Stochastic processes and the non-perturbative structure of the QCD vacuum

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Abstract

Based on a local Gaussian evaluation of the functional integral representation, a method is developed to obtain ground state functionals.

The method is applied to the gluon sector of QCD. For one of the functionals so obtained and using stochastic techniques one checks the consistency of the quantum theory, finiteness of the mass gap and the scaling relation in the continuum limit. The functional also implies strong chromomagnetic fluctuations which constrain the asymptotic spectrum on the fermion sector.

In the Appendices one discusses the Wentzell-Freidlin technique, a symmetry realization mode through reducible functionals and its relation to the failure of the cluster property and confinement.

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1. Introduction

There is a wide belief that quantum chromodynamics is the theory of strong interactions. However reliable predictions can be extracted from the theory only for a restricted class of experimentally observable quantities. For short-distance (large transverse momentum) phenomena, perturbation theory seems reliable and, for the static properties of hadrons, a few results have been obtained by lattice Monte Carlo calculations. However high energy small transverse momentum reactions, for example, are out of reach of both techniques.

Due to asymptotic freedom the (perturbative) physical picture of large transverse momentum reactions is well understood. For low energies however, even if future dedicated machines get around the small lattice limitation and are eventually able to crunch out the right numbers, some sort of understanding of the physical picture at low energies would be convenient. The physical picture is probably in the tapes containing thousands of computer-generated lattice configurations, but this is not very transparent for the common mortal.

The construction of approximations to the vacuum and other low-lying states in QCD has been attempted by many authors^[1-10] who used several approaches and conjectures. An approach that has been widely followed is the variational one, wherein a trial state is conjectured, dependent on some parameters, which are then adjusted to minimize the energy. However, as Feynman^[11] has pointed out, although the trial states are constructed to describe the low energy properties, the non-linear nature of the QCD couplings (and the large number of degrees of freedom) have as a consequence that the variational parameters tend to adjust themselves to the high frequency modes because they have the largest contribution to the energy. This in general leads to unrealistic values of the parameters as far as the low energy properties are concerned, which are exactly what we were trying to describe to begin with. In addition because one needs to compute high dimensional energy integrals, this essentially restricts the trial states to be Gaussian. These two limitations seriously restrict the usefulness of the variational approach in field theory¹.

In other approaches a systematic perturbative expansion of the vacuum is

¹A different point of view has recently been advocated by A. Neveu^[12]

constructed where typically the first term has the structure of the Abelian ground state. In the perturbative expansions the individual terms are not invariant under non-Abelian gauge transformations and non-perturbative features are hard to extract from the expansion. It has been proposed^[6] to improve the perturbative expansion by a so-called gauge invariant completion. I. e. order by order the expansion terms are replaced by gauge-invariant expressions which agree, to that order, with perturbation theory. A difficult consistency problem occurs however, because the gauge-invariant completion of a given order contains contributions of all higher orders.

The general conclusion is that methods to approximate the QCD vacuum are needed which do not rely on the minimization of the energy and that are gauge invariant. Furthermore they should also not rely on expansions on powers of the coupling constant to ensure that the leading terms already contain non-perturbative information. An attempt in this direction is presented in Section 2, based on a local Gaussian evaluation of the functional integral representation of the ground state.

When an approximation to the QCD vacuum is derived or conjectured, the second problem is what to do with it. In principle when the ground state is known, Green's functions and, by LSZ reduction, scattering amplitudes may be computed and the theory is completely defined. However one still has to compute functional integrals on the time-zero fields which, for non-trivial (non-Gaussian) ground states is not simple.

There is however another way to extract useful information from the ground state functionals $\psi_0(\phi)$ that relies on the interpretation of the measure $\psi_0^2(\phi)\{d\phi\}$ as the invariant measure of a stochastic process

$$d\phi = -L(\phi) dt + dW(t) \quad (1.1)$$

with drift

$$-L(\phi) = \frac{1}{\psi_0(\phi)} \frac{\delta \psi_0}{\delta \phi} \quad (1.2)$$

the generator of this process being the Hamiltonian operator. Consistency of the quantum theory defined by the ground state measure may then be rigorously formulated as a problem of closability of the associated energy form^[10] or as a problem of existence of solutions of the stochastic differential equation (1.1)^[13].

Furthermore the Dirichlet problem of the generator (i. e. the eigenvalue problem of the Hamiltonian) is related to the statistics of the exit time of the stochastic process

from the domain where boundary conditions are imposed. In particular the Wentzell-Freidlin theory^[14] and its associated large deviation estimates are especially adequate to characterize the mass gap and scaling properties of the continuum limit. The Wentzell-Freidlin technique seems in fact to be the most natural non-perturbative technique in the sense that quantities behaving like $\exp(-c/g^2)$ are the simplest ones to deal with. For details on this technique I refer to [14-17]. A summary of the main results and some applications may be found in [18].

Extensive use will be made of the ground state functional interpretation as the invariant measure of a stochastic process in the study of the properties of the ground state approximation constructed in Sect. 2.

The plan of the paper is the following. In Sect. 2 a method is developed to approximate the ground state functional which is then applied to the gluon sector of QCD. One of these approximations, which will be called QCD₀, is then shown to lead to a well-defined quantum theory, in the sense that the ground state functional is the density of a closable Dirichlet form^[19].

In Sect. 3 the interpretation of the ground state functional as the invariant measure of a stochastic process and the Wentzell-Freidlin technique are used to prove that QCD₀ has a finite mass gap and to derive its scaling properties in the continuum limit. Finally Sect. 4 deals with the structure of strong chromomagnetic vacuum fluctuations implied by the functional of QCD₀ and its effect on the asymptotic fermion spectrum.

The Appendices are intended to provide a background and clarification of some of the points touched upon on the paper. In some cases they probably go beyond the immediate needs of the paper. However, it seemed to me more appropriate to collect here this material, in a self-contained form, rather than to spread it over several publications.

Many results in the paper hold for any compact gauge group. However, whenever specific calculations were required, I have, for simplicity, used the SU(2) group.

2. Approximating the ground state functional

The starting point is the path integral representation of the ground state $\psi_0(x)$ as a sum over Euclidean paths pinned down, at time zero, to the x coordinates

$$\psi_0(x) \sim \int \mathfrak{P}x(\tau) \delta(x(0)-x) e^{-\int_0^\infty L_E(x(\tau), \dot{x}(\tau)) d\tau} \quad (2.1)$$

with $L_E(x, \dot{x}) = -c(\dot{x})^2 - V(x)$

It is more convenient to work with paths from $\tau = -\infty$ to $\tau = +\infty$ and this leads to

$$|\psi_0(x)|^2 = \frac{1}{N} \int \mathfrak{P}x(\tau) \delta(x(0)-x) e^{-\int_{-\infty}^\infty L_E(x(\tau), \dot{x}(\tau)) d\tau} \quad (2.2)$$

where the normalization factor N is the same functional integral without the delta function. We are going to make the change of variables

$$x(\tau) = x + z(\tau)$$

The delta function may be included in the exponential by

$$\delta(z(0)) = \left(\frac{1}{\sqrt{\pi}}\right) \int dy^i \exp\{i \int d\tau y^i \dot{z}^i(\tau) \delta(z)\}$$

Now add a term $z(\tau)J(\tau)$ to the Euclidean Lagrangian and separate the terms quadratic or less than quadratic in $z(\tau)$ from higher order terms

$$L_E(x(\tau), \dot{x}(\tau)) + z(\tau)J(\tau) = -V(x) - z(\tau) \left(-c\frac{\partial^2}{\partial \tau^2} + S(x)\right) z(\tau) - (\Gamma(x) - J(\tau)) \cdot z(\tau) + G(z(\tau)) \quad (2.3)$$

where $G(z(\tau))$ denotes the terms of order higher than two. Representing $G(z(\tau))$ by $G\left(\frac{\partial}{\partial J(\tau)}\right)$ and computing the Gaussian integral the following representation is obtained for $|\psi_0(x)|^2$

$$|\psi_0(x)|^2 = e^{\int d\tau G\left(\frac{\partial}{\partial J(\tau)}\right)} e^{\int d\tau (\Gamma(x) - J(\tau)) \frac{1}{-c\frac{\partial^2}{\partial \tau^2} + S(x)} (\Gamma(x) - J(\tau))} e^{\frac{1}{2\sqrt{c}} \int d\tau (\Gamma(x) - J(\tau)) \frac{1}{-c\frac{\partial^2}{\partial \tau^2} + S(x)} (\Gamma(x) - J(\tau))} \Big|_{J=0} \quad (2.4)$$

with

$$L = -\frac{i}{4\epsilon} \int d\tau \frac{1}{\sqrt{S(x)}} e^{-|\tau| \sqrt{\frac{S(x)}{\epsilon}}} (\Gamma(x) - J(\tau)) \quad (2.5a)$$

for $S(x) > 0$. L reduces to

$$L_0 = -\frac{i}{2} \frac{1}{\sqrt{S(x)}} \Gamma(x) \quad (2.5b)$$

in the limit $J \rightarrow 0$.

From (2.4), expanding $\exp(\int d\tau G(\frac{\partial}{\partial J}(\tau)))$, one may construct successive approximations to the ground state. Of particular interest is the leading term

$$|\psi_0(x)\rangle_{(0)}^2 = \exp\left\{-\frac{\sqrt{\epsilon}}{2} \Gamma(x) \frac{1}{\sqrt{S(x)}} \frac{1}{\sqrt{S(x)}} \Gamma(x)\right\} \quad (2.6)$$

This differs essentially from a perturbative estimate in that, at each point x of the wave function, a different expansion point is chosen, which is x itself. In the functional integral representation of the ground state the wave function is the integrated effect of paths coming from the infinite past to the point x . Because near x the difference $z(\tau) - x$ is small, the leading term will contain accurate information coming from the paths in the neighbourhood of x , and will be inaccurate only concerning non-harmonic contributions to the paths far away from x .

For non-Abelian gauge fields, $\mathcal{A}_1^\alpha(\tau, x)$ is defined to be the space-time Euclidean vector potential, $A_1^\alpha(x) = \mathcal{A}_1^\alpha(0, x)$ the time-zero field, $\mathfrak{B}_1^\alpha(\tau, x) = \epsilon_{ijk}(\partial_j \mathcal{A}_k^\alpha - \frac{g}{2} f_{\alpha\beta\gamma} \mathcal{A}_j^\beta \mathcal{A}_k^\gamma)$ the chromomagnetic field and $B_1^\alpha(x) = \epsilon_{ijk}(\partial_j A_k^\alpha - \frac{g}{2} f_{\alpha\beta\gamma} A_j^\beta A_k^\gamma)$ the time-zero chromomagnetic field.

