VARIATIONS ON KALUZA-KLEIN COSMOLOGY

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We investigate the cosmological consequences of having quantum fields living in a space with compactified dimensions. We will show that the equation of state is not modified by topological effects and so the dynamics of the universe remains as it is in the infinite volume limit. On the contrary the thermal history of the universe depends on terms that are associated with having non-trivial topology. In the conclusions we discuss some issues about the relationship between the $c = 1$ non-critical string-inspired cosmology and the result obtained with matter given by a hot massless field in $S^1 \times \mathbb{R}$.

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1 Introduction

As far as these authors know, the study of the influence of topology in relativistic cosmology as for its effects on the matter filling has been very limited. To our knowledge, it is in the context of Kaluza-Klein theories [1] where this subject has been studied for the first time by Randjbar-Daemi, Salam and Strathdee in [2].

The Kaluza-Klein philosophy has been assumed and, in a sense, transformed with the advent of String Theories [3]. The existence of extra dimensions is now a necessary ingredient. The main reason is that after seven years of febrile activity in the subject of String Theories, the only phenomenologically relevant string models [4] are those derived from the Heterotic String [5] (or of the heterotic type) which is formulated in ten dimensions. Furthermore, the string, being a one-dimensional object, sees the compactified dimensions in a richer way, because of the winding states which correspond to the wrapping of the string around the compact directions. One can believe in an extreme Kaluza-Klein philosophy in which all the spatial target dimensions are compactified at finite temperature so, in the imaginary time formalism, everything would happen as though no open dimension existed [6, 7].

A quantum string is not fully apart from Quantum Field Theory. In the canonical quantization approach we can describe the string by its field content, i.e., with every vibrational state of the string we associate a quantum field (analogue model). But, at the same time, that the string is something different from a collection of quantum fields was already evident in the old dual models because of the $s - t$ channel duality of the four point amplitude [8] (cf. also [9]).

In the string models of “everything” this difference can be evidentiated if we place our string in a topologically non-trivial space, because of the property of space-time duality [10] which has no counterpart in Field Theory. In the extreme Kaluza-Klein world described above, this implies that, assuming that the self-dual size is the minimum size of the Universe (of the order of the Planck scale), only those universes bigger than the Planck scale have physical meaning. On the other hand, recent investigations [11, 12, 13] have shown that at least in the large-size regime the string is described exclusively by its field content. If we put together this two facts we find that String Theory would seem to be nothing more than Field Theory plus a physical cut-off (a minimum accessible length at the self-dual radius). In the conclusions we will see that this statement does not hold completely.

The final objective of our work is to investigate how the string scenario can modify that of Kaluza-Klein presented in [2] in the extreme case described above. However we have realized that the subject of quantum fields in compact spaces as source for the gravitational field is a matter of interest on its own. Consequently we present here, along with a set of relevant issues about the thermodynamics of quantum fields in compact spaces, the cosmological implications
that can be extracted from the study of a two dimensional massless field coupled to Brans-Dicke gravity [14] (see also [15]) leaving the stringy modifications to be presented elsewhere [17]. In the conclusions we will discuss the relationship between a massless field in compact space at finite temperature and the $c = 1$ string model as a preparation for a numerical resolution of the $c = 1$ cosmology.

2 Two-dimensional Brans-Dicke equations

Since we are going to work in the two-dimensional case, we cannot use Einstein-Hilbert gravity because classically it is topological in two-dimensions. Then we are going to consider the Brans-Dicke extension [18] which is non-trivial in any case. The Brans-Dicke equations in an arbitrary number of dimensions can be written [14]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = \frac{8\pi}{\Phi}T_{\mu\nu} + \frac{\omega}{\Phi^2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2}g_{\mu\nu} \nabla_\sigma \Phi \nabla^\sigma \Phi \right) + \frac{1}{\Phi} \left( \nabla_\mu \Phi \nabla_\nu \Phi - g_{\mu\nu} \Box \Phi \right),$$

(1)

$$R - 2\Lambda = \frac{\omega}{\Phi^2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{2\omega}{\Phi} \Box \Phi,$$

(2)

where $T_{\mu\nu}$ is the stress tensor for the matter fields and $\Lambda$ is the cosmological constant. In two dimensions the Einstein tensor is identically zero, so the first equation simplifies

$$\frac{8\pi}{\Phi}T_{\mu\nu} + \frac{\omega}{\Phi^2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2}g_{\mu\nu} \nabla_\sigma \Phi \nabla^\sigma \Phi \right) + \frac{1}{\Phi} \left( \nabla_\mu \Phi \nabla_\nu \Phi - g_{\mu\nu} \Box \Phi \right) = \Lambda g_{\mu\nu}.$$  

(3)

For the metric we use a Friedmann-Robertson-Walker ansatz

$$ds^2 = -dt^2 + L^2(t)d\xi^2,$$

(4)

where the scale factor $L(t)$ depends only on time. We will take the energy-momentum tensor of the matter fields to have a perfect fluid form

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu}.$$ 

(5)

