COVARIANT PHASE-SPACE QUANTIZATION OF THE INDUCED 2D GRAVITY⁺

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Abstract

We study in a parallel way the induced 2D gravity and the Jackiw-Teitelboim model on the cylinder from the viewpoint of the covariant description of canonical formalism. We compute explicitly the symplectic structure of both theories showing that their (reduced) phase spaces are finite-dimensional cotangent bundles. For the Jackiw-Teitelboim model the base space (configuration space) is the space of conjugacy classes of the $PSL(2, R)$ group. For the induced 2D gravity the (reduced) phase space consists of two (identical) connected components that are isomorphic to the cotangent bundle of the space of conjugacy classes of the affine subgroup of $PSL(2, R)$. This result permits the physical Hilbert space of both theories to be determined exactly.

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1. Introduction

The quantization of Einstein's general relativity still remains as one of the most difficult and crucial problems in theoretical physics. Owing to the great difficulties appearing in the quantization of the gravitational field in four dimensions, the interest of studying lower-dimensional theories has been growing in the last few years. Moreover, in addition to providing insight into the problems that arise in the quantization of four-dimensional gravity, these simpler theories have their own physical and mathematical relevance.

In $2 + 1$ dimensions, general relativity becomes much simpler and can be solved exactly. In Witten's approach [1], the theory is formulated as a Chern-Simons theory with a $ISO(2, 1)$ gauge group. The physical phase space of the theory on an $R \times \Sigma$ space-time is the (finite-dimensional) moduli space $\mathcal{M}$ of flat $ISO(2, 1)$ connections on Σ (modulo gauge transformations). This phase space turns out to be a cotangent bundle, whose base space is the moduli space $\mathcal{N}$ of flat $SO(2, 1)$ connections. This special form of the canonical (symplectic) structure of the theory ($\mathcal{M} = T^*\mathcal{N}$) leads to a quantum Hilbert space given by the space of L^2 functions on $\mathcal{N}$ [1]. In this formulation the problem of renormalizability disappears although the conceptual issues of quantum gravity remain to be addressed. Also, for Chern-Simons theories with compact gauge group on $R \times \Sigma$, the computation of the symplectic structure ω on the (reduced) phase space, i.e. the moduli space of flat connections, is one of the main ingredients in the canonical quantization of the theory [2,3]. In fact, the symplectic structure ω obtained by pushing down the two-form (A_i is the gauge field and δ stands for the exterior derivative on phase space):

$$\frac{k}{8\pi} \int_{\Sigma} \text{Tr} \epsilon^{ij} \delta A_i \wedge \delta A_j , \quad (1.1)$$

turns out to be associated with the first Chern class of the determinant line bundle [4]. For an $SL(2, R)$ Chern-Simons theory the symplectic form is related with the Weil-Peterson Kähler form on the Teichmüller space of the surface Σ [5]. This way of proceeding, first solving the constraints and then quantizing, has been very useful in the understanding of the above topological field theories and can be regarded, in some sense, as the application of the pure (covariant) canonical formalism.

In $1 + 1$ dimensions, the Hilbert-Einstein action does not possess dynamical content since it is a topological invariant (the Euler class) of the space-time manifold. As a natural analogue of the vacuum Einstein equations with a cosmological constant, Jackiw and Teitelboim [6] proposed the equation

$$R = \frac{\Lambda}{2} . \quad (1.2)$$

This equation can be derived from a local variational principle if a scalar field Φ is present in the theory [6]:

$$S = \int \sqrt{g} \left(R - \frac{1}{2} \Lambda \right) \Phi . \quad (1.3)$$

The equation (1.2) can also be obtained from an action closely related to string theory, and this is a further motivation to study two-dimensional gravity. The 2D-gravity action induced by massless matter fields is given by [7,8]

$$S = \frac{c}{96\pi} \int \sqrt{g}(R\Box^{-1}R + \Lambda) \quad (1.4)$$

and yields the constant curvature equation (1.2).

The purpose of this paper is to carry out a canonical analysis of the generally covariant and metric-based field theory (1.4). We shall also study the theory (1.3) establishing a comparison with the induced 2D gravity (1.4). The first problem in trying to define the expression analogous to (1.1) for the action (1.4) is that we face now a non-local Lagrangian. In section 2, and based upon the covariant description of phase space [9,10], we introduce a general scheme to define the canonical structure of non-local field theories (see also [11]). In section 3, and applying the formalism of section 2, we work out the symplectic structure of the Jackiw-Teitelboim model (1.3) on a cylinder (the theory defined on an open spatial section has been analysed, in terms of the standard canonical approach, in [12]). Since the theory is generally covariant, the symplectic structure should be given in terms of diffeomorphism-invariant quantities. Our computation confirms this issue and we find an explicit expression for the symplectic structure leading to a finite-dimensional (reduced) phase space. More precisely, the (reduced) phase space is the cotangent bundle of the space of conjugacy classes of the $PSL(2, R)$ group, i.e. $T^*(PSL(2, R)/\text{ad}PSL(2, R))$. Because of the cotangent bundle nature of the phase space, the quantum Hilbert space then emerges as the space of L^2 class functions of the $PSL(2, R)$ group. As a byproduct of this result, we establish the complete equivalence between the (reduced) phase space of the Jackiw-Teitelboim model and that of the topological gauge theory model [13,14]. In section 4 we study the induced 2D gravity (1.4) on a cylinder in a way parallel to our analysis of the Jackiw-Teitelboim model. Here new features arise, as the fact that only hyperbolic and parabolic monodromies are permitted and the emergence of two disconnected components in the (reduced) phase space. The reduced phase space turns out to be the disjoint union of two identical copies of the cotangent bundle of the space of conjugacy classes of the affine ($\subset PSL(2, R)$) group. As in the Jackiw-Teitelboim model, this makes it possible to determine the Hilbert space of the theory exactly. Finally, in section 5 we state our conclusions.

2. Symplectic structure on the covariant phase space: canonical treatment of nonlocal theories

The phase space of a Lagrangian theory can be equivalently regarded in a manifestly covariant way: the phase space is the space of solutions of the classical equations of motion (modulo gauge transformations). This definition is, in general, equivalent to the standard one: the phase space is the space of the initial data q^i, p_j (restricted by the constraints equations for gauge theories). The symplectic form (and hence the Poisson bracket) is

then given by the standard formula

$$\omega = \sum_i \delta q^i \wedge \delta p_i . \quad (2.1)$$

In the covariant description of phase space the symplectic structure is given by [9,10]

$$\omega = \int_{\Sigma} \omega^{\mu} d\sigma_{\mu} , \quad (2.2)$$

where Σ is an initial value hypersurface and ω^{μ} (the so-called symplectic current) is a conserved and closed two-form ($\partial_{\mu}\omega^{\mu} = 0, \delta\omega^{\mu} = 0$). As we already mentioned δ refers to the exterior-derivative operator on phase space.

