{eff}}(t) \quad (4.27)$$

where the "effective Lagrangian" reads

$$\mathcal{L}_{\text{eff}}(t) = i\hbar \left(F(\vec{x}, t) \frac{\partial F^*(\vec{x}, t)}{\partial t} - F^*(\vec{x}, t) \frac{\partial F(\vec{x}, t)}{\partial t} \right) + E_0(t) + qF^*(\vec{x}, t)F(\vec{x}, t)V(\vec{x}) \quad (4.28)$$

and $E_0(t)$ is the time-dependent version of (4.5).

Worth noticing is that in going from a static to a time-dependent theory we have switched from a Hamiltonian to a Lagrangian description.

At this point it is again legitimate to employ the polar representation

$$F(\vec{x}, t) = f(\vec{x}, t) e^{i\theta(\vec{x}, t)} \quad (4.29)$$

which easily leads to

$$\mathcal{L}_{\text{eff}} = f^2 \dot{V} + \text{"old terms"} \quad (4.30)$$

In the above, the term explicitly indicated and proportional to $\dot{V}$ (see (2.2)) arises from the combination of the time derivative of the phase $\theta(\vec{x}, t)$ with the third term on the right-hand side in (4.28): notice that neither term is separately gauge invariant, but so it is their combination. Now the piece $f^2 \dot{V}$ embodies all what is needed to discuss the physics of a superconductor related to an electric field.

As an example we shortly describe the "Josephson effect" which shall be discussed in more detail in the Appendix. Here we simply consider the function between two superconductors, denoted "1" and "2," respectively, kept at a different voltage.

The Lagrangian of the system reads then, in a compactified notation,

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{12} \quad (4.31)$$

where

$$\mathcal{L}_1 = f_1^2 \dot{V}_1 + \text{"old terms"} \quad (4.32a)$$

$$\mathcal{L}_2 = f_2^2 \dot{V}_2 + \text{"old terms"} \quad (4.32b)$$

and

$$\mathcal{L}_{12} = \mathcal{L}_{21} \quad (4.32c)$$

Now since at the equilibrium the condition

$$\left(\frac{\partial \mathcal{L}}{\partial f_1^2} - \frac{\partial \mathcal{L}}{\partial f_2^2} \right)_{f_1=f_2} = 0 \quad (4.33)$$

must be fulfilled* then, by exploiting the symmetry (4.32c) of $\mathcal{L}_{12}$, one obtains

$$\dot{V}_1 = \dot{V}_2 \quad (4.34)$$

which expresses the Josephson law.

* As pointed out, e.g. in Ref. [3], the two densities f_1^2 and f_2^2 will always be constant and equal since the two sides of the junction are connected by a battery whose role is to keep the potential constant.

which essentially corresponds to the well-known expression of Ginzburg and Landau for the energy of a superconductor.

In a bulk superconductor without impurities and far from boundaries, because of translational invariance, $f(\vec{x})$ assumes a constant value* f_0 as it does $\vec{A}$.

Therefore, for a uniform system (4.7) reduces to

$$E_0^{\text{unif}} = \frac{q^2}{2mc^2} f_0^2 \vec{A}^2 + \vartheta(f_0^2) \quad (4.8)$$

and the minimization condition

$$\frac{\partial E_0^{\text{unif}}}{\partial f_0} = 0 \quad (4.9)$$

fixes the value of f_0 .

Using Eq. (4.2b), neglecting the (small) first term on the right-hand side of Eq. (4.8), one obtains

$$\mu = \dot{W}(f_0^2), \quad (4.10)$$

where the dot means derivative with respect to f^2 . On the other hand, the minimization of E_0^{unif} with respect to the electromagnetic field leads to

$$\vec{A} = 0 \quad (4.11)$$

in accord with (2.6).

As already mentioned, and as we shall show in the Appendix, Eq. (4.11) embodies both the Meißner effect and the magnetic flux quantization.

We wish now to study the corrections to the constant value approximation $f(\vec{x}) = f_0$ and $\vec{A} = 0$. For this purpose we set

$$f(\vec{x}) = f_0 + h(\vec{x}) \quad (4.12)$$

and insert this expression back into (4.7) keeping terms up to second order in h^2 only.

