Chern-Simons Theory and the Quark-Gluon Plasma

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Abstract

The generating functional for hard thermal loops in Quantum Chromodynamics is important in setting up a resummed thermal perturbation theory, so that all terms of a given order in the coupling constant can be consistently taken into account. It is also the functional which leads to a gauge invariant description of Debye screening and plasma waves in the quark-gluon plasma. We have recently shown that this functional is closely related to the eikonal for a Chern-Simons gauge theory. In this paper, this relationship is explored and explained in more detail along with some generalizations.

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1. Introduction

We have recently shown that the generating functional for hard thermal loops in Quantum Chromodynamics (QCD) is very closely related to the Chern-Simons (CS) gauge theory\(^1\). In this paper, we shall explore in further detail this remarkable connection between QCD at finite temperature and the CS theory.

Conceptually, the question we are considering is very simple. We consider QCD at temperatures well into the deconfinement phase. Thus we have a hot plasma of quarks and gluons. As is well known, one expects to have Debye screening in a plasma. If we consider an Abelian plasma of positive and negative charges \(\pm e\), viz. electrodynamics, screening can be understood using the classic argument of Debye. One considers the Poisson equation for the electrostatic potential of a test charge, say positive, in the plasma.

\[
-\nabla^2 A_0 = ne \left( \frac{e^{eA_0/T} - e^{-eA_0/T}}{e^{eA_0/T} + e^{-eA_0/T}} \right) \\
\approx \left( \frac{ne^2}{T} \right)
\]

(1a)

(1b)

where the right hand side is the charge density in the vicinity of the test charge. \(n\) is the average number density of particles. The exponentials are the Boltzmann factors giving the preferential accumulation of negative and depletion of positive charges in the vicinity due to the Coulomb forces, with proper normalization. The approximation (1b), which is valid for high temperatures, shows that the solutions have the screened Coulomb form \(\frac{1}{r} \exp(-m_D r)\) with a Debye screening mass \(m_D^2 = (ne^2/T)\). For a relativistic plasma, the qualitative features of this argument are valid and with \(n \sim T^3\), we expect \(m_D^2 \sim e^2 T^2\). And by calculating the photon propagator in thermal electrodynamics, one can indeed obtain this result\(^2\).

The above argument is presented in terms of potentials and as such, it does not seem to be gauge invariant. For the Abelian case, one can easily reformulate the arguments using only the gauge invariant electric and magnetic fields. However, in a non-Abelian plasma, such as the quark-gluon plasma, it is difficult to avoid the use of gauge potentials altogether. Further, since even the notion of the charge of a gluon has to be defined with respect to some chosen Abelian direction of the gauge group, it is clear that the simple
argument of equation (1) will have to be modified. The question of interest to us is: how do we obtain a gauge invariant description of Debye screening in a non-Abelian plasma (in terms of gauge potentials)? More specifically, we need a functional of the gauge potentials, \( \Gamma[A] \), which is generated by the statistical distributions and is effectively a gauge invariant mass term for the gauge fields. Screening, of course, is the static part of a more general problem, the dynamical part of which is the propagation of plasma waves. Such a \( \Gamma[A] \) will thus be important for the discussion of plasma waves as well.

Hard thermal loops are closely related to the above. They arise because of the need to carry out a partial resummation of perturbation theory in thermal QCD. The need for resummation is easily seen using the following argument due to Pisarski \(^3\). Consider the elementary polarization diagram of gluons, fig.(1a) and its first correction fig.(1b). Before the final loop integration over \( p \), the ratio of diagram (1b) to (1a) is \( \Pi(p)/p^2 \). Because of Debye screening, the polarization tensor \( \Pi(p) \) has a small \( p \)-expansion, \( \Pi(p) \sim g^2T^2 + g^2|p| + ... \) (\( g \) is the quark-gluon or gluon-gluon coupling constant.) We thus see that for the small \( p \)-regime of integration, i.e. \( p \leq g^2T^2 \), the naively higher order diagram (1b) is comparable to the lower order term (1a). Therefore, to be consistent to a given order in \( g \), one must sum over (1a), (1b), and a series of further insertions of \( \Pi(p) \). This resummation leads to new effective propagators, and also new effective vertices in general. The resummation can be summarized by saying that the new propagators and vertices arise from an action

\[
S = \int -\frac{1}{4}F^2 + \Gamma[A]
\]

where we add a term \( \Gamma[A] \) to the standard Yang-Mills action for gluons.

Going back to figures (1), we see that the regime of interest is when the momentum external to \( \Pi(p) \), viz. \( p \) is of the order \( gT \) or less. This holds in general and the momenta carried by the potentials in \( \Gamma[A] \) can be taken to be \( \lesssim gT \). General power counting arguments by Pisarski and Braaten show that \( \Gamma[A] \) is the sum of ‘hard thermal loops’ \(^4\). These are one-loop diagrams where the external momenta are \( \lesssim gT \) and the loop momentum is of the order of \( T \), i.e. relatively hard. An explicit form of \( \Gamma[A] \), the generating functional of these hard thermal loops, is necessary for the proper set up of thermal perturbation theory. Although of seemingly different origin, \( \Gamma[A] \) so defined will be the same as the functional
describing Debye screening and plasma waves. We shall see that it is also closely related to the Chern-Simons gauge theory.

The Chern-Simons (CS) action, it may be recalled, made its appearance in physics literature over a decade ago as a mass term for gauge fields in three dimensions. Studies since then have revealed a number of fascinating features of this action. The Abelian version can be used for spin transmutation, converting spin-zero bosons into anyons. The correlators of Wilson lines in a pure CS theory are related to the polynomial invariants of knot theory. Pure CS theory is also related to conformal field theory and Wess-Zumino-Witten (WZW) models in two dimensions. Chern-Simons-Higgs theories have an intriguing set of vortex solutions. Finally actions related to the CS action can be used for a Lagrangian description of selfdual gauge fields and integrable systems. Our observation gives another context, viz. the very physical context of the quark-gluon plasma, for the understanding of which the CS theory is very useful.

This paper is organized as follows. We begin with the explicit evaluation of some hard thermal loops. The two- and three-point functions are evaluated. These calculations have some overlap with the similar calculations of Frenkel and Taylor. However we organize and orient the calculations towards a more efficient realization of the CS connection. In section 3, generalizations to \(n\)-point functions using the gauge invariance of \(\Gamma[A]\) are considered. The strategy of using gauge invariance in this way is, to some extent, parallel to the work of Taylor and Wong. In section 4, we discuss the pure CS theory highlighting the eikonal and other features of relevance to the plasma problem. In section 5, we obtain \(\Gamma[A]\) in terms of the eikonal of the CS theory. In fact, equation (86) is the central result of our analysis. Plasma waves are briefly discussed in section 6; non-Abelian plasmons, we argue, must be understood as the propagating solutions of the effective action (88).

2. Evaluation of Hard Thermal Loops

We begin with the explicit evaluation of some of the hard thermal loops. We shall consider one-loop quark graphs with two and three external gluon lines; analysis of these diagrams will suffice to abstract many of the features that generalize to the \(n\)-point func-
tions. As mentioned in the introduction, there are many related but different formalisms for dealing with thermal corrections in a field theory. We shall use Minkowski space propagators with thermal averages for products of creation and annihilation operators. The calculation is conceptually the simplest in this formalism. The relevant part of the Lagrangian for the quark fields $q, \bar{q}$ is

$$\mathcal{L} = \bar{q} i\gamma \cdot (\partial + A) q$$

(3)

where $A_\mu = -i t^a A^a_\mu$ is the Lie algebra valued gluon vector potential, $t^a$ are hermitian matrices corresponding to the generators of the Lie algebra in the representation to which the quarks belong. We shall not explicitly display the quark-gluon coupling constant $g$, as it is easily recovered at any stage by $A \to g A$. The one-loop quark graphs are given by the effective action

$$\Gamma = -i \text{Tr} \log (1 + S \gamma \cdot A)$$

(4)

where

$$S(x, y) = \langle T q(x)\bar{q}(y) \rangle$$

(5)

is the quark propagator. Tr in (4) includes the functional trace as usual. The two-gluon and three-gluon terms in $\Gamma$ are given by

$$\Gamma^{(2)} = \frac{i}{2} \int d^4x d^4y \text{Tr} [\gamma \cdot A(x)S(x, y)\gamma \cdot A(y)S(y, x)]$$