Because one is dealing with approximations to the ground state measure, it is convenient to use the Schrödinger formulation for quantum fields^[20]. The temporal gauge $\mathcal{A}_3^0 = 0$ is chosen, the time-zero fields are the canonical variables, the chromoelectric fields the conjugate momenta and Gauss' law is imposed as a constraint, i.e. the wave functionals must be gauge invariant. Make the following change of variables

$$\mathcal{A}_1^\alpha(\tau, x) = A_1^\alpha(x) + \phi_1^\alpha(\tau, x)$$

The Euclidean Lagrangian is

$$L_E = -\frac{i}{2} \left(\frac{\partial}{\partial \tau} \phi_1^\alpha \right)^2 - \frac{i}{2} (\mathfrak{B}_1^\alpha)^2$$

To construct the ground state approximation according to Eq.(2.6), notice that

$$\mathfrak{B}_1^\alpha(\tau, x) = B_1^\alpha(x) + \epsilon_{ijk} D_j(A)_{\alpha\beta} \phi_k^\beta(\tau, x) - \frac{g}{2} \epsilon_{ijk} f_{\alpha\beta\gamma} \phi_j^\beta(\tau, x) \phi_k^\gamma(\tau, x) \quad (2.7)$$

with

$$D_j(A)_{\alpha\beta} = \partial_j \delta_{\alpha\beta} - g f_{\alpha\gamma\beta} A_j^\gamma(x) \quad (2.8)$$

Separating only the contributions of the local linear approximation to $\mathfrak{B}_1^\alpha(\tau, x)$ one obtains for the S , Γ and G functions of Eq.(2.3)

$$S_0(A(x))^{\beta\beta'}_{nn'} = \frac{1}{2} \epsilon_{nmk} D_m(A)^{\beta\alpha} \epsilon_{km'n'} D_n(A)^{\alpha\beta'} \quad (2.9a)$$

$$\Gamma_0(A(x))^\beta_n = \epsilon_{nmk} D_m(A)^{\beta\alpha} B_k^\alpha \quad (2.9b)$$

$$G_0(\mathcal{A}) = \frac{1}{2} \epsilon_{kmn} \epsilon_{km'n'} f_{\alpha\beta\gamma} \left\{ -g D_m(A)^{\alpha\beta} \mathcal{A}_n^\beta \mathcal{A}_{n'}^{\beta'} - \frac{g^2}{4} f_{\alpha\beta\gamma} \mathcal{A}_m^\beta \mathcal{A}_n^\gamma \mathcal{A}_{n'}^{\beta'} \right\} + \frac{g}{2} \epsilon_{nmk} f_{\alpha\beta\beta'} B_k^\alpha(A) \mathcal{A}_n^\beta \mathcal{A}_{n'}^{\beta'} \quad (2.9c)$$

If instead, one isolates all quadratic terms in L_E , including the cross terms between $-\frac{g}{2} \epsilon_{ijk} f_{\alpha\beta\gamma} \phi_j^\beta(\tau, x) \phi_k^\gamma(\tau, x)$ and $B_1^\alpha(x)$ the result is

$$S_1(A(x))^{\beta\beta'}_{nn'} = \frac{1}{2} \epsilon_{nmk} D_m(A)^{\beta\alpha} \epsilon_{km'n'} D_n(A)^{\alpha\beta'} + \frac{g}{2} \epsilon_{nmk} f_{\alpha\beta\beta'} B_k^\alpha(A) \mathcal{A}_n^\beta \mathcal{A}_{n'}^{\beta'} \quad (2.10a)$$

$$\Gamma_1(A(x))^\beta_n = \epsilon_{nmk} D_m(A)^{\beta\alpha} B_k^\alpha \quad (2.10b)$$

$$G_1(\mathcal{A}) = \frac{1}{2} \epsilon_{kmn} \epsilon_{km'n'} f_{\alpha\beta\gamma} \left\{ -g D_m(A)^{\alpha\beta} \mathcal{A}_n^\beta \mathcal{A}_{n'}^{\beta'} - \frac{g^2}{4} f_{\alpha\beta\gamma} \mathcal{A}_m^\beta \mathcal{A}_n^\gamma \mathcal{A}_{n'}^{\beta'} \right\} \quad (2.10c)$$

The ground state approximations obtained using (2.9) or (2.10) seem to be qualitatively similar. Because simpler expressions are obtained with (2.9) I will concentrate on this one, for which

$$\psi_0(A)_{(0)} = \exp\left\{-\frac{i}{2} \int d^3x B_k^\alpha(A(x)) \left(\frac{1}{\sqrt{R(A(x)) R(A(x))}} \right)_{kk'} B_{k'}^{\alpha'}(A(x)) \right\} \quad (2.11)$$

where the following operator has been defined

$$R(A)_{nn'}^{\alpha\alpha'} = \epsilon_{nmn'} D_m(A)^{\alpha\alpha'} \quad (2.12)$$

(2.11) is similar to the gauge invariant ansatz discussed in Ref.[10]. However unlike the ansatz of Ref[10], which was simply a non-Abelian generalization of the Abelian ground state, the functional (2.11), or (2.13) below, are leading terms in an exact representation of the (gluon sector) QCD vacuum. They all display qualitative features of physical relevance. Namely, as discussed in Ref.[10], the long-distance behaviour appears

associated to maximally disordered field configurations, whereas for high-momentum (short distances) the kernel approaches the free theory. The exponent of the exponential is negative semi-definite. Therefore the maximum is reached for the $B_k^\alpha(x)=0$ configurations, the functional being peaked at all homotopically non-equivalent classical vacua.

Using (2.10), the wave functional would be

$$\psi_0(A)_{(i)} = \exp \left\{ -\frac{i}{2} \int d^3x \, B(A)R(A) \left(R(A)R(A) - [D(A), D(A)] \right)^{-\frac{3}{2}} R(A)B(A) \right\} \quad (2.13)$$

where indices were suppressed to simplify the notation.

The quantum theory corresponding to the ground state approximation defined by (2.11) will be called QCD₀. It is mostly to the properties of this quantum theory that the remainder of this paper is dedicated. The first question is to show that the ground state functional (2.11) indeed defines a quantum theory. This will be done in two steps. First a lattice version of the stochastic process associated to (2.11) is constructed and a result of Fukushima^[20] used to show that it defines a closed Dirichlet form, hence a self-adjoint generator and a well-defined quantum theory. The second step which relates to the proof of the appropriate scaling for the continuum limit will be carried out in Sect. 3 after the behaviour of the mass gap is discussed.

In the construction of quantum theories using ground-state functionals^[10,13] $\psi_0 = \exp\{\sigma(A)\}$, these are always considered either as densities of a form

$$\mathfrak{E}(u,v) = \int \frac{\delta}{\delta A} u \cdot \frac{\delta}{\delta A} v \, e^{2\sigma(A)} \{dA\}$$

or as generating a stochastic process

$$dA = -L(A) dt + dW(t)$$

with drift $-L = \frac{\delta \sigma}{\delta A}$. Once closability of the form or existence of solutions to the stochastic differential equation and an ergodic invariant measure are proven, one is sure to have a positive self-adjoint Hamiltonian operator (the generator of the process). The measure then allows the reconstruction of a Hilbert space and one has a well-defined quantum theory.

A lattice regularization is used. Define the variables $\theta^\alpha(x, x+i)$ corresponding to a

rescaling of the fields $A_i^\alpha(x)$ and make the replacements

$$g \, a \, A_i^\alpha(x) \rightarrow \theta^\alpha(x, x+i) \doteq \theta_i^\alpha(x) \quad (2.14a)$$

$$g \, a^2 \, B_i^\alpha(x) \rightarrow \theta_i^\alpha(x) = \frac{1}{4} \gamma_{ijk} \left(\theta^\alpha(x+i, x+j+\hat{k}) - \frac{1}{2} f_{\alpha\beta\gamma} \theta^\beta(x, x+i) \theta^\gamma(x, x+\hat{k}) \right) \quad (2.14b)$$

where $\gamma_{ijk} = (\text{sign } i)(\text{sign } j)(\text{sign } k) \epsilon_{|i||j||k|}$ ($i, j, k \in \{1, 2, 3\}$), $\hat{i}$ denotes the unit lattice vector along the i -direction, x is a point in a three-dimensional lattice and a is the lattice spacing. The covariant derivative of a quantity $v^\alpha(x)$ transforming under the adjoint representation of the internal symmetry group becomes

$$\begin{aligned} (D_i)_{\alpha\beta} v^\beta(x) &\rightarrow \frac{1}{a} \left\{ \frac{1}{2} (v^\alpha(x+i) - v^\alpha(x-i)) - f_{\alpha\gamma\beta} \theta^\gamma(x, x+i) v^\beta(x) \right\} \\ &\doteq \frac{1}{a} (\mathfrak{D}_i)_{\alpha\beta} v^\beta(x) \end{aligned} \quad (2.15)$$

With these definitions the lattice version of (2.11) is

$$\psi_0(\theta)_{(o)} = \exp \left\{ -\frac{1}{2\pi g^2} \sum_x \int_0^\infty d\lambda \, \lambda^{-\frac{1}{2}} \beta_K^\alpha(x) \left(\frac{1}{\lambda + \mathfrak{D}_i \cdot \mathfrak{D}_i} \right) \beta_K^\gamma(x) \right\} \quad (2.16)$$

where the $\mathfrak{D}$ -operator is now

$$\mathfrak{D}_{nn'}^{\alpha\alpha'} = \epsilon_{nmn'} \mathfrak{D}_m(\theta)^{\alpha\alpha'}$$

and a standard integral representation for fractional powers of positive operators^[21] has been used

$$\frac{1}{\sqrt{\mathfrak{D}_i \cdot \mathfrak{D}_i}} = \frac{1}{\pi} \int_0^\infty d\lambda \, \lambda^{-\frac{1}{2}} \frac{1}{\lambda + \mathfrak{D}_i \cdot \mathfrak{D}_i} d\lambda$$

We may now state that

#1. For a finite lattice with periodic boundary conditions the ground state measure $\psi_0^2 d\theta$, with ψ_0 as in Eq.(2.16), characterizes a well-defined quantum theory"

The proof is the same as for the functional of Ref.[10] to which we refer for the details. The operator $\mathfrak{D}_i \cdot \mathfrak{D}_i$ plays now the same role as the operator $-\mathfrak{D}_i \cdot \mathfrak{D}_i$ in Ref.[10]. In a finite lattice with periodic boundary conditions $\mathfrak{D}_i \cdot \mathfrak{D}_i$ is a positive operator and the density of the ground state measure is bounded. Furthermore the singular set has zero measure and therefore the density satisfies the conditions of theorem 1 in Ref.[19]. The

corresponding energy form is closable, there is associated to it a diffusion process and, in the Hilbert space of square-integrable field configurations, its generator defines a positive self-adjoint Hamiltonian operator, i. e. a well-defined quantum theory is associated to $\psi_0^2 d\theta$.

