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Quantum anyon-like systems
with topological charges on
 $U(1)$ -bundles

Moscow — 1992

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Abstract

We investigate some class of locally Hamiltonian systems. To formulate a global Hamiltonian formalism a non-trivial principal $U(1)$ -bundle over multiparticle dynamical space is constructed. Lift of dynamics to the total space is considered for the models. Absence of a transposition operator lifting to the bundle space forces us to review a statistics theme from new standpoint. A topological invariant to characterize system properties is generalized by using geometrical methods and anyon analogy.

1 Introduction

A regular interest to planar physical systems in the context of fractional spin and statistics don't lost at the last time. Such objects to demonstrate the peculiar statistics belong among wide range of quantum systems, which can not be described a continuous wave function globally defined in the dynamical space. To study their dynamics the multivalued or singular functions [1] with a string should be used. One last glowing example of this sort is a quantum dynamics of anyons i.e. the particles obeying the fractional statistics [2, 3, 4, 5]. The descriptions of fractional quantum Hall effect [6] and two-dimensional electron gas excitations in a strong magnetic field such as "incompressible quantum fluid" [7] are based on the multivalued functions formalism also [8, 9].

It must be emphasized that absence of the non-ambiguous wave function is a result of nonexistence of the globally defined Hamiltonian for previously named systems. As is shown in [10, 11], the adequate description of these systems demands the evident generalization of path integral formalism. Another way of looking at it is to consider the self-adjoint extensions of initial Hamiltonian [12].

Notice that the examples of physical systems without globally defined Hamiltonian (when it can be defined only locally or on the limited class of "good" functions) are familiar. For some of these systems, the most natural description of dynamics is given by combining position coordinates with gauge coordinate by virtue of nontrivial principal G -bundle having dynamical space as a base and group G as a fibre.

In the muchstudied case that a charge-Dirac monopole system is discussed [13], the method based on a line bundle conception allows to exclude monopole string from consideration [14, 15]. Moreover, the dynamics lifted on a total space of Dirac monopole fibre bundle becomes Hamiltonian [16]. Essentially in this context that a physical system as a whole is assigned the topological invariant called topological charge and descended from bundles classification.

It shall be demonstrated that the geometry features of nontrivial $U(1)$ -bundle over multiparticle dynamical space must be taken into account in studies of the above-mentioned objects. The several peculiar properties to have the topological nature arise:

- all attempts to restrict of our treatment to the bundle base lead to the appearance of singularities of multiparticle wave function;
- the topological reasons forbid a decay (even virtual) of two- (multi-) particle system into the constituents if the initial two- (multi-) particle state is defined on the nontrivial bundle, because of the initial and final states are assigned to the different bundles;
- transposition operator will be shown to be lacking in total space because the question about particle statistics returns to interest.

Leaning upon the familiar example of charge-flux tube system, we extract and exactly formulate the main piquancy of the suggested approach — the structure of $U(1)$ -bundle. Going to the anyons theme, we apply the bundle conception to the dynamics analysis (it may be compared with another points of view when the “virtual flux tubes” are thought to exist in a dynamical space). It should be pointed out now that the term “anyon” has not the standart sense in the present paper. First and foremost we concentrate our attention on the possibility to specify *the permutation group action on the total bundle space* and induced action on global wave functions defined on it. After that we use some pictorial models appearing in the well-known anyon problems. But a braid group of constructed space is not studied.

We reduce more complicated planar multiparticle system to the simplest model of two-particle states attributed to $U(1)$ -bundle over torus $S^1 \times S^1$, when a circle S^1 is particle dynamical space. This example only demonstrate the meaning features of described states (Section 3). Section 4 contains the myltparticle systems analysis basing on a formalism of quasi-particles which are excitations of the collective degrees of freedom. In closing (Section 5) we introduce a new class of dynamical spaces. These spaces — Seifert bundles — serve to generalize configuration spaces considered in the previous Sections. To list all such bundles we use gemetrical treatment elaborated by Thurston [17] and the three-dimensional compact geometry classification.

2 Simplest examples and approaches

We start with the well-known Wilczeks’ example (see Ref. [18]): a point electric charge moving on a plane l in the external magnetic field of a thin

solenoid perpendicular to the plane. This system provides the needed material for the origin of fractional spin and statistics. It must be emphasized that three-dimensional physics is considered here, thus the initial assumption to be operated have treated the plane l as embedded in R^3 . Consequently, one could be called a quasi-planar system. For greatest generality, let us investigate a charge in an arbitrary magnetic field ¹, then dynamics is described on l by the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{x}}^2 + e\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}), \quad (1)$$

where e and m are the charge and mass of particle, interacting with the potential $\mathbf{A}$. If the potential $\mathbf{A}$ is chosen in the form

$$A^a = -\frac{\epsilon^{ab}}{2\pi} \frac{x^b}{r^2} \Phi(r), \quad a(b) = 1, 2; \quad (2)$$

this is a solenoid field case.

To consider the global properties of wave functions we calculate magnetic field $B = B_z$ ($z \perp l$) and flux Φ through l or any surface σ containing the solenoid

$$\oint_l B(\mathbf{x}) d\sigma.$$

By assuming that $\Phi \neq 0$, we produce conceptually a Dirac magnetic monopole in our theory. Although we shall restrict our consideration to dynamics on l , it is necessary to stress that the global space and the global behavior of solutions are changed essentially in the theory with a monopole (see Refs. [19]). Remember, an electric charge is quantized in this framework $eg = n/2$, where g is a monopole charge and n is integer. There are two approaches to formulate particle dynamics in magnetic field when a monopole is lacking:

1. the global treatment on l ; $\Phi = 0$ and it is impossible to describe one isolated solenoid with flux $\Phi_1 \neq 0$. Several solenoids with nonzeros fluxes Φ_i may exist with the constraint $\sum_i \Phi_i = 0$ only.
2. the local treatment in a bounded region U , $U \subset l$; thus all real systems (crystals, thin films and i.e.) are bounded in the space, the global

¹The solenoid field will be considered as the limiting case.

properties of wave functions should be corresponded to the space U other than a plane l . Then we may satisfy equality

$$\oint_U B(\mathbf{x}) d\sigma = \Phi_0,$$

where Φ_0 is arbitrary.