For an ordinary (first-order) field theory with a Lagrangian depending on a field Ψ^{α} and its first-order derivatives $\partial_{\mu}\Psi^{\alpha} \equiv \Psi_{\mu}^{\alpha}$, the symplectic current is given by [15]

$$\omega^{\mu} = \delta\Psi^{\alpha} \wedge \delta \frac{\partial\mathcal{L}}{\partial\Psi_{\mu}^{\alpha}} . \quad (2.3)$$

By using the Euler-Lagrange equations of motion and the antisymmetry of the exterior product, it is not difficult to check that (2.3) is a conserved current. Picking Σ to be the standard initial value hypersurface $t = 0$, (2.2) becomes (from now on the exterior product will be omitted):

$$\omega = \int_{t=0} \delta\Psi^{\alpha} \delta \frac{\partial\mathcal{L}}{\partial\Psi_0^{\alpha}} \equiv \int_{t=0} \delta\Psi^{\alpha} \delta\pi_{\alpha} , \quad (2.4)$$

which is the standard expression of the symplectic structure of an ordinary (first-order) field theory. For a gauge theory, the two-form (2.2) is degenerate. Nevertheless, one can obtain the true (non-degenerate) symplectic form by pushing down the two-form (2.2) on the physical phase space: the space of classical solutions modulo gauge transformations. Of course one could turn everything round and define a gauge-type symmetry whenever one faces a degenerate two-form (2.2).

On the bases of the covariant phase-space picture sketched above, we now present a general framework to define the canonical (symplectic) structure for arbitrary (in particular, non-local) field theories. First of all, let us concentrate on a first-order theory $\mathcal{L} = \mathcal{L}(\Psi^{\alpha}, \partial_{\mu}\Psi^{\alpha} \equiv \Psi_{\mu}^{\alpha})$. Under an arbitrary variation of the field $\Psi^{\alpha}(x)$:

$$\Psi^{\alpha}(x) \longrightarrow \Psi^{\alpha}(x) + \delta\Psi^{\alpha}(x) , \quad (2.5)$$

the variation of the Lagrangian is given by

$$\delta\mathcal{L} = \partial_{\mu}J^{\mu} + (E - L)_{\alpha} \delta\Psi^{\alpha} , \quad (2.6)$$

where $(E - L)_{\alpha} = 0$ are the Euler-Lagrange equations of motion implying the vanishing of the variation of the action functional, and $\partial_{\mu}J^{\mu}$ is the standard divergence term that does

not contribute to the variation of the action. However, it is just this term that captures the canonical structure of the theory. On phase space, (2.6) becomes

$$\delta \mathcal{L} = \partial_\mu j^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \Psi_\mu^\alpha} \delta \Psi^\alpha \right), \quad (2.7)$$

and interpreting now the term $\delta \Psi^\alpha(x)$ as a one-form on phase space, we can consistently reinterpret the variation δ in (2.6) as the exterior-derivative operator on phase space. This suggest to consider the one-form j^μ as a "potential" for the symplectic current, i.e.

$$\omega^\mu \equiv -\delta j^\mu. \quad (2.8)$$

From this general definition of the symplectic current, and taking into account that $\delta^2 = 0$, it is straightforward to verify that ω^μ satisfies the two basic properties: $\delta \omega^\mu = 0$ and $\partial_\mu \omega^\mu = 0$.

Now it is important to point out that these two basic properties are a direct consequence of the general expression (2.6) that is still valid for non-local field theories (the application to higher-derivative field theories have been given in [11]). This is the crucial point of our proposal to define the canonical structure of a non-local field theory and we shall apply this scheme to the action (1.4) in section 4.

We finish this section with two comments. If we add a total derivative to a Lagrangian $\mathcal{L} \longrightarrow \mathcal{L} + \partial_\mu A^\mu$, the two-form ω remains unchanged since the current j^μ is just modified by an exact form ($j^\mu \longrightarrow j^\mu + \delta A^\mu$). Although the definition of the two-form ω requires, in general, a Hodge operator, and hence a space-time metric as for the definition of the Lagrangian itself, no such background structure is needed for topological theories as is clear in (1.1).

Recently, the covariant phase-space formulation has been considered in connection with WZW-theories and quantum group symmetries[16].

3. Covariant phase space quantization of the Jackiw-Teitelboim model

In this section we shall study the Jackiw-Teitelboim model based upon the covariant phase-space approach introduced in the previous section. We should be able to compute explicitly the symplectic structure of the model, showing that the reduced phase space is a (finite-dimensional) cotangent bundle. This makes it possible to quantize the theory exactly. We shall present here this result in detail because, in addition to its own interest, it shows similar features to those we shall find in studying the induced 2D gravity.

We begin our analysis by writing the classical equations of motion of the model (1.3). The variation with respect to the scalar field leads to the constant-curvature equation

$$R = \frac{\Lambda}{2}, \quad (3.1)$$

and the equations for the scalar field Φ are equivalent to

$$\nabla_\mu \nabla_\nu \Phi + \frac{1}{4} \Lambda g_{\mu\nu} \Phi = 0 . \quad (3.2)$$

In general, the variation (or the exterior derivative) of the Lagrangian is given by

$$\delta \mathcal{L} = \sqrt{g} \left(R - \frac{\Lambda}{2} \right) \delta \Phi - \frac{1}{2} \sqrt{g} \left(\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \square \Phi - \frac{\Lambda}{4} g_{\mu\nu} \Phi \right) \delta g^{\mu\nu} + \partial_\alpha j^\alpha , \quad (3.3)$$

where the "potential current" j^α is

$$j^\alpha = \sqrt{g} \{ -\Phi (g^{\mu\nu} \nabla^\alpha \delta g_{\mu\nu} - g^{\alpha\mu} \nabla^\nu \delta g_{\mu\nu}) + \partial^\alpha \Phi g^{\mu\nu} \delta g_{\mu\nu} + \partial_\nu \Phi \delta g^{\nu\alpha} \} . \quad (3.4)$$

It is not difficult to check that, on phase space, $\partial_\alpha j^\alpha = 0$. Therefore, as in the general case, the symplectic current $\omega^\alpha = -\delta j^\alpha$ is a conserved current.