$p$ and $\rho$ are respectively the pressure and the energy density. By assuming that the Brans-Dicke field $\Phi$ is also a function of time alone, we can rewrite (2) and (3) in the form

$$2\dot{\Phi}^2 \frac{\dot{L}}{L} - 2\Lambda \dot{\Phi}^2 = -\omega \dot{\Phi}^2 + 2\omega \Phi \ddot{\Phi} + 2\omega \Phi \dot{\Phi} \frac{\dot{L}}{L},$$

(6)

$$\dot{\Phi}^2 - \frac{2}{\omega} \Phi \ddot{\Phi} \frac{\dot{L}}{L} = -\frac{16\pi}{\omega} \Phi \rho - \frac{2\Lambda}{\omega} \dot{\Phi}^2,$$

(7)

$$\Phi \ddot{\Phi} + \frac{1}{2} \omega \dot{\Phi}^2 = -8\pi \Phi p + \Phi^2 \Lambda.$$ 

(8)
It can be shown using (1) that the stress tensor for the matter fields has to be covariantly conserved, so we have the integrability condition
\[ \nabla_\mu T^{\mu \nu} = 0 . \] (9)

Using our ansatz for the metric this equation can be written as
\[ \dot{\rho} + \frac{\dot{L}}{L} (\rho + p) = 0 , \] (10)
which expresses the conservation of entropy in our universe. The four equations (6), (7), (8) and (10) are not independent. In fact, taking the time derivative in (7) and combining the result with (8) we see that (6) and (10) are indeed equivalent. We need one equation more in order to determine our dynamical system. This supplementary condition is supplied by the equation of state, which relates the pressure and the energy density \( p = p(\rho, \beta, L) \).

In order to get the equation of state, we usually begin with the Helmholtz free energy for the system, \( F(\beta, L) \). All the thermodynamics is given by this function so as to need only to compute derivatives of \( F(\beta, L) \). In fact, we could start from the very beginning using the Einstein-Hilbert-Brans-Dicke action
\[ S = \int d^2 x \sqrt{-g} \left[ \Phi (R - 2\Lambda) - \frac{\omega}{\Phi} \nabla_\mu \Phi \nabla^\mu \Phi \right] \] (11)
and adding the matter with an action which with our ansatz can be written as \( [2, 7] \)
\[ S_M = \int dt \sqrt{-g_{00}} F \left( \beta \sqrt{-g_{00}}, L \right) . \] (12)
It is easy to see that in our case the co-moving perfect fluid form of the stress-tensor gives the same field equations as this way of introducing hot matter.

With the ansatz for the metric, the Hilbert-Einstein-Brans-Dicke action is invariant under the replacement \([7]\)
\[ L \rightarrow \frac{1}{L} , \quad \Phi \rightarrow L^2 \Phi , \] (13)
which corresponds to the duality symmetry of String Theory \([16, 10]\) upon the identification of the Brans-Dicke (dilaton) field \( \Phi \) with the inverse of the string coupling constant squared. To be precise, the action (11) is invariant modulo a total derivative
\[ \Delta S = 2 \int dt \frac{d}{dt} \left( \Phi \dot{L} \right) . \] (14)
As a final comment, notice that the scale factor has been taken dimensionless. Putting a length dimension in \( L \) is completely equivalent to making the change of variables \( \xi \rightarrow \xi / \lambda \) where \( \lambda \) is the unit used to measure lengths. But of course the Einstein-Hilbert-Brans-Dicke action is
invariant under this change of coordinates. In the context of String Theory $\sqrt{\alpha'}$ will play the role of $\lambda$.

When we are working with the ordinary Einstein–Hilbert action, a non-zero vacuum energy $\Lambda_{\text{vac}} L$ can be seen as a contribution to the cosmological constant, since in that case the perfect-fluid energy-momentum tensor admits the decomposition $T_{\mu \nu} = \hat{T}_{\mu \nu} + \Lambda_{\text{vac}} g_{\mu \nu}$, where $\hat{T}_{\mu \nu}$ is the energy-momentum tensor without the vacuum energy. Then, as can readily seen from Einstein equations, $\Lambda_{\text{vac}}$ can be re-absorbed in a redefinition of the cosmological constant. In the case at hand, however, we have also the Brans-Dicke field which multiplies the energy-momentum tensor. Nevertheless, when this field acquire a vacuum expectation value $\langle \Phi \rangle$ we see from eqs. (1) and (2) that we can define an effective cosmological constant $\Lambda_{\text{eff}}$ given by

$$\Lambda_{\text{eff}} = \Lambda - \frac{8\pi}{\langle \Phi \rangle} \Lambda_{\text{vac}}$$

This is what one would regard as the physical (phenomenological) value for the cosmological constant.

### 3 Thermodynamics with compactified dimensions

We start this section computing the Helmholtz free energy for bosonic and fermionic fields in $S^1 \times \mathbb{R}$. We are going to see that, because of the fact that the only momentum is discrete, the well defined way of getting the thermodynamic potential is by computing directly a trace on the corresponding Fock space. In other words, as we will explain in the conclusions, the proper time representation of the Helmholtz free energy [2] is not a well defined quantity in this case. After all, for the time being, this representation looks like an unnecessary sophistication when, in the problem at hand, all the information we are interested in can be obtained using simpler methods.