(6a)

$$\Gamma^{(3)} = -\frac{i}{3} \int d^4x d^4yd^4z \text{Tr} [\gamma \cdot A(x)S(x, y)\gamma \cdot A(y)S(y, z)\gamma \cdot A(z)S(z, x)]$$

(6b)

where Tr denotes the trace over the color matrices $t^a$ and the Dirac matrices. The quark propagator is given by

$$S(x, y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \{ \Theta(x^0 - y^0)[\alpha_p e^{-ip \cdot (x-y)}\gamma \cdot p + \bar{\beta}_p e^{ip' \cdot (x-y)}\gamma \cdot p'] -$$

$$\Theta(y^0 - x^0)[\beta_p e^{-ip \cdot (x-y)}\gamma \cdot p + \bar{\alpha}_p e^{ip' \cdot (x-y)}\gamma \cdot p'] \}$$

(7)
where \( p^0 = |\vec{p}| \), \( p = (p^0, |\vec{p}|) \), \( p' = (p^0, -|\vec{p}|) \), and \( \Theta(x) \), of course, is the step function. Also

\[
\alpha_p = 1 - n_p, \quad \beta_p = n_p. \tag{8}
\]

The distribution functions \( n_p, \bar{n}_p \) are defined by the thermal averages

\[
\langle a_p^{\dagger \alpha r} a_p^{\beta s} \rangle = n_p \delta^{rs} \delta^{\alpha \beta}
\]

\[
\langle b_p^{\dagger \alpha r} b_p^{\beta s} \rangle = \bar{n}_p \delta^{rs} \delta^{\alpha \beta} \tag{9}
\]

where \( (a_p^{\dagger \alpha r}, a_p^{\alpha r}), (b_p^{\dagger \alpha r}, b_p^{\alpha r}) \) are the annihilation and creation operators for quarks and antiquarks respectively. \( \alpha, \beta \) are spin indices; \( r, s \) are color indices. For a plasma of zero fermion number, we can take

\[
n_p = \bar{n}_p = \frac{1}{e^{p^0/T} + 1} \tag{10}
\]

For a plasma with a nonzero value of fermion number, there is a chemical potential and correspondingly \( n_p, \bar{n}_p \) are not equal. In using the expression (7) for the propagator in (6), we encounter many terms corresponding to different ordering of the time arguments \( x^0, y^0, z^0 \). The strategy will be to carry out the time-integrations first, introducing convergence factors \( e^{\pm \epsilon x^0}, e^{\pm \epsilon y^0} \) etc., \( \epsilon \) small and positive, as required. The integrations give energy denominators and bring the result to a form where simplification appropriate to a hard thermal loop, viz. the loop momentum being hard (\( \sim T \)) and the external momenta being relatively soft (\( \sim gT \)), can be implemented easily.

**Simplification of the two-point function**

In using (7) for the quark propagator, we find four terms in \( \Gamma^{(2)} \) with \( x^0 > y^0 \) and four terms with \( y^0 > x^0 \). Writing

\[
A_\mu(x) = \int \frac{d^3k}{(2\pi)^4} e^{ikx} A_\mu(k) \tag{11}
\]

and carrying out the time-integrations we get

\[
\Gamma^{(2)} = -\frac{1}{2} \int d\mu(k) \int \frac{d^3q}{(2\pi)^3} \frac{1}{2p^0} \frac{1}{2q^0} \left[ T(p, q) \left( \frac{\alpha_p \beta_q}{p^0 - q^0 - k^0 + i\epsilon} - \frac{\alpha_q \beta_p}{p^0 - q^0 - k^0 + i\epsilon} \right) \right] +
\]

6
\[
T(p, q) = \text{Tr}[\gamma \cdot A(k)\gamma \cdot p \gamma \cdot A(k')\gamma \cdot q]
\]
where

\[
T(p, q) = \text{Tr}[\gamma \cdot A(k)\gamma \cdot p \gamma \cdot A(k')\gamma \cdot q]
\]

and

\[
d\mu(k) = (2\pi)^4 \delta^{(4)}(k + k') \frac{d^4k \cdot d^4k'}{(2\pi)^4}
\]

In (12), \(\bar{p} = \bar{q} + \bar{k}\). Since \(p^0 = |\vec{q} + \vec{k}| \approx q^0 + \vec{q} \cdot \vec{k}/q^0\) for \(|\vec{k}|\) small compared to \(|\vec{q}|\), the denominators in (12) involve \(k \cdot Q, k \cdot Q'\) and \(2q^0 + k \cdot Q, 2q^0 + k \cdot Q'\) where

\[
Q = (1, \frac{\vec{q}}{q^0}), \quad Q' = (1, -\frac{\vec{q}}{q^0})
\]

The \(i\epsilon\)’s in the denominators in (12) can be let go to zero at this stage. Of course, this requires that the kinematics be so chosen that \(k \cdot Q, k \cdot Q'\) and \(q^0\) are not zero. Making this choice and using \(\alpha, \beta\) from (8), we find, for the temperature-dependent part of \(\Gamma^{(2)}\),

\[
\Gamma^{(2)} = -\frac{1}{2} \int d\mu(k) \int \frac{d^3q}{(2\pi)^3} \frac{1}{2p^0 2q^0} \left[ (n_q - n_p) \frac{T(p, q)}{p^0 - q^0 - k^0} + (\bar{n}_q - \bar{n}_p) \frac{T(p, q)}{p^0 - q^0 + k^0} - (n_p + \bar{n}_q) \frac{T(p', q')}{p^0 + q^0 - k^0} - (\bar{n}_p + n_q) \frac{T(p', q')}{p^0 + q^0 + k^0} \right].
\]

We see that the result is linear in the distribution functions, a property that is expected to be true in general. Equation (16) agrees with Frenkel and Taylor. Some of the further simplification of (16) can be made along the lines of their calculation. For \(|\vec{k}|\) small compared to the loop momentum \(|\vec{q}|\), we have

\[
p^0 - q^0 - k^0 \approx -k \cdot Q, \quad p^0 - q^0 + k^0 \approx k \cdot Q'
\]
\[ p^0 + q^0 \pm k^0 \simeq 2q^0 \]  

(17)

\[ T(p, q) \simeq 8q^0 \text{tr}(A_1 \cdot QA_2 \cdot Q) \]

\[ T(p', q') \simeq 8q^0 \text{tr}(A_1 \cdot Q'A_2 \cdot Q') \]  

(18)

\[ T(p', q) \simeq T(p, q') \simeq 4q^0 \text{tr}(A_1 \cdot QA_2 \cdot Q + A_1 \cdot QA_2 \cdot Q' - 2A_1 \cdot A_2) \]

where \( A_1 = A(k), \ A_2 = A(k') \) and the remaining trace in the expressions for \( T \)'s, denoted by \( \text{tr} \), is over color indices. The difference of distribution functions can also be approximated as \( n_p - n_q \simeq \frac{dn}{dq} \bar{Q} \cdot \bar{k} \).

Using these results (16) simplifies to

\[ \Gamma^{(2)} = -\frac{1}{2} \int d\mu(k) \int \frac{d^3q}{(2\pi)^3} \text{tr} \left[ \left( \frac{dn}{dq} \frac{A_1 \cdot QA_2 \cdot Q}{k \cdot Q} - \frac{d\bar{n}}{dq} \frac{A_1 \cdot Q'A_2 \cdot Q'}{k \cdot Q'} \right) 2\bar{Q} \cdot \bar{k} \right. \]

\[ \left. - \frac{n + \bar{n}}{q^0} (A_1 \cdot QA_2 \cdot Q + A_1 \cdot QA_2 \cdot Q' - 2A_1 \cdot A_2) \right] \]  

(19)

We have the result

\[ \int d^3q \frac{dn}{dq} f(Q) = -\int d^3q \frac{2n}{q^0} f(Q) \]  

(20)

for any function \( f \) of \( Q \), or \( Q' \). We can further use \( 2\bar{Q} \cdot \bar{k} = k \cdot Q' - k \cdot Q \). Expression (19) then simplifies to

\[ \Gamma^{(2)} = -\frac{1}{2} \int d\mu(k) \int \frac{d^3q}{(2\pi)^3} \text{tr} \left[ \frac{n + \bar{n}}{q^0} (2A_1 \cdot QA_2 \cdot Q - A_1 \cdot QA_2 \cdot Q' - A_1 \cdot Q'A_2 \cdot Q + 2A_1 \cdot A_2) \right. \]

\[ \left. - \frac{n}{q^0} 2A_1 \cdot QA_2 \cdot Q \frac{k \cdot Q'}{k \cdot Q} - \frac{\bar{n}}{q^0} 2A_1 \cdot Q'A_2 \cdot Q' \frac{k \cdot Q}{k \cdot Q'} \right] . \]  