Classical solutions

Our next step to explicitly determine the phase space is to solve the classical equations of motion. It is well known that any two-dimensional metric can be written in the form

$$g = e^\phi f^* (\hat{g}(\tau)) , \quad (3.5)$$

where e^ϕ is a Weyl scaling, f^* represents the action of a diffeomorphism and $\hat{g}(\tau)$ is a fiducial metric depending only on the parameters τ_i of the moduli space of the surface. As in any generally covariant theory, the diffeomorphism transformations represent spurious degrees of freedom (playing the role of gauge transformations). The symplectic structure should therefore not depend on the f "parameters" of the general expression (3.5).

Using (3.5), the equation (3.1) turns out to be the Liouville equation for the Weyl scaling factor. The general solution, up to diffeomorphism for the metric field is

$$ds^2 = 2e^\phi dx^+ dx^- \quad (3.6)$$

where

$$\phi = \ln \frac{\partial_+ A \partial_- B}{\left(1 + \frac{\Lambda}{8} AB\right)^2} , \quad (3.7)$$

with $A(B)$ a function of $x^+(x^-)$ ($x^\pm = t \pm x$). Using (3.5) again, the equations for the scalar field take the form

$$T_{++} = \partial_+^2 \Phi - \partial_+ \Phi \partial_+ \phi = 0 \quad (3.8a)$$

$$T_{--} = \partial_-^2 \Phi - \partial_- \Phi \partial_- \phi = 0 \quad (3.8b)$$

$$T_{+-} = \partial_+ \partial_- \Phi + \frac{\Lambda}{4} \Phi e^\phi = 0 . \quad (3.8c)$$

From (3.8a,b) it is easy to see that

$$\partial_+ \Phi = R e^\phi \quad (3.9a)$$

$$\partial_- \Phi = C e^\phi, \quad (3.9b)$$

where $R(C)$ is a function of $x^-(x^+)$. Now, taking into account (3.8c) and (3.9) we obtain

$$\Phi = -\frac{4}{\Lambda} (\partial_- R + R \partial_- \phi) \quad (3.9a)$$

$$\Phi = -\frac{4}{\Lambda} (\partial_+ C + C \partial_+ \phi) . \quad (3.9b)$$

It is now convenient to redefine C and R as $C = c/\partial_+ A$ and $R = r/\partial_- B$. Inserting now the general solution (3.7) into (3.10), we arrive at

$$\Phi = -\frac{4}{\Lambda} \left(\frac{\partial_- r}{\partial_- B} - \frac{\Lambda}{4} r \frac{B}{(1 + \frac{\Lambda}{8} AB)} \right) \quad (3.11a)$$

$$\Phi = -\frac{4}{\Lambda} \left(\frac{\partial_+ c}{\partial_+ A} - \frac{\Lambda}{4} c \frac{A}{(1 + \frac{\Lambda}{8} AB)} \right) . \quad (3.11b)$$

To find the proper solutions we could argue that, since the field Φ is a generally covariant scalar, the function $r(c)$ should depend on the coordinates $x^+(x^-)$ through the function $B(A)$ only. So we can write $r = r(B)$, $c = c(A)$ and $\partial_- r = r' \partial_- B$, $\partial_+ c = c' \partial_+ A$, $\partial_-^2 r = r''(\partial_- B)^2 + r' \partial_-^2 B$, etc. We can now find an ordinary differential equation for the functions r and c substituting (3.11a,b) into (3.8a,b). We obtain

$$r''' = 0 \quad c''' = 0, \quad (3.12)$$

and, therefore, the general solution of the equation of motion is given, up to diffeomorphisms, by

$$\Phi = \frac{1}{1 + \frac{\Lambda}{8} AB} \left(\gamma \left(1 - \frac{\Lambda}{8} AB \right) + \alpha A + \beta B \right) \quad (3.13)$$

$$ds^2 = 2 \frac{\partial_+ A \partial_- B}{(1 + \frac{\Lambda}{8} AB)^2} dx^+ dx^-, \quad (3.14)$$

where γ, α, β are arbitrary constants.

Let us pause to analyse the general solution above on a space-time of the form $R \times S^1$. For a theory defined on a closed spatial section ($0 \leq x < 2\pi$, $-\infty < t < +\infty$) the requirement of periodicity of the metric, $\phi(t, x + 2\pi) = \phi(t, x)$, implies the well-known monodromy transformation properties for the functions A and B :

$$\sqrt{\frac{\Lambda}{8}}A(y+2\pi) = \frac{a\sqrt{\frac{\Lambda}{8}}A(y)+b}{c\sqrt{\frac{\Lambda}{8}}A(y)+d} \equiv M \left(\sqrt{\frac{\Lambda}{8}}A(y) \right) \quad (3.15a)$$

$$\sqrt{\frac{\Lambda}{8}}B(y-2\pi) = \frac{d\sqrt{\frac{\Lambda}{8}}B(y)-c}{-b\sqrt{\frac{\Lambda}{8}}B(y)+a} \equiv M^{-1T} \left(\sqrt{\frac{\Lambda}{8}}B(y) \right) \quad (3.15b)$$

where $ad - bc = 1$. Note that, since in (3.15) the common sign of a, b, c, d is irrelevant, the monodromy parameters form the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{+1, -1\}$. Let us concentrate now on the solution of the scalar field (3.13). The periodicity of Φ implies restrictions for the values of the constants γ , α , and β of (3.13). The unique well-defined solution for the scalar field on the cylinder is given by

$$\Phi = \frac{\lambda}{1 + \frac{\Lambda}{8}AB} \left((d-a)(1 - \frac{\Lambda}{8}AB) + 2c\sqrt{\frac{\Lambda}{8}}A + 2b\sqrt{\frac{\Lambda}{8}}B \right), \quad (3.16)$$

where λ is an arbitrary constant.

Canonical structure and quantization

In order to compute the canonical (symplectic) structure of the theory we have to insert the general solution (3.14,16) into the (time-component) of the symplectic current

$$\omega^0 = [-\delta\Phi \delta(\partial_+\phi + \partial_-\phi) + \delta(\partial_+\Phi + \partial_-\Phi) \delta\phi], \quad (3.17)$$

and push down the two-form

$$\omega = \int_0^{2\pi} \omega^0 dx \quad (3.18)$$

on the space of classical solutions.

