Sphalerons at finite mixing angle and singular gauges

Yves Brihaye

CERN, Geneva, Switzerland

Burkhard Kleihaus

Fachbereich Physik, University of Oldenburg, P.O. Box 2503, D-2900 Oldenburg, Germany

Jutta Kunz

Institute for Theoretical Physics, University of Utrecht, NL-3508 TA Utrecht, The Netherlands
and Fachbereich Physik, University of Oldenburg, P.O. Box 2503, D-2900 Oldenburg, Germany

(Received 6 October 1992)

The energy functional of the classical electroweak sphaleron at a finite mixing angle θ_W has a residual U(1) gauge symmetry. Several choices of gauge, which seem natural, lead to singular configurations for the sphaleron solution.

PACS number(s): 11.15.Kc

I. INTRODUCTION

Recently we have constructed the electroweak sphaleron at a finite mixing angle θ_W in the bosonic sector of the Weinberg-Salam theory [1,2]. While at zero values of θ_W the sphaleron is spherically symmetric, it becomes ellipsoidal at finite θ_W , owing to the nonvanishing coupling to the U(1) gauge field. At a finite mixing angle θ_W the dynamical degrees of freedom are then functions of two variables. In fact one finds a two dimensional gauge-Higgs theory with an Abelian gauge symmetry [1-4]. Fixing the gauge, however, in order to construct a regular classical sphaleron solution turns out to be a subtle problem.

The axial sphalerons have been constructed numerically in the Coulomb gauge [1,2]. In this gauge, the Higgs fields and the gauge potentials are well defined and regular everywhere. The energy density is also regular and the energy of the solution is finite. Here, we investigate

the regularity properties of the sphaleron solution in alternative gauges, the "hedgehog gauge," the "physical gauge," and the "radial gauge." These choices of gauge seem, *a priori*, natural, since they correspond to a "deformation" of the spherically symmetric sphaleron solution (for $\theta_W=0$) into its axially symmetric version (for $\theta_W \neq 0$). However, it appears that none of these gauges is appropriate to describe a regular sphaleron solution. The SU(2) functions necessary to perform the gauge transformation from the Coulomb gauge into those gauges are ill defined at the origin, and several components of the gauge potentials become singular.

II. ANSATZ

The appropriate *Ansatz* for the gauge and Higgs fields is invariant only under rotations about the z axis and parity reflections. Let us present the *Ansatz* [1-4] for the SU(2) gauge fields $W_i^a(\mathbf{r})$ by means of the matrix

$$W = \begin{pmatrix} \sin\phi \cos\phi [w_1^3(\rho, z) + w_3^1(\rho, z)] & \sin^2\phi w_1^3(\rho, z) - \cos^2\phi w_3^1(\rho, z) & \sin\phi w_2^3(\rho, z) \\ \sin^2\phi w_3^1(\rho, z) - \cos^2\phi w_1^3(\rho, z) & -\sin\phi \cos\phi [w_1^3(\rho, z) + w_3^1(\rho, z)] & -\cos\phi w_2^3(\rho, z) \\ \sin\phi w_2^3(\rho, z) & -\cos\phi w_2^3(\rho, z) & 0 \end{pmatrix} \quad (1a)$$

(rows correspond to isospin, columns to spin) and the *Ansätze* for the U(1) gauge field and the Higgs field by

$$\mathbf{A}(\mathbf{r}) = (\sin\phi a_3(\rho, z), -\cos\phi a_3(\rho, z), 0), \quad (1b)$$

$$\Phi(\mathbf{r}) = i((\tau^1 \cos\phi + \tau^2 \sin\phi)h_1(\rho, z) + \tau^3 h_2(\rho, z)) \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1c)$$