To formulate dynamics and study some peculiar features of the anyon-like systems on closed domain U we should construct the adequate mathematical language and illustrate one by an example of the above-mentioned system.

Concentrate our attention to fixed domain U , bounded by the inequalities

$$0 \leq x_1 \leq L, \quad 0 \leq x_2 \leq L,$$

where L are linear sizes of system. For simplicity, field B is chosen uniform now and flux through U

$$\frac{\Phi}{2\pi} = \frac{BL^2}{2\pi}$$

equals integer n . The case of rational Φ is fundamentally analogous, one will be considered in Section 5, but the irrational one is much more difficult for analysis. Choosing the magnetic potential $\mathbf{A}$ in the form

$$A_1 = -Bx_2, A_2 = 0, \quad (3)$$

we have Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 = -\frac{1}{2m} \{(\partial_1 - ieBx_2)^2 + \partial_2^2\}. \quad (4)$$

The problem to find the boundary condition is delicate. We may suppose that on boundary of U

1. the wave functions vanish or tend to a constant,
2. the wave functions are limited,
3. the boundary conditions are periodic in x_i ; all effects are believed to occur in a square L on a side. Then this square can be transformed to torus T^2 by identifying of opposite sides in pairs.

It is *not* a fruitful idea to compare together the different boundary conditions in detail, since we shall show that topological invariants are independent of them. Moreover, the distinctions of wave functions behavior on the boundary lose their meaning as value L increases. The first and second assumptions mean that a sphere serves as the particle dynamical space in accordance with Haldanes' consideration of Ref. [20]. Whole boundary is glued together to one point. In present paper the torus realization is chosen because of the multiparticle dynamics has more simpler form on $T^2 \times \dots \times T^2$. Torus T^2 locally conserves the Euclidean geometry of plane also.

There is one difficulty of such treatment. It is apparent that although magnetic field B is continuous (constant) on T^2 , the Hamiltonian (4) can't be globally defined on the torus thus potential $\mathbf{A}$ is ambiguous at $x_2 = 0 = L$ (see (3)). After going from a plane l to a torus the Hamiltonian system (4) transforms to nonhamiltonian one. To demonstrate this fact let us rewrite Lagrangian (1) to the form

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{y}}^2 + 2\pi n\dot{y}_1 y_2 \quad (5)$$

$$0 \leq y_{1,2} \leq 1$$

by the variables $y_i = x_i/L$. It is conserved by the transformation $x_1 \rightarrow x_1 + L$ or $y_1 \rightarrow y_1 + 1$, but it varies as $x_2 \rightarrow x_2 + L$ or $y_2 \rightarrow y_2 + 1$. Then the additional term of total derivative with respect to time $\frac{d}{dt}(2\pi n y_1)$ appears in (5). When the Hamiltonian formalism is used this means that the wave functions must satisfy the following boundary conditions on U

$$\begin{aligned} \Psi(y_1, y_2) \big|_{y_2=0} &= e^{2\pi i n y_1} \Psi(y_1, y_2) \big|_{y_2=1} \\ \Psi(y_1, y_2) \big|_{y_1=0} &= \Psi(y_1, y_2) \big|_{y_1=1} . \end{aligned} \quad (6)$$

Any more rigorous treatment requires the use of local sections instead of the globally defined wave function (with the boundary condition (6)). We are coming now to the question of reformulating by general geometric language. We would construct the total space P combining the position coordinates on T^2 with a gauge coordinate $U(1)$. This total space P is not the usual Cartesian product $T^2 \times U(1)$, since the wave function is not exactly periodical on a torus. P may be thought as the Cartesian product only locally, if U is some open subset of T^2 ($U \subset T^2$) then $U \times U(1) \subset P$. It shall assumed that torus is decomposed in the union of two cylinders

$$\begin{aligned} Cyl_\alpha &= \{y_1, y_2 \mid 0 \leq y_1 \leq 1, 0 - \delta \leq y_2 \leq 1/2 + \delta\} \\ Cyl_\beta &= \{y_1, y_2 \mid 0 \leq y_1 \leq 1, 1/2 - \delta \leq y_2 \leq 1 + \delta\}, \end{aligned}$$

their intersection is the union of two separated sets C_1 and C_2

$$\begin{aligned} C_1 &= \{y_1, y_2 \mid 0 \leq y_1 \leq 1, 0 - \delta \leq y_2 \leq 0 + \delta\} \\ C_2 &= \{y_1, y_2 \mid 0 \leq y_1 \leq 1, 1/2 - \delta \leq y_2 \leq 1/2 + \delta\}. \end{aligned}$$

The transition from the wavefunction Ψ_α on Cyl_α to the wavefunction Ψ_β on Cyl_β is determined by the transition function $F_{\alpha\beta}$ according with (6):

$$\begin{aligned} \Psi_\alpha(y_1, y_2) &= \Psi_\beta(y_1, y_2) \text{ on } C_1 \\ \Psi_\alpha(y_1, y_2) &= e^{i2\pi n y_2} \Psi_\beta(y_1, y_2) \text{ on } C_2. \end{aligned} \quad (7)$$