(21)

The angular integration in (21) over the directions of \( \vec{q} \) (or \( \vec{Q} \)) help simplify it further by virtue of

\[ \int d\Omega (2A_1 \cdot QA_2 \cdot Q - A_1 \cdot QA_2 \cdot Q' - A_1 \cdot Q'A_2 \cdot Q + 2A_1 \cdot A_2) = \int d\Omega (2A_1 \cdot QA_2 \cdot Q') \]  

(22)
Defining

\[ A_+ = \frac{A \cdot Q}{2}, \quad A_- = \frac{A \cdot Q'}{2} \]  \hspace{1cm} (23)

we can write (21) as

\[ \Gamma^{(2)} = -\frac{1}{2} \int d\mu(k) \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} 16 \text{tr} \left[ A_1 + A_2 - (n + \bar{n}) - \frac{n}{k \cdot Q} A_1 + A_2 - \frac{\bar{n}}{k \cdot Q'} A_1 - A_2 - \right] \]  \hspace{1cm} (24)

This expression becomes more transparent when written in coordinate space and finally using a Wick rotation to Euclidean space. We define the Green’s functions

\[ G(x_1, x_2) = \int \frac{e^{-i p \cdot (x_1 - x_2)}}{p \cdot Q} \frac{d^4 p}{(2\pi)^4} \] \hspace{1cm} (25)

\[ G'(x_1, x_2) = \int \frac{e^{-i p \cdot (x_1 - x_2)}}{p \cdot Q'} \frac{d^4 p}{(2\pi)^4} \]

In terms of the null vectors \( Q = (1, \vec{q}/q^0) \), \( Q' = (1, -\vec{q}/q^0) \), we can introduce the lightcone coordinates \((u, v, x^T)\) as

\[ u = \frac{Q' \cdot x}{2}, \quad v = \frac{Q \cdot x}{2}, \quad \vec{Q} \cdot \vec{x}^T = 0 \] \hspace{1cm} (26)

where \( \vec{Q} = \vec{q}/q^0 \). We then have \( Q \cdot \partial = \partial_u \), \( Q' \cdot \partial = \partial_v \). We shall also introduce a Wick rotation to Euclidean coordinates by the correspondence

\[ 2v \leftrightarrow z, \quad 2u \leftrightarrow \bar{z} \]

\[ \partial_u = Q \cdot \partial \leftrightarrow 2\partial_z, \quad \partial_v = Q' \cdot \partial \leftrightarrow 2\partial_{\bar{z}} \] \hspace{1cm} (27)

The Green’s functions in (25) are the continuations of the Euclidean functions

\[ G_E(x_1, x_2) = \frac{1}{2\pi i} \frac{\delta^{(2)}(x_1^T - x_2^T)}{z_1 - z_2} \]

\[ G'_E(x_1, x_2) = \frac{1}{2\pi i} \frac{\delta^{(2)}(x_1^T - x_2^T)}{\bar{z}_1 - \bar{z}_2} \] \hspace{1cm} (28)

This leads to the correspondence
\[
\begin{align*}
\frac{k \cdot Q'}{k \cdot Q} & \leftrightarrow \frac{1}{\pi} \frac{1}{\bar{z}_{12} \bar{z}_{21}} \\
\frac{k \cdot Q}{k \cdot Q'} & \leftrightarrow \frac{1}{\pi} \frac{1}{z_{12} z_{21}} \\
\frac{1}{k \cdot Q} & \leftrightarrow \frac{1}{2\pi i} \frac{1}{\bar{z}_{12}}
\end{align*}
\]
(29)

where \( z_{ij} = z_i - z_j \), etc. Equation (24) for \( \Gamma^{(2)} \) can then be written as

\[
\Gamma^{(2)} = -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} 16 \text{ tr} \left[ (n + \bar{n}) \int d^4x \ A_+(x) A_-(x) \\
- n \pi \int d^2x^T \frac{d^2z_1}{\pi} \frac{d^2z_2}{\pi} \frac{A_+(x_1) A_+(x_2)}{\bar{z}_{12} \bar{z}_{21}} - \bar{n} \pi \int d^2x^T \frac{d^2z_1}{\pi} \frac{d^2z_2}{\pi} \frac{A_-(x_1) A_-(x_2)}{\bar{z}_{12} \bar{z}_{21}} \right]
\]
(30)

We define the functional \( I(A_+) \) by the formula

\[
I(A_+) = i \sum \frac{(-1)^n}{n} \int d^2x^T \frac{d^2z_1}{\pi} \cdots \frac{d^2z_n}{\pi} \frac{\text{tr}(A_+(1) \cdots A_+(n))}{\bar{z}_{12} \bar{z}_{23} \cdots \bar{z}_{n1}}
\]

\[
= i \int d^2x^T \frac{d^2z_1}{\pi} \frac{d^2z_2}{\pi} \frac{\text{tr}(A_+(1) A_+(2))}{\bar{z}_{12} \bar{z}_{21}} + \ldots
\]
(31)

We shall show later that \( I(A_+) \) is related to the eikonal for a Chern-Simons theory. Equation (30) for \( \Gamma^{(2)} \) can finally be written as

\[
\Gamma^{(2)} = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} K^{(2)}[A_+, A_-]
\]
(32a)

where

\[
K[A_+, A_-] = -16 \left[ \frac{(n + \bar{n})}{2} \int d^4x \text{ tr} \left( A_+(x) A_-(x) \right) + n i\pi I(A_+) + \bar{n} i\pi \bar{I}(A_-) \right]
\]
(32b)

\( K^{(2)}[A_+, A_-] \) in (32a) denotes terms in \( K \) which are quadratic in \( A \). \( \bar{I} \) is obtained from \( I \) by \( z \leftrightarrow \bar{z} \).

Although we have introduced different distribution functions \( n, \bar{n} \) for quarks and antiquarks, it is only \( n + \bar{n} \) which is relevant at high temperatures. We see that, by virtue of

\[\int d\Omega \ I(A_+) = \int d\Omega \ \bar{I}(A_-),\]
we can write (32b) as
\[
K[A_+, A_-] = -16 \frac{n + \bar{n}}{2} \left[ \int d^4x \, \text{tr}(A_+(x)A_-(x)) + i\pi I(A_+) + i\pi \bar{I}(A_-) \right]. \tag{33}
\]

The integral over the magnitude of \( \vec{q} \) in (32) can be easily carried out. With zero chemical potential,

\[
\Gamma^{(2)} = \frac{-T^2}{12\pi} \int d\Omega \left[ \int d^4x \, \text{tr}(A_+ A_-) + i\pi I^{(2)}(A_+) + i\pi \bar{I}^{(2)}(A_-) \right]. \tag{34}
\]

**Simplification of the three-point function**

When expression (7) for the propagator is used in (6b) for the three-point function and the time-integrations are carried out we get a number of terms with different types of energy-denominators. The dominant terms will have differences of energies \( p^0, q^0, r^0 \) corresponding to the three propagators so that for a small \( k \), we get soft denominators. The most important terms will have differences of momenta in both the energy denominators. There are six orderings of the time labels \( x^0, y^0, z^0 \) and for each ordering we get two terms with both denominators soft; one of these terms will involve quark distribution factors \( \alpha, \beta \) and the other involves the antiquark distribution factors \( \bar{\alpha}, \bar{\beta} \). By looking at the exponentials from the propagators with maximal differences of momenta, we can easily write down these terms. As in the case of the two-point function we shall neglect the \( i\epsilon \)-factors. The result is then

\[
\Gamma^{(3)} = \frac{-i}{3} \int d\mu(k) \, \Gamma(k_1, k_2, k_3) \tag{35}
\]

\[
\Gamma(k_1, k_2, k_3) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2p^0} \frac{1}{2q^0} \frac{1}{2r^0} \left\{ T(p, q, r) \left[ \frac{\alpha_p \alpha_q \beta_r - \beta_p \beta_q \alpha_r}{(p^0 - q^0 - k^0_1)(q^0 - r^0 + k^0_3)} \right] + \frac{\beta_p \alpha_q \beta_r - \alpha_p \beta_q \alpha_r}{(p^0 - q^0 + k^0_2)(q^0 - r^0 + k^0_3)} + \frac{\beta_p \alpha_q \beta_r - \alpha_p \beta_q \alpha_r}{(p^0 - q^0 + k^0_2)(q^0 - r^0 - k^0_1)} \right\} + \frac{\bar{\alpha}_p \bar{\beta}_q \bar{\alpha}_r - \bar{\beta}_p \bar{\alpha}_q \bar{\beta}_r}{(p^0 - q^0 - k^0_1)(q^0 - r^0 + k^0_3)} + \frac{\alpha_p \bar{\beta}_q \bar{\beta}_r - \bar{\beta}_p \alpha_q \bar{\alpha}_r}{(p^0 - q^0 - k^0_1)(q^0 - r^0 + k^0_3)} \}
\tag{36}
\]
Using the expressions for $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ in terms of $n, \bar{n}$ we can further simplify these terms as