The key argument for the explicit computation of the symplectic structure is the following. Since we are dealing with a generally covariant theory we have assumed hitherto that its canonical structure depends on diffeomorphism-invariant only. This implies in particular that we should not find any sort of dependence with respect to the residual diffeomorphisms that appear in the general solution (3.14), (3.16) through the arbitrary functions A and B . The easiest way to eliminate this dependence is when ω^0 is a total derivative with respect to the x -coordinate of a two-form W :

$$\omega^0 = (\partial_+ - \partial_-)W. \quad (3.19)$$

Such a two-form exists and it reads as (from now on in this section we take $\frac{\Lambda}{8} = 1$):

$$\begin{aligned}
W = & \delta \left(\lambda(d-a) \frac{1-AB}{1+AB} \right) \delta \ln \frac{\partial_- B}{\partial_+ A} \\
& + \delta \left(\lambda(d-a) \frac{AB}{1+AB} \right) \delta \ln \frac{B}{A} \\
& + \delta \left(\lambda c \frac{A}{1+AB} \right) \delta \ln \frac{A^2 \partial_- B}{\partial_+ A} - \delta \left(\lambda b \frac{B}{1+AB} \right) \delta \ln \frac{B^2 \partial_+ A}{\partial_- B}
\end{aligned} \tag{3.20}$$

The two-form (3.18) now takes the form

$$\omega = W(2\pi) - W(0); \tag{3.21}$$

using (3.20) and (3.15), and after a long computation, we find that any dependence with respect to the A and B functions cancels out and we are left with the simple expression

$$w = 4\delta\lambda\delta|a+d|. \tag{3.22}$$

Obviously, we can rewrite (3.22) as

$$\omega = 4\delta\lambda\delta|Tr M|, \tag{3.23}$$

where M is the monodromy matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

We would like to comment on the consistency of the result (3.23). The functions A and B are not uniquely determined by a given solution $\phi(t, x)$ and $\Phi(t, x)$. Under the transformations

$$A \longrightarrow h(A) \tag{3.24a}$$

$$B \longrightarrow h^{-1T}(A), \tag{3.24b}$$

where h is a constant $PSL(2, R)$ matrix acting as a Möbius transformation, the solutions (3.14), (3.16) are form-invariant. However, the monodromy matrix M transforms as

$$M \longrightarrow h M h^{-1} \tag{3.25}$$

and we could have made use of this "gauge" ambiguity to bring M into the standard form:

i) hyperbolic monodromy: M is conjugate to a dilatation

$$\begin{bmatrix} e^p & 0 \\ 0 & e^{-p} \end{bmatrix} \tag{3.26a}$$

ii) parabolic monodromy: M is conjugate to a translation

$$\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \tag{3.26b}$$

iii) elliptic monodromy: M is conjugate to a rotation

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (3.26c)$$

The classification follows from the value of the trace of M : i) if $|\text{Tr}M| > 2$ we have an hyperbolic matrix, ii) if $|\text{Tr}M| = 2$ we have a parabolic matrix, and iii) if $|\text{Tr}M| < 2$ we have an elliptic matrix. The final expression of the symplectic form (3.23) just reflects the ambiguity of the matrix M under similarity transformations.

The symplectic structure (3.23) depends finally on diffeomorphism -invariant quantities and makes it easy to characterize the (reduced) phase space of the theory. It is the cotangent bundle of the space of conjugacy classes of the $PSL(2, R)$ group, i.e., $T^*(PSL(2, R)/\text{ad } PSL(2, R))$. In general, quantization requires finding a complete commuting subset of (phase-space) variables. Because of the cotangent nature of the phase space, the natural maximal set of commuting variables is provided by the "coordinates" of the configuration space. The Hilbert space is thus given by the space of square integrable functions on the configuration space. This conclusion is based on general principles of quantum mechanics and can be accomplished geometrically by choosing a real polarization [17] (see [18] for a group-theoretical point of view). In the present case, and since the configuration space $PSL(2, R)/\text{ad } PSL(2, R)$ is not a proper manifold—it has a singularity since it is spanned by three sets of adjoint orbits intersecting at the identity element—we shall better quantize the space $T^*(PSL(2, R))$ and impose, *a posteriori*, the "constraint" generated by the adjoint action. We arrive this way at a Hilbert space $\mathcal{H}$ given by the space of L^2 functions on $PSL(2, R)$, invariant under similarity transformations, i.e.

$$\mathcal{H} = \{ \Psi \in L^2(PSL(2, R)) / \Psi(hgh^{-1}) = \Psi(g) \} . \quad (3.27)$$

Using now the Peter-Weyl theorem [19], the quantum wave functions admit an expansion in characters

$$\Psi = \sum_{\alpha} \lambda_{\alpha} \chi_{\alpha} , \quad (3.28)$$

where $\lambda_{\alpha} \in \mathbb{C}$, and the sum runs over all (inequivalent) classes of irreducible representations of the $PSL(2, R)$ group.

Relation with the topological $PSL(2, R)$ gauge theory

Mimicking the Chern-Simons gauge-theory formulation of three-dimensional gravity [1], one can generalize the Jackiw-Teitelboim model in terms of a (topological) gauge theory with a de-Sitter ($SO(2, 1) \approx PSL(2, R)$) gauge group:

$$\int \text{Tr}(\varphi F) , \quad (3.29)$$

where $F = dA + A \wedge A$ is the curvature two-form and φ is a scalar multiplet in the adjoint representation of the gauge group. As usual, the metric-based model is contained in the gauge theory in the following sense. When the zweibein field $e_{\mu}^a, a = 0, 1$ (the

two components of the gauge field associated with the generators of translations) is non-degenerate, a metric field arises naturally $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ and after integrating out the fields e_μ^2 and φ^a , $a = 0, 1$ we are left with equations equivalent to (3.1) and (3.2). Therefore, the (reduced) phase space of the metric-based model is contained in the corresponding one of the gauge theory.