Transition function $F_{\alpha\beta}$ is defined as the mapping from intersection of domains into group $U(1)$. Then $F_{\alpha\beta}$ has the form on $C_1 \cup C_2$

$$\begin{aligned} (y_1, y_2) &\rightarrow 1 \text{ on } C_1, \\ (y_1, y_2) &\rightarrow e^{i2\pi n y_1} \text{ on } C_2. \end{aligned} \quad (8)$$

Thus, a state of the system is described by a pair of functions (each has its own domain) connected on the overlap $Cyl_\alpha \cap Cyl_\beta = C_1 \cup C_2$ by (7). This means that a complex-line fibre bundle with a base $B = T^2$ is realized, pair of functions Ψ_α, Ψ_β make up a section of this bundle. Alternatively, one can work with a principal fibre bundle P associated with a complex-line bundle in which case the pairs Ψ_α and Ψ_β are represented by a complex function Ψ on P [21]. For the purpose of its construction the Cartesian products should be considered and the points in $(Cyl_\alpha \cap Cyl_\beta) \times U(1)$ should be identified as follows

$$\begin{aligned} (y_1, y_2; g)_\alpha &\equiv (y_1, y_2; g)_\beta \text{ on } C_1 \times U(1), \\ (y_1, y_2; g)_\alpha &\equiv (y_1, y_2; \exp(2\pi i n y_1) g)_\beta \text{ on } C_2 \times U(1), \end{aligned}$$

where the elements of set $Cyl_\alpha \times U(1)$ are on the left side of the identity and the elements of set $Cyl_\beta \times U(1)$ are on the right side of one. This construction is readily illustrated by Figure 1.

Finally, the constituents of discussed principal bundle are:

- the total space P ;
- action $U(1)$ on P which in local coordinates on $U \times U(1)$ has the form $(u, g) \rightarrow (u, g e^{ia})$, where $u \in U, g \in U(1), e^{ia} \in U(1)$;

- the projection π of P to the base B $\pi : P \rightarrow P/U(1) = B$;
- the distinction between different n lies in the choice of the class of functions $\Psi : P \rightarrow \mathcal{C}$, which is determined by the equivariant condition

$$\Psi(pe^{ia}) = e^{ina}\Psi(p), \quad (p \in P).$$

Last item is known to ensure that the global dynamics to be defined on P transform to initial one (5) on the base. By other mean we may consider the family of fibre bundles P_n over torus where P_n is a quotient space of $P = P_1$ by the action of discrete subgroup $\mathcal{Z}_n$ of $U(1)$ with generator $e^{2\pi i/n}$. The different integers n to be concerned with the unequivalent $U(1)$ -bundles P_n are the first Chern indexes of the associated complex-line fibre bundles.

From the homotopy standpoint, the total space P_n is dictated by the homotopy class of transition function. In the torus case the set of homotopy classes $[S^1, S^1]$ ² classifies all unequivalent fibre bundles over T^2 . Thus, the transition function may be deformed but if the homotopy class $[S^1, S^1]$ is conserved then the bundle have not been changed. The case of $n = 0$ is specific the bundle P_0 is trivial i.e. it is decomposed in the Cartesian product $T^2 \times U(1)$.

Now one helpful mathematical trick should be received proper attention. Varying the magnetic field (flux remains fixed and integer), we modify (8) as follows

$$\begin{aligned} (y_1, y_2) &\rightarrow e^{i2\pi n f(y_1)} \text{ on } C_2, \\ (y_1, y_2) &\rightarrow \mathbf{1} \text{ on } C_1, \end{aligned} \tag{9}$$

where

$$f(y_1) = \begin{cases} y_1/\epsilon & 0 \leq y_1 \leq \epsilon \\ 1 & \epsilon \leq y_1 \leq 1 \end{cases}.$$

It is obvious that the homotopy class of transition function is not exchanged. New bundle to be constructed by (9) is equivalent to previous one. Figure 1 demonstrates such possibility. In the limit $\epsilon \rightarrow 0$ the potential ambiguity is shrank to sole point Q on C_2 . The case of solenoid field (2) is realizing now. Excluding the point Q , the magnetic field vanishes on a torus and there is the flat connection on $U(1)$ -bundle over $T^2/\{Q\}$. Q is the point of intersection of plane l and the solenoid possessing flux $\Phi/2\pi$. It is necessary

²Here, the first S^1 is the homotopic equivalent of C_2 and the second S^1 is the group $U(1)$.

to remember once again that the solenoid field with an integer (up to 2π) flux to be examined on the closed 2D-manifold is a limiting case of arbitrary field described in the terms of nontrivial $U(1)$ -bundles.

Now that we have presented the fibre bundles P_n , we focus upon anyons theme. Firstly, some starting hints should be dropped. Using the virtual flux tubes concept [7] we are led to the *anyons* Lagrangian [5] ³

$$\mathcal{L}_{anyon} = \sum_i \frac{1}{2} m (\dot{\mathbf{y}}^i)^2 + \mathcal{L}_{int} , \quad (10)$$

$$\mathcal{L}_{int} = \sum_{i \neq j} \frac{\Phi}{2\pi} \frac{d}{dt} \Omega(\mathbf{y}^i(t) - \mathbf{y}^j(t)) , \quad (11)$$

where Ω is the polar angle to depend on the relative positions of pairs of particles

$$\Omega[\mathbf{y}] = \arctan\left(\frac{y_2}{y_1}\right), \text{ here } \mathbf{y} = (y_1, y_2).$$

The sums run over particles, and the points $\mathbf{y}^i = \mathbf{y}^j$ are excluded from configuration space ⁴.

