TRANSVERSE DISTRIBUTION EFFECTS IN BEAMSTRAHLUNG

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Abstract

Beamstrahlung is studied for bunches with densities that are transversely non-uniform. By treating an interesting special case analytically, one elicits the consequences of the new stationary phase conditions. Focusing effects are found to be relatively unimportant on the global radiated rate. Results are compared with the previously known case of uniform cylindrical bunches.

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1 Introduction

Over the past few years, interest in electron-positron colliders operating in the TeV energy range and higher has prompted a detailed study of beamstrahlung in the deep quantum regime [1]. The specific features thus ascertained are very relevant to the design of such machines.

The first studies of the radiative effects expected in bunch-bunch collisions under high-energy and high-density conditions were conducted with the approximation of uniform cylindrical bunches [1, 2]. Specific features associated with more realistic longitudinal bunch shapes and with fluctuations in longitudinal bunch densities were later analysed [3]. Questions associated with the variations of the transverse density of the bunch, however, have not so far been addressed in a systematic way. This deserves attention since one may a priori expect considerable changes from the radiation met with cylindrical bunches. One may perhaps suffer from some strong focalization effects, and resulting enhancements of a rainbow type, as radiation occurs in zones of different transverse densities.

One may also find interest in the radiation which occurs when the electron misses the bunch of positrons while traversing the intense field region associated with the bunch. This latter case is very similar to that met in channelling, with its intense radiation by electrons (positrons) traversing a crystal parallel to one of the lattice planes [4].

The reader familiar with beamstrahlung calculations is well aware of the technical difficulties encountered when one departs from the cylindrical approximation. Most of the radiation originates from the zone of stationary phase. For an electron traversing a cylindrical bunch, with a small disruption factor, the stationary phase conditions impose an impact parameter r , which is directly determined by the transverse momentum taken by the traversed bunch, Δ_T . Namely:

$$r = R^2 \frac{\Delta_T}{2N\alpha} , \quad (1)$$

where N is the number of particles in the bunch. The integration over transverse co-ordinates, which is part of the calculation of the radiation amplitude, can then be simply performed by the method of stationary phase, and the relevant impact parameter value can be substituted in the equation for stationary phase in the longitudinal direction [2].

If one now considers a transverse bunch shape, with a density which rises and falls in some continuous way, the one-to-one relation between recoil momentum and impact parameter is lost. In the simplest case, there are now two solutions whereby the electron classically goes through the same average field and radiates the same way, but at two different impact parameters.

The uniform cylindrical bunch case is sufficiently simple for an analytic treatment to be carried out completely. We study here an example of a transversely non-uniform bunch which offers the same advantage. This turns out to be possible with the density distribution

$$\rho(r) = \frac{1}{\pi R^2} \left(\frac{R^2}{R^2 + r^2} \right)^2 , \quad (2)$$

which is here normalized to unity. This density distribution is closer to a realistic bunch shape than a uniform cylinder. The fall-off at large r , being algebraic, may not be quite fast enough. Thus, in this respect, realistic bunch shapes are actually bracketed by two cases which can both be treated analytically.

This sheds more light on the key features of the radiation process and the results obtained can be used as a guide for numerical calculations based on arbitrary transverse shapes.

Since we here wish to focus on the specific features associated with the two solutions found for the impact parameter, we simplify the rest of the problem as much as possible, considering a uniform bunch in the longitudinal direction and treating the motion of the electron with a Klein-Gordon equation. The results, however, are such that the extra features associated with varying longitudinal densities and with the use of the Dirac equation, which have been studied in the case of a uniform transverse density so far [1], can be readily extended to the new situation, at least when some realistic approximations are made.

In order to avoid duplication, we make extensive use here of the paper quoted in Ref. [2], which should be considered as a companion to the present one. We follow the same notations and refer extensively to the relations derived in that paper.