As a byproduct of our analysis of the Jackiw-Teitelboim model, it is now easy to see that both theories possess indeed the same (reduced) phase space. In fact, the configuration space $PSL(2, R)/\text{ad} PSL(2, R)$ coincides with the configuration space of the gauge theory (3.29) (i.e. the moduli space of flat $PSL(2, R)$ connections):

$$\begin{aligned} PSL(2, R)/\text{ad} PSL(2, R) &\approx \text{Hom}(Z, PSL(2, R))/\text{ad} PSL(2, R) \\ &\approx \text{Hom}(\pi^1(R \times S^1), PSL(2, R))/\text{ad} PSL(2, R). \end{aligned} \quad (3.30)$$

Moreover, in terms of the Schrödinger representation of the gauge theory, the formula (3.28) admits a simple interpretation:

$$\Psi(A) = \sum_{\alpha} \lambda^{\alpha} Tr_{\alpha}(U) \quad (3.31)$$

where U is the variable

$$U = P \exp \oint A, \quad (3.32)$$

and Tr_{α} stands for the trace of U in the representation α ($Tr_{\alpha}U$ are the Wilson loop variables). The gauge-invariant functional (3.31) is the general solution to the quantum constraints.

4. Covariant phase space quantization of the induced 2D gravity

Let us begin our analysis of the 2D gravity by considering first the action (1.4) in the light-cone gauge $ds^2 = dx^+ dx^- + h(dx^+)^2$ ($x^{\pm} = t \pm x$):

$$S = \frac{c}{24\pi} \int \partial_-^2 h \Delta^{-1} \partial_- h, \quad (4.1)$$

where $\Delta \equiv \partial_+ - h\partial_-$. The variation of the gauge-fixed Lagrangian is given by

$$\delta \mathcal{L} = [2\partial_-^2 (\Delta^{-1} \partial_- h) - (\partial_- \Delta^{-1} \partial_- h)^2] \delta h + \partial_{\mu} j^{\mu}, \quad (4.2)$$

where

$$\begin{aligned} j^+ &= -(\partial_- h + h\partial_- \Delta^{-1} \partial_- h) \delta(\Delta^{-1} \partial_- h) - 2\partial_- \Delta^{-1} \partial_- h \delta h \\ j^- &= \partial_- \Delta^{-1} \partial_- h \delta(\Delta^{-1} \partial_- h) \end{aligned} \quad (4.3)$$

According to the formalism of section 2 we can reinterpret the currents (4.3) as one-forms on phase space and define the symplectic form for the hypersurface $t = 0$ as

$$\omega = -\frac{c}{24\pi} \int dx \delta(j^- + j^+) . \quad (4.4)$$

On the other hand, the equation of motion coming from (4.2) (i.e. $T \equiv \delta\mathcal{L}/\delta h = 0$), can be rewritten as

$$\{f; x^-\} = 0 , \quad (4.5)$$

where $\{f; x^-\}$ is the Schwarzian derivative and $\Delta^{-1}\partial_-h = \ln\partial_-f$. As is well known, the general solution to (4.5) is

$$f = \frac{A(x^+)x^- + B(x^+)}{C(x^+)x^- + D(x^+)} , \quad (4.6)$$

where $AD - CB = 1$. Now it is easy to obtain the standard equation ($h = \partial_+f/\partial_-f$) [8]

$$\partial_-^3 h = 0 . \quad (4.7)$$

Inserting the solution (4.6) into (4.4) we arrive finally at the following expression:

$$\omega = \frac{c}{6\pi} \int dy \{ \delta(B'D - BD') \delta(C/D) - \frac{1}{D^2} \delta D \delta D' \} . \quad (4.8)$$

It is not very difficult to check that ω in (4.8) is indeed the exterior derivative of the Maurer-Cartan one-form associated with the central generator of the $SL(2, R)$ -Kac-Moody group with central charge $k = c/6$. In fact, the form ω is invariant under constant $SL(2, R)$ transformations and then defines a non-degenerate symplectic form on the coadjoint-orbit $LSL(2, R)/SL(2, R)$. A global parametrization of this space is given by the currents appearing in the expression

$$h = j_{(+)}(x^+) - 2j_{(0)}(x^+)x^- + j_{(-)}(x^+)(x^-)^2 , \quad (4.9)$$

where

$$\begin{aligned} j_{(+)} &= B'D - BD' \\ j_{(-)} &= A'C - AC' \\ j_{(0)} &= -1/2(B'C + A'D - BC' - AD') \end{aligned} . \quad (4.10)$$

These $SL(2, R)$ currents are nothing other than the "hidden" $SL(2, R)$ currents of Ref. [8] which now appear in a clear way because the two-form (4.8) is just the group/geometric symplectic form associated with the $SL(2, R)$ Kac-Moody group (see also [20]).

In applying general principles of quantization, and in addition to the renormalization of the central term and of the scaling dimensions obtained from standard arguments, one should conclude that the quantum theory of the gauge-fixed Polyakov action can be described in terms of irreducible, highest-weight representations of the $SL(2, R)$ Kac-Moody group. However there are no unitary, standard highest-weight representations with a nonzero central charge [21]. From the physical point of view, one could argue that the

physical (unitary) Hilbert space must be obtained as a constrained subspace given by the conditions $\hat{T}_{+-} = 0$, $\hat{T}_{--} = 0$, coming from the remaining equations of motions of the covariant Lagrangian (1.4). Partial results were obtained, using the BRST formulation, in Ref. [22]. It is at this point that the pure (covariant) canonical formalism could emerge as a powerful approach. The formalism does not require any sort of gauge-fixing at all. We can solve all the classical equations of motion and work out the symplectic form in terms of the coordinates parametrizing the space of solutions. Once an explicit result is found, one could quantize the theory by using, if it is possible, any sort of group/geometric method. Next, and this is the main goal of this paper, we shall apply this program to the generally covariant action (1.4).

After a long computation, the variation (or the exterior derivative) of the Lagrangian of (1.2) can be rewritten in the standard form (for the sake of simplicity we shall drop a factor $c/48\pi$ from the action (1.4) and the corresponding expressions for the two-form ω):

$$\delta\mathcal{L} = \frac{\sqrt{g}}{2} T_{\mu\nu} \delta g^{\mu\nu} + \partial_\alpha j^\alpha, \quad (4.11)$$

where

$$\begin{aligned} T_{\mu\nu} = & \nabla_\mu(\Box^{-1}R)\nabla_\nu(\Box^{-1}R) - 2\nabla_\mu\nabla_\nu(\Box^{-1}R) - \frac{1}{2}g_{\mu\nu}\nabla^\alpha(\Box^{-1}R)\nabla_\alpha(\Box^{-1}R) \\ & + g_{\mu\nu}(2R - \frac{1}{2}\Lambda) \end{aligned} \quad (4.12)$$

(we have made use of the two-dimensional identity $R_{\mu\nu} = 1/2g_{\mu\nu}R$) and

$$\begin{aligned} j^\alpha = & \sqrt{g}\{ -\Box^{-1}R(g^{\mu\nu}\nabla^\alpha\delta g_{\mu\nu} - g^{\alpha\mu}\nabla^\nu\delta g_{\mu\nu}) + \partial^\alpha(\Box^{-1}R)g^{\mu\nu}\delta g_{\mu\nu} \\ & + \partial_\nu(\Box^{-1}R)\delta g^{\alpha\nu} + g^{\alpha\beta}\partial_\alpha(\Box^{-1}R)\partial_\beta(\Box^{-1}R)\}. \end{aligned} \quad (4.13)$$

Note that the first term of (4.13) is proportional to the symplectic current potential of standard general relativity [9].