Parameter $\Phi/2\pi$ is known to indicate the peculiar behavior of wave function under permutation of pair of indistinguishable particles

$$\psi(\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^d) = e^{-i\Phi} \psi(\mathbf{y}^2, \mathbf{y}^1, \dots, \mathbf{y}^d).$$

Thus the values Φ and $\Phi + 2\pi k$ ($k \in \mathbb{N}$) are not distinct. In the strict sense lagrangian (10) is functional to be appropriate for path integral using only, and an effective particle interaction of the form (11) is nonlocal. We shall demonstrate that for the integer $\Phi/2\pi$ such form of interaction is only the limiting case of more general situation (compare with above example) which may be formulated on the nontrivial $U(1)$ -bundle P over multiparticles configuration space $T^2 \times \dots \times T^2$. As is easy to see, in this instance there is the fundamental distinction between cases of Φ , $\Phi + 2\pi$, ..., since the total spaces of respective bundles will differ. Moreover, the transposition operator

³We restrict our attention to a torus $0 \leq y_i = x_i/L \leq 1$ as before.

⁴ The next usual procedure [22], when the symmetric points (under the symmetric group of d object action, S^d) are identified in configuration space (i.e. $(\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^d) \equiv (\mathbf{y}^2, \mathbf{y}^1, \dots, \mathbf{y}^d)$), is ignored in the present paper.

is proved to be absent on these non-trivial bundles ($\Phi \neq 0$). Our conclusions supplement the results found by the use of path integral formalism and analysis of irreducible unitary representations of the fundamental group $\pi_1(Q)$ [23]; it is impossible to detect $U(1)$ -bundles by the latter approach.

Let us inspect Lagrangian (10) for two particle case. Particles are marked by a and b . It is convenient usually to make the transition to coordinates relative to the center of mass. As known there is no a way to do it on the closed manifold $T^2 \times T^2$. To obviate appearing barrier we construct some covering of torus T^4 which locally gives the desired transition. Namely, if $T^4(y_1^a, y_2^a, y_1^b, y_2^b) = \mathcal{R}^4 \pmod{\mathcal{Z} \oplus \mathcal{Z} \oplus \mathcal{Z} \oplus \mathcal{Z}}$ and $T^4(R_1, R_2, r_1, r_2) = \mathcal{R}^4 \pmod{\mathcal{Z} \oplus \mathcal{Z} \oplus \mathcal{Z} \oplus \mathcal{Z}}$ then covering map g is

$$g : T^4(R_1, R_2, r_1, r_2) \rightarrow T^4(y_1^a, y_2^a, y_1^b, y_2^b), \quad (12)$$

where $\mathbf{y}^a = \mathbf{R} + \mathbf{r}$ and $\mathbf{y}^b = \mathbf{R} - \mathbf{r}$. Ignoring the coordinate $\mathbf{R}$ dependence of Lagrangian we shall concentrate our attention on the relative coordinate $\mathbf{r}$ and torus $T^2(r_1, r_2)$. We shall demonstrate that if non-trivial bundle P_n over $T^2(r_1, r_2)$ is realized as the dynamical space where a global Lagrangian is defined and continuous then the Cartesian product $P_n \times T^2(R_1, R_2)$ gives the suitable space for “anyon” description. The possibility of problem reduction to one spartial dimension is clearly visible now. We shall do this in the next section.

3 Two particles on the torus $S^1 \times S^1$

Consider the scalar particles system in the 1D-space S^1 . Any $U(1)$ -bundle over S^1 is trivial, therefore the gauge coordinate dependence of solutions is not essential. The dynamical space of d particles system is a torus $T^d = S^1 \times \dots \times S^1$. Tori to have the dimension $d \geq 2$ possess non-trivial cohomology group $H^2(T^d, \mathcal{Z})$ and thus in line with the theorem of bundles classification, there are non-trivial principal $U(1)$ -bundles over base T^d . The group $H^2(T^d, \mathcal{Z})$ may be mapped one-to-one on a set of classes of equivalent principal $U(1)$ -bundles.

Let y^a and y^b are the coordinates on a torus T^2 , $(y^a, y^b) \in \mathcal{R}^2 \pmod{\mathcal{Z}, \mathcal{Z}}$. The two-particle Hamiltonian is assumed to be

$$H = H_0 + \mathcal{U}, \quad (13)$$

where interaction term $\mathcal{U}$ (possibly nonlocal) is defined only locally (see Eq.4 for comparison), and

$$H_0 = -\frac{1}{2m_a}\partial_a^2 - \frac{1}{2m_b}\partial_b^2. \quad (14)$$

The wave function is replaced by the local sections ψ_α and ψ_β defined in the domains Cyl_α and Cyl_β respectively. We fix the transition functions $g_{\alpha\beta}(y^a, y^b)$ in an intersection $Cyl_\alpha \cap Cyl_\beta$ and postulate

$$\psi_\alpha(y^a, y^b) = g_{\alpha\beta}(y^a, y^b)\psi_\beta(y^a, y^b).$$

Choosing the transition function as for (8), we resume a discussion of fibre bundles P_n features in a new physical context.

The regular question may be asked. How does our current framework correlate with the popular theme of statistics in one dimension [24]? To compare these treatments together, we should digress for a while to discuss some extensive details. It may be emphasized only that configuration spaces are quite different (see footnote on the page 10). The usual identification procedure, as we suggest, is in rather poor agreement with the quantum mechanics spirit.