2 Stationary Phase Conditions

We take the radial bunch density (2) and calculate the potential for a bunch of length L_b :

$$eV(r, z) = \begin{cases} 0 & z < -L_b/2 \\ \frac{N\alpha}{L_b} \ell n \frac{R^2 + r^2}{R^2} & |z| < L_b/2 \\ 0 & z > L_b/2 \end{cases} . \quad (3)$$

As usual [2, 3], we work in the rest-frame of the bunch and only the electric field is relevant. It is radial with a value

$$E(r) = \frac{Ne}{4\pi L_b} \frac{2r}{R^2 + r^2} . \quad (4)$$

It goes through a maximum at $r = R$, but the value of the field there is half of that met with a uniform bunch with the same global charge [2]. As a result, the maximum transverse momentum transferred by the bunch is only half of the value $\frac{2N\alpha}{R}$ found for a uniform cylindrical bunch.

We next calculate the phase of the electron wave function using Eqs. (7) and (8) of Ref. [2]. We obtain, for $|z| < \frac{L_b}{2}$:

$$\begin{aligned} \phi_i &= k_i z - \frac{N\alpha}{L_b} \left(z + \frac{L_b}{2} \right) \left\{ \ell n \frac{R^2 + r^2}{R^2} + \frac{2}{3k_i} \frac{N\alpha}{L_b} \left(z + \frac{L_b}{2} \right)^2 \frac{r^2}{(R^2 + r^2)^2} \right\} \\ \phi_f &= \vec{k}_f \cdot \vec{x} + \frac{N\alpha}{L_b} \left(\frac{L_b}{2} - z \right) \ell n \frac{R^2 + r^2}{R^2} + \frac{1}{2k_f} \left\{ \frac{2k_{fx} N\alpha}{L_b} \frac{x}{R^2 + r^2} \left(\frac{L_b}{2} - z \right)^2 \right. \\ &\quad \left. + \frac{4}{3} \left(\frac{N\alpha}{L_b} \right)^2 \left(\frac{L_b}{2} - z \right)^3 \frac{r^2}{(R^2 + r^2)^2} \right\} . \end{aligned} \quad (5)$$

We work with a distorted wave Born approximation and it is the phases of the initial and final electron wave functions which enter the calculation of the radiation amplitude. These expressions show similarities to and differences from those previously obtained in the case of a cylindrical bunch and which are given by Eq. (11) of Ref. [2]. Here the xz plane is chosen as the bending plane, defined by the momenta $\vec{k}_i$ and $\vec{k}_f$ of the incoming and outgoing electrons with $\vec{k}_i = k_i \hat{z}$.

The phase of the radiation amplitude, which was given by relation (13) of Ref. [2], is now found to be ($|z| < \frac{L_b}{2}$):

$$\phi = \left(k_i z - \vec{k}_f \cdot \vec{x} - \vec{k}_\gamma \cdot \vec{x} - N\alpha \ell n \frac{R^2 + r^2}{R^2} \right)$$

$$\begin{aligned}
& + \frac{1}{2k_f} \left\{ \frac{-2k_{fx}N\alpha}{L_b} \frac{x}{R^2 + r^2} \left(\frac{L_b}{2} - z \right)^2 \right. \\
& \left. - \frac{4}{3} \left(\frac{N\alpha}{L_b} \right)^2 \left(\frac{L_b}{2} - z \right)^3 \frac{r^2}{(R^2 + r^2)^2} \right\} \\
& - \frac{2}{3k_i} \left(\frac{N\alpha}{L_b} \right)^2 \left(z + \frac{L_b}{2} \right)^3 \frac{r^2}{(R^2 + r^2)^2} .
\end{aligned} \tag{6}$$

There are similar expressions for $z < -\frac{L_b}{2}$ and $z > \frac{L_b}{2}$, which are not explicitly written since we shall focus on the bulk of the radiation obtained for $|z| < \frac{L_b}{2}$.