First, let us consider a free massless scalar field. The partition function $Z(\beta)$ is defined by

$$Z(\beta) = \text{Tr} \ e^{-\beta H} ,$$

where $H$ is the hamiltonian of the system and $\beta$ is the inverse temperature. To directly evaluate this quantity we start by noticing that the Fock space of a bosonic field in $S^1 \times \mathbb{R}$ is the direct product of the Hilbert space for $n$-particles. The one-particle excitations of the system have momenta

$$k = \frac{2\pi n}{L} ,$$

with $L$ the length of the compactified dimension and $n \in \mathbb{Z}$. We introduce creation-annihilation
operators $a_n, a_n^+$ such that the normal-ordered hamiltonian is given by

$$H = \sum_{n \in \mathbb{Z}} \frac{2\pi|n|}{L} a_n^+ a_n.$$  \hfill (18)

The prime in (18) indicates that the term in the sum with $n = 0$ is omitted. This is done because the only state with $p = 0$ is the vacuum state, since we are dealing with scalar massless particles. Using these creation-annihilation operators, we construct the completely symmetrized states $|\{l_n\}\rangle$

$$|\{l_n\}\rangle = \prod_{n \in \mathbb{Z}}' \frac{1}{\sqrt{n!}} (a_n^+)^{l_n} |0\rangle,$$  \hfill (19)

which span the whole Fock space. The action of the $a_n, a_n^+$ operators on these states can be easily determined

$$a_n^+ |\ldots, l_n, \ldots\rangle = \sqrt{l_n + 1} |\ldots, l_n + 1, \ldots\rangle,$$

$$a_n |\ldots, l_n, \ldots\rangle = \sqrt{l_n} |\ldots, l_n - 1, \ldots\rangle.$$  \hfill (20)

Knowing the Fock space we are prepared to compute the trace in (16) to give

$$\text{Tr} \ e^{-\beta H} = \sum_{\{l_n\}} \langle \{l_n\} | \exp \left( -2\pi \frac{\beta}{L} \sum_{n \in \mathbb{Z}}' |n| a_n^+ a_n \right) |\{l_n\}\rangle$$

$$= \prod_{n \in \mathbb{Z}}' \sum_{l_n} e^{-2\pi \beta |n| l_n} = e^{\frac{\beta}{L} \eta} \eta^{-2} \left( \frac{i\beta}{L} \right),$$  \hfill (21)

where we have used the Dedekind $\eta$-function [19] $\eta(\tau) = e^{\frac{\pi i}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau})$. The Helmholtz free energy for the bosonic field $F_B(\beta, L)$ is then given by

$$F_B(\beta, L) = \frac{\pi}{6L} + \frac{2}{\beta} \ln \eta \left( \frac{i\beta}{L} \right).$$  \hfill (22)

Let us remark that computing the $\beta \to \infty$ limit we get

$$F_B(\beta, L) \to \frac{\pi}{6L} - \frac{\pi}{6L} = 0.$$  \hfill (23)

This is a simple consequence of having considered the normal-ordered hamiltonian (18), since then $H|0\rangle = 0$ and no Casimir energy is present.

In order to get the equation of state, we have to obtain both the energy density and the pressure. The first quantity is defined from the Helmholtz free energy so as to give

$$\rho(\beta, L) = \frac{1}{L} \frac{\partial}{\partial \beta} [\beta F(\beta, L)] = \frac{\pi}{6L^2} - \frac{\pi}{6L^2} E_2 \left( \frac{i\beta}{L} \right),$$  \hfill (24)
where \( E_2(\tau) \) is a normalized Eisenstein series [19]

\[
E_2(\tau) = 1 + \frac{6}{\pi^2} \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2} = -\frac{12i\eta'(\tau)}{\pi \eta(\tau)}.
\] (25)

In the case of the pressure, we have

\[
p(\beta, L) = -\frac{\partial}{\partial L} F(\beta, L) = \frac{\pi}{6L^2} - \frac{\pi}{6L^2} E_2 \left( \frac{i\beta}{L} \right).
\] (26)

Then, we find that our massless scalar field has the equation of state

\[
p(\beta, L) = \rho(\beta, L).
\] (27)

Of course, we can reobtain the Helmholtz free energy (22) using path integrals, since the partition function for a quantum field in a \( d \)-dimensional Minkowski space-time can be represented as a Euclidean path integral for the theory in \( \mathbb{R}^{d-1} \times S^1 \) with the length of the circle fixed to \( \beta \) [20, 21]. The boundary conditions we are taking for bosonic fields are periodic along the compactified dimension \( (S^1) \) whereas for fermionic fields antiperiodic ones are chosen in order to recover Fermi statistics. Computing the euclidean path integral for a free scalar field one gets that the Helmholtz free energy is given by [21]

\[
F_B(\beta) = \frac{1}{\beta} \sum_k \ln \left( 1 - e^{-\beta \omega_k} \right),
\] (28)

where \( \omega_k = \sqrt{m^2 + k^2} \) and we sum over the momenta \( k \). In our case (a massless scalar field in \( S^1 \times \mathbb{R} \)) we have momenta of the form (17) and then

\[
F_B(\beta, L) = \frac{1}{\beta} \sum_{n \in \mathbb{Z}}' \ln \left( 1 - e^{-2\pi \sqrt{m^2 + n^2}} \right).
\] (29)