To reformulate dynamics on the bundle P_n a handy coordinatization of total space is required. The Heisenberg group $\mathcal{H}$ (i.e the group of real upper triangular matrices of third order) is proved to cover all nontrivial fibre bundles over torus T^2 [17]. The total space P_1 is the quotient of $\mathcal{H}$ by the discrete group Γ , where

$$\begin{pmatrix} 1 & y^a & \varphi \\ 0 & 1 & y^b \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{H}, \quad y^{a,b}, \varphi \in \mathcal{R};$$

$$\begin{pmatrix} 1 & N_1 & N_3 \\ 0 & 1 & N_2 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma, \quad N_i \in \mathcal{Z}.$$

Projection $\mathcal{H}$ to an Euclidean plane $l = E^2(y^a, y^b)$ defines, when quotienting $\mathcal{H}$ by Γ and $E^2(y^a, y^b)$ by $\mathcal{Z} \oplus \mathcal{Z}$, projection π

$$\pi: \mathcal{H}/\Gamma \rightarrow E^2/\mathcal{Z} \oplus \mathcal{Z},$$

giving the $U(1)$ -bundle structure over $T^2(y^a, y^b) = E^2(y^a, y^b)/\mathcal{Z} \oplus \mathcal{Z}$. A commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\hat{\pi}} & E^2 \\ p \downarrow & & p' \downarrow, \end{array} \quad (15)$$

$$P = \mathcal{H}/\Gamma \xrightarrow{\pi} T^2$$

(where the down arrows denote the covering maps) clearly shows a bundle projection. According with (15), dynamic equations on the P_n over T^2 may be rewritten as equations on an universal covering space $\mathcal{H}$ with additional condition of wave function invariance under discrete group Γ action.

However, from this point on we shall use another realization of the universal covering space $\tilde{P}$ of bundle space [17]. The space $\tilde{P}$ is a twisted product of $\mathcal{R}$ and E^2 . Isometry group $Isom(\tilde{P})$ is an extension of the group of isometries of E^2 by $\mathcal{R}$

$$0 \rightarrow \mathcal{R} \rightarrow Isom(\tilde{P}) \rightarrow Isom(E^2) \rightarrow 1.$$

For more bright exhibition of $Isom(\tilde{P})$, let us consider $\mathcal{R}^3$ with a certain 2-plane field τ , constructed as below. At points $\mathbf{y} \in \mathcal{R}^3$ with coordinates (y^a, y^b, y) a plane of field τ is spanned by two vectors

$$\mathbf{e}_a - \frac{y^b}{2}\mathbf{e} \quad \text{and} \quad \mathbf{e}_b + \frac{y^a}{2}\mathbf{e}.$$

The group $Isom(\tilde{P})$ contains the affine maps of $\mathcal{R}_3$ of special sort. They may be thought of as formed by the lifting of Euclidean isometries of plane $\{\mathbf{y} \in \mathcal{R}^3 \mid y = 0\}$ to an affine map of $\mathcal{R}_3$, preserving field τ and distances along lines parallel to the axis $\mathcal{R} \cdot \mathbf{e}$. A discrete transformation group Λ to be generated by two generators

$$(y^a, y^b, y) \rightarrow (y^a + 1, y^b, y + y^b/2),$$

$$(y^a, y^b, y) \rightarrow (y^a, y^b + 1, y - y^b/2)$$

is a subgroup of $Isom(\tilde{P})$. It is easy to show that the quotient space $\tilde{P}/\Lambda$ is the desired non-trivial principal $U(1)$ -bundle over T^2 . Here, the analog of diagram (15) takes place also.

A tangent vector

$$\tilde{V} = -\frac{i}{2\pi}\partial, \quad (\partial = \frac{\partial}{\partial y}),$$

is defined at every point on $\tilde{P}$. The vector field $\tilde{V}$ is invariant under action of discrete group Λ , because it is projected on P_1 by covering map p where the vertical (tangent to fibres) vector field $V = p_*\tilde{V}$ exists. Field V determines an action of group $U(1)$ on principal bundle space P_1 .

There is the $U(1)$ -invariant measure $d\mu$ on P . Under local trivialization in coordinates $U_\alpha \times U(1)$ this measure is uniquely dictated by condition $p_*d\mu = dy^a \wedge dy^b$ and have the form of volumes product.

A complex functions space $\mathfrak{S}$ on P_1 is shown to be decompose into direct sum $\mathfrak{S} = \oplus_n \mathfrak{S}_n$ of orthogonal subspaces $\mathfrak{S}_n$ of n -equivariant functions. In fact, the analog of Fourier analysis on a wave functions space takes place on a principal $U(1)$ -bundle. The space of wave functions corresponding to a topological charge $n \neq 0$

$$\mathfrak{S}_n = \{\psi(u) \mid \psi(ue^{i\alpha}) = e^{in\alpha}\psi(u)\}, \quad (u \in P),$$

is orthogonal to that for $m \neq n$, where the scalar product is

$$\langle \psi', \psi'' \rangle = \int_P \psi'^*(u) \psi''(u) d\mu. \quad (16)$$

Every function $\psi_n \in \mathfrak{S}_n$ on $P = P_1$ is yet the function defined on the bundle $P_n = P/\mathcal{Z}_n$ (see previous section). The integers $n \in \mathcal{Z}$ are naturally consistent with the elements of homology group $H^2(T^2, \mathcal{Z}) = \mathcal{Z}$.

