CM-P00057523

Archives

CERN-TH.6826/93

Monte Carlo Simulation with Fluctuating Boundary Conditions

Martin Hasenbusch

Theory Division, CERN, Geneva 23, Switzerland

Abstract

I study the finite-size behaviour of the ratio of partition functions with periodic and antiperiodic boundary conditions in the vicinity of the critical coupling of the 3D Ising model. This ratio is obtained as a simple expectation value in an ensemble where I average over both kinds of boundary conditions. Based on such simulations I obtain $\beta_c = 0.221650(5)$ and $\nu = 0.630(2)$. These results are compared with fourth-order cumulant results obtained in the same Monte Carlo simulations.

CERN-TH.6826/93
March 1993

1 Introduction

Critical exponents are characteristic parameters for universality classes of statistical systems undergoing a second-order phase transition. In three dimensions these exponents have been calculated quite accurately by approximate methods such as the analysis of high- and low-temperature series expansions [1], the field theoretic renormalization group [2], and the ϵ -expansion [3]. But it seems hard to improve the accuracy of these methods since the effort to compute higher-order coefficients of the series increases exponentially.

With increasing computer power and the invention of non-local algorithms [4, 5], Monte Carlo simulations became competitive in the determination of critical parameters.

The phenomenological renormalization group approach to critical phenomena as proposed by Binder [6] has become a standard method in Monte Carlo studies of critical systems. The dimensionless reduced fourth-order cumulant $U_L = 1 - \frac{1}{3} \frac{\langle m^4 \rangle_L}{\langle m^2 \rangle_L^2}$, where m is the magnetization of the system, converges to a non-trivial fixed point with increasing lattice size L at the critical temperature. Hence crossings of U_L , plotted against the temperature for different L , provide estimates for the critical temperature, and the slope of U_L at the critical temperature gives information on the critical exponent ν of the correlation length.

In this paper I will show that the ratio of partition functions $\frac{Z_{xx}}{Z_p}$ of a system with periodic (p) and antiperiodic (ap) boundary conditions can be employed in a similar fashion. As a consequence of scale invariance $\frac{Z_{xx}}{Z_p}$ has a non-trivial fixed point at the critical temperature. In the numerical study of the three-dimensional Ising model, which I present below, it turned out that this quantity converges even faster to its fixed point than the fourth-order cumulant U_L .

My study of $\frac{Z_{xx}}{Z_p}$ was first motivated by the investigation of the surface tension of Ising models in the low-temperature phase. There the surface tension can be defined as the thermodynamic limit of minus the logarithm of $\frac{Z_{xx}}{Z_p}$, divided by the surface area [7].

In ref. [8] I explain how $\frac{Z_{xx}}{Z_p}$ can be obtained in a single Monte Carlo simulation. The ratio $\frac{Z_{xx}}{Z_p}$ can be expressed as an expectation value in a system where it is summed over the possibility of periodic and antiperiodic boundary conditions. This system has the partition function $Z = Z_{ap} + Z_p$. It can be efficiently updated with a modified version [8] of the cluster algorithm [4, 5].

This paper is organized as follows. First I introduce the model with periodic and antiperiodic boundary conditions. Then I discuss how the ratio of the partition functions $\frac{Z_{xx}}{Z_p}$ is related to the spectrum of the transfer matrix and the implications of scale invariance. I describe the Monte Carlo algorithm for the system allowing periodic and antiperiodic boundary conditions. Finally I present my Monte Carlo simulations at T_c on lattices up

to 64^3 and exact results for lattices of size 2^3 , 3^3 and 4^3 . These results are compared with other recent Monte Carlo studies of the 3D Ising model [9, 10].

2 The Model

I consider a simple cubic lattice with extension L in the x - and y -direction and with extension t in the z -direction. The uppermost layer of the lattice is regarded as the lower neighbouring plane of the lowermost one. An analogous identification is done for the other two lattice directions. The Ising model is defined by the Hamiltonian

$$H(s, bc) = - \sum_{\langle ij \rangle} J_{\langle ij \rangle} s_i s_j, \quad (1)$$

where $\langle ij \rangle$ is a pair of nearest-neighbour lattice sites. When periodic boundary conditions are employed, then $J_{\langle ij \rangle} = 1$ for all nearest-neighbour pairs. When antiperiodic boundary conditions are employed, then $J_{\langle ij \rangle} = -1$ for bonds $\langle ij \rangle$ connecting the lowermost and uppermost plane of the lattice, while all other nearest-neighbour pairs keep $J_{\langle ij \rangle} = 1$.