To discuss the continuum limit of this lattice quantum theory one needs to control the transformation law of physical quantities when the lattice spacing tends to zero. This is achieved through the characterization of the behaviour of the mass gap (the first excited level of the Hamiltonian) in the next section.

3. The mass gap

The purpose of this section is to prove that the QCD_0 ground state measure (on the lattice) implies a finite mass gap at small but non-zero coupling and to derive its scaling behaviour when the lattice spacing tends to zero.

The existence of a finite mass gap in QCD is related to the question of confinement. The statement of confinement contains two parts:

- (i) All observables are colour neutral.
- (ii) All physical states are colour singlets.

The first part is a trivial consequence of the existence of an exact non-Abelian gauge symmetry. The second part may or may not be of "kinematical" origin, in the sense that it does not depend on the particular dynamical details of the theory (see the discussion in Appendix A).

Whether or not the search for dynamical proofs of confinement is useful, it is certainly necessary to look for its manifestations. In particular, when constructing approximations to QCD, it is essential to check whether they are consistent with some of the consequences of confinement. An important consequence is the existence of a mass gap in the gluon sector. If massless gluons cannot appear in the asymptotic fields, then there should exist a finite gap in the energy spectrum above the ground state of the Yang-Mills Hamiltonian and the first excited state is a massive excitation (glueball). Otherwise if the gap is arbitrarily small it is hard to see why a massless gluon cannot propagate to infinity. This latter situation occurs in QED and so some relevant difference must be found between the mass gap in Abelian and non-Abelian gauge theories. An appropriate setting to discuss this question is the Wentzell-Freidlin large

deviation theory.

In the lattice formulation the exact Hamiltonian associated to the functional (2.16), i. e. the Hamiltonian for which it is the exact ground state, is

$$H = \frac{g^2}{2a} \sum_{x,i,\alpha} \left\{ -\frac{\partial}{\partial \theta_1^\alpha(x)} + L_1^\alpha(x) \right\} \left\{ \frac{\partial}{\partial \theta_1^\alpha(x)} + L_1^\alpha(x) \right\} \quad (3.1)$$

where

$$-L_1^\alpha(x) = \frac{1}{\psi_0} \frac{\partial \psi_0}{\partial \theta_1^\alpha(x)} \quad (3.2)$$

Making the unitary transformation $H \rightarrow H' = \psi_0^{-1} H \psi_0$ one obtains

$$-a H' = \frac{g^2}{2} \sum_{x,i,\alpha} \frac{\partial}{\partial \theta_1^\alpha(x)} \frac{\partial}{\partial \theta_1^\alpha(x)} + \sum_{x,i,\alpha} b_1^\alpha(x) \frac{\partial}{\partial \theta_1^\alpha(x)} \quad (3.3)$$

with

$$b_1^\alpha(x) = -g^2 L_1^\alpha(x) = g^2 \frac{\partial}{\partial \theta_1^\alpha(x)} (\ln \psi_0) \quad (3.4)$$

The operator in (3.3) has the standard form of a second-order elliptic operator which is the generator of a diffusion process

$$d\theta_1^\alpha(x) = b_1^\alpha(x) dt + g dW_1^\alpha(t) \quad (3.5)$$

with drift $b_1^\alpha(x)$ and diffusion coefficient g . This provides a characterization of the eigenvalue problem of the Hamiltonian through stochastic techniques. Consider the

$$-a H'u = \lambda u \quad (3.6)$$

in a domain D with boundary condition $u=0$ on ∂D . From the theory of positive operators there is, at least, one positive eigenvalue, the eigenfunction space of the smallest positive eigenvalue λ_1 being one-dimensional. λ_1 is called the principal eigenvalue. In the lattice theory it would be $\lambda_1 = a m$, m being called the mass gap. There is a probabilistic characterization of the principal eigenvalue λ_1 in terms of the first exit time τ from D of the process defined by (3.5), namely

$$\lambda_1 = \sup \{ \lambda \geq 0 ; \sup_{x \in D} E_x e^{\lambda \tau} < \infty \} \quad (3.7)$$

E_x denotes the statistical expectation when the starting point of the process is x . Eq.(3.7) may be used for example as a reliable tool for the numerical evaluation of the

mass gap ^[22]. Of more interest here is the fact that near the continuum limit one is in the weak coupling regimen. For small g it follows from (3.5) that one is in the "weak noise" limit. Then analytical results can be obtained for the mass gap from the Wentzell-Freidlin estimates ^[14]

$$\exp\left\{\frac{-I(s,x,\partial D)-h}{2g^2}\right\} < P_x\{\tau < s\} < \exp\left\{\frac{-I(s,x,\partial D)+h}{2g^2}\right\} \quad (3.8a)$$

for any $h>0$ and sufficiently small g . $P_x\{\tau < s\}$ is the probability for the first exit time τ to be smaller than s when the process starts from x . The functional $I(s,x,\partial D)$ is

$$I(s,x,\partial D) = \inf_{x \in \psi} I_{0,s}(x) \quad (3.8b)$$

$$\psi_s = \left\{x \in C_{0,s} : x(0)=x ; \min_{0 \leq s' \leq s} d(x(s'), \partial D) = 0\right\} \quad (3.8c)$$

with

$$I_{T_1,T_2}(x) = \frac{1}{2} \int_{T_1}^{T_2} \left(\frac{dx}{ds} - b(x(s)) \right)^2 ds \quad (3.8d)$$

i. e. $I(s,x,\partial D)$ is the infimum of $I_{T_1,T_2}(x)$ over all continuous paths that, starting from x , hit the boundary ∂D of the domain D in time less than or equal to s .

From (3.7-8) it is clear that the $g \rightarrow 0$ behaviour of the mass gap is controlled by the nature of the deterministic dynamical system

$$\frac{d\theta_1^\alpha(x)}{dt} = b_1^\alpha(x) \quad (3.9)$$

The case of most interest here is when the following two conditions are verified :

- (a) There are a certain number of ω -limit sets K_i of (3.9) inside the domain D , with all points in each set K_i being equivalent for the functional I , i. e. $I(s,x,y)=0$ if both $x,y \in K_i$.
- (b) $b, \nu > 0$ on ∂D , ν being the inward normal to ∂D .

Let $V_i = \inf I(s,x,y)$ for $x \in K_i$ and $y \in \partial D$ and

$$V^* = \max(V_1, \dots, V_r)$$

$$V_* = \min(V_1, \dots, V_r)$$

Then

$$\lim_{g \rightarrow 0} (-g^2 \ln \lambda_1(g)) \leq V^* \quad \lim_{g \rightarrow 0} (-g^2 \ln \lambda_1(g)) \geq V_*$$

In particular if there is only one V_i

$$\lambda_1(g) \asymp m(g) \asymp \exp\left(-\frac{V}{g^2}\right) \quad (3.10)$$

the symbol $\asymp$ meaning logarithmic equivalence in the sense of large deviation theory.

For systems of the type (3.5), when the drift is the gradient of a function $(\ln \psi_0)$, the minimizing paths are those that exactly follow the flow or exactly oppose it. Therefore if, for example, there is only one attractive fixed point in D , V is given (up to a factor) by the smallest difference between $\ln \psi_0$ at the fixed point and at the boundary ∂D . If this difference is $\neq 0$ then, at small but finite g , λ_1 will be different from zero and approaches zero with g according to (3.10).

The situation described above does not apply to the vacuum of Abelian gauge fields which, using Fourier-transformed transverse fields, may be written

$$\psi_0[A] = \exp\left\{-\frac{1}{2} \int d^3k \ A_1^T(-k) |k| A_1^T(k)\right\} \quad (3.11)$$

Fix the boundary ∂D of the eigenvalue problem at some distance from the origin $A_1=0$, for example $D=\{A : |A| < M\}$. Now, for any M and any ϵ arbitrarily small, there is always a sufficiently small k such that on a path along k one has $I(s, \{A=0\}, \partial D) < \epsilon$, i. e. the boundary cannot be separated from the point $\{A=0\}$ by the functional I . Notice that for transverse fields the points in the boundary, that we are considering, are not gauge equivalent to $\{A=0\}$. Therefore the conditions (a) and (b) above are not satisfied. This is consistent with the fact that Abelian gauge fields have no positive mass gap. For the non-Abelian gauge fields, with dynamics defined by the QCD_0 functional, the situation is seen to be different, because choosing the boundary at a finite distance from the configuration $A=0$ (and its gauge equivalent ones), the boundary is separated from this point by the functional I .

Like in the Abelian case, the paths of smallest variation of the exponent of the wave functional are those corresponding to gauge potentials that are constant over space. For definiteness consider the gauge group to be $SU(2)$. Consider the matrix $M_{ij} = \sum_{\alpha} \theta_1^\alpha \theta_j^\alpha$, $\theta_1^\alpha = \theta^\alpha(x, x+i)$ being a constant lattice gauge potential. M , being a symmetric matrix, may be diagonalized by a space rotation. In the new coordinate axis the three $SU(2)$ -vectors θ_1^α , θ_2^α and θ_3^α are orthogonal. Without loss of generality, the internal $SU(2)$ -space coordinates may be chosen such that

$$\theta_1^\alpha = (a_1, 0, 0) \quad ; \quad \theta_2^\alpha = (0, a_2, 0) \quad ; \quad \theta_3^\alpha = (0, 0, a_3) \quad (3.12a)$$

to which correspond the chromomagnetic lattice fields

$$\beta_1^\alpha = (-a_2 a_3, 0, 0) \quad ; \quad \beta_2^\alpha = (0, -a_3 a_1, 0) \quad ; \quad \beta_3^\alpha = (0, 0, -a_1 a_2) \quad (3.12b)$$

We have to compute now the exponent of (2.16) for this field configuration (Recall that, up to a constant, the exponent of (2.16) is the potential of the deterministic motion (3.9)) The result is

$$\begin{aligned} \sigma(a_1, a_2, a_3) = & -\frac{N}{2\pi g^2} \int_0^\infty d\lambda \lambda^{-\frac{1}{2}} \left\{ (a_1 a_2 a_3)^2 (a_1^2 + a_2^2 + a_3^2) + \lambda \left(a_1^4 (a_2^2 + a_3^2) + a_2^4 (a_1^2 + a_3^2) + a_3^4 (a_1^2 + a_2^2) \right) + \right. \\ & \left. \lambda^2 (a_2^2 a_3^2 + a_1^2 a_3^2 + a_1^2 a_2^2) \right\} \left\{ 4(a_1 a_2 a_3)^2 + \lambda (\lambda + a_1^2 + a_2^2 + a_3^2)^2 \right\}^{-1} \end{aligned} \quad (3.13)$$

N is the number of sites on the lattice.