The stationary conditions with respect to the transverse co-ordinates are controlled by the terms in the first parenthesis of Eq. (6) [2]. They contribute to the total phase the quantity

$$\begin{aligned}
\phi_0 &= (k_i - k_{fz} - h_{\gamma z})z - k_{fx}x - k_{\gamma x}x - k_{\gamma y}y \\
&\quad - N\alpha \ell n \frac{R^2 + r^2}{R^2} .
\end{aligned} \tag{7}$$

The points of stationary phase are given by the equations:

$$\begin{aligned}
\frac{x}{R^2 + r^2} &= -\frac{k_{fx} + k_{\gamma x}}{2N\alpha} \\
\frac{y}{R^2 + r^2} &= -\frac{k_{\gamma y}}{2N\alpha} .
\end{aligned} \tag{8}$$

A simpler linear relation was bound to the uniform cylindrical bunch (Eq. (14) of Ref. [2]). We now have two solutions

$$\begin{aligned}
x &= -\frac{N\alpha(k_{fx} + k_{\gamma x})}{\Delta_T^2} \left(1 \pm \sqrt{1 - \frac{\Delta_T^2 R^2}{(N\alpha)^2}} \right) \\
y &= -\frac{N\alpha k_{\gamma y}}{\Delta_T^2} \left(1 \pm \sqrt{1 - \frac{\Delta_T^2 R^2}{(N\alpha)^2}} \right) ,
\end{aligned} \tag{9}$$

(here $\Delta_t = \sqrt{(h_{fx} + h_{\gamma x})^2 + h_{\gamma y}^2}$) hence the doubling of stationary points previously mentioned. They coincide for maximum recoil, when $\Delta_T = \frac{N\alpha}{R}$, or $r = R$, and it is this merging of the two solutions which has to be handled carefully. This corresponds to the domain of maximum radiation.

With $\Delta_L = k_i - k_{fz} - k_{\gamma z}$ (Eq. (16) of Ref. [2]), we can write (7), under the stationary conditions, as

$$\begin{aligned}
\phi_0 &= \Delta_L z + N\alpha \left(1 \pm \sqrt{1 - \frac{\Delta_T^2 R^2}{(N\alpha)^2}} \right) \\
&\quad + N\alpha \ell n \frac{1}{2} \left(1 \mp \sqrt{1 - \frac{\Delta_T^2 R^2}{(N\alpha)^2}} \right)
\end{aligned} \tag{10}$$

With ϕ given by (6) and ϕ_0 given by (10), and substituting r in (6) with either of the solutions (9), we find that, whilst ϕ and ϕ_0 depend on which solution is chosen, the value of

$$\phi_1 = \phi - \phi_0 \tag{11}$$

is independent of the solution which is selected. One finds

$$\begin{aligned}\phi_1 = & \frac{1}{2k_f} \left(\frac{h_{fx}\Delta_{Tx}}{L_b} \left(\frac{L_b}{2} - z\right)^2 - \frac{\Delta_T^2}{3L_b^2} \left(\frac{L_b}{2} - z\right)^3 \right) \\ & - \frac{1}{6k_i} \frac{\Delta_T^2}{L_b^2} \left(z + \frac{L_b}{2}\right)^3 .\end{aligned}\quad (12)$$

Since the z dependence of ϕ_0 is not related to the solution chosen, this means that the z dependence of the phase of the amplitude is the same for both solutions. There is therefore only one stationary condition in the longitudinal direction. This is the first important result.

The next step consists of performing the integration over the radial co-ordinates, granting the conditions imposed by stationary phases (9). In the uniform bunch case this was particularly simple. The only relevant term in the phase was $-N\alpha \frac{r^2}{R^2}$ (Eq. (13) of Ref. [2]) which, up to a global phase independent of z , gave under integration a factor $\pi R^2/N\alpha$.