We drop again the zero momentum term in the sum because, as we said, the vacuum is the only state with zero momentum. In fact, the inclusion of this term in the sum would produce a logarithmic singularity. This would look like an infinite entropy (degeneration) for the fundamental state (zero temperature limit). Introducing it would be crooked. It is straightforward to get (22) by converting the sum (29) into an infinite product and making use of the definition of \( \eta(\tau) \).

The result (22) is independent of the particular form of the Helmholtz free energy. In fact, it is easy to see that as long as \( \beta F(\beta, L) \) is a function of \( \beta/L \) alone

\[
\beta F(\beta, L) = f \left( \frac{\beta}{L} \right),
\] (30)
we can compute formally the density and the pressure with the following result

\[ p(\beta, L) = \rho(\beta, L) = \frac{1}{L^2} f' \left( \frac{\beta}{L} \right) . \]  

(31)

So we obtain the same equation of state, namely \( p = \rho \). This functional dependence of \( \beta F(\beta, L) \) is the only one possible as long as we have no other dimensionful parameter entering in the theory. Had we a mass term, we could have \( \beta F(\beta, L, m) = f(\beta/L, \beta m, L m) \). As we will see later the term \( L m \) does not appear for the free theory. Later, we will make some comments about the interacting theory.

For the fermionic field we will use the result of the Euclidean path integral computation in \( \mathbb{R}^{d-1} \times S^1 \) where, as mentioned, we have to impose antiperiodic boundary conditions over the fermionic field along the circle \( S^1 \) of length \( \beta \). In the case we are dealing with the Helmholtz free energy turns out to be

\[ F_F(\beta, L) = -\frac{1}{\beta} \sum_{n \in \mathbb{Z}}' \ln \left( 1 + e^{-2\pi \frac{\beta}{L} |n|} \right) , \]  

(32)

where the zero momentum contribution has been omitted. Again it is quite easy to rewrite the last expression in a more manageable form

\[ F_F(\beta, L) = -\frac{\pi}{6L} - \frac{2}{\beta} \ln \frac{\eta \left( i \frac{2\beta}{L} \right)}{\eta \left( i \frac{\beta}{L} \right)} . \]  

(33)

This expression can be checked to be correct by using the relation between the free energy for a bosonic and a fermionic free field of the same mass [22]

\[ F_F(\beta) = F_B(\beta) - 2F_B(2\beta) , \]  

(34)

which stems from the simple mathematical identity \((1-x)(1+x) = (1-x^2)\) and then no mention to the proper time representation and to any Jacobi theta function gymnastics is needed. Since \( \beta F(\beta, L) \) depends only on \( \beta/L \) the equation of state is the same as that for the boson field

\[ p(\beta, L) = \rho(\beta, L) = -\frac{\pi}{6L^2} + \frac{\pi}{3L^2} E_2 \left( \frac{2\beta}{L} \right) - \frac{\pi}{6L^2} E_2 \left( \frac{\beta}{L} \right) . \]  

(35)

We see that the Helmholtz free energy goes to zero when \( \beta \) goes to infinity. This is again because the hamiltonian of the fermionic field is normal-ordered and then annihilates the vacuum state setting the zero-point energy to zero.

It is of some interest to check that the expressions obtained so far for the bosonic and fermionic fields recover their correct values in the decompactification limit \( L \to \infty \). First,
we note that if we naively take this limit in the expression for the free energy of the bosonic and fermionic fields this quantity diverges. The reason is that in the infinite-volume limit the quantity which has physical sense is not the total free energy but the free energy density. We can compute the limit of this quantity using $\eta \left( -1/\tau \right) = \sqrt{-i\tau} \eta(\tau)$. In the case of the bosonic field we have
\[
\frac{1}{L} F_B(\beta, L) = \frac{\pi}{6L^2} + \frac{1}{\beta L} \ln \frac{L}{\beta} + \frac{2}{\beta L} \ln \eta \left( \frac{L}{\beta} \right) \longrightarrow -\frac{\pi}{6\beta^2}.
\]
(36)

To compute the pressure and the energy density in this limit we use an analogous inversion relation for $E_2(\tau)$ [19]:
\[
E_2 \left( -1/\tau \right) = \tau^2 E_2(\tau) - 6i\tau/\pi.
\]
Applying this formula we have the following result
\[
p(\beta, L) = \rho(\beta, L) \longrightarrow \frac{\pi}{6\beta^2}.
\]
(37)

For the fermionic field the situation is quite the same. The decompactification limit of the free energy density is
\[
\frac{1}{L} F_F(\beta, L) \longrightarrow -\frac{\pi}{12\beta^2}
\]
(38)
and
\[
p(\beta, L) = \rho(\beta, L) \longrightarrow \frac{\pi}{12\beta^2}.
\]
(39)

Let us then recapitulate the results obtained so far. We find that the introduction of a compactified spatial dimension in our two dimensional space-time does not modify the equation of state. On the contrary, the dependence of the energy density and the pressure with the temperature changes drastically. This will be important when studying cosmological solutions of the two dimensional Brans-Dicke equations.