To decide whether the mass gap is finite or not, one has to look for neutral paths from the point $a_1, a_2, a_3 = 0$ to the boundary of the Dirichlet problem. The integrand in (3.13) is positive for generic a_1, a_2, a_3 and is zero only if two among these variables vanish. Therefore in the $\mathbf{R}^3$ space of a_1, a_2, a_3 the integral is strictly different from zero except along the axis. Thus it would seem that there is a neutral path along the axis. However all points along each one of the axis are gauge equivalent to the origin and should be identified with it. Therefore when the boundary is placed at some finite distance from the origin (which means at a finite distance from the axis) there are no neutral paths and, according to the discussion above, the mass gap is positive. One has therefore proved that

At small but finite g , QCD_0 has a finite mass gap.

The fact that the small noise estimates give a specific form for the dependence of the mass gap on g , at small g , provides a way to discuss the scaling of am and $g=g(a)$ when the lattice spacing a goes to zero. The result is

The mass gap of QCD_0 scales as $am \sim \exp\{-c/g'\}$ when $a \rightarrow 0$, $g(a) \rightarrow 0$.

The statement is again a consequence of the discussion above. Notice that the occurrence of the power -2 in the g -dependence of the exponent is a consequence of the fact that the drift $-L$ of the lattice process is independent of g , a necessary condition discussed in Ref.[18]. The specific form $\exp\{-c/g^2\}$ follows then from the fact that the ω -limit set of the deterministic system (3.9) is attractive in the domain of the Dirichlet problem. This is fulfilled because $g^2 \ln\{\psi_0\}$ is negative definite with maxima in the $\beta=0$ field configurations.

From $am \sim \exp\{-c/g^2\}$ one sees that for the physical mass gap m to remain fixed when $a \rightarrow 0$, one should require $g^2(a) \sim |\frac{c}{\log a}|$. Therefore, when $a \rightarrow 0$, $g(a) \rightarrow 0$ consistent with the use of the small noise estimates.

Estimates using large deviation theory are only accurate up to logarithmic equivalence. In particular they are not sensitive to power factors in front of the exponentials. In Appendix B we display this fact, in a simple example, by comparing exact asymptotic expressions with those obtained by the Wentzell-Freidlin technique. The example is also used to explain why the technique is not appropriate to estimate the wave functional of the lowest excited state. A better way to estimate this wave functional at small values of the coordinates and weak coupling would be to solve the equation

$$-\sum_{x,1,\alpha} b_1^\alpha(x) \frac{\partial}{\partial \theta_1^\alpha(x)} u\{\theta\} = \lambda u\{\theta\}$$

4. Vacuum background fluctuations and the fermion sector

The ground state structure derived in Sect.2 is the leading term of an exact (non-perturbative) representation of the QCD vacuum. We will now see that it provides an unambiguous description of a chromomagnetic vacuum structure.

The chromomagnetic structure of the QCD vacuum has been the object of many conjectures and phenomenological models, ranging from the search for low energy configurations [23-25] to the QCD sum rules [26] and the parametrization of hadronic properties as functions of the vacuum correlators of a fluctuating vacuum [27,28]. The discussions of background chromomagnetic configurations in the QCD vacuum faced the problem of showing both that they minimize the energy and that they are

stable. No such problem arises in QCD_0 . Because the ground state measure itself defines the theory and it has been shown to lead to a closable energy form, the theory (and the QCD_0 vacuum) are necessarily stable. The picture that emerges is that the magnetic structure of the vacuum is not to be found in some stable background field configuration, but in the nature of the fluctuations of the ground state process. No stability problem of the background fields arises because they are simply fluctuations of the vacuum and what is stable is the vacuum itself as a stochastic process associated to a closed Dirichlet form.

Consider again the exponent (3.13) of the ground state measure computed for the constant fields (3.12). For large volume (large N) and weak coupling ($g(a) \rightarrow 0$), the measure is dominated by those configurations that lead to values of the exponent near zero. The exponent (3.13) is never positive and is zero only when two of the variables vanish. In one of the coordinate planes, $a_1 = 0$ for example, it reduces to

$$\sigma(0, a_2, a_3) = -\frac{N}{2g^2} \frac{a_2^2 a_3^2}{\sqrt{a_2^2 + a_3^2}} \quad (4.1)$$

This simple function is plotted in Fig.1. It vanishes along the axis and grows monotonically away from them. However comparing (4.1) and (3.12b) one concludes that $\sigma(0, a_2, a_3)$ may be made arbitrarily small while, at the same time, β_1^α is arbitrarily large. It suffices to consider for any $|\beta_1^\alpha| = M$, $a_2 = M/\delta$ and $a_3 = \delta$. Then, for small δ , σ is proportional to δM and may be made as small as we like without changing β_1^α . This means that there are vacuum fluctuations with arbitrarily large chromomagnetic fields and that all magnitudes are equally probable. Notice that these are non-trivial fluctuations because, although they correspond to points very near the axis in Fig.2, they are not in the gauge orbit of the origin.

The dynamics of these fluctuations must in fact be the dominant contribution of the constant fields to the continuum theory because for large volume and small $g(a)$ only values near the axis will not be suppressed. Because the points along the axis are gauge equivalent to the origin, and the main contributions are all concentrated near the axis, one could think of getting rid of the gauge degeneracy and, by a change of coordinates, concentrate the measure at the origin. This transformation would however be very singular because, the integral of $\exp\{2\sigma(a_2, a_3)\}$ in a ball of radius r behaving like $r^{1/2}$, the transformed quantity, if concentrated at the origin, would not be a measure. Hence it seems more convenient to work with the gauge degenerate form

(3.13).

The effect of the constant gauge field fluctuations in the fermion sector has a simple interpretation. For large N and small $g(a)$ the gluonic background corresponding to the constant fields is approximated by

$$\mu = \delta(a_1)\delta(a_2) + \delta(a_2)\delta(a_3) + \delta(a_3)\delta(a_1) \quad (4.2)$$

One is led to a description where the integrated effect of all possible (equiprobable) background fields has to be taken into account. For each fixed background an irreducible fermion field algebra is constructed and, to take into account all backgrounds, the appropriate framework is a direct integral of Hilbert spaces. The field algebra is then reducible. For details of the description of quantum theories in the *reducible functional* context, I refer to Appendix C. In this case because each one of the irreducible spaces is characterized by a fixed gauge potential, the effective fermion theory over the gauge field background is even a quantum field with degenerate Lorentz vacuum in the sense of Appendix D.

Strocchi^[29] has shown that, contrary to positive metric fields, gauge theories could lead to the breakdown of the cluster property (a necessary condition for confinement) even with a unique vacuum state. However, for this to occur, the space-time translations have to fail to be measures on the light-cone. A situation where failure of the cluster property may occur, and that is easier to analyse, is the case where the vacuum functional is reducible. Therefore when a gauge theory (with unique vacuum) is shown to be equivalent to a reducible functional theory, the failure of the cluster property is easy to analyse.

The considerations in the previous paragraph are, of course, relevant only to the complete theory because for the effective description we are considering, of the fermions on a gauge background, the only way the cluster property may ever fail is through the reducible functional mechanism. This comes about because the background is not a quantum dynamical variable and, if the fermions have a mass, the strong spectral condition on the Wightman functions implies the cluster property, even with the indefinite metric^[29].

Consider now one of the components of the fermion field algebra over the gauge background defined by (4.2), for example the background $a_1 = a_2 = 0$ and $a_3 \neq 0$. The spinors are solutions of

$$i\gamma_\mu(\partial^\mu + iA_\mu^a \frac{\lambda_a}{2})\psi = m\psi$$

and the energy-momentum operator becomes

$$\begin{aligned} P^\mu = & \sum_p (p^\mu + A_3 \delta^{\mu 3}) b_{ps}^{(1)} b_{ps}^{(1)} + (p^\mu - A_3 \delta^{\mu 3}) d_{ps}^{(1)} d_{ps}^{(1)} \\ & + (p^\mu - A_3 \delta^{\mu 3}) b_{ps}^{(2)} b_{ps}^{(2)} + (p^\mu + A_3 \delta^{\mu 3}) d_{ps}^{(2)} d_{ps}^{(2)} \end{aligned} \quad (4.3)$$

where (1) and (2) denote up and down colour components.

It is clear that, for elementary fermion excitations, the spectrum condition fails because for any A_3 there is a choice of p^μ such that $(p^\mu + A_3 \delta^{\mu 3})^2 < 0$. We therefore fall under one of the cases analysed in Appendix C, where the cluster property fails because of the failure of the spectrum condition. However for composite fields formed of quark-antiquark pairs of the same type and diquarks or antidiquarks of opposite types

$$(b_{ps}^{(i)} d_{p's'}^{(i)} ; (b_{ps}^{(1)} b_{p's'}^{(2)}), (d_{ps}^{(1)} d_{p's'}^{(2)})) \quad (4.4)$$

the spectral condition is verified, the cluster property holds and excitations with these quantum numbers may have asymptotic fields. Forming SU(2) invariant combinations of the composites (4.4), the same conclusions are obtained for the other components of the reducible functional generated by (4.2).

The gauge group considered throughout the paper was SU(2) and the fact that the asymptotic fermion fields are found to be of the quark-quark, diquark and antidiquark types, is an SU(2)-dependent feature. For SU(3) the basic mechanisms discussed in the paper will carry through. The essential difference, that deserves further study, is the nature of the vacuum fluctuations, which now belong to several non-equivalent classes. This comes about because, contrary to the SU(2) case, the action of SU(3) on its adjoint space has two strata and furthermore there is a continuous infinity of vectors in the generic stratum that cannot be rotated to one another^[30].