Let us further introduce the functions $F_i(\rho, z)$ via

$$w_1^3(\rho, z) = \frac{2z}{gr^2} F_1(\rho, z), \quad w_2^3(\rho, z) = -\frac{2\rho}{gr^2} F_2(\rho, z), \quad (2a)$$

$$w_3^1(\rho, z) = -\frac{2z}{gr^2} F_3(\rho, z), \quad w_3^2(\rho, z) = \frac{2\rho}{gr^2} F_4(\rho, z);$$

$$h_1(\rho, z) = \frac{\rho}{r} F_5(\rho, z), \quad h_2(\rho, z) = \frac{z}{r} F_6(\rho, z), \quad (2b)$$

$$a_3(\rho, z) = \frac{2\rho}{g'r^2} F_7(\rho, z). \quad (2c)$$

Here the trivial θ dependence has been taken out, so in

the spherical limit we obtain $F_1(r, \theta) = F_2(r, \theta) = F_3(r, \theta) = F_4(r, \theta) = f(r)$, $F_5(r, \theta) = F_6(r, \theta) = h(r)$, and $F_7(r, \theta) = 0$, where the functions $f(r)$ and $h(r)$ correspond to those of Ref. [5].

III. U(1) GAUGE FREEDOM

The above *Ansatz* leads to an axially symmetric energy functional [1,2], which is still invariant under U(1) gauge transformations generated by

$$U = \exp[i\gamma(\rho, z)(\tau^1 \sin\phi - \tau^2 \cos\phi)] \quad (3)$$

(analogous to the energy functional of multimono- poles [3,4]). The functions $w_i^j(\rho, z)$ and $h_i(\rho, z)$ represent a two-dimensional Abelian gauge theory coupled to two Higgs doublets. Under the gauge transformation (3) the two-dimensional gauge field $(w_1^3(\rho, z), w_2^3(\rho, z))$ transforms in the standard way,

$$\begin{bmatrix} w_1^3 \\ w_2^3 \end{bmatrix} = \begin{bmatrix} w_1^3 \\ w_2^3 \end{bmatrix} + \frac{2}{g} \begin{bmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial z} \end{bmatrix} \gamma, \quad (4a)$$

while the Higgs doublets $(h_1(\rho, z), h_2(\rho, z))$ and $(w_1^3(\rho, z), w_2^3(\rho, z) - 1/g\rho)$ transform as

$$\begin{bmatrix} h_1' \\ h_2' \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad (4b)$$

$$\begin{bmatrix} w_1^3 \\ w_2^3 - 1/g\rho \end{bmatrix} = \begin{bmatrix} \cos 2\gamma & \sin 2\gamma \\ -\sin 2\gamma & \cos 2\gamma \end{bmatrix} \begin{bmatrix} w_1^3 \\ w_2^3 - 1/g\rho \end{bmatrix}. \quad (4c)$$

In order to construct the sphaleron solution, one has to fix a gauge. In the following we will consider several seemingly obvious choices of gauge.

A. Coulomb gauge

In Refs. [1,2] we presented most of our results for the sphaleron in the Coulomb gauge, which is obtained by choosing the gauge condition

$$G_{gf} = \partial_\rho w_1^3 + \partial_z w_2^3 = 0. \quad (5)$$

In this gauge there are seven unknown functions. But by making use of the gauge degree of freedom, one may try to reduce the number of functions by one.

B. Hedgehog gauge

When choosing the hedgehog gauge,

$$G_{gf} = \rho h_2(\rho, z) - z h_1(\rho, z) = 0, \quad (6a)$$

i.e., $F_5(\rho, z) = F_6(\rho, z)$, there are only six unknown functions. The Higgs field then assumes the hedgehog form

$$\Phi'(\mathbf{r}) = U_h \Phi(\mathbf{r}) = i\tau \cdot \hat{\mathbf{r}} L(\rho, z) \frac{v}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (6b)$$

where $\hat{\mathbf{r}}$ denotes the unit vector.

C. Physical gauge

The physical gauge is obtained from the hedgehog gauge by the transformation

$$U_p = -i\tau \cdot \hat{\mathbf{r}}. \quad (7a)$$

Here the Higgs field assumes its vacuum expectation value asymptotically:

$$\Phi'(\mathbf{r}) = L(\rho, z) \frac{v}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (7b)$$

D. Radial gauge

Another simple choice of gauge with six unknown functions is the radial gauge $\mathbf{r} \cdot \mathbf{W}^a = 0$, leading to

$$G_{gf} = \rho w_1^3(\rho, z) + z w_2^3(\rho, z) = 0, \quad (8)$$

i.e., $F_1(\rho, z) = F_2(\rho, z)$.