In the case in which, for example, we have a compact space with intrinsic curvature (like in [2]) it is not necessary to know the exact form of the free energy because like in the bosonic field in $S^1 \times \mathbb{R}$ the functional dependence on $\beta$ and the volume is enough to get the equation of state. For example, in the case of a massless bosonic field in $S^2 \times \mathbb{R}$ it is easy to see that the field can be expanded in terms of spherical harmonics
\[
\phi(t, \theta, \varphi) \sim \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} e^{i\varphi} \sqrt{l(l+1)} Y_l^m(\theta, \varphi) + h.c.
\]
(40)

where the normalization is chosen to get the correct commutation relations for the creation-annihilation operators and $R$ is the radius of the sphere $S^2$. We see then that the momenta of the excited states are given by
\[
k = \frac{\sqrt{l(l+1)}}{R}.
\]
(41)
Then, applying (28), we get the Helmholtz free energy
\[
F_B(\beta, R) = \frac{1}{\beta} \sum_{l=1}^{\infty} (2l + 1) \ln \left( 1 - e^{-\beta \sqrt{R(l+1)}} \right).
\] (42)

Since \( R = \sqrt{V/4\pi} \) we can compute the equation of state by simply noticing that \( \beta F(\beta, V) = f(\beta/\sqrt{V}) \) with the result \( \rho = 2p \) as in the \((2 + 1)\)-dimensional uncompactified case.

Moreover, we can generalize this construction to the case in which we have a field (bosonic or fermionic) in a \( T^{d-1} \times \mathbb{R} \) where \( T^{d-1} \) is a \((d - 1)\)-dimensional torus with lengths \( L_i \). In this case the momenta of the particles are labeled by a set of \((d - 1)\) integers \((n_1, \ldots, n_{d-1})\) such that the momentum in the \( i \)-th direction is equal to
\[
k_i = \frac{2\pi n_i}{L_i}.
\] (43)

By using either (28) or (32) we see that
\[
\beta F(\beta, L_1, \ldots, L_{d-1}, m) = f \left( \frac{\beta}{L_1}, \ldots, \frac{\beta}{L_{d-1}}, \beta m \right).
\] (44)

If we denote the total volume by \( V = L_1 \ldots L_{d-1} \) we can write the energy density as
\[
\rho = \sum_{i=1}^{d-1} \frac{1}{VL_i} \partial_i f \left( \frac{\beta}{L_i}, \beta m \right) + \frac{m}{V} \partial_d f \left( \frac{\beta}{L_i}, \beta m \right),
\] (45)
where \( \partial_i \) indicates the partial derivative with respect to the \( i \)-th entry of \( f \) and \( \partial_d \) that with respect to the last one. The pressure in the \( i \)-th direction is given by
\[
p_i = \frac{1}{VL_i} \partial_i f \left( \frac{\beta}{L_i}, \beta m \right)
\] (46)
so the equation of state is
\[
\rho = \sum_{i=1}^{d-1} p_i + \frac{m}{V} \partial_d f \left( \frac{\beta}{L_i}, \beta m \right).
\] (47)

For an isotropical system all the \( p_i \)'s are equal, so we have
\[
\rho = (d - 1)p + \frac{m}{V} \partial_d f \left( \frac{\beta}{L_i}, \beta m \right).
\] (48)

We are going to show that this equation of state is the same we get in the infinite volume limit. To see that, let us use the proper time expression for the Helmholtz free energy [9, 23] which is well defined
\[
F_{B(F)} \left( \beta, V, \frac{1}{m} \right) = -\frac{V}{2^{d+1} \pi^{d/2}} \int_0^{\infty} ds \ s^{-1-d/4} \left[ \theta_{3(4)} \left( 0 \left| \frac{i\beta^2}{2\pi s} \right. \right) - 1 \right] e^{-s^{1/2}}.
\] (49)
It is easy to check the homogeneity properties of this function

\[ F_{B(F)} \left( \lambda \beta, \lambda V, \frac{\lambda}{m} \right) = \lambda^{1-d} F_{B(F)} \left( \beta, V, \frac{1}{m} \right). \]  

(50)

Then, applying Euler’s theorem for homogeneous functions we get

\[ \rho = (d - 1)p + \frac{m}{V} \frac{\partial F_{B(F)}}{\partial m}, \]

(51)

which agrees with (48). Here we have used the fact that the free energy is a homogeneous function of degree one with respect to the volume and then \( p = -F/V \). One can also get (47) by defining the infinite volume \( V = L_1 \ldots L_{d-1} \) where each \( L_i \) goes to infinity. The second term in the right-hand side of (51) makes the equation of state for massive bosons and fermions different from one another. Incidentally, when we set \( m = 0 \) and \( d = 2 \) we recover the equation of state for the massless field in \( S^1 \times \mathbb{R} \). Furthermore, this is the equation of state for a massive field in \( d \)-dimensions with an arbitrary number of them compactified.