There is the tantamount definition of subspace $\mathfrak{S}_n$ as the eigenfunctions space of operator V :

$$\psi_n \in \mathfrak{S}_n \Leftrightarrow V\psi_n = n\psi_n. \quad (17)$$

Every function ψ_n on P may be lifted into the Λ -invariant function $\Psi_n = p^*\psi_n$ on $\tilde{P}$. It is obvious that

$$V\psi_n = n\psi_n \Leftrightarrow \tilde{V}\Psi_n = n\Psi_n,$$

$$\int_P \psi'_n(u) \psi''_m(u) d\mu = \int_{I^3} \Psi'_n(y) \Psi''_m(y) dy^a dy^b dy = \delta_{nm},$$

where I^3 denotes unit cell.

Topology and geometry ⁵ of total space P_n are shown may be understood with the help of exact sequences of groups (extensions) [25]. The problem of dynamic equation formulation on bundle requires to determine a connection 1-form ω on P . For simplicity, the connection possessing a constant positive curvature may be chosen (if bundle is non-trivial the flat connections are absent). It is easy to check that 1-form

$$\tilde{\omega} = 2\pi i \left(\frac{y^b}{2} dy^a - \frac{y^a}{2} dy^b + dy \right) \quad (18)$$

on $\tilde{P}$ defines the desired connection ω , $p^*\omega = \tilde{\omega}$. Such simplest connection allows us to operate with a set of well-known images. Now, the covariant derivatives of function on P are determined by two horizontal vectors h_a and h_b (i.e. $\omega(h_{a,b}) = 0$) and the following equalities are valid

$$\pi_* h_a = \partial_a, \quad \pi_* h_b = \partial_b,$$

where π denotes the bundle projection (15). An existence of covering map p is indicative of two vectors $\tilde{h}_a$ and $\tilde{h}_b$ on $\tilde{P}$ such that $h_{a,b} = p_* \tilde{h}_{a,b}$

$$\tilde{h}_a = \partial_a - \frac{y^b}{2} \partial, \quad \tilde{h}_b = \partial_b + \frac{y^a}{2} \partial \quad \text{on } \tilde{P}.$$

The Heisenberg algebra realization is clear also. One is based on the commutation relations

$$[h_a, h_b] = 2\pi i V, [h_a, V] = [h_b, V] = 0.$$

Here, the former nonzeros commutator demonstrates that the space P possesses a constant positive curvature, and the flat connections are absent ⁶. The curvature independence of point in P guarantees that the lifted algebra displays a finite dimension.

Comprehensive mention should be made of dynamics on a total space P_n . The globally defined two-body Hamiltonian H_P is formed from (14) by

⁵We have touched on the metric properties of total space only for illustrative purposes, although this is separate interesting problem.

⁶It is in agreement with the usual treatment of the gauge fields as the connections on a principal fibre bundle. But this situation is more wide than the gauge theory realization, because a phase factor is always present in the quantum theory.

fixing of connection ω and substituting of the horizontal vectors h_a and h_b for derivatives $\partial_{a,b}$. Differential operators V , $-ih_{a,b}$ and H_P are self-ajoint while (16). The choice of admissible Hamiltonian is restricted by condition

$$H_P \mathfrak{S}_n \subset \mathfrak{S}_n$$

or else $[H_P, V] = 0$, a topological charge is conserved then and dynamics to be defined by H_P on P can be projected (only locally) into a base of bundle.

In classical theory such approach permits us to investigate dynamics on a base as a projection of degenerated Lagrangian dynamics on P . Note once again that on a base dynamics is not a Lagrangian. When being chosen connection (18), the function

$$\mathcal{L}_B = \frac{m_a(\dot{y}^a)^2}{2} + \frac{m_b(\dot{y}^b)^2}{2} + \pi n y^b \dot{y}^a - \pi n y^a \dot{y}^b, \quad (19)$$

to be used instead of a Lagrangian is ambiguous on the torus $T^2(y^a, y^b)$. The action which it determines is a multiple-valued functional of path [26] and it has to regard the corresponding variation principle only locally. Alternatively, on the bundle space $P = \tilde{P}/\Lambda$ a Lagrangian

$$\mathcal{L}_P = \frac{m_a(\dot{y}^a)^2}{2} + \frac{m_b(\dot{y}^b)^2}{2} + \pi n (y^b \dot{y}^a - y^a \dot{y}^b + 2y) \quad (20)$$

is globally defined and respective action is single-valued. If a suitable local section is fixed the Lagrangian $\mathcal{L}_P$ transforms into $\mathcal{L}_B$. The Euler-Lagrange equation ruling the motion over the phase coordinate y has identical form $0 = 0$, i.e. it is completely undetermined. After the Legendre transformation, in which canonical momenta are defined as follows

$$\pi^a = \frac{\partial \mathcal{L}_P}{\partial \dot{y}^a}, \quad \pi^b = \frac{\partial \mathcal{L}_P}{\partial \dot{y}^b}, \quad \pi = \frac{\partial \mathcal{L}_P}{\partial \dot{y}} = 2\pi n,$$

we obtain the Hamiltonian H_P , determined on the constraint surface $\pi = \frac{\partial \mathcal{L}_P}{\partial \dot{y}} = 2\pi n$. In retrospect, the whys and wherefores of the quantum problem (13) and equivariance condition (17) have been touched on.