Appendix A. Confinement and gauge invariance

The confinement dogma covers two distinct statements :

- (i) All observables are colour neutral.
- (ii) All physical states are colour singlets.

For fields transforming under a non-Abelian gauge group, once it is assumed that

gauge invariance is an exact symmetry, the first statement is a simple manifestation of the existence of a non-Abelian superselection rule^[29]. Let $\{Q_a\}$ be the set of colour charges that generates the (global) gauge group G and $\mathcal{O}$ a local observable. Computing the commutator between physical states ψ, ϕ

$$\langle \psi | [Q_a, \mathcal{O}] | \phi \rangle = \langle \psi | [\int d^3x J_a^0, \mathcal{O}] | \phi \rangle = \langle \psi | [\int d^3x (J_a^0 - \partial_j F_a^{j0}), \mathcal{O}] | \phi \rangle = 0 \quad (A.1)$$

where the second equality follows from locality of $\mathcal{O}$ and the third from Gauss' law

$$\partial_j F_a^{j0} = j_a^0 + g f_{abc} A_{1b} F_c^{i0} = j_a^0$$

acting on physical states. The term j_a^0 denotes the non-gluonic charge.

Eq.(A.1) implies that all local observables, in the physical space, commute with the colour charges, i. e. G is a non-Abelian superselection rule. In particular it implies that (local) colour charges cannot be observable quantities. Therefore the fact that colour charges are not observable is not a dynamical question in the sense that it does not depend on the detailed dynamics of non-Abelian gauge theory, but simply on the fact that current conservation occurs in a particular form, namely the current is the divergence of an antisymmetric tensor. Unobservability of colour charges is therefore a trivial consequence of non-Abelian gauge symmetry. The deep question is of course why there is an exactly conserved colour gauge symmetry.

It is possible that the second part of the confinement dogma, the existence of just colour singlets, may also be a "kinematical" consequence of gauge symmetry, although the situation here is not so clear. The existence of a non-Abelian superselection rule implies that the superselection sectors are labelled by the eigenvalues of the Casimir operators. For all except the singlet sector, there is more than one vector corresponding to the same physical state. Hence if non-singlet states were to exist, their description would imply a departure from the usual quantum mechanical framework. Namely there would not exist a complete commuting set of observables and the description of scattering experiments, for example, would require special care because the computed matrix elements would depend on the initial and final vector representatives chosen among the physically equivalent multiplet vectors^[31]. A description using direct integral spaces^[32] or some other form of averaging over initial and final physically equivalent vectors would be mandatory to obtain unambiguous predictions.

If colour is an unbroken symmetry, the question of confinement is not whether

any coloured states are going to be found, because colour charges are unobservable anyway, but whether the colourless objects one sees are real singlets or some sort of balanced admixture of hidden colour states.

Because the cluster property of quantum fields plays a crucial role in the existence of the asymptotic limits of field operators, the exclusion of all but the singlet states must be related to the failure of the cluster property. It has indeed been proved^[29] that unlike the situation in positive metric quantum field theory, the indefinite metric in local covariant gauge field theory allows the violation of the cluster property (without non-uniqueness of the vacuum). Although this remains a distinct possibility it has however not been proved that failure of the cluster property indeed follows from the dynamics of QCD.

The question may however not be related to the dynamics of a particular theory. Think for a moment of a more familiar non-Abelian superselection rule, that of permutation symmetry of identical particles^[33]. Here all observables are required to commute with any permutation of identical particles. As a consequence the physical space is divided into superselection sectors labelled by Young diagrams. Again, except in the symmetric and antisymmetric sectors, there would be more than one vector corresponding to the same physical state for any other sector. It is clear then that, from permutation symmetry alone, parastatistics are as likely as Bose or Fermi statistics and no simple dynamical mechanism has been found to justify its apparent non-existence.

What is usually done is to require the existence in the physical spaces of a complete commuting set of observables^[34], which automatically excludes all but the one-dimensional sectors. But if existence of a complete commuting set of observables is imposed as an axiom on physical spaces then, together with exact conservation of gauge colour symmetry, one is automatically restricted to colour singlet states in the physical spaces. In this sense both confinement and absence of parastatistics would be phenomena of "kinematical" origin, in the sense that they are direct consequences of the symmetries and general principles of theory construction, rather than detailed consequences of the dynamics. Of course deeper reasons for the principles and symmetries may yet be found, but probably they lie at a different theoretical level, not at the level of specification of the dynamical laws.

Another "almost-kinematical" way in which the second part of the confinement

principle may be implemented is through a symmetry realization mode with reducible functionals (see Appendix C).

Appendix B. The Dirichlet problem and the Wentzell-Freidlin method. Comparison with exact solutions

Consider the eigenvalue problem

$$\left\{ -\frac{\epsilon^2}{2} \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} \right\} u(x) = \lambda u(x) \quad (B.1)$$

in the domain $D = \{x : |x| \leq M\}$ with the boundary condition $u(\pm M) = 0$, ϵ being a small parameter. If λ is not an integer two independent solutions of (B.1) are

$$u_{\pm}(x) = \exp\left(\frac{x^2}{2\epsilon^2}\right) D_{\lambda}\left(\pm \sqrt{\frac{2}{\epsilon}} x\right)$$

where $D_{\lambda}(x)$ is a parabolic cylinder function. A symmetric solution is

$$u(x) = \exp\left(\frac{x^2}{2\epsilon^2}\right) \left\{ D_{\lambda}\left(\sqrt{\frac{2}{\epsilon}} x\right) + D_{\lambda}\left(-\sqrt{\frac{2}{\epsilon}} x\right) \right\} \quad (B.2)$$

When ϵ is small the argument of D_{λ} in the boundary of the interval $[-M, M]$ is very large and one may use the asymptotic expansion of the parabolic cylinder functions to obtain

$$u(M) \simeq \left(\frac{\sqrt{2}M}{\epsilon}\right)^{\lambda} \left\{ 1 + \frac{\lambda(1-\lambda)}{4\left(\frac{M}{\epsilon}\right)^2} + \dots \right\} - \frac{\pi\lambda}{\Gamma(1-\lambda)} \sqrt{\frac{2}{\epsilon}} e^{\left(\frac{\sqrt{2}M}{\epsilon}\right)^2} \left\{ 1 + \frac{(\lambda+1)(\lambda+2)}{4\left(\frac{M}{\epsilon}\right)^2} + \dots \right\} \quad (B.3)$$

Enforcing the boundary condition $u(\pm M) = 0$ one obtains for small ϵ

$$\lambda_1 \simeq \frac{M}{\epsilon} e^{-\frac{M^2}{\epsilon^2}} \quad (B.4)$$

for the smallest non-zero eigenvalue.

The stochastic process associated with this Dirichlet problem is

$$dx_t = -x dt + \epsilon dW_t \quad (B.5)$$

Estimating the lowest positive eigenvalue by the Wentzell-Freidlin technique (see 3.8–3.10) one obtains

$\lambda_1 \sim e^{-\frac{M^2}{\epsilon^2}}$ (B.6)
 i. e. one obtains the exponential but not the power factor $\frac{M}{\epsilon\sqrt{\lambda}}$. This is typical of the large deviation estimates which are only accurate in the logarithmic equivalence sense.

The solutions $u(x)$ of the more general problem

$$K u(x) + c(x) u(x) = f(x) \quad \text{with} \quad u(x)|_{x \in \partial D} = \psi(x) \quad (B.7)$$

where K is an elliptic operator, have the stochastic representation

$$u(x) = -E_x \int_0^\tau f(x_t) e^{\lambda t} dt + E_x \int_0^\tau c(x_s) ds e^{\lambda \tau} \quad (B.8)$$

where τ is the first hitting time of the boundary ∂D and E_x denotes the expectation value for the process starting from x . For the particular case of (B.1), Eq.(B.8) reduces to

$$u(x) = E_x \psi(x_\tau) e^{\lambda \tau} \quad (B.9)$$

which according to (3.7) is only valid for $\lambda < \lambda_1$. One might think that a limiting procedure $\lambda \rightarrow \lambda_1$ $\psi(x_\tau) \rightarrow 0$ and large deviation techniques would lead to an estimate of the wave function corresponding to the lowest non-zero eigenvalue. However, as explained below, because large deviation methods reproduce accurately only the exponential factors, this is not a very useful technique for the wave functionals.

For the case of a process with one deterministic global attractor as in (B.1) and according to the Wentzell-Freidlin estimates (3.8), the process at small ϵ is well approximated by a Markov chain where:

- Transition probability from x to the boundary $\partial D = p_x \sim e^{-\frac{V(\partial D) - V(x)}{\epsilon^2}}$
- Transition probability from x to the neighbourhood of the attractor $= 1 - p_x$
- Transition probability from the attractor to $\partial D = p_0 \sim e^{-\frac{V(\partial D) - V(0)}{\epsilon^2}}$
- Transition probability from the attractor to itself $= 1 - p_0$
- Transition probability from the boundary to the boundary $= 1$

For (B.1), $V(x) = x^2$. Then

$$E_x(e^{\lambda \tau}) = e^{\lambda p_x + (1-p_x)p_0} \{ e^{2\lambda} + (1-p_0)e^{3\lambda} + \dots \} \\ = e^{\lambda p_x} + \frac{(1-p_x)p_0 e^{2\lambda}}{1-(1-p_0)e^{\lambda}}$$

$E_x(e^{\lambda \tau})$ is finite only for $(1-p_0)e^{\lambda} < 1$ which, according to (3.7), leads exactly to the estimate (3.10) and (B.6) for λ_1 . For the wave function $u(x)$ corresponding to λ_1 we may consider a boundary condition $1 - (1-p_0)e^{\lambda}$ which vanishes when $\lambda \rightarrow \lambda_1$. Then

$$u_1(x) \sim c_1 e^{-\frac{V(\partial D) - V(x)}{\epsilon^2}} + c_2$$

Comparing with (B.3) one sees that the exponential behaviour is reproduced but that the power behaviour entering in c_1 and c_2 is not reachable by this technique.

In some cases^[15,17] it has been shown that when $\epsilon \rightarrow 0$ the solutions of (B.7) with $K = \frac{\epsilon^2}{2} \Delta + b \cdot \nabla$ approach the solutions of the corresponding equation with $K = b \cdot \nabla$ and the same boundary conditions. For the particular case of (B.1) this leads to $u \sim x^\lambda$ which is accurate at small x .