In order to obtain the ratio of partition functions $\frac{Z_{xx}}{Z_p}$, I sum over both periodic and antiperiodic boundary conditions. The partition function of this system is given by

$$Z = \sum_{bc} \sum_{s_i = \pm 1} \exp(-\beta H(s, bc)), \quad (2)$$

where β is the inverse temperature. The fraction of configurations with antiperiodic boundary conditions is given by the ratio

$$\begin{aligned} \frac{Z_{ap}}{Z} &= \frac{\sum_{s_i = \pm 1} \exp(-\beta H(s, ap))}{Z} \\ &= \frac{\sum_{bc} \sum_{s_i = \pm 1} \exp(-\beta H(s, bc)) \delta_{bc, ap}}{Z} \\ &= \langle \delta_{bc, ap} \rangle. \end{aligned} \quad (3)$$

An analogous result can be found for periodic boundary conditions. Now we can express the ratio $\frac{Z_{xx}}{Z_p}$ as a ratio of observables in this system,

$$\frac{Z_{xx}}{Z_p} = \frac{\frac{Z_{xx}}{Z}}{\frac{Z_p}{Z}} = \frac{\langle \delta_{bc, x, ap} \rangle}{\langle \delta_{bc, x, p} \rangle}, \quad (4)$$

and it is hence accessible in a single Monte Carlo simulation, provided the boundary conditions can be updated efficiently.

3 The Ratio of Partition Functions and the Spectrum of the Transfer Matrix

First let us express the partition functions of the periodic and antiperiodic Ising system in terms of the transfer matrix T . The antiperiodic boundary

conditions are represented by a spin-flip operator P , which flips the sign of all spins in a given z -slice.

The partition function of the periodic system is given by

$$Z_p = \text{Tr} T^t \quad (5)$$

while the partition function of the antiperiodic system is given by

$$Z_{ap} = \text{Tr} T^t P, \quad (6)$$

where t is the length of the lattice in the z -direction. Since the operators T and P commute, they have a common set of eigenfunctions. Say the eigenvalues of T are λ_i and those of P are p_i . The possible values of p_i are 1 and -1. The partition functions take the form

$$Z_p = \sum_i \lambda_i^t \quad (7)$$

$$\text{and} \quad Z_{ap} = \sum_i \lambda_i^t p_i. \quad (8)$$

Let us consider the ratio of the partition functions at the critical coupling. We can assume that scale invariance is valid for sufficiently large lattices. This means that those correlation lengths that are sufficiently large compared with the lattice spacing scale with the size L of the lattice [11, 12]:

$$\frac{1}{-\ln(\frac{\lambda_i}{\lambda_0})} = \xi_i = c_i L, \quad (9)$$

where λ_0 is the eigenvalue of the ground state. Hence $(\frac{\lambda_i}{\lambda_0})^L$ should not depend on L at β_c . If we choose $t = L$, then the ratio of the partition functions

$$\frac{Z_{ap}}{Z_p} = \frac{\sum_i \lambda_i^L p_i}{\sum_i \lambda_i^L} = \frac{\sum_i (\frac{\lambda_i}{\lambda_0})^L p_i}{\sum_i (\frac{\lambda_i}{\lambda_0})^L} \quad (10)$$

should also not depend on L at β_c . Of course the number of eigenvalues λ_i increases with increasing lattice size L . But I assume that the new λ_i that appear with increasing L are small compared with those that already have a counterpart for smaller L . In the numerical simulation discussed below I will use this property to determine the critical coupling β_c of the 3D Ising model.

4 The Algorithm

I shall now describe an efficient algorithm to update the above system, where the type of boundary condition is a random variable [8]. The simplest idea to alter the boundary conditions is to propose a collective change of the

couplings $J_{\langle ij \rangle}$ between sites in the uppermost plane and sites in the lowermost plane from 1 to -1 or vice versa in a single Metropolis step. With a high probability, most of the spins s_i and s_j have the same sign in the case of periodic boundary conditions and a different sign in the case of antiperiodic ones. Hence the acceptance rate of such a Metropolis step would be extremely small. This simple algorithm does not take into account that the physical interface can be built anywhere in the system and, what is even more important, that the interface wildly fluctuates close to the critical point.