We would like to note, in passing, that the expression $\Box^{-1}R$ in formulae (4.12)(4.13) can be seen, equivalently, as an auxiliary scalar field $\Phi \equiv \Box^{-1}R$. In so doing, our scheme implies, in a natural way, the equivalence between the theory (1.4) and the one defined by the local action

$$\frac{1}{2} \int \sqrt{g}(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + 2R\Phi + \Lambda), \quad (4.14)$$

since the equation of motion and the symplectic current coincide for both actions (the action (4.14) was the starting point in the works of Ref. [23]). We have to point out that the general scheme of section 2 can be applied to any non-local theory, irrespective of whether it can be converted into a local one.

Classical solutions

Let us now solve the equations of motion $T_{\mu\nu} = 0$. By taking the trace of $T_{\mu\nu}$ we arrive at the equation of motion that implies the constancy of the curvature (i.e. the Liouville equation):

$$R = \frac{\Lambda}{2} . \quad (4.15)$$

We shall use again the general decomposition (3.5) for the metric field. Let us recall that the general solution, up to diffeomorphism, to the equation above is

$$ds^2 = 2e^\phi dx^+ dx^- , \quad (4.16)$$

where

$$\phi = \ln \frac{\partial_+ A \partial_- B}{(1 + \frac{\Lambda}{8} AB)^2} , \quad (4.17)$$

with $A(B)$ a function of $x^+(x^-)$. Using (4.16), the equation

$$\Phi = \square^{-1} R \quad (4.18)$$

implies then

$$\partial_+ \partial_- \phi = -\partial_+ \partial_- \Phi , \quad (4.19)$$

the general solution of which is given by

$$\Phi = -\phi + f_+(x^+) + f_-(x^-) \quad (4.20)$$

[do not confuse the function f_+ and f_- with the general diffeomorphism f of (3.5)]

The remaining equations of motion are

$$T_{++} = (\partial_+ \Phi)^2 - 2\partial_+^2 \Phi + 2\partial_+ \phi \partial_+ \Phi = 0 \quad (4.21a)$$

$$T_{--} = (\partial_- \Phi)^2 - 2\partial_-^2 \Phi - 2\partial_- \phi \partial_- \Phi = 0 . \quad (4.21b)$$

Inserting (4.20) into (4.21) we obtain

$$(\partial_+ f_+)^2 - 2\partial_+^2 f_+ = (\partial_+ \phi)^2 - 2\partial_+^2 \phi \quad (4.23a)$$

$$(\partial_- f_-)^2 - 2\partial_-^2 f_- = (\partial_- \phi)^2 - 2\partial_-^2 \phi . \quad (4.23b)$$

Now it is convenient to introduce the functions $R = R(x^-), C = C(x^+)$ defined by

$$f_+ = \ln \partial_+ A + C \quad (4.24a)$$

$$f_- = \ln \partial_- B + R \quad (4.24b)$$

Making the substitution (4.24) in (4.23), and using the general solution (4.17), we arrive at:

$$(\partial_+ C)^2 - 2\partial_+^2 C + 2\frac{\partial_+^2 A}{\partial_+ A}\partial_+ C = 0 \quad (4.25a)$$

$$(\partial_- R)^2 - 2\partial_-^2 R + 2\frac{\partial_+^2 B}{\partial_+ B}\partial_+ R = 0. \quad (4.25b)$$

In terms of the functions R and C , the form of the solution for the scalar field reads

$$\Phi = \ln \left(1 + \frac{\Lambda}{8} AB \right)^2 + C + R, \quad (4.26)$$

but the covariance of the scalar field implies that the function $R(C)$ should depend on $x^- (x^+)$ through the function $B(A)$ only. So that we can write $R = R(B)$, $C = C(A)$, and $\partial_- R = R' \partial_- B$, $\partial_+ C = C' \partial_+ A$, $\partial_-^2 R = R'' (\partial_- B)^2 + R' \partial_-^2 B$, etc. It is now easy to find the solutions for the equations (4.25):

$$R = -\ln(\bar{\beta}B + \beta) \quad (4.27a)$$

$$C = -\ln(\bar{\alpha}A + \alpha) \quad (4.27b)$$

where $\bar{\alpha}, \alpha, \bar{\beta}, \beta$ are arbitrary constants. Inserting (4.27) into (4.26), we obtain the general solution for the scalar field

$$\Phi = \ln \frac{(1 + \frac{\Lambda}{8} AB)^2}{(\bar{\alpha}A + \alpha)^2 (\bar{\beta}B + \beta)^2}. \quad (4.28)$$

Now we shall study the general solution (4.28), (4.17) on a cylinder. As far as the solution of the metric field is concerned, the analysis is the same as in the Jackiw-Teitelboim model. However we will find novel features when the scalar field Φ is required to be single-valued. The periodicity of Φ , $\Phi(t, x + 2\pi) = \Phi(t, x)$, restricts the permitted value of the arbitrary constant entering expression (4.28). It is not difficult to verify that the general solution on the cylinder is given by (from now on we shall make the simplification $\frac{\Lambda}{8} = 1$):

$$\Phi = \ln \lambda c^2 b^2 \frac{(1 + AB)^2}{\left(2cA + (d - a) \pm \sqrt{(d - a)^2 + 4bc}\right)^2 \left(2bB + (d - a) \pm \sqrt{(d - a)^2 + 4bc}\right)^2}, \quad (4.29)$$

where λ is an arbitrary positive constant.

Surprisingly, (4.29) yields a non-trivial constraint for the monodromy parameters

$$(d - a)^2 + 4bc \geq 0, \quad (4.30)$$

or, which is the same,

$$|\text{Tr}M| \geq 2 . \quad (4.31)$$

This implies that the elliptic monodromies are forbidden and, therefore, only hyperbolic and parabolic monodromies enter in the (reduced) phase space of the theory. Moreover an interesting feature of the solution (4.29) is the double sign of the square root. This will lead to a (reduced) phase space with two disconnected components. Furthermore, and in contrast with the Jackiw-Teitelboim model, the solutions of the scalar field (4.29) are not invariant under the $PSL(2, R)$ transformations (3.24), (3.25).