$$\Gamma(k_1, k_2, k_3) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2p^0} \frac{1}{2q^0} \frac{1}{2r^0} \left[ T(p, q, r) \left\{ \frac{n_p}{(p^0 - r^0 - k_1^0)(p^0 - q^0 + k_2^0)} + \frac{n_q}{(p^0 - q^0 + k_2^0)(r^0 - q^0 - k_3^0)} + \frac{n_r}{(p^0 - r^0 - k_1^0)(q^0 - r^0 + k_3^0)} \right\} \right]$$

$$- T(p', q', r')[\frac{\bar{n}_p}{(p^0 - r^0 + k_1^0)(p^0 - q^0 - k_2^0)} + \frac{\bar{n}_q}{(p^0 - q^0 + k_2^0)(r^0 - q^0 - k_3^0)} + \frac{\bar{n}_r}{(p^0 - r^0 + k_1^0)(q^0 - r^0 - k_3^0)}] \right\}. \tag{37}$$

In these equations, $\bar{\rho} = \bar{q} - \bar{k}_2$ and $\bar{r} = \bar{q} + \bar{k}_3$ and

$$d\mu(k) = (2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3) \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \tag{38}$$

The numerators in (36,37) are given by

$$T(p, q, r) = \text{Tr} \left[ \gamma \cdot A_1 \gamma \cdot p \gamma \cdot A_2 \gamma \cdot q \gamma \cdot A_3 \gamma \cdot r \right] \simeq 16q^0 \text{tr}(A_1 \cdot QA_2 \cdot QA_3 \cdot Q) \tag{39}$$

where $A_1 = A(k_1)$, etc.

Using the approximation as in (17) for the energy-denominators, we get

$$\Gamma(k_1, k_2, k_3) = \int \frac{d^3q}{(2\pi)^3} \left\{ -2 \text{tr}(A_1 \cdot QA_2 \cdot QA_3 \cdot Q) \left[ \frac{n_p}{k_1 \cdot Q k_2 \cdot Q} + \frac{n_q}{k_2 \cdot Q k_3 \cdot Q} + \frac{n_r}{k_3 \cdot Q k_1 \cdot Q} \right] \right\}$$

$$+ 2 \text{tr}(A_1 \cdot QA' \cdot QA' \cdot Q') \left[ \frac{\bar{n}_p}{k_1 \cdot Q k_2 \cdot Q} + \frac{\bar{n}_q}{k_2 \cdot Q k_3 \cdot Q} + \frac{\bar{n}_r}{k_3 \cdot Q k_1 \cdot Q} \right]. \tag{40}$$

Further, we can approximate the distribution functions $n_p, n_r$ as

$$n_p \simeq n_q + \frac{dn}{dq^0}(-\bar{k}_2 \cdot \bar{Q}) \simeq n_q + \frac{1}{2} \frac{dn}{dq^0}(k_2 \cdot Q - k_2 \cdot Q')$$

$$n_r \simeq n_q + \frac{dn}{dq^0}(\bar{k}_3 \cdot \bar{Q}) \simeq n_q + \frac{1}{2} \frac{dn}{dq^0}(k_3 \cdot Q' - k_3 \cdot Q) \tag{41}$$

When these expressions are used in (40), terms with no $\frac{dn}{dq^0}$ factor is zero by momentum conservation. With one partial integration over $q^0 = |\bar{q}|$, the remaining terms simplify as

$$\Gamma(k_1, k_2, k_3) = -32 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} \left\{ n \text{tr}(A_1 + A_2 + A_3) \left[ \frac{k_2 \cdot Q'}{k_1 \cdot Q k_2 \cdot Q} - \frac{k_3 \cdot Q'}{k_1 \cdot Q k_3 \cdot Q} \right] + \bar{n} \text{tr}(A_1 - A_2 - A_3) \left[ \frac{k_2 \cdot Q}{k_1 \cdot Q' k_2 \cdot Q'} - \frac{k_3 \cdot Q}{k_1 \cdot Q' k_3 \cdot Q'} \right] \right\}. \tag{42}$$
By using the correspondence rules (29) we can write (42) and \( \Gamma^{(3)} \) of (35) as

\[
\Gamma^{(3)} = -\frac{16\pi}{3} \int \frac{d^3q}{(2\pi)^3 \cdot 2q^0} \frac{1}{d^2x} \frac{d^2z_1}{\pi} \frac{d^2z_2}{\pi} \frac{d^2z_3}{\pi} \left[ \frac{n \text{tr}(A_+(x_1)A_+(x_2)A_+(x_3))}{\bar{z}_{12} \bar{z}_{23} \bar{z}_{31}} + \frac{n \text{tr}(A_-(x_1)A_-(x_2)A_-(x_3))}{\bar{z}_{12} \bar{z}_{23} \bar{z}_{31}} \right] + \frac{n \text{tr}(A_+(x_1)A_+(x_2)A_+(x_3))}{\bar{z}_{12} \bar{z}_{23} \bar{z}_{31}} \]

\[
= \int \frac{d^3q}{(2\pi)^3 \cdot 2q^0} (16) \left[ n \pi I^{(3)}(A_+) + \bar{n} \pi \bar{I}^{(3)}(A_-) \right],
\]

(43)

where \( I^{(3)} \) denotes the terms of cubic order in the \( A \)'s in the expansion (31) of \( I(A) \). Again by virtue of \( \int d\Omega I(A_+) = \int d\Omega I(A_-) \), the result depends only on \( n + \bar{n} \). We can then combine equations (32) for the two-point function and (43) for the three-point function as

\[
\Gamma = \int \frac{d^3q}{(2\pi)^3 \cdot 2q^0} K[A_+, A_-]
\]

(44)

where \( K \) is given by equation (33) and we retain terms up to the cubic order in \( A \)'s.

We shall now rewrite this result in a slightly different way, with a view to generalizations. Using the identity

\[
\int d\Omega \text{tr}(A_+A_-) = -\frac{1}{12} \left[ 2\pi A_0^a A_0^a - \int d\Omega (A_+^a A_+^a) \right]
\]

(45)

and \( \int d\Omega \bar{I}(A_-) = \int d\Omega I(A_+) \), we can write

\[
\Gamma = \frac{1}{(2\pi)^3} \int q dq \left[ \left( n + \bar{n} \right) \right] \left[ \int d^4x \ 2\pi A_0^a A_0^a - \int d\Omega d^4x (A_+^a A_+^a) - 4\pi i I(A_+) \right].
\]

(46)

Notice that \( \Gamma \), except for the \( A_0^a A_0^a \) term, depends on \( A_\mu \) only through the combination \( A_+ = A \cdot Q/2 \). (Here \( q = |\vec{q}| \).)

It is possible to interpret the various terms in equation (37) in terms of certain forward scattering amplitudes. If we use an interaction Hamiltonian \( H_{int} = \gamma \cdot A \gamma \cdot P \) and calculate the forward scattering amplitude, using standard time-dependent perturbation theory, for a quark of momentum \( p \), we get the first term in (37). The sum of this amplitude over spin and colors gives the traces. Integration over all momenta, distributed according to \( n_p \), gives the thermal loop contribution. The other terms in (37) can be similarly interpreted in
terms of permutations of the external gluons. (This interpretation generalizes to four-point and higher functions as well.)