Appendix C. Symmetries, reducible functionals, cluster property and confinement

In practically all field theoretical models, in current use, the notion of spontaneous symmetry breaking plays a central role. This means that the vacuum is one of a set of degenerate ground states, each one violating some of the original symmetries of the Lagrangian. This structure reveals itself in the formation of vacuum condensates in the sense that an operator, with non-trivial transforming properties under a (Lagrangian) symmetry operation, acquires a non-vanishing vacuum expectation value. The operator that condenses may be an elementary one as in the case of the $\langle \phi \rangle$ scalar condensate of the Higgs mechanism or a composite operator like the fermion bilinear $\langle \bar{q}q \rangle$ associated to the spontaneous breaking of chiral symmetry.

There are however cases where a condensate is formed, which is not invariant under the symmetry operation, and nevertheless it turns out that the physical system still preserves the symmetry. The classical example is the BCS model of superconductivity, where $U(1)$ symmetry is spontaneously broken although charge is

conserved in the physical systems. In this model the ground state of the space $\mathfrak{K}_\alpha$, associated to an irreducible representation of the field algebra, is the *sharp phase* state $|\alpha\rangle$ for which^[35]

$$\langle\alpha|\psi(x)\psi(o)|\alpha\rangle = \phi_o e^{i2\alpha} \quad (C.1)$$

In general any product with different numbers of destruction (ψ) and creation (ψ^*) operators has a non-zero vacuum expectation value in the α -state. A gauge transformation

$$\psi \rightarrow \psi e^{i\theta} \quad \psi^* \rightarrow \psi^* e^{-i\theta} \quad (C.2)$$

brings the field representation $\mathfrak{K}_\alpha$ into one in the space $\mathfrak{K}_{\alpha+2\theta}$. It means that, in the space of the representation $\mathfrak{K}_\alpha$ (in $\mathfrak{K}_\alpha$) of the field algebra, there is no unitary operator U_θ implementing the (first-order) gauge transformation.

For charge to be conserved, there should be a unitary transformation U_θ such that

$$U_\theta A U_\theta^{-1} = A \quad (C.3)$$

for all observables A . Eq. (C.3) would then require the observables to have equal numbers of ψ 's and ψ^* 's. However, if the operator U_θ does not exist in $\mathfrak{K}_\alpha$, there is no reason to constrain the observables to have such a structure. What one usually does, to avoid the breaking of charge conservation in the physical processes, is to require the observables to have equal numbers of ψ 's and ψ^* 's anyway, while, at the same time, one sticks to the $\mathfrak{K}_\alpha$ representation. The restrictive condition on the observables appears therefore as an additional constraint, meaning that the specification of the $\mathfrak{K}_\alpha$ representation is not a complete definition of the theory.

Alternatively the model may be formulated using explicitly invariant *sharp particle number*^[35] states $|\Omega\rangle$ for which the vacuum functional $W_\Omega(a) = \langle\Omega|a|\Omega\rangle$ is

$$W_\Omega(\psi^*(x_1)\dots\psi^*(x_m)\psi(y_1)\dots\psi(y_n)) = \langle\alpha|\psi^*(x_1)\dots\psi^*(x_m)\psi(y_1)\dots\psi(y_n)|\alpha\rangle \delta_{m,n} \quad (C.4)$$

The field algebra representation $\mathfrak{K}_\Omega$, built using $|\Omega\rangle$ as a generating vector, is now (infinitely) reducible. In this case the unitary operator U_θ exists and $U_\theta|\Omega\rangle = |\Omega\rangle$,

$W_{\theta\Omega} = W_\Omega$. It is now possible to use Eq.(C.3) and the requirement of equal numbers of creation and annihilation operators for the observables is a consequence of the representation, not an additional constraint.

In conclusion, one finds that there are three ways to realize an exact symmetry in a quantum model:

First, the *manifest* way when the vacuum functional is irreducible and invariant. The phenomenological consequences of the symmetry are, in this case, degenerate multiplets and selection rules in transitions.

Second, the *spontaneously broken* or *hidden symmetry* way when the vacuum is irreducible but not invariant. The experimentally accessible consequences of a symmetry realized in this way are low energy theorems, Goldstone bosons and (or) the Higgs mechanism.

Finally, in the third case, which I will call the *reducible functional realization* or *restored mode*, one has a vacuum functional that is invariant but not irreducible. The reducible functional actually restores the spontaneously broken symmetry which is again exactly implemented in the physical space. Nevertheless this realization mode is quite different from the *manifest symmetry* one, in that the effect of the condensates has distinct phenomenological implications. These are the gap (mass generation) mechanism, the composite collective modes, as well as some exact quantitative relations between these phenomena^[36]. As we see below, this symmetry realization mode may also preclude the appearance of certain quantum numbers in the spectrum of elementary excitations. Confinement of certain quantum numbers is therefore a "kinematical" consequence of this symmetry realization mode. This comes about through the violation of the cluster property.

In (positive metric) quantum field theory if: (i) the fields $\phi(x)$ transform covariantly under a unitary representation of the space-time translations; (ii) the generators of the space-time translations satisfy the spectrum condition ($P_0 \geq 0, P^\mu P_\mu \geq 0$); (iii) the fields commute at space-like separations and (iv) the vacuum is unique, then the cluster property

$$\lim_{\lambda \rightarrow \infty} \{ \langle \phi(x_1) \dots \phi(x_j) \phi(y_1 + \lambda a) \dots \phi(y_k + \lambda a) \rangle - \langle \phi(x_1) \dots \phi(x_j) \rangle \langle \phi(y_1) \dots \phi(y_k) \rangle \} = 0$$

(a being a spacelike vector) holds^[37].

If the cluster property fails for a field ϕ , then the state $\phi(f)|0\rangle$ has no asymptotic limit and this field is confined. In the reducible functional mode the vacuum is not unique and the cluster property may fail. However, the cluster property does not necessarily fail for all composite fields that can be formed from the basic fields and one has to analyse in detail the conditions of failure of the cluster property. From the proof of Araki, Hepp and Ruelle^[37] two situations may be readily identified:

(a) Denote by $\tilde{h}_{12}(p)$ the Fourier transform of

$$h_{12}(\xi) = \langle B_1(x_1) B_2(x_2) \rangle - \langle B_1(x_1) \rangle \langle B_2(x_2) \rangle$$

and by $\tilde{h}(p)$ the Fourier transform of the corresponding commutator

$$h(\xi) = \langle [B_1(x_1), B_2(x_2)] \rangle$$

To extract in a unique manner the positive frequency part $\tilde{h}_{12}(p)$ from the Jost-Lehmann-Dyson representation for $\tilde{h}(p)$ this one should have zero measure for the point $\{p=0\}$. From

$$\tilde{h}_{12}(p) dp = (2\pi)^{-2} d\langle B_1(0) E(p) B_2(0) \rangle$$

one sees that if the vacuum is not unique the cluster property may fail, but only if the B-fields connect different vacua.

(b) Another scenario that leads to the violation of the cluster property is the violation of the spectrum condition in each one of the irreducible components of the field algebra. Then the proof of the cluster property breaks down because the Jost-Lehmann-Dyson representation cannot be used. This is the situation analysed in Sect.4 and Appendix D.

I elaborate a little further on the *reducible functional realization* of symmetry operations. Consider first the case of a compact symmetry group G commuting with the Poincaré group. Let W_0 be a vacuum functional of class Q_0 in the sense of Vasil'ev^[31], i. e.

1) W_0 satisfies a strong spectral condition (the energy-momentum continuous spectrum is concentrated in the forward light-cone and separated from the $p^0=0$ point; $p^2 \geq m^2 > 0$, $p^0 > 0$).

2) W_0 generates a representation $\mathfrak{R}_{W_0}$ of the field algebra $\mathcal{A}$ with unique vacuum.

Let G be a group of automorphisms of the field algebra. One assumes that the functional W_0 is invariant under the action of a subgroup K of G but that there are elements $g \in G$ for which W_g , $\{W_g(a) = W_0(ga), a \in \mathcal{A}\}$, is not equivalent to W_0 , i. e. $W_0 \rightarrow W_g$ cannot be reduced to a unitary transformation in $\mathfrak{R}_{W_0}$.

Because the field algebra automorphisms of G commute with the Poincaré group the energy-momentum operators in the representations $\mathfrak{R}_{W_0}$ and $\mathfrak{R}_{W_g}$ have the same spectrum. Now one defines the functional $\bar{W}$ by

$$\bar{W} = \int \frac{W_g}{G/K} d\mu(g) \quad (C.5)$$

where μ is a G -invariant measure.

To construct the representation $\mathfrak{R}_{\bar{W}}$ of $\mathcal{A}$ associated to the functional $\bar{W}$, first one uses the standard GNS reconstruction for each W_g . One obtains the representation $\mathfrak{R}_{W_g}$

$$\tau_{W_g}(a) \phi_{W_g}(b) = \phi_{W_g}(ab) \quad a, b \in \mathcal{A} \quad (C.6)$$

where $\phi_{W_g}(b) \in \mathcal{A}/N_{W_g}$, $N_{W_g} = \{a \in \mathcal{A}; W_g(a^\dagger a) = 0\}$ being the null space of W_g .

The Hilbert space $\mathfrak{H}_{W_g}$ is the completion of $\mathcal{A}/N_{W_g}$ for the scalar product

$$(\phi_{W_g}(a), \phi_{W_g}(b)) = W_g(a^\dagger b) \quad (C.7)$$

Then one considers a space $\mathfrak{H}_{\bar{W}}$ which is the direct integral of the spaces $\mathfrak{H}_{W_g}$ with measure μ .

$$\mathfrak{H}_{\bar{W}} = \int_{\oplus} \mathfrak{H}_{W_g}(g) d\mu(g) \quad (C.8)$$

$\mathfrak{H}_{\bar{W}}$ consists of all vector functions $z = \{z_g\}$, $z_g \in \mathfrak{H}_{W_g}$, on G/K which are in $L^2(G/K, \mu)$. The scalar product is

$$(z, z') = \int \frac{(z_g, z'_g)}{G/K} d\mu(g) \quad (C.9)$$

As a consequence of the group compactness the special type of vector functions

$$\phi_{\bar{W}}(a, 1) = \{\phi_{W_g}(a)\} \quad (C.10)$$

is in $\mathfrak{H}_{\bar{W}}$. The space

$$L_{\bar{W}}(1) = \{\phi_{\bar{W}}(a, 1) ; a \in A\} \quad (C.11)$$

is dense in $\mathfrak{K}_{\bar{W}}$ as a consequence of the strong spectral condition^[31]. In $L_{\bar{W}}(1)$ one defines operators $r_{\bar{W}}(a)$ by

$$r_{\bar{W}}(a) \phi_{\bar{W}}(b, 1) = \phi_{\bar{W}}(ab, 1) \quad (C.12)$$

These operators form a densely defined representation of the field algebra $\mathfrak{A}_{\bar{W}} = \{r_{\bar{W}}(a) ; a \in A\}$. $\phi_{\bar{W}}(1, 1)$ is a generating vector for the representation $\mathfrak{A}_{\bar{W}}$. In $\mathfrak{K}_{\bar{W}}$ all the automorphisms of G are now unitarily implemented.