How can we deal with the restriction (4.21)? Unlike the way we proceeded in the Jackiw-Teitelboim model, we can use here the ambiguity of the monodromy matrix M , coming from the invariance of the solutions (4.17), under $PSL(2, R)$ transformations (3.24), (3.25), to bring the allowed matrices (verifying (4.31)) into an affine subgroup of $PSL(2, R)$ (we use the same letters as in the original matrix elements of M , and $a > 0$):

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \quad (4.32)$$

Obviously this determines M up to a similarity transformation generated by the affine subgroup of $PSL(2, R)$. For our purpose it is convenient to keep the monodromy matrix fixed to the affine subgroup (4.32).

It is not difficult to determine directly the general solution of the scalar field with the affine monodromy (4.32). The solutions are

$$\Phi = \ln \lambda \frac{(1 + AB)^2}{((d - a)A + b)^2 B^2} \quad (4.33a)$$

and

$$\Phi = \ln \lambda \frac{(1 + AB)^2}{(bB + (d - a))^2} , \quad (4.33b)$$

where $d = a^{-1}$. Therefore we face a (reduced) phase space with two disconnected components associated with the two different kinds of solutions (4.33a) and (4.33b). We can easily understand the appearance of two connected sectors in terms of the double sign in (4.29). Let us analyse with more detail the solutions (4.29). Since the elliptic monodromy is not permitted, we can restrict our analysis to the case $c \rightarrow 0$. We have

$$\begin{aligned} 2cA + (d - a) \pm \sqrt{(d - a)^2 + 4bc} = \\ 2cA + (d - a) \pm (d - a) \left(1 + \frac{2bc}{(d - a)^2} + \theta(c)\right) , \end{aligned} \quad (4.34)$$

and in accordance with the double sign in (4.34) we have the solutions

$$\Phi = \ln \frac{\lambda b^2 (d - a)^2}{4} \frac{(1 + AB)^2}{((d - a)A + b + \theta(c))^2 (2bB + \theta(c))^2} \quad (4.35a)$$

and

$$\Phi = \ln \lambda c^2 b^2 \frac{(1 + AB)^2}{(2(d - a) + 2cA + \theta(c))^2 (2bB + 2(d - a) + \theta(c))^2} . \quad (4.35b)$$

At $c = 0$ the solutions (4.35a) turn out to be equivalent to (4.33a) and, redefining $\lambda c^2 b^2 / 16(d - a)^2 \rightarrow \lambda$ in (4.35b) we obtain the second class of solutions (4.33b).

It is interesting to note now that, although the general expression (4.29) of the solutions for the scalar field is not invariant under $PSL(2, R)$ transformations (3.24) (3.25), the solutions (4.33) are indeed form-invariant under the affine subgroup. We then face a situation similar to that found for the Jackiw-Teitelboim model, but we have to replace the $PSL(2, R)$ group by its affine subgroup. We shall return to this important point later on.

Canonical structure and quantization

To work out the canonical (symplectic) structure of the theory, we have to push down the two-form

$$\omega = \int_0^{2\pi} \omega^0 dx \quad (4.36)$$

on the space of classical solutions. The time component of the symplectic current for the induced 2D gravity is given by

$$\begin{aligned} \omega^0 &= -\delta(j^+ + j^-) \\ &= [-\delta\Phi\delta(\partial_+ + \partial_-)(\Phi + \phi) + \delta(\partial_+ + \partial_-)\Phi\delta\phi] . \end{aligned} \quad (4.37)$$

The general covariance of the theory should be recovered by finding a symplectic form depending only on diffeomorphism-invariant quantities. As in the Jackiw-Teitelboim model, the residual diffeomorphism dependence of the general solution represented by the arbitrary functions A and B should thus disappear in the resulting expression for the symplectic structure. The easiest way to enforce this condition is that ω^0 be a total derivative with respect to the x -coordinate of a two-form W :

$$\omega^0 = (\partial_+ - \partial_-)W . \quad (4.38)$$

Fortunately, as in the Jackiw-Teitelboim model, such a two-form exists and is given by [for the sake of completeness we shall give the expression of W for the solutions (4.28)]:

$$\begin{aligned} W &= \delta \ln \frac{A_+}{B_-} \left(\frac{\bar{\beta}B + \beta}{\bar{\alpha}A + \alpha} \right)^2 \delta \ln \frac{(1 + AB)^2}{(\bar{\alpha}A + \alpha)^2 (\bar{\beta}B + \beta)^2} \\ &\quad + \delta \ln(1 + AB)^2 \delta \ln \frac{A}{B} \left(\frac{\bar{\beta}B + \beta}{\bar{\alpha}A + \alpha} \right)^2 + \delta \ln \frac{(\bar{\alpha}A + \alpha)^2}{\bar{\beta}^2} \delta \ln \frac{\bar{\alpha}^2}{(\bar{\beta}B + \beta)^2} . \end{aligned} \quad (4.39)$$

Therefore, the two-form ω can be computed from the expression

$$\omega = W(2\pi) - W(0) . \quad (4.40)$$

Inserting the solutions (4.33) into (4.40) and taking into account the (affine) monodromy properties of A and B we find, after a long but straightforward computation, an expression that does not depend on the functions A or B . For the sector of solutions (4.33a) we obtain

$$\omega = 4 \frac{\delta \lambda}{\lambda} \frac{\delta a}{a} , \quad (4.41a)$$

and for (4.33b) we get

$$\omega = -4 \frac{\delta \lambda}{\lambda} \frac{\delta a}{a} . \quad (4.41b)$$

It is clear from the above expressions that the two connected components of the (reduced) phase space are isomorphic despite the apparently different structure of the solutions (4.33a) and (4.33b) [this asymmetry comes from the arbitrary selection $c = 0$, instead of $b = 0$, in selecting an affine subgroup of $PSL(2, R)$]. Moreover, the symplectic form is invariant under (affine) similarity transformations of the (affine) monodromy matrix. In fact, the parameter "a" is now the continuous invariant parameter of the adjoint action of the affine group playing the same role as the invariant $Tr M = |a + d|$ in the Jackiw-Teitelboim model [see (3.26)]. We can conclude, therefore, that the physical phase space of the induced 2D gravity is the disjoint union of two (identical) copies of the cotangent bundle of the space of conjugacy classes of the affine group G :

$$T^*(G/\text{ad}G) \cup T^*(G/\text{ad}G) . \quad (4.42)$$

It is a simple exercise to characterize the space of conjugacy classes of the affine group. They are given by:

$$i) \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad ii) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad iii) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad (4.43)$$

The parameter "a" is an invariant and characterizes uniquely each adjoint orbit for $a \neq 1$. When $a = 1$ the orbits are also labelled by the sign of the "b" parameter (we have chosen in (4.43) the representatives $b = 0, 1, -1$).