For the potential

$$\vartheta(f^2) = -\mu f^2 + W(f^2) \quad (4.13)$$

the expansion

$$\vartheta(f^2) = -\mu f_0^2 + W(f_0^2) + 2h^2 b f_0^2 \quad (4.14)$$

is easily found, whatever the explicit form of $W(f^2)$ might be. In (4.14) we have set $b = \ddot{W}(f = f_0)$. We accordingly obtain

$$E_0^{\text{unif}} + \delta E_0 = -\frac{\hbar^2}{2m} \partial_i h \partial_i h + \frac{q^2}{2mc^2} f_0^2 \vec{A}^2 - \mu f_0^2 + W(f_0^2) + 2h^2 b f_0^2 + \frac{1}{2} (\text{rot } \vec{A})^2 \quad (4.15)$$

* Note that deep inside the superconductor $F(\vec{x})$ "does not" take a constant value, since then also θ would be fixed as well: gauge invariance would be broken.

which we want to minimize with respect to θ , $\vec{A}$ and $h(\vec{x})$. In the minimization procedure with respect to θ only the second term on the right-hand side of (4.15) enters (since $\text{rot } \vec{A} = \text{rot } \vec{A}$). One thus gets

$$\text{div } \vec{A} = 0 \quad (4.16)$$

which helps in minimizing with respect to $\vec{A}$.

Indeed in this case the equation one should consider is easily found to be

$$\text{rot } \delta \vec{A} \cdot \text{rot } \vec{A} + \frac{q^2}{mc^2} f_0^2 \delta \vec{A} \cdot \vec{A} = 0 \quad (4.17)$$

A partial integration over the whole space together with the assumption that both $\vec{A}$ and h vanish at infinity lead then to the equation

$$\Delta \vec{A} + \frac{1}{\lambda^2} \vec{A} = 0 \quad (4.18)$$

if (4.16) is taken into account.

Following the same procedure we get for h the equation

$$\Delta h + \frac{1}{\xi^2} h = 0 \quad (4.19)$$

In Eq. (4.18)

$$\lambda = \frac{1}{q f_0} \sqrt{mc^2} \quad (4.20a)$$

is the penetration length and in Eq. (4.19)

$$\xi = \frac{\hbar}{2 f_0} \sqrt{\frac{1}{mb}} \quad (4.20b)$$

is the coherence length.*

They determine the size of the spatial region, close to the surface and inside the superconductor, where f is not constant and the magnetic field is not zero, respectively.

In the analogy we have outlined in Section II, $1/\lambda$ corresponds to the mass of the vector boson (the photon), $1/\xi$ to the mass of the Higgs particle.

It is remarkable that both lengths are very small because of the factor f_0^2 which appears in the denominator: we thus most clearly see how the Bose-Einstein condensation "protects" superconductivity.

Also worth stressing is that the ratio

$$\frac{\lambda}{\xi} = \frac{2mc}{\hbar q} \sqrt{b} \quad (4.21)$$

depends on the forces at work in the superconductor, but not upon the order parameter f_0 .

* In the standard Ginzburg and Landau treatment one simply takes $W(f^2) = \frac{1}{2} f^4$ and the coherence length is conventionally defined as $\sqrt{2} \xi$.

In other words, the coherent states force the quantum many-body system to be as classical as possible. They are obtained by introducing the unitary operator

$$U = \exp \left\{ - \int (F^*(\vec{x})\psi(\vec{x}) - F(\vec{x})\psi^\dagger(\vec{x})) d\vec{x} \right\} \quad (3.12)$$

where F is a c -number function. We accordingly define

$$|G\rangle = U|0\rangle \quad (3.13)$$

with $|0\rangle$ being the lowest state of the uncorrelated system ("bare vacuum").

In order to evaluate the expectation value in the coherent state of products of field operators, we use the identity

$$\langle G|\psi\psi\cdots|G\rangle = \langle 0|\psi_c\psi_c\cdots|0\rangle \quad (3.14)$$

where the coherent field operators are

$$\psi_c(\vec{x}) = U^{-1}\psi(\vec{x})U = F(\vec{x}) + \psi(\vec{x}) \quad (3.15a)$$

and

$$\psi_c^\dagger(\vec{x}) = U^{-1}\psi^\dagger(\vec{x})U = F^*(\vec{x}) + \psi^\dagger(\vec{x}) \quad (3.15b)$$

We thus realize that the coherent field operator is most conveniently split into two components; it can in fact be viewed as the old "quantum" field operator in the presence of the "classical" background field $F(\vec{x})$.

As a consequence, by taking its expectation value in the bare vacuum, since obviously $\langle 0|\psi(\vec{x})|0\rangle = 0$, we get

$$\langle G|\psi(\vec{x})|G\rangle = \langle 0|\psi_c(\vec{x})|0\rangle = F(\vec{x}), \quad (3.16)$$

namely the definition (3.9) of the order parameter.