The problem

$$E\psi_n = - \left(\frac{1}{2m_a} h_a^2 + \frac{1}{2m_b} h_b^2 \right) \psi_n, \\ V\psi_n = n\psi_n$$

on P can be replaced by the equivalent one on $\tilde{P}$ for function $\Psi_n = p^*\psi_n$. For connection (18), we have

$$E\Psi_n = - \left[\frac{1}{2m_a} \left(\partial_a - \frac{y^b}{2} \partial \right)^2 + \frac{1}{2m_b} \left(\partial_b + \frac{y^a}{2} \partial \right)^2 \right] \Psi_n, \quad (21)$$

$$\tilde{V}\Psi_n = n\Psi_n \quad (22)$$

$$\lambda\Psi_n = \Psi_n, \quad \lambda \in \Lambda. \quad (23)$$

Condition (23) is equivalent to

$$\Psi_n(y^a, y^b, y) = \Psi_n(y^a + 1, y^b, y + \frac{y^b}{2}) = \Psi_n(y^a, y^b + 1, y - \frac{y^a}{2}). \quad (24)$$

To reduce the equation (21) we take into account (22), then

$$\Psi_n(y^a, y^b, y) = \Phi_n(y^a, y^b) \exp(2\pi i n y)$$

and equations obtained for Φ_n are

$$E\Phi_n(y^a, y^b) = - \left[\frac{1}{2m_a} (\partial_a - i\pi n y^b)^2 + \frac{1}{2m_b} (\partial_b + i\pi n y^a)^2 \right] \Phi_n(y^a, y^b), \quad (25)$$

$$\Phi_n(y^a, y^b) = \Phi_n(y^a + 1, y^b) \exp(i\pi n y^b) = \Phi_n(y^a, y^b + 1) \exp(-i\pi n y^a). \quad (26)$$

Spectrum of the problem (25) is well-known. We dwell on the description of set of eigenfunction that must satisfy the condition (26). To transform (26) let us consider two commutative operators T_1 and T_2 , which actions under the functions of variables y^a, y^b are

$$(T_1 f)(y^a, y^b) = f(y^a + 1, y^b) \exp(i\pi n y^b),$$

$$(T_2 f)(y^a, y^b) = f(y^a, y^b + 1) \exp(-i\pi n y^a).$$

Then (26) takes the following form

$$(T_1 \Phi_n)(y^a, y^b) = \Phi_n(y^a, y^b), \quad (T_2 \Phi_n)(y^a, y^b) = \Phi_n(y^a, y^b).$$

The fulfilment of this condition means that the function $\Psi_n(y^a, y^b, y)$ on the covering space $\tilde{P}$ may be thought as a single-valued function on the bundle space P_n .

The spectral problem (25)-(26) have been discussed in another physical context by Novikov et.al. [27]. This discussion to be based on the Atiyah-Singer index theorem allows to extract the desired result that the energy level degeneracy is equal to $|n|$ ($n \neq 0$). The complete set of eigenfunctions

$$\mathfrak{S} = \{\psi_{nNs}(u), u \in P \mid p^* \psi_{nNs} = \Psi_{nNs}\}$$

is parametrized by two integers $N = 0, 1, 2, \dots$ and $s = 0, 1, \dots, |n| - 1$ in addition to a topological charge n .

$$H_P \psi_{nNs}(u) = E_{nNs} \psi_{nNs}(u), \quad (27)$$

where $E_{nNs} = \frac{2\pi|n|}{\sqrt{m_a m_b}}(N + \frac{1}{2})$.

The ground state is $|n|$ -fold degenerate in the present model as noted above. To explain the reason of such phenomena the new operator t_1 may be entered into consideration:

$$(t_1 f)(y^a, y^b) = f(y^a + 1/n, y^b) \exp(i\pi y^b), \quad (n > 0).$$

Because $[t_1, T_1] = [t_1, T_2] = [t_1, H_P] = 0$ and $t_1^n = T_1$, every energy level is a composition of n sublevels, which are the eigenfunctions of operator t_1 with eigenvalues $e^{i2\pi s/n}$.

The wave functions of ground state to be lifted to universal covering space are

$$\Psi_{n0s}(y^a, y^b, y) = A_{n0s} \exp(i\pi n y^a (y^b + i y^a) + 2\pi i n y) \Theta \left(y^b + i y^a - \frac{s}{n}, \frac{i}{n} \right), \quad (28)$$

here $m_a = m_b = 1$, $n \geq 0$,

$$t_1 \Psi_{n0s} = \exp(i2\pi s/n) \Psi_{n0s},$$

and Θ denotes Θ -function of Jacobi:

$$\Theta(z, \tau) = \sum_{l \in \mathbb{Z}} \exp(i\pi l^2 \tau + 2\pi i l z).$$

The wave functions of excited states may be obtained by the consistent action of operator

$$a^+ = \frac{1}{2\sqrt{\pi V}}(-ih_a + h_b), \quad (n > 0),$$

here V is a configuration volume.

Equations (21)-(23) of two body dynamics are rewritten in coordinates R and r relative to the center of mass. In line with the remarks on the page 11 this transformation may be produced only locally on a circle. By the using of the double covering of the torus $T^2(y^a, y^b)$ we have

$$g : T^2(R, r) \rightarrow T^2(y^a, y^b), \quad g((R, r)) = (R - r, R + r).$$

The mapping g can be treated as a mapping from the torus $T^2(R, r)$ into the base of $U(1)$ -bundle P_1 , let us induce the bundle $P_2 = g^*P_1$ over $T^2(R, r)$ by standart procedure [28]. The fibre bundle P_2 shows up as a quotient space $(R, r, \varrho)/\Delta$, where the discrete group Δ is defined by two generators

$$\begin{aligned} (R, r, \varrho) &\rightarrow (R + 1, r, \varrho), \\ (R, r, \varrho) &\rightarrow (R, r + 1, \varrho - 2R). \end{aligned}$$