When we deal with an interacting theory we have to proceed with more care. Let us consider for example an interacting massless scalar field with dimensionless coupling constant \( \lambda \) living in \( T^{d-1} \times \mathbb{R} \). In this case, because of the divergences appearing in the Feynman diagrams containing loops, the renormalization procedure lead us to consider an effective, scale dependent coupling \( \lambda_{eff}(\mu) \). Since we are considering the system at finite temperature we see that the effective coupling constant entering in our equations has to depend on both \( \beta \) and \( L_i, \lambda_{eff}(\beta, L_i) \).

From dimensional arguments, we see that

\[ \beta F(\beta, L_i, \lambda_{eff}) = f \left[ \frac{\beta}{L_i}, \lambda_{eff}(\beta, L_i) \right] \]

(52)

Now we can compute both the energy density and the pressure and we get the following equation of state

\[ \rho = \sum_{i=1}^{d-1} p_i + \frac{1}{\beta V} \left[ \gamma_{\beta}(\beta, L_i) + \sum_{i=1}^{d-1} \gamma_{L_i}(\beta, L_i) \right] \frac{\partial f}{\partial \lambda_{eff}} \]

(53)

where

\[ \gamma_{\beta}(\beta, L_i) = \beta \frac{\partial \lambda}{\partial \beta} \]

\[ \gamma_{L_i}(\beta, L_i) = L_i \frac{\partial \lambda}{\partial L_i} \]

(54)

are a kind of finite temperature beta functions. The case of a massive field can be treated in a similar fashion.
Having all the ingredients needed, we can fulfill our other objective which is to determine the dynamics and the thermal history of our toy-universe. If we substitute $p = \rho$ in equations (7), (8) and (10) and set $\Lambda = 0$ we find the following equations

$$\dot{\Phi}^2 - \frac{2}{\omega} \Phi \frac{\dot{L}}{L} = -\frac{16\pi}{\omega} \Phi \rho,$$

(55)

$$\Phi \ddot{\Phi} + \frac{1}{2} \omega \dot{\Phi}^2 = -8\pi \Phi \rho,$$

(56)

$$\dot{\rho} + \frac{2}{L} \dot{L} \rho = 0.$$  

(57)

The equation (57) can be easily integrated to give

$$\rho(t) = \frac{\rho_0 L_0^2}{L^2(t)}$$

(58)

where the subscript zero indicates the initial values for the variables. Subtracting (55) from (56), we get a first integral, namely,

$$\frac{d}{dt}(L(t) \dot{\Phi}(t)) = 0$$

(59)

so we have

$$L(t) \ddot{\Phi}(t) = L_0 \dot{\Phi}_0.$$  

(60)

Using this first integral, we finally obtain a differential equation for the Brans-Dicke field $\Phi(t)$

$$-\frac{\ddot{\Phi}}{\Phi^2} = \frac{\omega}{2\Phi} + \frac{8\pi \rho_0}{\dot{\Phi}_0^2}$$

(61)

that can be solved numerically. The first part of this equation can be related with the first derivative of the scale factor $L(t)$ to get

$$\dot{L}(t) = \frac{\omega \dot{\Phi}_0 L_0}{2\Phi(t)} + \frac{8\pi \rho_0 L_0}{\dot{\Phi}_0}.$$  

(62)

From this last equation we see that whenever the Brans-Dicke field grows big enough, the first term in the right-hand side of (62) can be neglected and our universe will expand linearly with slope $8\pi \rho_0 L_0/\dot{\Phi}_0$. The range in which this approximation is faithful depends strongly on the initial values for the dynamical variables $\rho_0$, $\Phi_0$, $\dot{\Phi}_0$ and $L_0$. We see that, whenever $\omega < 0$ we have that $L(t)$ reaches a minimum value.

To study the thermal history of our toy-universe, we make use of equation (58) and the results obtained in section 3 for the energy density of a massless field. In the case of a massless
bosonic field, we have that, according to equation (24), the thermal history $\beta(t)$ is given by the solution of the following transcendental equation

$$\rho_0 L_0^2 = -\frac{\pi}{6} E_2 \left( i \frac{\beta(t)}{L(t)} \right)$$  \hspace{1cm} (63)$$

where $L(t)$ is given by the solution of (62). Here we have introduced the Casimir energy that was subtracted in section 3. In the case of a fermionic field the equation to solve is a bit more complicated, namely

$$\rho_0 L_0^2 = \frac{\pi}{3} E_2 \left( i \frac{2\beta(t)}{L(t)} \right) - \frac{\pi}{6} E_2 \left( i \frac{\beta(t)}{L(t)} \right).$$  \hspace{1cm} (64)$$

From these equations we can extract a qualitative conclusion about the behavior of $\beta(t)$. Since the left-hand side of equations (63) and (64) is a constant, we must have that

$$\beta(t) = C_{B(F)} \times L(t)$$  \hspace{1cm} (65)$$

where the value of $C_{B(F)}$ is determined from the differential equations themselves together with the thermodynamics. In the uncompactified limit, these constants can be easily gotten from the results of section 3, namely, $C_B = \sqrt{2} C_F = \sqrt{\pi/(6\rho_0 L_0^2)}$.