3. Generalizations

We shall now consider some of the general features of the calculations presented so far. In generalizing to four-point and higher functions, it is easy to see which kind of terms will dominate. We need the maximal number of soft denominators. This will come from the time-orderings which give maximal number of differences of momenta in the exponential for the propagators. In an \(n\)-point function, we can get \((n-1)\) such terms. The denominators simplify to products of \(k \cdot Q\)'s (and \(k \cdot Q'\)'s for the antiquark distributions). The general result has the structure

\[
\Gamma = \frac{i^n}{n} \int \frac{d^3p_1}{(2\pi)^3} \prod_i \frac{1}{2p_i^0} \left[ \frac{T(p_1, p_2, \cdots, p_n) n_{p_1}}{k_2 \cdot Q \cdots k_{(n)} \cdot Q} + \text{cyclic} \right] d\mu(k_1, \cdots, k_n)
\]

(47)

and a similar term for the antiquark distributions. (We have renamed \(k\)'s compared to (40).) The numerator, in the kinematic regime of interest, viz. \(k\)'s small compared to \(p\), becomes \(2^{n+1} p_0^n A_1 \cdot QA_2 \cdot Q \cdots A_n \cdot Q\). We can thus expect the final result to depend on \(A\)'s only through \(A \cdot Q\) for quarks (and \(A \cdot Q'\) for the antiquarks). Since \(\int d\Omega f(A \cdot Q) = \int d\Omega f(A \cdot Q')\), in fact we can take all higher terms to be a function of \(A \cdot Q\). Further, since

\[
\sum_{\text{cyclic}} \frac{1}{k_2 \cdot Q k_3 \cdot Q \cdots k_n \cdot Q} = 0
\]

(48)

by conservation of energy and momentum, the nonzero contributions involve \(\frac{dn}{d\mu}\) and hence the quantity is proportional to \(\int \frac{d^3p}{(2\pi)^3} \frac{n_{p}}{2p^0} \) or \(T^2\). We may then write the result for higher order terms in the form

\[
\Gamma_{\text{higher}} = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2q^0} \frac{(n + \bar{n})}{2} W(A \cdot Q).
\]

(49)

Combining this with (46), we see that we can write \(\Gamma\) as

\[
\Gamma = \frac{1}{(2\pi)^3} \int q dq \frac{(n + \bar{n})}{2} \left[ \int d^4x 2\pi A_0^a A_0^a + \int d\Omega W(A \cdot Q) \right].
\]

(50)
The key observation here is that, including all higher powers of \( A_\mu \), \( W \) depends on \( A_\mu \) only through \( A \cdot Q \).

Consider now the contribution of gluon loops. The diagram (4) will generate an \( A_0^a A_0^a \)-term, as in the case of quark loops. For the higher functions, the dominant terms will come from the three-gluon vertices. The basic vertex has the form

\[
V_{\alpha,\beta} = 2iA \cdot p \delta_{\alpha\beta} + (2iA \cdot k \delta_{\alpha\beta} + 4ik_\alpha A_\beta)
\]

\[
\simeq 2iA \cdot p \delta_{\alpha\beta}
\]

where \( p \) will be taken as the loop momenta and \( A \) denotes the external gluon of momentum \( k \). The color structure is given by \( A = -iT_{bc}^a A^a = -f_{bc}^a A^a \), where \( f_{bc}^a \) are the structure constants of the gauge group. We then see that gluon loops contribute the same way as quark loops except that the color trace is over the \( T^a \)'s which are matrices in the adjoint representation. Now, from the structure of the higher terms in \( A \cdot Q \) we see that, although we have factors like \( \text{tr}(T_{a_1} T_{a_2} \cdots T_{a_n}) \), all the matrix products can be reduced by the symmetry properties of the spacetime part to commutation rules and hence products of structure constants except for one last trace over two \( T \)'s. Thus the ratio of the gluon loops to quark loops is given by \( C_A/C_F \) where for any representation \( R \), \( \text{tr}(t^a t^b)_R = C_R \delta^{ab} \). For gauge group \( SU(N) \) with quarks in the fundamental representation, we get \( C_A/C_F = 2N \).

The fact that there are only two physical polarizations of the gluon is taken care of by assigning distribution functions only to the transverse physical polarizations. Equivalently we may subtract a ghost-loop contribution. (Naively, (51) leads to \( 2^n \text{tr} \delta_{\mu\nu} = 2^n \cdot 4 \) for an \( n \)-th order term. Actually, since there are only two physical polarizations we get \( 2^{n+1} \) as for fermions.) Further there are no separate \( n \) and \( \bar{n} \) contributions for gluons. Putting all this together we get for the gluon contribution

\[
\Gamma = 2N \frac{1}{2\pi} \int \frac{n_g}{2} dq d\Omega \left[ \int 2\pi A_0^a A_0^a d^4 + \int d\Omega \ W(A \cdot Q) \right]
\]

(52)

where \( n_g \) is the gluon distribution

\[
n_g = \frac{1}{e^{q/T} - 1}.
\]
Strictly speaking we have not proved that $W(A \cdot Q)$ for gluon loops in (52) is the same as $W(A \cdot Q)$ of the quark loops in (50). The vertex (51) only shows that the higher order contributions, for gluon loops also, depend only on $A \cdot Q$. The strategy will be to determine $W(A \cdot Q)$ by gauge invariance of the $\Gamma$. $W$ is thus determined by the gauge transformation properties of $A_0^a A_0^a$ and therefore will be the same for quark loops and for gluon loops. With zero chemical potential for the quarks we have

$$\int q dq \, \frac{(n + \bar{n})}{2} = \frac{\pi^2 T^2}{12}$$

$$\int q dq \, n_g = \frac{\pi^2 T^2}{6} \tag{54}$$

We can combine (52) and (50) for $N_F$ flavors of quarks as

$$\Gamma = (N + \frac{1}{2} N_F) \frac{T^2}{12 \pi} \left[ \int d^4 x \, 2 \pi A_0^a A_0^a + \int d \Omega \, W(A \cdot Q) \right]. \tag{55}$$

Our arguments so far have been confined to one-loop diagrams. The crucial result which can be proved by a power-counting analysis of diagrams is that only one-loop diagrams give hard thermal contributions. Thus in constructing $\Gamma$ only one-loop diagrams need be considered \(^4\). The other crucial property of $\Gamma$ which helps us to proceed further is that $\Gamma$, considered as a functional of $A_{\mu}$, is invariant under gauge transformations of $A_{\mu}$ and further that $\Gamma$ is independent of the gauge-fixing used for gluon propagators in the internal lines. This property can be established by an analysis of the Ward identities. A set of identities for the gauge dependence of the generating functional for one-particle irreducible or proper vertices can be derived using the BRST symmetry. The thermal power counting rules, applied to the diagrammatic expansion of various terms in these identities, show that the gauge dependence can affect only terms which are of the order of $T$ or less, in a high $T$-expansion. The leading term which is of order $T^2$, viz. the generator of hard thermal loops, is gauge invariant and independent of the gauge-fixing chosen for the internal lines \(^15\). We shall not repeat this analysis here but notice that since only one-loop diagrams are important for $\Gamma$, the thermal contribution is really classical and hence the BRST Ward identities can be expected to reduce to a statement of gauge invariance.

We know that, apart from the $A_0^a A_0^a$-term, $\Gamma$ can be taken to depend on $A \cdot Q$ with a final integration over the orientations of $\vec{Q}$. This result, viz. (55), holds to all orders in
$A_\mu$. Of course, explicitly, we have calculated $W(A \cdot Q)$ to cubic order in $A_\mu$. We can now use gauge invariance along the lines of reference 12, to determine $W(A \cdot Q)$. The condition for gauge invariance of $\Gamma$ is

$$\int d\Omega \delta W = 4\pi \int d^4x \dot{A}_0^a \omega^a$$

(56)

where $\dot{A}_0$ is the time-derivative of $A_0^a$ and $\omega = -it^a \omega^a$ is the parameter of the gauge transformation, i.e. $\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega]$. Equation (56) can be realized by

$$\delta W = \int d^4x A^a \cdot Q \omega^a.$$  

(57)

One can check that (57) is indeed the way gauge invariance is realized by analysis of the diagrams. It is clearly so for the two- and three-point functions from our explicit calculations. We now rewrite (57), using

$$\delta W = -\int d^4x(Q \cdot \partial \frac{\delta W}{\delta(A \cdot Q)} + [A \cdot Q, \frac{\delta W}{\delta(A \cdot Q)}])^a \omega^a$$

(58)

as

$$\frac{\partial f}{\partial u} + [A \cdot Q, f] + \frac{1}{2} \frac{\partial(A \cdot Q)}{\partial v} = 0,$$

(59)

where

$$f = \frac{\delta W}{\delta(A \cdot Q)} + \frac{1}{2} A \cdot Q.$$  

(60)

We have used the lightcone coordinates from equation (26). We now introduce the fields

$$a_z = \frac{1}{2} A \cdot Q = A_+ \quad a_{\bar{z}} = -f = -\frac{1}{2} \frac{\delta W}{\delta a_z} - a_z$$

(61)

and also make the Wick rotation, as in (27). The condition of gauge invariance, equation (59), then becomes

$$\partial_{\bar{z}} a_z - \partial_z a_{\bar{z}} + [a_{\bar{z}}, a_z] = 0.$$  

(62)

If $a_z$, $a_{\bar{z}}$ are thought of as the gauge potentials of another gauge theory, we see that equation (62) is the vanishing of the field strength or curvature $F_{z\bar{z}}$. The gauge theory whose equations of motion say that the field strengths vanish is the Chern-Simons theory. We shall therefore turn to a digression on the Chern-Simons theory, returning to (62)
and its solution in section 5. Of course, an understanding of Chern-Simons theory is not absolutely essential to solving (62). (For a solution in a nonthermal context, see ref.16.) One can simply solve (62) and regard Chern-Simons theory as an interpretation of the mathematical steps along the way. However Chern-Simons theory does illuminate many of the nice geometrical properties of the final result and is a worthwhile digression.