These problems can be resolved by using a modification [8] of a standard cluster algorithm [4, 5]. First the bonds are deleted with the standard probability

$$p_d = \exp(-\beta(1 + J_{\langle ij \rangle} s_i s_j)), \quad (11)$$

or else they are frozen. After deleting or freezing the bonds of the system one searches for an interface of deleted bonds that completely cuts the lattice in the z -direction. If there is such an interface, the spins between the bottom of the system and this interface and the sign of the coupling $J_{\langle ij \rangle}$ connecting top and bottom are flipped. This is a valid update, since the bonds in the interface are deleted and the value of $J_{\langle ij \rangle} s_i s_j$ for i in the lowermost and j in the uppermost plane is not changed when we alter the sign of $J_{\langle ij \rangle}$ and s_i . If the given configuration has antiperiodic boundary conditions then there is an interface of bonds with $J_{\langle ij \rangle} s_i s_j = -1$. Bonds with $J_{\langle ij \rangle} s_i s_j = -1$ have $p_d = 1$. Hence there is always a interface of deleted bonds when one starts with antiperiodic boundary conditions. This means that the performance of the algorithm is only limited by the probability to have configurations with antiperiodic boundary conditions, which is equal to $\frac{Z_{ap}}{Z_p + Z_{ap}}$.

This algorithm can be generalized to $O(N)$ -invariant spin models with $N > 1$. Here one has to consider a complete set of orthogonal spin components simultaneously. The freeze or delete decision can be done for the single components independently. When there is a cutting interface of deleted bonds for all the spin components at the same time, one is then allowed to change the type of boundary condition.

In my simulations I alternate this boundary flip update with standard single-cluster updates [5].

5 Numerical Results

On small lattices I calculated the partition functions Z_p and Z_{ap} exactly. Besides their physical meaning these results provide a test-ground for the Monte Carlo simulations.

5.1 Exact Results on Small Lattices

For 2^3 , 3^3 and 4^3 lattices I computed the partition functions Z_p and Z_{ap} exactly. This can be done as follows. First one writes the partition functions

as

$$Z = \sum_{\mathbf{S}} n_{\mathbf{S}} \exp(-\beta E). \quad (12)$$

Since there is only a rather small number of different energies E , once the number of spin configurations $n_{\mathbf{S}}$ with energy E is found, it is easy to compute the result for the partition function for any β . For $L = 2$ and 3, I generated all configurations on the computer and counted the $n_{\mathbf{S}}$. This took several minutes on a workstation. But this naïve method extrapolates to about 10^6 years for the 4^3 lattice. Based on a checkerboard decomposition of the lattice, Pearson [13] managed to solve the problem in about 1500 seconds on a CDC 7600 for the lattice with periodic boundary conditions. I now also calculated the $n_{\mathbf{S}}$ for the system with antiperiodic boundary conditions in one direction employing Pearson's method. The results for the ratio $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ for $L = 2, 3$ and 4 are given in fig. 1. The discussion of the results is given in the following, together with the discussion of the Monte Carlo results.

5.2 Monte Carlo Results

I simulated the 3D Ising model on a simple cubic lattice of size L^3 with $L = 4, 8, 16, 32$ and 64, with boundary conditions as dynamical variables at $\beta_0 = 0.221652$, which is the estimate for the critical coupling of ref. [10]. On each lattice I performed 10000 measurements. A measurement was recorded after $L/2$ single-cluster updates [5] plus one boundary flip update.

I measured the energy

$$E = - \sum_{\langle ij \rangle} J_{\langle ij \rangle} s_i s_j, \quad (13)$$

the magnetization

$$m = \frac{1}{L^3} \sum_i s_i, \quad (14)$$

and the type of boundary condition. These data were used to calculate the ratio of the partition function with antiperiodic and periodic boundary conditions $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ and the fourth-order cumulant $U_L = 1 - \frac{1}{3} \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}$ in the neighbourhood of the critical point. For the evaluation of the fourth-order cumulant of course only the magnetizations of configurations with periodic boundary conditions were averaged. For the extrapolation to couplings β other than the simulation point β_0 , I used the reweighting formula [14]

$$\langle A \rangle(\beta) = \frac{\sum_i A_i \exp((-\beta + \beta_0) E_i)}{\sum_i \exp((-\beta + \beta_0) E_i)}, \quad (15)$$

where i labels the configurations generated with the Boltzmann distribution at β_0 . The statistical errors I computed from Jackknife binning on the final result of the extrapolation. In fig. 2 I compare the Monte Carlo data for $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ for $L = 4$ with the exact result for $L = 4$. The Monte Carlo curve agrees within one standard deviation with the exact result, giving me confidence in the simulation and the extrapolation programs. In fig. 3 I plotted the

Monte Carlo results for the fourth-order cumulant for $L = 4, 8, 16, 32$ and 64. In fig. 4 I give the corresponding results for the ratio of partition functions $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$. The crossings of the curves for different L are well covered in the valid β -range for reweighting for both quantities. For $L = 64$ for example, the extrapolation for U_L and $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ seems to become unreliable for $|\beta - 0.221652| > 0.001$, while the crossings of $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ and U_L for $L = 32$ and 64 are both contained in $|\beta - 0.221652| < 0.00001$.