All this situation changes as soon as we add a vacuum energy density, i.e., a constant to $F(\beta, L)/L$. Where this constant might come from is an irrelevant question at this moment. Needless to say that we have in mind a string as a source for the matter fields in order to justify its addition. What happens now is that the proportionality between $\beta(t)$ and $L(t)$ is broken because of the appearance of this constant on the right-hand side of (63) and (64) with the corresponding signs (negative for the bosonic case and positive in the fermionic one). After all, the equation of state is no longer $p = \rho$ but $p = \rho + \text{constant}$ so the dynamics of our universe changes. In fig. 1 we plot the temperature vs. time for the situation in which the constant is absent (C2) and when it is turned on (C1).

In fig. 2, $\rho_0 L_0^2$ vs. the value of the quotient $\beta/L$ is plotted. The curve $F1$ corresponds to compact space with a Casimir energy of $\pi/(6L^2)$. The only effect of this vacuum energy is that of shifting the possible values of $\rho_0 L_0^2$. $F2$ is the curve for the regular $\mathbb{R}^2$ universe at finite temperature while $F3$ is the curve for the compactified universe after subtracting the above-mentioned Casimir energy. Comparing $F2$ and $F3$ we notice that, since $L$ is the same for both universes as a function of the cosmic time, the effect of the periodic boundary conditions in the spatial dimension is to produce, for a given size, a universe hotter than in the open space case.

Essentially the same applies to the bosonic case (fig. 3). Now, we see that the Casimir energy is negative, so in principle we can allow negative values for $\rho_0$ as it is shown in curve $B1$. Now we find again that the universe with the spatial dimension compactified in a circle is
hotter than its uncompactified counterpart \((B2)\) as it is evident from comparing the curve \(B2\) with \(B3\) in which we have subtracted the (negative) Casimir energy.

When the cosmological constant \(\Lambda\) is turned on, the equations governing the dynamics are

\[
\frac{d}{dt}(L\dot{\Phi}) = 2\Lambda\Phi L \tag{66}
\]

\[
\ddot{\Phi} + \frac{1}{2\omega}\frac{\dot{\Phi}^2}{\Phi} = -8\pi\rho + \Phi\Lambda \tag{67}
\]

\[
\dot{\rho} + 2\rho \frac{\dot{L}}{L} = 0. \tag{68}
\]

From these equations we see that, since the cosmological constant \(\Lambda\) is always multiplied by the Brans-Dicke field \(\Phi(t)\), when this is small enough the solutions are the same as those corresponding to a vanishing cosmological constant. In the case of equation (66) we have besides the product \(\Lambda\Phi L\). This means that, in order to get a behavior similar to the \(\Lambda = 0\) case we must fulfill two conditions, namely, \(\Lambda\Phi \ll 1\) and \(\Lambda\Phi L \ll 1\). This is the case for small \(t\) (see fig. 4) in which the curves corresponding to the \(\Phi\) field for \(\Lambda > 0\) and \(\Lambda < 0\) coincide with that of the \(\Lambda = 0\) case. For the values of \(t\) for which the above approximations cannot be made we get a splitting of the curves corresponding to the three cases (see fig. 5). For \(\Lambda > 0\) the Brans-Dicke field grows as \(\exp(t\sqrt{2\Lambda})\) whereas in the case of a negative cosmological constant the \(\Phi\) field reaches a maximum after which it begins to decrease. The marginal case \(\Lambda = 0\) gives a logarithmic growing for \(\Phi(t)\). All the plots and the reasoning have been made taking \(\omega = -1\) having in mind the \(c = 1\) non-critical string (see for example [3]).

The behavior of \(L(t)\) can be analyzed along the lines of those for \(\Phi(t)\). For large \(t\) we get again a splitting of the curves (fig. 7). The effect of a positive cosmological constant is to produce an asymptotically static universe. On the contrary when \(\Lambda < 0\) the universe inflates. For short times in which the universe is close to its minimum length the three cases are undistinguishable. As a final comment, it is worth noticing that the plots have been made taking \(\dot{\Phi}_0 > 0\). Owing to the fact that the equations are invariant under time reversal, the solutions with \(\dot{\Phi}_0 < 0\) correspond to traveling backwards in time in the solutions plotted.