An insight into the role of the latter into the theory of superconductivity is obtained by considering the one-particle propagator (two point function)

$$\begin{aligned} \langle G|\psi(\vec{x})\psi^\dagger(\vec{y})|G\rangle &= \langle 0|\psi_c(\vec{x})\psi_c^\dagger(\vec{y})|0\rangle \\ &= F(\vec{x})F^*(\vec{y}) + \langle 0|\psi(\vec{x})\psi^\dagger(\vec{y})|0\rangle \end{aligned} \quad (3.17)$$

where the second term on the right-hand side vanishes for large separation distances $|\vec{x} - \vec{y}|$. The asymptotic behavior of (3.17) signals the presence of the off-diagonal long-range order (ODLRO)*

Finally, let us consider the expectation value of the density operator ρ in the true vacuum

$$\begin{aligned} \langle G|\rho|G\rangle &= \langle G|\psi^\dagger(\vec{x})\psi(\vec{x})|G\rangle \\ &= F(\vec{x})F^*(\vec{x}) + \langle 0|\psi^\dagger(\vec{x})\psi(\vec{x})|0\rangle. \end{aligned} \quad (3.18)$$

We recognize in the first term on the right-hand side the particle density in the macroscopically occupied quantum state; accordingly the average number N_0 of condensed particles will be

$$N_0 = \int d\vec{x} F^*(\vec{x})F(\vec{x}) \cong N. \quad (3.19)$$

* For a discussion of the ODLRO, see the work of Anderson,⁴ Penrose and Onsager⁹ and Yang.¹⁰

III. A SIMPLE FIELD THEORETICAL MODEL

A. The Static Case

In order to better appreciate the significance of the scheme developed in the previous sections, we consider a scalar boson field ψ , associated with particles of mass m and charge q , dropping, to start with, the time dependence. This field of course interacts with the electromagnetic field A_i according to the model grand canonical Hamiltonian density, valid in the non-relativistic approximation,

$$H = -\frac{1}{2mc^2} (D_i\psi)^\dagger (D_i\psi) + \vartheta(\psi^\dagger\psi) + \frac{1}{2}(\text{rot } \vec{A})^2 \quad (4.1)$$

with D_i being the standard covariant derivative

$$D_i = \hbar c \frac{\partial}{\partial x_i} - iq A_i \quad (4.2a)$$

and

$$\vartheta(\psi^\dagger\psi) = -\mu\psi^\dagger\psi + W(\psi^\dagger\psi) \quad (4.2b)$$

the potential term in which the chemical potential μ is embodied.

In our approach, ψ (in Eq. (4.1)) is meant to represent the "boson" field of a Cooper pair, so that $q = -2e$ ($-e$ being the electron charge). In this picture $|0\rangle$ just corresponds to a state where no Cooper pairs are present.

Now, in the static case, which we consider first, we should study

$$\langle G|H|G\rangle = \langle 0|H_c|0\rangle \quad (4.3)$$

where

$$H_c = U^{-1}HU = -\frac{1}{2mc^2} (D_i\psi_c)^\dagger (D_i\psi_c) + \vartheta(\psi_c^\dagger\psi_c) + \frac{1}{2}(\text{rot } \vec{A})^2. \quad (4.4)$$

Our aim here is to explore the impact of condensation on the energy of a superconductor.

With the plausible assumption that, at the mean values level, the contribution arising from the classical order parameter $F(\vec{x})$ dominates over the one due to the quantum field $\psi(\vec{x})$, from (4.3) and (4.4) we get

$$E_0 = -\frac{1}{2mc^2} (D_i F)^\dagger (D_i F)^* + \vartheta(F F^*) + \frac{1}{2}(\text{rot } \vec{A})^2. \quad (4.5)$$

Now, according to previous discussions, we are entitled to set

$$F(\vec{x}) = f(\vec{x}) e^{i\theta(\vec{x})}; \quad (4.6)$$

therefore, from (4.5) we obtain

$$E_0 = -\frac{\hbar^2}{2m} \partial_i f \partial_i f + \frac{q^2}{2mc^2} f^2 \vec{A}^2 + \vartheta(f^2) + \frac{1}{2}(\text{rot } \vec{A})^2 \quad (4.7)$$

III. THE COHERENT STATES

In the previous Section we emphasized the crucial role played by the phase θ in the setting up of a theoretical framework for superconductivity. We wish now to face the challenging task of providing a precise and unambiguous meaning to θ , whose importance in the theory of superconductivity can hardly be overvalued.