In table 1 the couplings of the crossings of curves for the fourth-order cumulant and $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ with subsequent L are summarized. The estimates for β_c from the two quantities converge from different sides as the lattice size is increased. This improves the control on systematic errors on the final estimate of the infinite volume critical temperature. The estimates for β_c from $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ seem to converge much faster than those from U_L as L increases. For the $L = 32$ and 64 crossings the β_c estimates from both quantities agree within error bars. Because of the faster convergence of the estimate from $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ I give $\beta_c = 0.221650(5)$ as my final estimate for the critical coupling. This value is consistent within 1σ with the value $\beta_c = 0.221652(3)$ obtained by a MCRG study [10] and within 2σ with $\beta_c = 0.2216595(26)$ found in a Monte Carlo finite-size scaling study [9]. A summary of results obtained in other Monte Carlo simulations, by the analysis of high-temperature series expansions, and by transfer matrix methods can be found in refs. [9, 10].

To leading order the approach of the crossing β_{cross} to β_c is given by [6]

$$\beta_{\text{cross}} = \beta_c + g L^{-(\omega+1/\nu)}, \quad (16)$$

where ω is the correction to scaling exponent. A fit according to that ansatz including the crossings of U_L for $L = 4$ and 8, $L = 8$ and 16, and $L = 16$ and 32 gives $\omega + 1/\nu = 2.32(26)$ when using $\beta_c = 0.221650$. A similar fit for the crossings of $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ is useless since the estimates of β_c from $L = 8$ up to $L = 64$ are consistent within error bars.

The fourth-order cumulant at the critical coupling is supposed to converge to a fixed point as [6]

$$U_L(\beta_c) = U_{\infty}(\beta_c) (1 + c L^{-\omega}), \quad (17)$$

where ω is the correction to scaling exponent. In the MCRG simulation of ref. [10], $\omega = 0.8 - 0.85$ is found, the resummed ϵ -expansion yields $\omega = 0.78(4)$ and $\omega = 0.81(4)$ depending on the resummation scheme, and the resummed perturbation theory at fixed dimension provides $\omega = 0.79(3)$ [15]. In table 2 I give the values for the fourth-order cumulant U_L and $\frac{Z_{\text{ant}}}{Z_{\text{per}}}$ at β_c . Using $L = 4, 8, 16$ and 32, I performed a fit according to eq. (17). Using $\omega = 0.8$ fixed, I obtain $U_{\infty}(\beta_c) = 0.464(2)$, while also varying ω , I obtain $\omega = 0.73(25)$ and $U_{\infty}(\beta_c) = 0.462(6)$. This result can be compared with $U_{\infty}(\beta_c) \approx 0.47$ obtained in ref. [9] and $U_{\infty}(\beta_c) = 0.44(2)$ obtained by Binder [6] in 1981. $\frac{Z_{\text{ant}}}{Z_{\text{per}}}(\beta_c)$ shows a much weaker dependence on L . Hence a similar fit seems to be useless. As infinite volume value I quote the result for $L = 32$ and $L = 64$, $\frac{Z_{\text{ant}}}{Z_{\text{per}}}(\beta_c) = 0.546(3)$.