4. Chern-Simons Theory

The Chern-Simons theory is a gauge theory in two space (and one time) dimensions. The action is given by

$$S = \frac{k}{4 \pi} \int_{M \times [t_i, t_f]} d^3x \ e^{i\mu\alpha} \text{tr}(a_\mu \partial_\nu a_\alpha + \frac{2}{3} a_\mu a_\nu a_\alpha).$$

(63)

Here $a_\mu$ is the Lie algebra valued gauge potential, $a_\mu = -it^a a^a_\mu$. We shall consider $SU(N)$ gauge group in what follows. $k$ is a constant whose precise value we do not need to specify at this stage. We shall consider the spatial manifold to be $\mathbb{R}^2$, or $\mathbb{C}$ since we shall be using complex coordinates $z = x + iy$, $\bar{z} = x - iy$. (Actually, we have sufficient regularity conditions at spatial infinity that we may take $M$ to be the Riemann sphere.) The equations of motion for the theory are

$$F_{\mu\nu} = 0.$$  

(64)

The theory is best analyzed, for our purposes, in the gauge where $a_0$ is set to zero. In this gauge, the equations of motion (64) tell us that $a_z$, $a_{\bar{z}}$ are independent of time, but must satisfy the constraint

$$F_{zz} \equiv \partial_za_z - \partial_za_{\bar{z}} + [a_z, a_{\bar{z}}] = 0.$$  

(65)

This constraint is just the Gauss law of the CS gauge theory. It can be solved for $a_{\bar{z}}$ as a function of $a_z$, at least as a power series in $a_z$. The result is

$$a_{\bar{z}} = \sum (-1)^{n-1} \int \frac{d^2z_1}{\pi} \ldots \frac{d^2z_n}{\pi} \frac{a_z(z_1, \bar{z}_1)a_z(z_2, \bar{z}_2) \ldots a_z(z_n, \bar{z}_n)}{(\bar{z} - \bar{z}_1)(\bar{z}_1 - \bar{z}_2) \ldots (\bar{z}_n - \bar{z})}.$$  

(66)

This can be easily checked using $\partial_z(\frac{1}{z - z'}) = \pi \delta^{(2)}(z - z')$. 

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In the $a_0 = 0$ gauge, the action becomes

$$S = \frac{ik}{\pi} \int dt d^2 x \ \text{tr}(a_z \partial_0 a_z).$$

(67)

This shows that $a \bar{z}$ is essentially canonically conjugate to $a_z$. In fact in carrying out a variation of $S$, we find the surface term $\theta(t_f) - \theta(t_i)$, where

$$\theta = \frac{ik}{\pi} \int_M d^2 x \ \text{tr}(a \bar{z} \delta a_z).$$

(68)

(We assume $a \bar{z} \delta a_z$ to vanish at spatial infinity.) $\theta$ is the canonical one-form of the CS theory. (This is so by definition; the canonical one-form in any theory can be defined by $\theta$, where the surface term in the variation of the action is $\theta_f - \theta_i$, the subscripts referring to the final and initial data surfaces. This is an old result going back to the theory of canonical transformations and Hamilton-Jacobi theory. In more modern times, for example, Schwinger’s action principle is essentially based on this statement. For some recent references, see ref.17.) $\theta$ is the analogue of $p_i dx^i$ of point-particle mechanics; for an action $S = \int dt dx [\frac{m \dot{x}^2}{2} - V(x)]$, we would find $\theta = m \dot{x} i \delta x^i = p_i \delta x^i$. We can make another variation of $\theta$, antisymmetrized with respect to the variation $\delta a_z$, denoted by the wedge product sign, and write

$$\omega \equiv \delta \theta = \frac{ik}{\pi} \int_M d^2 x \ \text{tr}(\delta a \bar{z} \wedge \delta a_z)$$

$$= \frac{1}{2} \int d^2 x d^2 x' \ \omega_{ab}(x, x') \ \delta \xi^a(x) \wedge \delta \xi^b(x')$$

(69)

where

$$\omega_{ab}(x, x') = -\frac{ik}{2\pi} \begin{pmatrix} 0 & \delta(x - x') \delta^{ab} \\ -\delta(x - x') \delta^{ab} & 0 \end{pmatrix}$$

(70)

and $\delta \xi^a = (\delta a \bar{z}^a, \delta a_z^a)$. $\omega$ defined by (69,70) is called the symplectic structure and is of course the analogue of $dp_i dx^i$ of particle mechanics. The inverse of $\omega_{ab}$ gives the Poisson brackets, the commutators being $i$ times the Poisson brackets. For our case we get

$$[\xi^a(x), \xi^b(x')] = i(\omega^{-1})^{ab}(x, x')$$

or

$$[a \bar{z}^a(x), a_z^b(x)] = \frac{2\pi}{k} \delta^{ab} \delta(2)(x - x').$$

(71)
One does not have to go through the symplectic structure to arrive at (71). One could simply use the fact that, from (67) the canonical momenta are $\pi = a\bar{z}$, $\bar{\pi} = 0$. This is thus a constrained system in the Dirac sense and using the theory of constraints one can derive (71). The procedure of using $\omega_{ab}(x, x')$ is quicker.

In the expression (68) for $\theta$, $a\bar{z}$ is independent of $a z$. We can however express $a\bar{z}$ as a function of $a z$ via the constraint (65) or equivalently (66) and functionally integrate $\theta$. In other words, we define $I(a z)$ by

$$\delta I = \frac{ik}{\pi} \int d^2 x \, \text{tr}[a\bar{z}(a z) \delta a z].$$

The solution for $I$ is given by

$$I = ik \sum \frac{(-1)^n}{n} \int \frac{d^2 z_1}{\pi} \cdots \frac{d^2 z_n}{\pi} \frac{\text{tr}(a z_1, \bar{z}_1) \cdots a z_n(z_n, \bar{z}_n)}{\bar{z}_{12} \bar{z}_{23} \cdots \bar{z}_{n-1 n} \bar{z}_{n1}}.$$

The quantity $I$ has a rather simple interpretation. For one-dimensional point-particle mechanics, $\theta$, as we mentioned earlier, is given by $pdx$. $p$ is independent of $x$ to begin with, but we can express it as a function of $x$ via a constraint such as of fixed energy, e.g. $\frac{p^2}{2m} + V(x) = E$. Integral of $\theta = pdx$ then gives Hamilton’s principal function or the eikonal, familiar as the exponent for the WKB wave functions of one-dimensional quantum mechanics. We have an analogous situation with (65) expressing $a\bar{z}$ as a function of $a z$. $I$ is thus an eikonal of the Chern-Simons theory.