Things are now more involved. In view of (11) and (12), the quantity to integrate is $\exp(i\phi_0)$. We therefore consider the integral

$$\begin{aligned}\mathcal{I}_o &= \int_0^\infty r dr \int_{-\pi}^\pi d\theta e^{i\phi_0} \\ &= e^{i\Delta_L z} \int_0^\infty r dr \int_{-\pi}^\pi d\theta \exp \left\{ -i(k_{fx} + k_{\gamma x})r \cos \theta \right. \\ &\quad \left. - k_{\gamma y} r \sin \theta - N\alpha \ln \frac{R^2 + r^2}{R^2} \right\} \\ &= 2\pi e^{i\Delta_L z} \int_0^\infty r dr \left(\frac{R^2 + r^2}{R^2} \right)^{-iN\alpha} J_0(r\Delta_T) .\end{aligned}\quad (13)$$

The remaining integral is known [5]. One finds

$$\mathcal{I}_o = 2\pi R^2 e^{i\Delta_L z} \left(\frac{R\Delta_T}{2} \right)^{iN\alpha-1} K_{iN\alpha-1}(R\Delta_T) \frac{1}{r(iN\alpha)} .\quad (14)$$

Including now ϕ_1 (12), we can write the radial integral over the full phase factor as

$$\begin{aligned}\mathcal{I} &= 2\pi R^2 \frac{1}{\Gamma(iN\alpha)} \left(\frac{R\Delta_T}{2} \right)^{iN\alpha-1} K_{iN\alpha-1}(R\Delta_T) \\ &\quad \exp \left\{ i\Delta_L z + \frac{1}{2k_f} \left(-\frac{k_{fx}\Delta_{Tx}}{L_b} \left(\frac{L_b}{2} - z\right)^2 - \frac{\Delta_T^2}{3L_b^2} \left(\frac{L_b}{2} - z\right)^3 \right) \right. \\ &\quad \left. - \frac{1}{2k_i} \frac{\Delta_T^2}{3L_b^2} \left(z + \frac{L_b}{2}\right)^3 \right\} .\end{aligned}\quad (15)$$

The radiation matrix elements (Eq. (33) of Ref. [2]) are calculated at the stationary points. It is then clear that the integral $\mathcal{I}$ is the one relevant for the full amplitude, summing over radial co-ordinates while including both solutions in the same expression.

We can now compare (15) the corresponding expression in the uniform density case. The z -dependent parts of the phase are identical (Eq. (21) of Ref. [2]). This is our second important result.

This result implies that the stationary phase conditions and the corresponding integrals over z , with the related Airy functions, can proceed as previously [2]. One can read the relevant

results from Ref. [2]. The overall $\frac{\pi R^2}{N\alpha}$ factor is, however, now replaced by a more complicated expression according to the substitution

$$\frac{\pi R^2}{N\alpha} \rightarrow 2\pi R^2 \frac{1}{\Gamma(iN\alpha)} \left(\frac{R\Delta_T}{2}\right)^{iN\alpha-1} K_{iN\alpha-1}(R\Delta_T). \quad (16)$$

This implies that the phase-space integrals, met in the calculation of the radiation rate, are now far more involved.

3 The Radiation Rate

We can read the expression for the full radiation rate I from Ref. [2], where it is given by Eqs. (51) and (56). We divide it however by $\left(\frac{\pi R^2}{N\alpha}\right)^2$, since this simple factor is no longer present:

$$I \left(\frac{N\alpha}{\pi R^2}\right)^2 = \frac{\alpha}{\pi} \mathcal{N} \left(\frac{L_b}{k_i}\right)^{\frac{1}{3}} \int_0^1 \left(\frac{1-X}{X}\right)^{\frac{2}{3}} dX \int_0^{\frac{2N\alpha}{R}} \Delta_T^{\frac{5}{3}} d\Delta_T. \quad (17)$$

Here N is a numerical factor, which was found to be

$$\mathcal{N} = 6^{5/3}(\sqrt{\pi})^{-1}\Gamma(5/6)/\Gamma(5/3)\left(\frac{1}{216} + \frac{1}{72}\right) = 0.26.$$

It results from the integration over Airy functions. The quantity I/R^2 is, as it should be, dimensionless. The photon takes a fraction X of the incident electron energy.

The rate in the present case can be obtained from (17) including the modulus square of the right-hand side of (16) under the integral over Δ_T .

The net change in the radiation rate is only this new integral over Δ_T . The structure of the integrand includes the focusing effects.

The coherent radiation length ℓ_c [2] is otherwise the same as the one obtained in the uniform case since the z dependence of the phase remains the same. The spectrum (X dependence) also remains the same.