Next I studied the slope of the fourth-order cumulant $dU/d\beta$ and the slope of the ratio of partition functions $d^2_{xx}/d\beta$. The derivatives of the observables entering these quantities I evaluated according to

$$\frac{d < A >}{d\beta} = < AE > - < A > < E > . \quad (18)$$

The statistical errors were again computed via a Jackknife binning procedure. In table 3 I give the slopes for $\beta = 0.22165$ and the maximal value of the slopes. I did not calculate an error for the β_{max} . Also the errors I quote for the maximal slope do not take into account the uncertainty in β_{max} . The critical behaviour of the derivative of the U_L with respect to β is given by

$$\frac{dU}{d\beta} |_{\beta_c} = c_1 L^{1/\nu} (1 + c_2 L^{-w}) . \quad (19)$$

I expect a similar behaviour for $d^2_{xx}/d\beta$. I tried to extract estimates for the exponent ν following the above equation. Estimates including various lattice sizes are given in table 6. Corrections to the leading scaling behaviour can only be detected for $L \leq 4$ within the given statistical accuracy for both $dU/d\beta$ and $d^2_{xx}/d\beta$. As my final estimate for the critical exponent, I quote $\nu = 0.630(2)$, the result of the fit for $d^2_{xx}/d\beta$ including $L = 8$ up to $L = 64$.

This value can be compared with $\nu = 0.624(2)$ of ref. [10] and $\nu = 0.6289(8)$ given in ref. [9]. The resummed ϵ -expansion gives $\nu = 0.6310(15)$ and $\nu = 0.6300(15)$ depending on the applied resummation scheme [15]. From the analysis of high-temperature series expansions Adler obtains $\nu = 0.630(4)$ and Nickel and Rehr $\nu = 0.6300(15)$ [1]. A more complete summary of values for ν obtained by various methods can be found in refs. [9, 10].

Finally let me shortly comment on the performance of the algorithm and my implementation. 32 single cluster updates plus one boundary It takes about 12 sec on a SPARC 10-41 workstation, to make 32 single cluster updates plus one boundary flip update of the $L = 64$ lattice. The boundary update takes about 33% of the CPU time. This is essentially the price one is paying for computing $\frac{d^2_{xx}}{d\beta}$. The integrated autocorrelation times for the energy density and the magnetic susceptibility are between 1 and 2 in units of measurements for all the lattice sizes I considered. The autocorrelation of the type of boundary condition alters its sign. As a consequence the integrated autocorrelation time for the type of boundary condition is even smaller than 0.5.

6 Conclusion

I have shown that the analysis of the ratio of partition functions $\frac{d^2_{xx}}{d\beta}$ on finite lattices can be used to determine critical parameters of the 3D Ising model. It turns out that the estimates for the critical coupling β_c obtained in this way converge even faster with increasing lattice size than the estimates obtained from the fourth-order cumulant U_L . Considering the ratio

of partition functions in addition to the fourth-order cumulant improves the control on systematic errors, since the strength and the sign of corrections to scaling are different for the two quantities. An extension of the Monte Carlo algorithm to $O(N)$ models with $N > 1$ is possible.

Acknowledgements

I would like to thank K. Pinn and U. Wolff for discussing the subject of this paper with me. The simulations were performed on the IBM RISC system/6000 of the Regionales Hochschulrechenzentrum Kaiserslautern (RHRK) and on the SUN-cluster of the CERN TH Division.

References

- [1] As examples for work concerning the 3D Ising model:
J. Adler, J.Phys. A16 (1983) 3585
B.G. Nickel and J.J. Rehr, J.Stat.Phys. 61 (1990) 1
- [2] E. Brezin, J.C. Le Guillou and J. Zinn-Justin in: Phase Transitions and Critical Phenomena, Vol. 6, eds.: C. Domb, M. S. Green, Academic Press, London, 1976
J.C. Le Guillou and J. Zinn-Justin, Phys. Rev. B21 (1980) 3976
J. Zinn-Justin, in: Phase Transitions, Cargèse 1980, eds.: M. Lévy et al., Plenum Press, New York, 1982
- [3] K.G. Wilson and M.E. Fisher, Phys.Rev.Lett 28 (1972) 240
- [4] R. H. Swendsen, J.-S. Wang, Phys. Rev. Letters 58 (1987) 86
- [5] U. Wolff, Phys. Rev. Letters 62 (1989) 361
- [6] K. Binder, Phys. Rev. A25 (1982) 1699
- [7] R. K. Mon, D. Jasnow, Phys. Rev. A30 (1984) 670
- [8] M. Hasenbusch, Kaiserslautern University preprint KL-TH 92-16, to be published in J.Phys.I
- [9] A.M. Ferrenberg and D.P. Landau, Phys. Rev. B44 (1991) 4081
- [10] C.F. Baillie, R. Gupta, K.A. Hawick and G.S. Pawley, Phys. Rev. B45 (1992) 10438
- [11] M. P. Nightingale, Physica A83 (1976) 561
- [12] V. Privman and M. E. Fisher, Phys. Rev. B30 (1984) 322
- [13] R. B. Pearson, Phys. Rev. B26 (1982) 6285
With other methods, even a $5 \times 5 \times 4$ lattice has been solved:
G. Bhanot and S. Sastry, J.Stat.Phys. 60 (1990) 333.