IV. RESULTS AND DISCUSSION

A. Coulomb gauge

Regular finite-energy solutions of the field equations with the imposed symmetries require the following boundary conditions for the functions $F_i(r, \theta)$ in the Coulomb gauge [1,2]:

$$r=0: F_i(r, \theta)|_{r=0} = 0, \quad i=1, \dots, 7, \quad (9)$$

$$r \rightarrow \infty: F_i(r, \theta)|_{r \rightarrow \infty} = 1, \quad i=1, \dots, 6, \quad F_7(r, \theta)|_{r \rightarrow \infty} = 0.$$

In this gauge the functions $F_i(r, \theta)$ are well defined and regular everywhere, and so is the energy density. Both energy density and functions $F_i(r, \theta)$ are exhibited in [1,2] for $0 \leq \theta_w < \pi/2$.

B. Hedgehog gauge

In the hedgehog gauge we did not manage to construct regular solutions, subject to the above boundary conditions (9). In fact, several functions $F_i(r, \theta)$, ($i=1, 2, 3$) had ill-defined derivatives at the origin and the energy density diverged there [2]. To understand this phenomenon at first sight surprising, let us now try to gauge rotate the regular sphaleron solution of the Coulomb gauge into the hedgehog gauge. The gauge transformation is given by [2]

$$\tan\gamma(r, \theta) = -\frac{\cos\theta \sin\theta [F_5(r, \theta) - F_6(r, \theta)]}{\sin^2\theta F_5(r, \theta) + \cos^2\theta F_6(r, \theta)}. \quad (10)$$

In the Coulomb gauge close to the origin the functions $F_5(r, \theta)$ and $F_6(r, \theta)$ rise linearly with r :

$$r \rightarrow 0: F_5(r, \theta) \approx rp, \quad F_6(r, \theta) \approx r(p + \epsilon). \quad (11)$$

For small mixing angle θ_w , the difference in the slopes of $F_5(r, \theta)$ and $F_6(r, \theta)$ close to the origin, i.e., ϵ , is small [1,2]. Keeping θ fixed in Eq. (10) and expanding $\gamma(r, \theta)$

for small values of r and of ϵ/p , we find

$$\gamma(r, \theta) = \frac{\epsilon}{2p} \sin 2\theta + O(\epsilon^2) + O(r^2). \quad (12a)$$

Therefore several of the SU(2) gauge-field functions are not well defined at the origin:

$$\begin{aligned} r=0: F'_1(r, \theta)|_{r=0} &= \frac{\epsilon}{p} \cos 2\theta + O(\epsilon^2), \\ F'_2(r, \theta)|_{r=0} &= \frac{\epsilon}{p} \cos 2\theta + O(\epsilon^2), \end{aligned} \quad (12b)$$

$$\begin{aligned} r=0: F'_3(r, \theta)|_{r=0} &= \frac{\epsilon}{p} + O(\epsilon^2), \\ F'_4(r, \theta)|_{r=0} &= 0 + O(\epsilon^2). \end{aligned} \quad (12c)$$

These conditions are to be contrasted with the boundary conditions (9). Note that the Higgs-field functions $F_5(r, \theta)$ and $F_6(r, \theta)$ remain well defined, since they transform linearly and vanish at the origin. The values of ϵ/p are exhibited in Fig. 1 as a function of the mixing angle for the parameters $g=0.65$, $M_W=80$ GeV, and $M_H=M_W$. The transformed fields satisfying (12) lead to a regular energy density.

C. Physical gauge

In the physical gauge the SU(2) gauge-field functions $F_i(r, \theta)$, $i=1, \dots, 4$, change according to

$$\begin{aligned} F'_i(r, \theta) &= 1 - F_i(r, \theta), \quad i=1, 2, \\ F'_3(r, \theta) &= 1 - F_3(r, \theta) + 2 \sin^2 \theta (F_3(r, \theta) - F_4(r, \theta)), \\ F'_4(r, \theta) &= 1 - F_4(r, \theta) - 2 \cos^2 \theta (F_3(r, \theta) - F_4(r, \theta)), \end{aligned} \quad (13)$$

with respect to the hedgehog gauge, thus, the functions are not well defined either.