As in the Jackiw-Teitelboim model, and because of the cotangent bundle nature of the reduced phase space, the quantum Hilbert space $\mathcal{H}$ is given by the space of square integrable functions on the configuration space. The spectrum consists of a continuous sector $L^2(R)$ associated with $i)$ and a two-dimensional discrete sector associated with the set of orbits $ii)$ and $iii)$. Moreover, since the reduced phase space has two (identical) disconnected components (4.42), the Hilbert space is then of the form

$$\mathcal{H} = \mathcal{H}^{(+)} \otimes \mathcal{H}^{(-)} , \quad (4.44a)$$

where

$$\mathcal{H}^{(+)} = \mathcal{H}^{(-)} = L^2(R) \oplus C^2 . \quad (4.44b)$$

The physical states are then characterized by (we restrict our attention to one component of $\mathcal{H}$)

$$\Psi(\ln a) + \alpha_1 | +1 \rangle + \alpha_{-1} | -1 \rangle \quad (4.45)$$

where $\Psi \in L^2(R)$ and $\alpha_1, \alpha_{-1} \in C$. The scalar product is given by

$$\langle \phi; \beta_1, \beta_{-1} | \Psi; \alpha_1, \alpha_{-1} \rangle = \int d(\ln a) \phi^* \Psi + \beta_1^* \alpha_1 + \beta_{-1}^* \alpha_{-1} \quad (4.46)$$

The spectrum of the induced 2D gravity also admits, as the Jackiw-Teitelboim theory, a nice "Peter-Weyl" interpretation. The affine group has the following Lie algebra

$$[B, P] = iP, \quad (4.47)$$

but, unlike the Heisenberg-Weyl algebra, (4.47) admits two unitary, inequivalent, (infinite-dimensional) irreducible representations, one for which $P > 0$ and the other for which $P < 0$ [24]. Moreover, if $P = 0$ we also have an infinite-dimensional space of inequivalent (one-dimensional) irreducible representations characterized by the distinct eigenvalues of the B operator. All these irreducible representations are in one-to-one correspondence with the orbits (4.43) and, therefore, with the states of one of the two identical components of the Hilbert space of the theory.

5. Conclusions and final comments

In this paper we have developed a detailed, systematic and novel analysis of the canonical structure of the induced 2D gravity and the Jackiw-Teitelboim model. Our approach is based on the covariant definition of phase space as the space of classical solutions and it also possesses the virtue of being gauge-independent. We have thus avoided the conventional canonical analysis in terms of quantum constraint equations (BRST technique, etc). After characterizing the general solution to the equations of motion, we have computed explicitly the symplectic structure finding that the (reduced) phase space does not have local degrees of freedom and turns out to be a (finite-dimensional) cotangent space for both theories. For the Jackiw-Teitelboim model the reduced phase space is the cotangent bundle of the space of conjugacy classes of the $PSL(2, R)$ group. This space admits an interpretation in terms of a related topological gauge-theory model and our result can be used to establish the complete equivalence between the two models. Surprisingly the canonical structure of the induced 2D-gravity theory is quite similar to that of the Jackiw-Teitelboim model. Essentially we only have to replace the $PSL(2, R)$ group by its affine subgroup. In fact only the hyperbolic and parabolic monodromies admit classical solutions. This feature is due to the appearance, in solving the equations of motion, of an algebraic equation whose discriminant is proportional to $|\text{Tr} M|^2 - 4$. Moreover, the double sign of the square root produces a non-connected reduced phase space that consists indeed in two identical connected components isomorphic to the cotangent bundle of the space of conjugate classes of the affine subgroup of $PSL(2, R)$. These results allow us to quantize

both theories exactly. The Hilbert space arises as the space of square integrable class functions of the corresponding group and, equivalently, the physical states are in one-to-one correspondence with the space of (inequivalent) unitary, irreducible representations. For the induced 2D gravity we find that each component of the physical Hilbert space has a continuous sector associated with the one-dimensional representations of the affine group and a discrete sector —consisting in two independent states— associated with the infinite-dimensional representations of the affine group.

Since our approach exploits the structure of the space of classical solutions it has an interesting resemblance with the canonical treatment of $2 + 1$ dimensional gravity carried out by Witten[1]. In the present paper we have studied, however, generally covariant theories constructed with a metric field as the basic dynamical variable instead of a gauge field as is usual in the Chern-Simons formulation of $2 + 1$ dimensional gravity [1]^(*).

We would like to point out that in performing the covariant phase-space quantization of the induced 2D gravity we have not find any kind of restriction for the coupling constant c . However, it is interesting to note that a constraint similar to that relating the matter central charge and the level k of the $SL(2, R)$ Kac-Moody algebra [26,23] appears in a stage of our analysis, but here produces a non-trivial restriction for the monodromy parameters truncating then the reduced phase space and hence the physical Hilbert space of the theory. It is not clear to us whether such a restriction of the coupling constant might appear in a further stage of the present approach (like the analysis of observables). Nevertheless, we have to remark that in quantizing a theory with gauge degrees of freedom there are two ways of proceeding. One can either first solve the constraints and then quantize (the approach of this paper) or first quantize and then impose the constraints as quantum operators. Although in most situations both approaches lead to the same quantum theory there is no a general proof of the equivalence between them. From a general point of view this ambiguity is indeed a reflection of the quantization ambiguity in the selection of the observable algebra to quantize a theory (see for instance [27]).

Finally we would like to emphasize that the covariant definition of phase space has been crucial to providing the definition of the canonical structure in the (non-trivial) case of a non-local theory (see section 2). An intriguing question is to extend the formalism to space-time manifolds that do not admit a space-time splitting $R \times \Sigma$, and even to consider the space-time topology as additional degrees of freedom in a canonical setting.

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^(*) Some interesting results in $2 + 1$ dimensional gravity based on the standard Hilbert-Einstein action can be found in [25]

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