If we deal with a one-particle system then we have no problem in writing for its wave function

$$\varphi = |\varphi| e^{i\theta} \quad (3.1)$$

and the Higgs combinations $\hat{A}$ and $\hat{V}$ appear automatically in the Schrödinger equation obeyed by a particle moving in an external electromagnetic field. If, however, one wishes to study a many-particle system (in our case the number of particles is of the order of 10^{23}) then second quantization must be introduced. In so doing it is wise to proceed cautiously.

The first step is to consider the system as described by a classical complex field ψ , whose dynamical behavior is ruled by an extension, possibly non-linear, of the Schrödinger equation.

As we shall see in the following, the field ψ is closely connected with the order parameter introduced by Landau in his treatment of superconductivity and superfluidity and whose significance has been splendidly discussed by Feynman.³

It suffices here to recall that while the square modulus of (3.1) yields the "probability" density of finding a particle in the unit volume, the square modulus

$$\rho = \psi^* \psi \quad (3.2)$$

furnishes the "particle" density, i.e. the number of particles a measurement would find in the unit volume. This point has been deeply analyzed in Ref. [3]. Moreover the term $\mathcal{L}_t$ in the classical Lagrangian density containing the time derivative of the fields reads

$$\mathcal{L}_t = i \frac{\hbar}{2} \left(\frac{\partial \psi^*}{\partial t} \psi - \psi^* \frac{\partial \psi}{\partial t} \right), \quad (3.3)$$

thus entailing the equal-time Poisson bracket (we omit, for simplicity, to display explicitly the time variable)

$$[\psi(\vec{x}_1), \psi^*(\vec{x}_2)]_{\text{PB}} = i\delta(\vec{x}_1 - \vec{x}_2). \quad (3.4)$$

Hence also in this case it is perfectly legitimate to introduce the polar representation for the classical field ψ , namely

$$\psi = |\psi| e^{i\theta} \quad (3.5)$$

and one is naturally led to consider, in parallel with (3.4), the Poisson bracket

$$[\rho(\vec{x}_1), \theta(\vec{x}_2)]_{\text{PB}} = i\delta(\vec{x}_1 - \vec{x}_2) \quad (3.6)$$

which, when integrated over $\vec{x}_1$, yields

$$[N, \theta(\vec{x})]_{\text{PB}} = i \quad (3.7)$$

where $N = \int \rho(\vec{x}) d\vec{x}$ represents the total number of particles. As is well-known, (3.7) plays a key role in Anderson's⁴ approach to superfluidity.

At this point we have to take the second step in our procedure, which brings us into the domain of second quantization, a framework allowing one to account for the discrete nature of the physical observable represented by the particle numbers.

Second quantization makes use of the correspondence principle to transform the Poisson bracket (3.4) into the commutator

$$[\psi(\vec{x}_1), \psi^\dagger(\vec{x}_2)] = i\hbar\delta(\vec{x}_1 - \vec{x}_2), \quad (3.8)$$

the field operators being in the Schrödinger representation.

However, it has been very clearly shown that an analogous correspondence does not hold for the Poisson brackets (3.6) and (3.7). The basic reason for this occurrence lies in the difficulty of finding a Hermitian quantum operator associated with the phase θ .^{*}

It therefore appears that to incorporate $\hat{A}$ and $\hat{V}$ in the second quantization framework is a task which is hard to accomplish.

A way out of the impasse is offered by the recognition that, as already mentioned, our classical field should be viewed as an order parameter, which is defined as the expectation value of the quantum field operator in the ground state $|G\rangle$ (the physical vacuum)

$$F(\vec{x}) = \langle G | \psi(\vec{x}) | G \rangle \quad (3.9)$$

and for this quantity we are indeed allowed to set

$$F = f e^{i\theta} \quad (3.10)$$

The existence of $F(\vec{x})$ stems from the spontaneous breaking of the electromagnetic gauge invariance: indeed would $|G\rangle$ be an eigenstate of the charge then the expectation value (3.9) would vanish.

Its introduction into the formalism is best achieved by means of the coherent states.⁸ These are known to minimize the product of the root mean square deviations. For example, in the case of the number operator, for large N (when θ can be accurately defined!) one has

$$\Delta N \Delta \theta \Big|_{\text{CS}} = \frac{1}{2}. \quad (3.11)$$

* For a discussion of this important issue in the simple case of a harmonic oscillator, we refer the reader to the papers of Susskind and Glogower,⁵ Carruthers and Nieto⁶ and Jackiw.⁷ In these works it is suggested that the difficulty can be circumvented by considering rather than the phase θ , the functions $\sin \theta$ and $\cos \theta$. However, the use of the latter prevents the introduction into the formalism of the quantities $\hat{A}$ and $\hat{V}$, which are essential for displaying the all important gauge invariance of the theory.