The change from the Klein-Gordon case to the Dirac case, neglecting as is legitimate in practice the helicity-flip contribution, is again simply a change in the spectrum [1]. The leading effect associated with a non-uniform longitudinal distribution is again obtained as in the case of a transversely uniform distribution [3], introducing a local coherent radiation length $\ell_c(z)$. One can even factorize the effects of longitudinal and transverse density fluctuations as long as the parameter R in Eq. (2) does not vary appreciably over ℓ_c , while still varying appreciably over L_b .

We can now proceed with the calculation of the radiation rate combining (16) and (17) according to the procedure which we spelled out.

This gives

$$I = \frac{\alpha}{\pi} \mathcal{N} \left(\frac{L_b}{k_i}\right)^{1/3} \int_0^1 \left(\frac{1-X}{X}\right)^{2/3} dX (2\pi R^2)^2 \int_0^{N\alpha/R} \Delta_T^{5/3} \left| \left(\frac{R\Delta_T}{2}\right)^{iN\alpha-1} \frac{1}{\Gamma(iN\alpha)} K_{iN\alpha-1}(R\Delta_T) \right|^2 d\Delta_T. \quad (18)$$

Before moving ahead with this non-trivial integral, we remark that it has, as it should, the dimension of an area. In the case of a uniform density, it is proportional to πR^2 and one obtains the radiated intensity per incoming electron, normalizing it to one electron incident over the bunch cross-section, namely dividing it by πR^2 .

In the present case, the bunch spreads to large transverse distances (2) though with a rapidly decreasing density. The most realistic normalization is obtained in terms of an incident electron with a probability density distributed in r according to the same bunch density profile. This corresponds to the situation with good bunch overlap.

If the normalization is not properly handled, the integral in (18) will diverge because of the small Δ_T (large r) behaviour of the integrand. This will not occur if an extra weight, associated with the probability of presence of the incident electron, falls at large r .

We thus substitute (9) in (2) to obtain the electron density associated with a particular recoil Δ_T . This gives:

$$\frac{R^2}{R^2 + r^2} = \frac{1}{1 + \left(\frac{N\alpha}{R\Delta_T}\right)^2 \left(1 \mp \sqrt{1 - \frac{(R\Delta_T)^2}{(N\alpha)^2}}\right)} \quad (19)$$

for the small and large r solutions, respectively. To simplify the notation, we introduce

$$N\alpha = n, \quad \Delta_T R = x. \quad (20)$$

The quantity n is large (typically 10^6), while x varies from 0 to n .

The presence of two weights, associated with the two values of r corresponding to the same value of Δ_T raises a difficulty. This can be remedied by identifying two different pieces in the production rate (18) which correspond to the small r and large r contribution, respectively, and weighting them differently according to (19).

The two weights are

$$\frac{1}{\pi R^2} \left(\frac{1}{1 + \left(\frac{n}{x}\right)^2 \left(1 \mp \sqrt{1 - \left(\frac{x}{n}\right)^2}\right)} \right)^2 = \frac{1}{\pi R^2} \frac{1}{4} \left(\frac{x}{n}\right)^4 \left(1 \mp \sqrt{1 - \left(\frac{x}{n}\right)^2}\right)^{-2} \quad (21)$$

We may get a first look at this separation between two contributions in the production rate by checking the overall normalization in the cases of very small bending, when x is small.

We use the relation

$$K_{in-1}(x) = \frac{\pi}{2 \sin(in-1)\pi} \left(I_{-in+1}(x) - I_{in-1}(x) \right), \quad (22)$$

with, for small x

$$I_{-in+1}(x) \simeq \left(\frac{x}{2}\right)^{-in+1} \frac{1}{\Gamma(-in+2)}. \quad (23)$$

Since the integrand (18) includes a $\left(\frac{x}{2}\right)^{in-1}$ factor, the first term only will result in a non-vanishing result upon integration. The contribution associated with the second one will indeed be damped out because of the rapid oscillations.