[14] C. Itzykson and J.-M. Drouffe, Statistical field theory, Vol. 2 (Cambridge Press, New York, 1991).

[15] J.C. Le Guillou and J. Zinn-Justin, J.Physique **50** (1989) 1365

Table 1: β_{cross} for the fourth-order cumulant and $\frac{z_{xx}}{z_r}$. The numbers for the $L = 2, 3$ and $L = 3, 4$ crossings of $\frac{z_{xx}}{z_r}$ stem from the exact partition functions.

L	β_{cross} of U_L	β_{cross} of $\frac{z_{xx}}{z_r}$
2,3		0.21761
3,4		0.22123
4,8	0.22265(11)	0.22140(10)
8,16	0.221864(40)	0.221636(37)
16,32	0.221683(16)	0.221648(13)
32,64	0.221656(6)	0.221650(5)

Table 2: The fourth-order cumulant U_L and $\frac{z_{xx}}{z_r}$ at $\beta = 0.22165$. The values for $\frac{z_{xx}}{z_r}$ without error-estimate stem from the exact partition functions. For $L = 16, 32$, and 64 in addition to the statistical error of the quantities at $\beta = 0.22165$ I give the uncertainty of the quantities due to the uncertainty in the estimate of the critical coupling $\beta_c = 0.221650(5)$.

L	U_L	$\frac{z_{xx}}{z_r}$
2		0.5700
3		0.5541
4		0.5520
4	0.4946(10)	0.5544(22)
8	0.4823(11)	0.5477(22)
16	0.4737(12)(2)	0.5466(21)(6)
32	0.4701(14)(8)	0.5460(23)(18)
64	0.4680(14)(26)	0.5460(27)(46)

Table 3: Slope of the fourth-order cumulant U_L at $\beta = 0.22165$ and the maximum slope.

L	Slope	β_{max}	Slope
4	5.97(6)	0.2048	6.93(9)
8	18.77(18)	0.2177	20.41(27)
16	58.97(54)	0.22033	63.96(78)
32	172.5(2.0)	0.22125	183.2(2.7)
64	524.8(7.0)	0.22154	551.3(8.6)

Figure Captions

Figure 1: The exact result for the ratio of the partition functions $\frac{Z_{ex}}{Z_p}$ for the lattice sizes $L = 2, 3$ and 4 is plotted as a function of the coupling β .

Figure 2: The exact result (solid line) and the Monte Carlo (dashed line) result are compared for the lattice size $L = 4$. The dotted lines give the statistical error of the Monte Carlo result.

Figure 3: The Monte Carlo result for the fourth-order cumulant U_L (solid lines) for $L = 4, 8, 16, 32$ and 64 is given as a function of the coupling β . The statistical error is given by the dotted line.

Figure 4: Same as in the previous figure, but for the ratio of partition functions $\frac{Z_{ex}}{Z_p}$.

Table 4: Slope of the ratio $\frac{Z_{ex}}{Z_p}$ at $\beta = 0.22165$ and the maximum slope.

L	Slope	β_{max}	Slope
2	3.66	0.2269	3.67
3	7.68	0.2309	7.85
4	12.76	0.2284	13.20
4	12.67(9)	0.2288	13.14(8)
8	39.80(30)	0.22434	41.94(25)
16	120.4(1.0)	0.22259	127.7(9)
32	357.1(2.7)	0.22199	382.0(2.7)
64	1090.(9.)	0.22176	1170.(9.5)

Table 5: The critical exponent ν from the slope of the fourth-order cumulant at $\beta = 0.22165$ and the slope of $\frac{Z_{ex}}{Z_p}$ at $\beta = 0.22165$. The results for $L = 2, 3$ and $L = 3, 4$ stem from the exact partition functions.

L 's used	ν from $U_L(\beta_c)$	ν from $\frac{Z_{ex}}{Z_p}$
2,3		0.547
3,4		0.567
4,8	0.605(7)	0.606(6)
8,16	0.605(7)	0.626(6)
16,32	0.646(10)	0.638(8)
32,64	0.623(13)	0.621(8)
8,16,32,64	0.625(3)	0.630(2)
16,32,64	0.636(5)	0.630(4)

Fig. 1

Fig. 2

Fig. 3

Fig. 4