$I$ is in fact the Wess-Zumino-Witten action $^{18}$. We can write the gauge potential $a z$ as $a z = -\partial z U U^{-1}$ where $U$ is in general not unitary; it is an $SL(N, C)$ matrix for gauge group $SU(N)$. Notice that since $\partial z$ has an inverse by virtue of $\partial_z \frac{1}{(z-z')} = \pi \delta^{(2)}(z-z')$, such a $U$ can be constructed for any $a z$, at least as a power series in $a z$. $I$ can then be written as $I = -ik S_{WZW}(U)$ where

$$S_{WZW}(U) = \frac{1}{2\pi} \int_M d^2 x \, \text{tr}(\partial_z U \partial_{\bar{z}} U^{-1}) - \frac{i}{12\pi} \int_{M^3} d^3 x \, \epsilon^{\mu\nu\alpha} \text{tr}(U^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\alpha U).$$

As usual the second term involves an extension of $U$ into a three-dimensional space. We take $M^3 = M \times [0, 1]$ with $U(z, \bar{z}, 0) = 1$, $U(z, \bar{z}, 1) = U(z, \bar{z})$. The relationship to $I$ is easily seen by considering variations of $S_{WZW}$. Under the variation $U \rightarrow e^{\varphi} U \simeq (1 + \varphi) U$, 20
we find $\delta a_z = -D_z \varphi = - (\partial_z \varphi + [a_z, \varphi])$ and

$$\delta S_{WZW} = \frac{1}{\pi} \int d^2x \, \text{tr}(\partial_{\bar{z}} \varphi a_z).$$  \hspace{1cm} (75)$$

Partial integrating and using $F_{\bar{z}z} = 0$ and $\delta a_z = -D_z \varphi$, we find that $S_{WZW}$ obeys (72) except for a factor $(-ik)$, thus identifying $I = -ik S_{WZW}$.

Another quantity of geometrical significance for the CS theory is the Kähler potential. The phase space of the CS theory, viz. the function-space of the potentials $a_z, a_{\bar{z}}$ is actually a Kähler manifold, i.e. it is a complex manifold with a metric or distance function defined in terms of a potential. Specifically,

$$\parallel \delta a \parallel^2 = \frac{k}{2\pi} \int_M d^2x \, \delta a_z^a \delta a_{\bar{z}}^a = \int d^2x d^2x' \left[ \frac{\delta}{\delta a_z^a(x)} \frac{\delta}{\delta a_{\bar{z}}^a(x')} K \right] \delta a_z^a(x) \delta a_{\bar{z}}^a(x').$$  \hspace{1cm} (76a)$$

The potential functional $K$ for the metric is the Kähler form. Alternatively we may define $K$ by

$$\omega = -i \int d^2x d^2x' \left[ \frac{\delta}{\delta a_z^a(x)} \frac{\delta}{\delta a_{\bar{z}}^a(x')} K \right] \delta a_z^a(x) \wedge \delta a_{\bar{z}}^a(x').$$  \hspace{1cm} (76b)$$

We find that

$$K = \frac{k}{2\pi} \int d^2x \, a_z^a a_{\bar{z}}^a + h(a_z) + \overline{h(a_{\bar{z}})}$$  \hspace{1cm} (77)$$

where $h$ is an arbitrary function of the argument indicated. In general, there is nothing to dictate any preferred choice for $h$; but for the CS theory, because there is the additional structure of gauge transformations, we can choose $h$ so as to make $K$ gauge invariant. $h$ is then proportional to the eikonal $I$ and the gauge invariant Kähler potential is

$$K = -\frac{1}{\pi} \left[ k \int d^2x \, \text{tr}(a_z a_{\bar{z}}) + i\pi I(a_z) + i\pi \tilde{I}(a_{\bar{z}}) \right].$$  \hspace{1cm} (78)$$

Finally notice that the eikonal $I$ of (72) may be regarded as the expansion in powers of $a_z$ of the logarithm of the functional determinant of $D_z = \partial_z + a_z$; i.e. $I = (-ik) \log \det D_z = -ik \, \text{Tr} \log D_z$. The Kähler potential $K$ of (78) is then given by $-k \text{Tr} \log(D_z D_{\bar{z}})$. The expansion of this expression in powers of the potential obviously gives $I$ and $\tilde{I}$. The extra term $\int \frac{1}{\pi} \text{tr}(a_z a_{\bar{z}})$ is precisely the local counterterm needed to give a gauge invariantly regulated meaning to $\text{Tr} \log(D_z D_{\bar{z}})$.  \hspace{1cm} 19
5. The Action for the Hot Quark-Gluon Plasma

We now return to equations (61,62) for the quark-gluon plasma. We can rewrite (61) as

\[ \delta W = 4 \int d^4 x \, \text{tr}(a \bar{a} \delta a) - \int d^4 x \, a^a \bar{a}^a. \]  

(79)

Comparing with (72), we see that, since \( a \bar{a}, a \) obey the constraint (62), the solution is related to the eikonal \( I \). The difference here is that \( a \bar{a}, a \) depend on all four coordinates \( x_\mu \), not just \( z, \bar{z} \). However, there are no derivatives with respect to the transverse coordinates \( x^T \) in (62) and hence the solution for \( a \bar{a} \) in terms of \( a \) is the same as in (66), with \( a \) depending on \( x^T \) in addition to \( z, \bar{z} \). The argument of all \( a \) factors is the same for \( x^T \), i.e.

\[ a \bar{a} = \sum (-1)^{n-1} \int \frac{d^2 z_1}{\pi} \ldots \frac{d^2 z_n}{\pi} \frac{a(z_1, \bar{z}_1, x^T) a(z_2, \bar{z}_2, x^T) \ldots a(z_n, \bar{z}_n, x^T)}{(\bar{z} - \bar{z}_1) \bar{z}_1 \ldots \bar{z}_n - \bar{z}). \]  

(80)

For the eikonal we get the same expression as (73) but with integration over the transverse coordinates, i.e.

\[ I = i k \sum \frac{(-1)^n}{n} \int d^2 x^T \frac{d^2 z_1}{\pi} \ldots \frac{d^2 z_n}{\pi} \frac{\text{tr}(a(x_1) \ldots a(x_n))}{\bar{z}_1 \bar{z}_2 \ldots \bar{z}_n}. \]  

(81)

Since it is not relevant to the present discussion, we have, for the moment, set \( k = 1 \). We then find the solution to (79) as

\[ W = -4 \pi i I(a) - \int a^a \bar{a}^a d^4 x. \]  

(82)

From (55), \( \Gamma \) is given by

\[ \Gamma = (N + \frac{1}{2} N_F) \frac{T^2}{12 \pi} \left[ \int d^4 x \, 2 \pi A_0^a A_0^a - \int d^4 x d\Omega \, A_+^a A_+^a - 4 \pi i I(A_+) \right]. \]  

(83)

We can now use identity (45) in reverse to write

\[ \Gamma = -(N + \frac{1}{2} N_F) \frac{T^2}{6 \pi} \int d\Omega \left[ \int d^4 x \, \text{tr}(A_+ A_-) + i \pi I(A_+) + i \pi \bar{I}(A_-) \right] \]  

(84a)

\[ = (N + \frac{1}{2} N_F) \frac{T^2}{6} \int d\Omega \, K(A_+, A_-) \]  

(84b)

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where \( K \) is given by (78) with the additional integration over coordinates \( \vec{x}^T \) transverse to \( \vec{Q} \). If we write \( A_+ = -\partial_\vec{z} U U^{-1} \) and \( A_- = U^{\dagger -1} \partial_\vec{z} U \), we can write (84) in terms of \( S_{WZW} \) as
\[
\Gamma = -(N + \frac{1}{2} N_F) \frac{T^2}{6} S_{WZW}(U^{\dagger} U).
\] (Again, a suitable additional integration over the transverse coordinates is understood.)

From equation (55) or (59) it may seem that our solution is ambiguous up to the addition of a purely gauge invariant term. But for hard thermal loops, the additional structure that \( W \) depends only on \( a_\vec{z} = \frac{1}{2} A \cdot Q \) tells us that the gauge invariant piece must obey \( D_\vec{z} \frac{\delta W}{\delta a_\vec{z}} = 0 \). Since \( \partial_\vec{z} \) is invertible, at least perturbatively, there is no nontrivial solution to this equation. Thus (83) or (84) is the unique solution and \( \Gamma \) so defined must indeed be the generator of hard thermal loops.

In the last section, we noted that \( K(A_+, A_-) \) can be considered as \( \text{Trlog}(D_\vec{z} D_\bar{\vec{z}}) \). Since \( D_\vec{z} \) and \( D_\bar{\vec{z}} \) are the chiral Dirac operators in two dimensions, \( \text{Trlog}(D_\vec{z} D_\bar{\vec{z}}) \) is clearly the photon mass term of the Schwinger model, for Abelian gauge fields. More generally, for non-Abelian fields as well, we can consider \( \text{Trlog}(D_\vec{z} D_\bar{\vec{z}}) \) as a gauge invariant mass term. It is perhaps fitting that the gauge invariant Debye screening mass term in four-dimensional QCD is given by suitable integrations of such a two-dimensional mass term (with, of course, the additional \( x^T \)-dependence).