In the quantity

$$2\pi R^2 \frac{1}{\Gamma(in)} \left(\frac{x}{2}\right)^{in-1} K_{in-1}(x),$$

we can therefore only retain, at small x

$$\pi R^2 \frac{\pi}{\sin(in-1)} \frac{1}{\Gamma(in) (-in+1)\Gamma(-in+1)} = \pi R^2 \frac{-i}{n+i} . \quad (24)$$

Up to a constant phase, we thus get back the uniform density value which is on the left-hand side of (16). This is the proper answer since we have picked only the very small impact-parameter contribution and it should be independent of the bunch shape since $\rho(0) = \frac{1}{\pi R^2}$ in both cases.

The large impact-parameter contribution is also present but, because of the very rapid oscillations of the amplitude when x is small and n is large, it does not contribute appreciably to the rate.

This drastic separation is, however, not general and both contributions are usually on similar footings. Yet we thus see how one can separate two contributions from their different phases in the amplitude, and how one could weight differently, with different incident electron densities, two contributions to the radiation rate.

We now proceed with the calculation of (18).

We first separate out the constant term in the integrand, namely

$$\left| \frac{1}{\Gamma(in)} \right|^2 = \frac{1}{\Gamma(in)\Gamma(-in)} = \frac{n \sinh \pi n}{\pi} . \quad (25)$$

Since n is large, it can be written as:

$$\left| \frac{1}{\Gamma(in)} \right|^2 \simeq \frac{n}{\pi} e^{\pi n} . \quad (26)$$

We have to calculate (18), or more particularly, using (20), the quantity

$$Q = R^{-8/3} \int_0^1 x^{5/3} \left| \left(\frac{x}{2} \right)^{in-1} K_{in-1}(x) \right|^2 dx \quad (27)$$

It is useful to rewrite it as [6]

$$Q = R^{-8/3} \int_0^1 x^{5/3} \left| \left(\frac{x}{2} \right)^{ip} \sqrt{2\pi} (p^2 - x^2)^{-1/4} e^{(-1/2)p\pi} \sin(pCh^{-1} \frac{p}{x} - \sqrt{p^2 - x^2} + \frac{\pi}{4}) \right|^2 dx \quad (28)$$

with $p = n + i$.

This clearly exhibits a very small factor $e^{-\pi n}$ which compensates the very large factor in (26).

Since n is very large (10^6) we can keep only the first derivative to the argument of the sine, ψ , to express it in terms of n . Because

$$\frac{d}{dp} (pCh^{-1} \frac{p}{x} - \sqrt{p^2 - x^2}) = Ch^{-1} \frac{p}{x} .$$

This gives $\psi = \psi_0 + i\psi_1$, with

$$\begin{aligned} \psi_0 &= nCh^{-1} \frac{n}{x} - \sqrt{n^2 - x^2} + \frac{\pi}{4} \\ \psi_1 &= Ch^{-1} \frac{n}{x} . \end{aligned} \quad (29)$$

We thus have, for (28), an expression valid at large n

$$Q = 2\pi R^{-8/3} e^{-n\pi} \int_0^1 x^{5/3} \left| \left(\frac{x}{2} \right)^{in-1} (n^2 - x^2)^{-1/4} \sin(\psi_0 + i\psi_1) \right|^2 dx \quad (30)$$

with

$$\left| \sin(\psi_0 + i\psi_1) \right|^2 = \frac{1}{2} (Ch2\psi_1 - \cos 2\psi_0) . \quad (31)$$

Because of the rapid oscillations of the phase, the cosine term gives a vanishing contribution upon integration. What remains is the contribution of the $Ch2\psi_1$ term. This can be written as the sum of two terms. From the definition (29), the former one is proportional to

$$\begin{aligned} e^{2\psi_1} &= \exp \left(2 \ln \frac{n}{x} + \sqrt{\left(\frac{n}{x} \right)^2 - 1} \right) \\ &= \frac{n^2}{x^2} \left(1 + \sqrt{1 - \left(\frac{x}{n} \right)^2} \right) , \end{aligned} \quad (32)$$

whereas the latter one is proportional to

$$e^{-2\psi_1} = \frac{n^2}{x^2} \left(1 - \sqrt{1 - \left(\frac{x}{n} \right)^2} \right) . \quad (33)$$

These two terms are each associated with a particular phase factor in (31) which can be used to trace the origin of the radiation, associating it either with the small r or with the large r solution of (9). This follows the separation obtained in (22) before the low x approximation was introduced. One thus finds that the factors (32) and (33) are respectively associated with the large r and small r solutions.