Instead of realizing the space $\mathfrak{K}_{\bar{W}}$ as a direct integral an equivalent construction is possible in terms of Hilbert bundles^[38,39].

If the automorphisms, that are not unitarily implemented in the original irreducible representation $\mathfrak{A}_{W_0}$, do not commute with the Poincaré group, the above construction has to be slightly modified. Of particular interest for our purposes is the case where each one of the irreducible vacuum copies is translationally invariant, but not invariant under the whole homogeneous Lorentz group. Already at the level of the irreducible representations $\mathfrak{A}_{W_g}$ the situation should be analysed with care because a non-Poincaré invariant vacuum requires a generalization of the usual quantum field theory axioms. The spectrum of the energy-momentum operator is not the same in each $\mathfrak{A}_{W_g}$ and questions like a theorem by Borchers, which is sometimes quoted as meaning that a translational invariant vacuum is also Poincaré invariant, have to be dealt with. This is done in Appendix D.

Appendix D. Quantum Field Theory with degenerate Lorentz vacuum

Lorentz invariance of the vacuum has been one of the most unchallenged axioms of quantum field theory. However it is well known that the isotropy group of the four-vector $p^\mu = 0$ has many inequivalent irreducible representations besides the trivial identity representation.

According to standard theory^[40], the automorphisms $\Theta(\Lambda)$ of the algebra of observables, corresponding to the Lorentz transformations, are conserved symmetries of

a physical model if they satisfy the following two conditions :

- a) The energy-momentum operator P transforms under $\Theta(\Lambda)$ as a four-vector operator.
- b) The automorphisms $\Theta(\Lambda)$ have an explicit representation by unitary operators in the Hilbert space $\mathfrak{K}$ of the model.

Therefore a degenerate Lorentz vacuum is perfectly consistent with exact conservation of Lorentz invariance if the energy-momentum operator has the right tensor properties and if the usual *unique vacuum* is replaced by a *vacuum subspace* carrying a unitary representation of the Lorentz group.

THE AXIOMS

Except for the modifications needed to suit the proposed generalization, the axioms are taken to be as close as possible to those of standard quantum field theory.

AXIOM 1 : There is a Hilbert space $\mathfrak{K}$ carrying a continuous unitary representation $U(\Lambda, a)$ of $ISL(2, C)$. The energy-momentum spectrum of $U(1, a)$ in $\mathfrak{K}$ lies in the forward light-cone $V^+ = \{P_\mu P^\mu > 0 ; P^0 > 0\}$ (*spectrum condition*).

AXIOM 2 : For each function f in Schwartz' space $\mathcal{S}$ there are operators $\phi_\lambda(f)$ (λ running over a finite set) with dense domain D in $\mathfrak{K}$, the same for all λ and f . D is invariant under each $\phi_\lambda(f)$ and every $U(\Lambda, a)$. The Hermitian conjugate $\phi_\lambda^\dagger(f)$ of each field is included among the ϕ_λ 's.

AXIOM 3 : (*Lorentz covariance*) The fields transform under $ISL(2, C)$ according to

$$U(\Lambda, a) \phi_\lambda(f) U^{-1}(\Lambda, a) = M_{\lambda\lambda'}(\Lambda) \phi_{\lambda'}(f_{\Lambda, a}) \quad (D.1)$$

where $f_{\Lambda, a}(x) = f((\Lambda)^{-1}(x - a))$ and M is a finite-dimensional representation of $SL(2, C)$.

AXIOM 4 : (*Causality*) If the supports of f and g are space-like separated, then

$$[\phi_\lambda(f), \phi_{\lambda'}(g)]_\pm = 0 \quad (D.2)$$

The $\pm$ sign is independent of f and g , $+$ for some pairs of fields, $-$ for the rest.

AXIOM 5 : (*Existence of the vacuum*) There is, at least, one translation invariant state

$$U(1,a) \Omega = \Omega \quad (D.3)$$

AXIOM 6 : The subspace V_0 of states satisfying axiom 5 (*vacuum subspace*) carries a unitary representation of $SL(2,C)$ and $V_0 \in D$. The closure of the set $\{\phi_{\lambda_1}(f_1) \dots \phi_{\lambda_n}(f_n) \Omega_1\}$ where $\Omega_1 \in V_0$ contains D .

To ensure that one has included enough fields to reach all physical states by operating on the vacuum, Wightman's theory requires a cyclic vacuum. Here one has allowed for the possibility that the set, obtained by operating with the field algebra on the vacuum subspace, is overcomplete.

That the axioms are compatible follows trivially from the fact that they are weaker than Wightman's axioms. They allow for all quantum systems that are compatible with the usual Wightman axioms. The real question is whether they allow for other systems as well.

Consider the consequences of axiom 6. The irreducible unitary representations (UIR) of $SL(2,C)$ are (Naimark's notation^[41]) denoted by (k_0, c) where:

- a) k_0 is an arbitrary non-negative integer or half-integer and c is purely imaginary (*principal series*), or
- b) $k_0 = 0$ and c is a real number $-1 < c < 1$ (*complementary series*).

An irreducible (not necessarily unitary) representation of $SL(2,C)$ is finite dimensional when $c^2 = (k_0 + n)^2$ for some integer n . Therefore the only irreducible unitary and finite dimensional representation is the trivial one $(0,0)$. If the vacuum is Lorentz degenerate it is infinitely degenerate. Hence there are two classes of quantum field theories implied by axiom 6 which are characterized either by

$$(k,c) = (0,0) \quad (\text{unique vacuum}), \quad \text{or}$$

$$(k,c) \neq (0,0) \quad (\text{infinitely degenerate vacuum}).$$

In a vacuum subspace V_0 the vectors may be characterized by angular momentum quantum numbers^[41]

$$J_3 |k m\rangle = m |k m\rangle \quad J^2 |k m\rangle = k(k+1) |k m\rangle \quad (D.4)$$

Another way to label the vectors, in a Lorentz degenerate vacuum subspace, is through the generalized eigenvectors of some tensor operator (see below).

The UIR's of $SL(2,C)$ being subspace-irreducible, if one chooses a vector Ω_0 in an

arbitrary basis the set $\{U(\Lambda) \Omega_0 ; \Lambda \in SL(2,C)\}$ is dense in V_0 . Therefore, by the covariance axiom 3, all diagonal Wightman functions (W-F) may be expressed in W-F's on the standard vacuum Ω_0 .

$$\langle \Omega | \phi_{\lambda_1}(f_1) \dots \phi_{\lambda_n}(f_n) | \Omega \rangle = \langle U(\Lambda) \Omega_0 | \phi_{\lambda_1}(f_1) \dots \phi_{\lambda_n}(f_n) | U(\Lambda) \Omega_0 \rangle$$

By covariance the right-hand side is a linear combination of $(W-F)_0$'s. On the other hand the non-diagonal W-F's may always be reduced to $\langle \Omega | \phi_{\lambda_1}(f_1) \dots \phi_{\lambda_n}(f_n) | \Omega_0 \rangle$.

To check the compatibility of the axioms for situations other than the unique vacuum case (for which the free fields provide a consistency check), the best thing to do is to construct a specific model, even trivial, which satisfies explicitly the axioms. Define a model in the following way:

The dynamical field variable is a spinor field $\psi(x)$ such that

$$\psi(x) = \psi'(x) e^{-i\bar{q} \cdot x} \quad (D.5)$$

where $\psi'(x)$ is a mass m spinor operator obeying the free Dirac equation, and $\bar{q}$ is an operator that commutes with $\psi'(x)$ and transforms as a four-vector under the Lorentz group. The specification of the model is completed by stating that the energy momentum operator is

$$P^\mu = \sum_{p,s} \left((-\bar{q}^\mu - p^\mu) b_{ps}^\dagger b_{ps} \Theta(\bar{q}^2 + p \cdot \bar{q}) + (\bar{q}^\mu + p^\mu) b_{ps}^\dagger b_{ps} \Theta(-\bar{q}^2 - p \cdot \bar{q}) + (-\bar{q}^\mu + p^\mu) d_{ps}^\dagger d_{ps} \right) \quad (D.6)$$

where $b_{ps}^\dagger$, b_{ps} and $d_{ps}^\dagger$, d_{ps} are particle and antiparticle creation and annihilation operators associated to the momentum expansion of the free field $\psi'(x)$. The spectrum of the operator $\bar{q}$ is assumed to lie inside the backward light-cone. Let us check the structure of this model and its compatibility with the axioms. Covariance and causality (axiom 3 and 4) are an obvious consequence of the free Dirac spinor nature of $\psi'(x)$ and the transformation properties of $\bar{q}$. However $\psi(x)$ is not simply a gauge transformation of a free spinor because $\bar{q}$ is an operator.

The structure of the vacuum subspace follows from the assumed energy-momentum operator (D.6). The translation invariant states are annihilated by

$$\Theta(\alpha^2 + p \cdot g) b_{ps}^\dagger + \Theta(-\alpha^2 - p \cdot g) b_{ps} |> = d_{ps} |> = 0 \quad (D.7)$$

Therefore they are parametrized by the eigenvalues of g .

$$g^\mu | \alpha > = \alpha^\mu | \alpha > \quad (D.8)$$

From the four-vector transformation $U(\Lambda) g^\mu U(\Lambda) = (\Lambda^{-1})^\mu_\lambda g^\lambda$ one writes

$$U(\Lambda) | \alpha > = | \Lambda \alpha > \quad (D.9)$$

and this defines a unitary representation if an invariant normalization is chosen for the states $| \alpha >$.

$$< \alpha' | \alpha > = | \alpha^0 | \delta^3(\alpha' - \alpha) \quad (D.10)$$

The states $| \alpha >$ are generalized eigenstates of g and do not belong to the Hilbert space $\mathcal{H}$. Normalizable states are obtained from

$$| h > = \int \frac{d^3 \alpha}{|\alpha^0|} h(\alpha) | \alpha > \quad (D.11)$$

where $h(\alpha)$ is a square integrable function on the lower hyperboloid $\alpha^2 > 0$, $\alpha^0 < 0$ (spectrum assumptions on g). Associated to the vacuum subspace $V_0 \in \mathcal{H}$ there is a triplet of spaces

$$V' \{ | \alpha > \} \supset V_0 \{ | h > \} \supset V \{ | h >_{\text{fast decrease}} \}$$

where V' is dual to V . Matrix elements $< \alpha' | \dots | \alpha >$ are tempered distributions over the space of functions of fast decrease on the hyperboloid. V_0 carries, as required, an unitary representation of $SL(2, C)$

$$U(\Lambda) | h > = | \Lambda h > \quad (D.12)$$

with $\Lambda h(\alpha) = h(\Lambda^{-1} \alpha)$.