We close this section with two remarks on \( \Gamma \). The CS theory (63) violates parity. We see that this parity violation disappears, as indeed it should, by integration over the orientations of \( \vec{Q} \), for the QCD case. Alternatively, expression (84a) is manifestly parity-symmetric with \( A_+ \leftrightarrow A_- \), \( Q \leftrightarrow Q' \) under parity. Secondly, instead of setting \( k = 1 \) and then having a prefactor \( (N + \frac{1}{2} N_F) \frac{T^2}{6} \) in equation (84), we could simply choose \( k = (N + \frac{1}{2} N_F) \frac{T^2}{6} \). For WZW actions \( S_{WZW}(U) \) for which \( U \) has a non-Abelian unitary part, any action we use must be an integer times \( S_{WZW} \). This has to do with the fact that there are homotopically nontrivial maps from the three-sphere \( S^3 \) into the space of matrices \( U \), characterized by the fact that the third homotopy group \( \Pi_3 \) of the space of matrices \( U \) is the set of integers \(^{\text{18}}\). In our case, \( \Gamma \) involves \( S_{WZW}(U^{\dagger} U) \) or \( S_{WZW}(H) \) where \( H \) is hermitian. The space of hermitian matrices has trivial \( \Pi_3 \) and so, as expected, there is no argument for quantization of the coefficient, which is \( (N + \frac{1}{2} N_F) \frac{T^2}{6} \) for us.
We have given the coefficient of $K$ in (84) for a plasma with zero chemical potential. For the more general case, we can write $\Gamma$ as

$$\Gamma = \int \frac{1}{2q} \frac{d^3q}{(2\pi)^3} [Nn_g + \sum_{i=1}^{N_c} (\frac{n_i + \bar{n}_i}{2})] \left[ \int d^4x \text{tr}(A_+A_-) + i\pi I(A_+) + i\pi \bar{I}(A_-) \right].$$

(86)

6. Plasma Waves and Debye Screening

We have obtained the effective action $\Gamma$ which is the generator of hard thermal loops. As is standard in other contexts of resummations of perturbation theory, we can develop the resummed perturbative expansion of thermal QCD as follows. We introduce a splitting of the Yang-Mills action as

$$S = S_0 - c\Gamma$$

$$S_0 = \int -\frac{1}{4} F^2 + \Gamma$$

(87)

We define propagators and vertices and start off the perturbative expansion using $S_0$. $c\Gamma$ will be treated as a ‘counterterm’, nominally one order higher in the thermal loops than $S_0$. Eventually of course $c$ is taken to be 1, so that we are only achieving a rearrangement of terms in the perturbative expansion. (As usual we must have gauge fixing and ghost terms. We have also not displayed the quark terms; for the quark terms, see ref.20.) Using this procedure one can calculate quantities which require resummations such as the gluon decay rate in the plasma.

$\Gamma$ also gives us an effective action for low energy gluon fields

$$S_{eff} = \int -\frac{1}{4} F^2 + \Gamma.$$ 

(88)

(Although $S_{eff}$ looks like $S_0$ in (87), the interpretation is different. $S = S_0 - c\Gamma$ in (87) generates the thermal perturbation theory. Momenta for fields in $S_0$, for example, can be very high. Calculations starting with (87) lead to a low energy effective action (88); the latter is, of course, useful only for small momenta.)

Long wavelength and low frequency plasma waves are the classical solutions of the effective theory (88). It also includes effects such as the screening of Coulomb fields.
These features can be seen by examining the Abelian case or electrodynamics. In this case, the terms in $\Gamma$ which are cubic or higher order in $A_\mu$ are zero and in terms of the Fourier components of $A_\mu$, we can write

$$S_{\text{eff}} = \frac{1}{2} \int A_\mu(-k)M^{\mu\nu}(k)A_\nu(k)\frac{d^4k}{(2\pi)^4}$$  \hspace{1cm} (89a)$$

where

$$M^{\mu\nu} = (-k^2g^{\mu\nu} + k^\mu k^\nu) + \frac{NF\varepsilon^2T^2}{12\pi} \left[ 4\pi\delta^{\mu0}\delta^{\nu0} - \int d\Omega \frac{k_0}{k \cdot Q} Q^\mu Q^\nu \right].$$  \hspace{1cm} (89b)$$

$k^\mu M_{\mu\nu} = 0$ in accordance with the requirement of gauge invariance. We have restored the coupling constant $e$ at this stage. We can split $A_\mu$ into a gauge dependent part and gauge invariant components as

$$A_\mu = k_\mu \Lambda(k) + \alpha_\mu + \beta_\mu$$  \hspace{1cm} (90)$$

where $\Lambda$ shifts under gauge transformations and $\alpha_\mu$, $\beta_\mu$ are gauge invariant. We take

$$\alpha_0 = \left( \frac{\vec{k}^2}{k^2 - \vec{k}_0^2} \right) \phi \hspace{1cm} \alpha_i = \frac{k_0}{\sqrt{k^2}} e^{(3)}_i \left( \frac{\vec{k}^2}{k^2 - \vec{k}_0^2} \right) \phi$$

$$\beta_0 = 0 \hspace{1cm} \beta_i = e^{(\lambda)}_i a_\lambda, \hspace{1cm} \lambda = 1, 2.$$  \hspace{1cm} (91)$$

(Here we are considering fields off-shell and so $k^0 \neq \sqrt{\frac{\vec{k}}{T}}$.) The $e_i$’s form a triad of spatial unit vectors which may be taken as

$$e^{(3)}_i = \frac{k_i}{\sqrt{k^2}}, \hspace{1cm} i = 1, 2, 3,$$

$$e^{(1)} = (\epsilon_{ij} \frac{k_j}{\sqrt{k^2_T}}, 0), \hspace{1cm} e^{(2)} = (\frac{k_3 k_i}{\sqrt{k^2_T k^2}}, -\frac{\sqrt{k^2_T}}{k^2}, \hspace{1cm} i = 1, 2, \hspace{1cm} (92)$$

where $k^2_T = k^2_1 + k^2_2$. Notice that $k_i e^{(\lambda)}_i = 0$, $\lambda = 1, 2$. $\phi$ and $a_\lambda$ are the gauge invariant degrees of freedom in (90). When the mode decomposition (90) is used in (89) we get

$$S_{\text{eff}} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[ \beta_j(-k)(\frac{\vec{k}^2 \delta_{ij} - k_i k_j}{k^2})M^T(k)\beta_j(k) + \phi(-k)M^L(k)\phi(k) \right] + \int \phi J^0 + \beta_i J^i$$  \hspace{1cm} (93)$$
where

\[ M^T(k) = k_0^2 - \vec{k}^2 - \frac{N_F e^2 T^2}{6} \left[ \frac{k_0^2}{k^2} + \left( 1 - \frac{k_0^2}{k^2} \right) \frac{1}{2|\vec{k}|L} \right] \]

\[ M^L(k) = \vec{k}^2 + \frac{N_F e^2 T^2}{3} \left( 1 - \frac{1}{2|\vec{k}|L} \right) \tag{94} \]

\[ L = \log\left( \frac{k_0 + k}{k_0 - k} \right). \tag{95} \]

We have also included an interaction term with a source \( J_\mu \) in (93); i.e. we include \( \int A_\mu J^\mu \) and simplify it using (90). From (93) we see that the interaction between charges in the plasma is governed by \((M^L)^{-1}\) which shows the Debye screening with a Debye mass \( m_D = \sqrt{\frac{N_F e^2 T^2}{3}} \). The action (93) can also give free wavelike solutions. The dispersion rules for these plasma waves would be \( M^T = 0 \) for the transverse waves and \( M^L = 0 \) for the longitudinal waves \(^{21}\).

Non-Abelian plasma waves can be defined in a similar way as propagating solutions to the equations of motion given by (88). If one truncates \( \Gamma \) to the term quadratic in \( A_\mu \), these are essentially the same as the Abelian plasma waves with \( m_D^2 = (N + \frac{1}{2} N_F) \frac{g^2 T^2}{3} \) and \( (N + \frac{1}{2} N_F) \frac{g^2 T^2}{6} \) in \( M^T(k) \) rather than \( \frac{N_F e^2 T^2}{6} \). However such a truncation of \( \Gamma \) loses the full gauge invariance. In the approximation of small \( g A_\mu \), i.e. oscillations of very small amplitudes, this lack of gauge invariance may not be very serious. However, in general, one must really seek propagating solutions to the equations of motion keeping all of \( \Gamma \). These will be the genuine plasma oscillations in the non-Abelian case. We do not have any such solutions as yet, but some of them might coincide with those found by Kajantie and Montonen \(^{22}\).

References