One may trace the ratio between the two factors to the different variations of the field with r , with a sharper rise than the decline.

It is a welcome feature of our bunch profile (2) that this ratio is just the inverse of that of the relevant densities (21) with which we found that it was proper to weight each contribution.

With this weighting performed, the term $\frac{1}{2}Ch2\psi_1$, as it appears in (31), should be replaced by

$$\frac{1}{2}Ch2\psi_1 \rightarrow \frac{1}{\pi R^2} \frac{1}{4} \left(\frac{x}{n} \right)^2 , \quad (34)$$

which gives, for (30)

$$Q = \frac{R^{-14/3}}{8n^2} e^{-n\pi} \int_0^1 x^{5/3} \frac{1}{\sqrt{n^2 - x^2}} dx . \quad (35)$$

The $x(\Delta_T)$ distribution is singular at $\Delta_t = \frac{N\alpha}{R}$, as expected since the field passes through its maximum for the impact parameter associated with this transferred momentum. However, the singularity can be safely integrated upon. There is no strong rainbow effect with an important enhancement of the rate.

This result justifies an approach whereby the two contributions associated with the two stationary points would be separately calculated through stationary phase integrals, something which can be used for more realistic forms of the density but at the price of numerical integrations.

This is our third important result. The transverse spread of the density does not increase in any important way the beamstrahlung yield, as one could have a priori feared as the result

of a strong rainbow effect associated with a more singular term in (35). The simple sum of an $r < R$ and an $r > R$ contribution is a good approximation.

The integral in (35) can be written as a beta function:

$$Q = \frac{R^{-14/3}}{16n^2} e^{-n\pi} \sqrt{\pi} \frac{\Gamma(4/3)}{\Gamma(11/6)} . \quad (36)$$

Combining it with (26) completes the calculation of the integral over Δ_T (18) taking into account the weighting (21) which normalizes the result to a unit flux. The overall n dependence is $n^{2/3}$. The overall R dependence is $R^{-2/3}$. We can thus factorize all quantities with dimensions into a dimensionless factor

$$\frac{L_b}{\ell_c} = \left(\frac{L_b}{k_i} \right)^{1/3} \left(\frac{N\alpha}{R} \right)^{2/3} . \quad (37)$$

This is the global factor, as it appears in relation (58) of Ref. [2], when the rate is also normalized to a unit incident flux.

4 Conclusions

The study of beamstrahlung with bunches with a sizeable spreading of their transverse distribution has been done in a specific case which can be treated analytically. This complements previous analytic studies limited so far to a uniform charge distribution with a sharp boundary.

We have shown how the new stationary phase conditions with a doubling of the solution can be treated and we have calculated the radiation rate. The new focusing properties do not result in an appreciable enhancement.

In order to actually compare energy losses in the cylindrical case and in the present one, the machine luminosity should be kept the same. With the present bunch profile (2), the parameter R should then be taken as $R_0/\sqrt{3}$, where R_0 is the radius of the luminosity-equivalent cylindrical bunch. For the same luminosity one then finds that the ratio of beamstrahlung energy losses is given by

$$\frac{I}{I_0} = \frac{\sqrt{\pi}}{3} 6^{1/3} \frac{\Gamma(4/3)}{\Gamma(11/6)} \simeq 1.02 . \quad (38)$$

It is extremely close to 1.

The transverse spreading of the bunch does not increase the beamstrahlung energy loss provided that the luminosity is kept the same at fixed N . One can conclude that previous treatments of beamstrahlung [1],[2],[3] can be safely used in the design of high-energy electron-positron colliders, irrespectively of the transverse bunch profile.

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