D. Radial gauge

In this gauge we did not manage to construct regular solutions either, subject to the boundary conditions (9). Therefore let us try to gauge-rotate the regular sphaleron solution of the Coulomb gauge into the radial gauge. The condition that determines the gauge function $\gamma(r, \theta)$ is

$$\begin{aligned} F_1(r, \theta) + r \partial_r \gamma(r, \theta) \tan \theta + \partial_\theta \gamma(r, \theta) \\ = F_2(r, \theta) - r \partial_r \gamma(r, \theta) \cot \theta + \partial_\theta \gamma(r, \theta). \end{aligned} \quad (14)$$

Since $\gamma(r, \theta)$ is determined (up to a trivial constant) by its derivatives, we must impose boundary conditions when integrating $\gamma(r, \theta)$. Requiring that $F'_1(r \rightarrow \infty, \theta) = 1$, for instance, leads to

$$\gamma(r, \theta) = \frac{1}{2} \sin 2\theta (I(r, \theta) - K(\theta)); \quad (15a)$$

$$\begin{aligned} I(r, \theta) &= \int_0^r \frac{F_2(r', \theta) - F_1(r', \theta)}{r'} dr', \\ K(\theta) &= \int_0^\infty \frac{F_2(r', \theta) - F_1(r', \theta)}{r'} dr'. \end{aligned} \quad (15b)$$

Thus at the origin again several of the SU(2) gauge field

FIG. 1. The function ϵ/p defined in Eq. (11) is shown as a function of the mixing angle θ_w for $M_H=M_W$. At $\theta_w=0.5$ we find $\epsilon/p = -0.006/0.437 = -0.014$. Also shown is the function $K(\theta)$ defined in Eq. (15) as a function of the mixing angle θ_w for $M_H=M_W$. The solid and the dashed curves represent the angles $\theta=0$ and $\theta=\pi/2$, respectively.

functions are not well defined:

$$r=0: F'_1(r, \theta)|_{r=0} = -\cos 2\theta K(\theta) - \frac{1}{2} \sin 2\theta \partial_\theta K(\theta), \quad (15c)$$

$$r=0: F'_3(r, \theta)|_{r=0} = -K(\theta), \quad F'_4(r, \theta)|_{r=0} = 0. \quad (15d)$$

In Fig. 1 $K(\theta)$ is exhibited as a function of the mixing angle for the parameters $g=0.65$, $M_W=80$ GeV, and $M_H=M_W$.

Let us conclude now. It has often appeared technically difficult to construct axially symmetric, regular, finite-energy configurations, e.g., multimonopoles or $B > 1$ Skyrmons [6], and the axially symmetric sphaleron is no exception in this respect [1,2]. As in the case of multimonopoles, one is dealing with a gauge theory when constructing the axially symmetric sphaleron, so that the question of gauge fixing is present. However, unlike the case of the monopoles, there are so far no analytical tools that might be used as a guiding line. For the axially symmetric sphaleron the technical problems are therefore threefold. (1) We are dealing with a system of nonlinear partial differential equations. (2) Analytical solutions (whatever limit we consider) are absent and there are no mathematical tools available, such as topological classes or Backlund transformations. (3) We are facing the problem of making a proper choice of gauge. In this last respect we have demonstrated here that seemingly natural choices of gauge can lead to ill-defined fields when one is dealing with axially symmetric configurations and therefore functions of two variables. In fact the gauge transformation from a regular gauge to such a singular gauge itself is then ill defined. Assuming that a regular solution exists, then inspection of the necessary gauge transformation to the chosen gauge reveals whether this choice of gauge also allows for a regular solution. But even if the solution is not regular, it does have a well-defined finite-energy density, provided that an adequate choice of boundary conditions is made.

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