For $| h >$, $| h' > \in V_0$ the Wightman functions $< h' | \psi_{\lambda_1}(f_1) \dots \psi_{\lambda_n}(f_n) | h >$ may be constructed from $< \alpha' | \psi_{\lambda_1}(x_1) \dots \psi_{\lambda_n}(x_n) | \alpha >$, a tempered distribution over the direct product of the space of fast decrease functions in $4n$ variables and the space of fast decrease functions on the hyperboloid.

$$< h' | \psi_{\lambda_1}(f_1) \dots \psi_{\lambda_n}(f_n) | h > = \int \frac{d^3 \alpha'}{|\alpha'^0|} \frac{d^3 \alpha}{|\alpha^0|} d^4 x_1 \dots d^4 x_n h'^*(\alpha') h(\alpha) f(x_1) \dots f(x_n) < \alpha' | \psi_{\lambda_1}(x_1) \dots \psi_{\lambda_n}(x_n) | \alpha >$$

Because g commutes with $\psi(x)$

$$< \alpha' | \psi_{\lambda_1}(x_1) \dots \psi_{\lambda_n}(x_n) | \alpha > = | \alpha^0 | \delta^3(\alpha' - \alpha) < \alpha' | \psi_{\lambda_1}(x_1) \dots \psi_{\lambda_n}(x_n) | \alpha > \quad (D.13)$$

In this sense, the field structure is infinitely reducible. The model however is not a trivial superposition of free fields as can be checked easily by computing, for example, the Fourier transform $S(p)$ of the time-ordered two-point function $-i < \alpha | T(\psi_\lambda(x') \bar{\psi}_\lambda(x)) | \alpha >$

$$S(p) = \frac{\gamma_\mu (p - \alpha)^\mu + m}{p^2 - 2p \cdot \alpha + \alpha^2 - m^2 + i\epsilon} \quad (D.14)$$

$S(p)$ no longer has a pole on a well-defined $(\text{mass})^2$ value.

Actually, a Lorentz degeneracy of the vacuum of this type together with the spectrum condition (axiom 1) has some non-trivial consequences. The squared mass of a state created either by b_{ps} or $b_{ps}^\dagger$ is

$$(p + \alpha)^2 = m^2 + (\alpha^0)^2 - 2p^0 |\alpha^0| \quad (D.15)$$

There is always a p^0 such that $(p + \alpha)^2 < 0$. Therefore the physical Hilbert space $\mathcal{H}$ cannot coincide with the closure of the set $\{ \psi_{\lambda_1}(x_1) \dots \psi_{\lambda_n}(x_n) | h >, | h > \in V_0 \}$ because this contains states that do not satisfy the spectrum condition. Recall that, in axiom 6, we only require the domain $D \subset \mathcal{H}$ to be contained in this set, not to coincide with its closure. A Hilbert space, satisfying the spectrum condition, is obtained by forming, from $b^\dagger b$ and $d^\dagger d$, the following composite operators^[42]

$$A_{ps}^\dagger = \{ \frac{1}{\sqrt{2}} (d^\dagger d_- + b b^\dagger) - i b d_-^\dagger - \sqrt{2} b d_-^\dagger d_- b^\dagger \} \Theta(\alpha^2 + p \cdot g)$$

$$B_{ps}^\dagger = \{ \frac{1}{\sqrt{2}} (d^\dagger d_- + b^\dagger b) - i b^\dagger d_-^\dagger - \sqrt{2} b^\dagger d_-^\dagger d_- b \} \Theta(-\alpha^2 - p \cdot g)$$

where $d_- = d_{p,-s}$ and $b = b_{ps}$.

These operators and its adjoints satisfy

$$A | \alpha > = B | \alpha > = 0 \quad ; \quad [A, A^\dagger] | \alpha > = [B, B^\dagger] | \alpha > = | \alpha > \quad \text{or} \quad 0$$

The last property implies that $A^\dagger | \alpha >$ and $B^\dagger | \alpha >$ are eigenstates of the Hamiltonian. Restricted to a space $\mathcal{H}\{A, B\}$ generated from the vacua $| \alpha >$ and any number of $A^\dagger$ and $B^\dagger$ excitations the energy-momentum operator is exactly

$$P^\mu = \sum_{p,s} (-2\alpha^\mu A_{ps}^\dagger A_{ps} + 2p^\mu B_{ps}^\dagger B_{ps})$$

From $\mathcal{H}\{A, B\}$ one obtains a Hilbert space fulfilling the spectrum condition with a

proper domain D for the fields. The interplay of the Lorentz degeneracy of the vacuum and the spectrum condition implies that the model describes a theory with two boson excitations of masses $2\sqrt{\alpha^2}$ and $2m$.

There is a result by Borchers^[43] which is sometimes quoted as meaning that translational invariant zero energy states are also invariant under the whole inhomogeneous Lorentz group, i. e. all vacua would necessarily transform under the trivial representation of $SL(2, C)$. Actually what is proved (theorem 3 in Ref. 43) is that if one of the vacua is Lorentz invariant, then all others that are connected to this one by non-vanishing Wightman functions are also Lorentz invariant. This is not the case here because none of the generalized states $|\alpha\rangle$ is Lorentz invariant.

Let us nevertheless apply the reasoning of Borchers' theorem to a Wightman function in our example, for instance to

$$S(\Lambda, x' - x) = -i \langle \Lambda f | T(\psi_{\lambda}(x') \bar{\psi}_{\lambda}(x)) | f \rangle = \int \frac{d^3 \alpha}{|\alpha|} f^*(\Lambda^{-1} \alpha) f(\alpha) \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x' - x)} \left(\frac{\gamma_{\mu}(p - \alpha)^{\mu} + m}{p^2 - 2p \cdot \alpha + \alpha^2 - m^2 + i\epsilon} \right)_{\lambda' \lambda}$$

Consider Lorentz boosts along a particular direction. As Borchers points out, it is enough to consider this case.

$$\Lambda(\rho) = \begin{bmatrix} \text{ch } \rho & \text{sh } \rho & 0 & 0 \\ \text{sh } \rho & \text{ch } \rho & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this case we would have a function $S(\rho, \xi)$, analytic in $\xi = x' - x$. However it cannot be analytic in ρ unless it vanishes identically. In fact, let us choose a $f(\alpha)$ of compact support in the hyperboloid. Then, there is always an R such that the intersection of the support of $f(\Lambda^{-1}(\rho) \alpha)$ for $\rho > R$ and $f(\alpha)$ is empty. $S(\rho, \xi)$ being equal to zero in the half line $\rho > R$, it must vanish identically if it is analytic in ρ . Hence requiring analyticity in ρ one reaches the conclusion that the non-diagonal Wightman function vanishes, but not that $|\Lambda f\rangle = |f\rangle$.

Having implicitly assumed the existence of one Lorentz invariant vacuum, the

Wightman functions considered by Borchers are not of the type $\langle \Lambda f | \dots | f \rangle$ but $S(\Lambda, \xi_1 \dots \xi_n) = \langle \Lambda f | \dots | 0 \rangle$, $|0\rangle$ being invariant for all transformations of the Lorentz group. From the invariance of $|0\rangle$ it then follows

$$S(\Lambda, \xi_1 \dots \xi_n) = S(1, \Lambda^{-1} \xi_1 \dots \Lambda^{-1} \xi_n) \quad (\text{D.16})$$

and analyticity in $\xi_1 \dots \xi_n$ requires analyticity in $\Lambda^{-1} \xi_1 \dots \Lambda^{-1} \xi_n$, i. e. analyticity in the direct product of the complex Lorentz group and the extended primitive domain. It is then easy to see that $S(\Lambda, \xi_1 \dots \xi_n)$ must be independent of Λ implying $|\Lambda f\rangle = |f\rangle$.

In our example independence of Λ is also trivially verified but it does not require $|\Lambda f\rangle = |f\rangle$ because the fact that g commutes with the b and d operators implies that $\langle \Lambda f | \dots | f \rangle$ vanishes for $\Lambda \neq 1$. In general however, in a theory with degenerate Lorentz vacuum, one might want to have some of the functions of type $\langle \Lambda f | \dots | f \rangle$ for $\Lambda \neq 1$ different from zero. As long as none of the $|f\rangle$'s is invariant for the whole Lorentz group, Eq.(D.16) does not hold and therefore, not being forced to require analyticity in Λ , one is not led into contradiction with Borchers' result.

As a final remark let me point out that a construction similar to Borchers' algebra

$$\mathfrak{B} = C \oplus \mathfrak{F}(R^*) \oplus \mathfrak{F}(R^0) \oplus \dots$$

may also be carried out for degenerate Lorentz vacuum theories.

Let the vacuum subspace V_0 be a space $\mathfrak{T}(\alpha)$ of square integrable functions on a homogeneous space of the Lorentz group and let $\alpha = \{\alpha_i\}$ be a coordinate frame in the homogeneous space. Then one defines the vector space

$$\mathcal{A} = \overline{\mathfrak{T}}^*(\alpha) \otimes \mathfrak{B} \otimes \overline{\mathfrak{T}}(\alpha)$$

where $\overline{\mathfrak{T}}(\alpha)$ is the space $\mathfrak{T}(\alpha)$ completed by the constant functions. The elements $1 \otimes \{1\} \otimes f(\alpha)$ may be identified with creation operators of the states in V_0 . In this way one formulates the theory in a way similar to the invariant vacuum case. The main difference is the replacement of the algebra $\mathfrak{B}$ by the vector space $\mathcal{A}$.

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Figure caption

Fig. 1 - The function $\sigma(0, a_2, a_3)$, Eq. (4.1)

FIG. 1