5 Conclusions and outlook

We have concluded that the presence of compactified dimensions has no influence on the dynamics of the universe, since this does not modify the equation of state of the matter. On the other hand, the functional relation between the pressure, the density, and the temperature changes. As a consequence, the universe with compactified spatial dimensions is hotter than the regular \(\mathbb{R}^2\) universe.
A point to clarify is that of the relationship between the Helmholtz free energy and the would-be corresponding toroidal compactification in the massless case. In a first sight it is clear that the relationship is broken because the would-be related compactification, namely, the partition function for a massless particle on a torus, should be invariant under the exchange \( \beta \leftrightarrow L_i \). A bridge joining both calculations would be the proper time representation of the Helmholtz free energy. The main question is whether that representation exists in this case and is well defined. It is easy to apply the expression given in [2] for \( \beta F(\beta) \) to the case of \( S^1 \times \mathbb{R} \) (this is easily generalizable to any number of compactified dimensions) to get

\[
\beta F(\beta) = L \frac{d}{dt} \left( \frac{1}{2\Gamma(-t)} \int_0^\infty ds \, s^{-t-1} e^{-s \theta_3(0 \mid \frac{4\pi^2 s}{\beta^2})} \right)_{t=0}
\]

where the second equation follows from the first one by formally taking the derivative. Both integrals are ultraviolet divergent. We can fix the problem by subtracting the vacuum energy corresponding to the limit in which \( \beta \to \infty \) and \( L \to \infty \) simultaneously, i.e.,

\[
- \frac{L \beta}{4\pi} \int_0^\infty \frac{ds}{s^2} e^{-s \theta_3(0 \mid \frac{2\pi^2 s}{L^2})} e^{-s m^2/2} \tag{70}
\]

Setting \( m = 0 \) in (69) we add another divergence in the \( s \to \infty \) limit. Being only concerned about mathematics one can fix again the problem by subtracting at the same time

\[
- \frac{1}{2} \int_0^\infty \frac{ds}{s} \tag{71}
\]

Physically, this divergence is the result of including the second-quantized version of the zero momentum state that, when computing the trace, is suppressed because, when \( m = 0 \), the only state with zero momentum is the vacuum state. If we try to keep on track of this expression by commuting the integral with the sums in order to extract the regulated finite result, we find that the output of this manipulation does not depend on both \( \beta \) and \( L \). The situation changes when an ultraviolet cutoff in proper time is included in the second part of (69) with \( m = 0 \) because in that case the formal integration produces an expression which at least depends on \( \beta \) and \( L \). For this phenomenon to occur it is crucial the form of the integration measure, \( ds/s \), since there is no jacobian induced by a linear change of variables and, in the absence of a cutoff, the limits of integration are not modified either. Using this ultraviolet lower limit for the proper time we still have to subtract the infrared divergence. Instead of putting a cutoff we can also try dimensional regularization, and at the same time subtract (70) from (69) with \( m = 0 \) to fix the ultraviolet divergence. Doing so we get

\[
\beta F(\beta, L) = \ln(\mu L) + 2 \ln \left[ \eta \left( \frac{L}{\beta} \right) \right] \tag{72}
\]
where we have subtracted the term in $1/(d-1)$ together with other constant terms and $\mu$ is the energy scale introduced when dimensionally regularizing. This expression enjoys the invariance under the exchange $\beta \leftrightarrow L$ (although it does not give the partition function of a massless boson in $S^1 \times S^1$). We are now very close to a string variation of our massless field. If in the right hand side of (72) we make the replacement $\beta \rightarrow (2\pi)^2/(\mu^2 \beta)$ and add the resulting new term to (72) we get, upon the identification $\sqrt{\alpha'} = \mu^{-1}$, what would be the one-loop free energy for the $c = 1$ model in a $S^1 \times \mathbb{R}$ target [24, 25, 7]. By construction this new expression in invariant under the replacements $\beta \leftrightarrow L$ and the transformation $\beta \rightarrow (2\pi)^2/(\mu^2 \beta)$. An immediate consequence is that the equation of state is no longer $p = \rho$ [17]. After all, strings are not equivalent to quantum fields. It seems that they are equivalent to fields with a kind of ultraviolet cutoff but not at the self-dual point; what we have is a cutoff in proper time provided by modular invariance. However, the Helmholtz free energy gotten from the dimensionally regularized massless boson using this procedure presents unphysical thermodynamical properties. For example it gives a negative infinite entropy in the low temperature limit. This lead us to claim that the equivalence between the Helmholtz free energy and the toroidal compactification is also broken for the $c = 1$ non-critical string [17].

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References


Figure 1: Temperature vs. $t$ with and without a vacuum energy.

Figure 2: $\rho_0 L_0^2$ vs. $\beta/L$ for a massless fermionic field.
Figure 3: $\rho_0 L^2$ vs. $\beta/L$ for a massless bosonic field.

Figure 4: $\Phi$ field vs. $t$ for negative time
Figure 5: $\Phi$ vs. $t$ for several values of the cosmological constant

Figure 6: $L(t)$ vs. $t$ for negative time
Figure 7: Scale factor $L(t)$ vs. $t$ for several values of the cosmological constant.