The mapping $\hat{g}$

$$\hat{g} : (R, r, \varrho)/\Delta \rightarrow \tilde{P}/\Lambda; \quad \hat{g}(R, r, \varrho) = \left(R - r, R + r, \varrho + (R + \frac{1}{2})(r - \frac{1}{2}) \right)$$

determines correctly a bundles morphism, which locally is a transformation to the coordinates relative to the center of mass. Equations (25)-(26) are transformed on $(R, r, \varrho)/\Delta$ as follows

$$E\Phi_{2n}(R, r) = - \left[\frac{1}{2M}(\partial_R - i\pi n(4r - 1))^2 + \frac{1}{2\mu}(\partial_r - i\pi n)^2 \right] \Phi_{2n}(R, r), \quad (29)$$

$$\Phi_{2n}(R, r) = \Phi_{2n}(R + 1, r) = \Phi_{2n}(R, r + 1) \exp(-4\pi i n R). \quad (30)$$

As evident from (30), the system states are described by series

$$\Psi_{2n,s}(R, r, \varrho) = \exp(4\pi i n \varrho) \sum_k \exp(i(k + s)R) f_k(r), \quad [k \equiv s \pmod{4\pi n}],$$

and total momentum k is *not conserved*⁷. This means that the proper two-particle interaction can't be separated from an external field on the non-trivial bundle P_n (or $(R, r, \varrho)/\Delta$), thus the connection ω contains the both indivisible constituents of interaction.

⁷This feature deals with the small number of particles only. It may be shown, that there are some non-trivial $U(1)$ -bundles over three-(and more) particle's configuration space $S^1 \times \dots \times S^1$, in which the total momentum is conserved.

One general property of two- (multi-) particle dynamics on non-trivial bundles to be independent of a connection choice should be underlined once more. Namely, if we pick out two configuration characterized by different topological numbers n and n' , there are not transitions between them, because of orthogonality of $\mathcal{F}_n$ and $\mathcal{F}_{n'}$ and a topological charge conserves. Thus the configuration of interacting particles with $n \neq 0$ can't decay into free ones ($n = 0$).

We now turn to a discussion of statistics for the present model. It is obvious, that an invariance of dynamic equations (25)-(26) under two indistinguishable particle transposition σ exists on the trivial sector ($n = 0$) only. If $n \neq 0$, the transposition to conserve the bundle P structure can't be defined by topological reason. To see it independently of a connection choice, one question should be tackled. Is there a mapping $\hat{\sigma} : P \rightarrow P$, which is the lift of base mapping σ to the total space? The lift $\hat{\sigma}$ of σ is described by following commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\hat{\sigma}} & P \\ \pi \downarrow & & \pi' \downarrow, \\ T^2 & \xrightarrow{\sigma} & T^2 \end{array} \quad (31)$$

where $(y^a, y^b) \xrightarrow{\sigma} (y^b, y^a)$ and $\hat{\sigma}$ is a morphism of bundles, i.e. it must satisfy two conditions:

1. $\sigma\pi = \pi'\hat{\sigma}$ — $\hat{\sigma}$ is fibre-to-fibre map,
2. $\hat{\sigma}(ug) = (\hat{\sigma}(u))g$, $u \in P$, $g \in U(1)$ — $\hat{\sigma}$ conserves the action of structure group.

A violation of the former condition brings out us from the class of considered topological objects – fibre bundles. A violation of the second one (the first is believed to be valid) leads to nonconservation of topological charge n , since the function ψ_n equivariance is not conserved under the action $\hat{\sigma}^* : \psi_n(u) \rightarrow (\hat{\sigma}^*\psi_n)(u)$.

The problem of discrete symmetries lifting to a total bundle space is very intriguing. One have been actively discussed [29, 30, 31, 32] in monopole and two dyons context. For example, using the topological frameworks, it was

pointed out that the notion of space reflection is ill-defined when physical states are defined on a fibre bundle describing a charge-Dirac monopole system. Basing on the same approach, we remark that the mapping $\sigma : S^1 \times S^1 \rightarrow S^1 \times S^1$ interchanges the circles in the Cartesian product and inverses a torus orientation. The induced homomorphism $\sigma^* : H^2(T^2, \mathcal{Z}) \rightarrow H^2(T^2, \mathcal{Z})$ is a multiplication by -1 . As a result, the desired mapping $\hat{\sigma}$ does not exist, since the elements h and $-h$ of homology group $H^2(T^2, \mathcal{Z})$ represent the different classes of unequivalent $U(1)$ -bundles. With the loss of condition 2), we may obtain the mapping $\hat{\sigma}^c : P \rightarrow P$, which satisfies the condition 1) and inverses a sign of topological charge of ψ_n

$$\psi_n(u) \rightarrow (\hat{\sigma}^{c*}\psi_n)(u) \in \mathfrak{F}_{-n}. \quad (32)$$

It is impossible to investigate the wave function symmetry or antisymmetry under transposition operator $\hat{\sigma}$ action because the operator $\hat{\sigma}$ is absent. This remarkable fact means that we can't obtain not only the particles obeying the fractional statistics but fermions and bosons as well. Let us direct one way around this problem. Although $\hat{\sigma}^c$ does not commute with Hamiltonian (21), the property (32) allows us to determine the permutation group action on a direct sum $\mathfrak{F}_{-n} \oplus \mathfrak{F}_n$:

$$\begin{pmatrix} \psi_n \\ \psi_{-n} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\sigma}^{c*}\psi_n \\ \hat{\sigma}^{c*}\psi_{-n} \end{pmatrix}.$$

Here operator $\Sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \hat{\sigma}^{c*}$ ($\Sigma^2 = 1$) commutes with modified Hamiltonian

$$H_{new} = \begin{pmatrix} H_n & 0 \\ 0 & H_{-n} \end{pmatrix},$$

which is determined on $\mathfrak{F}_{-n} \oplus \mathfrak{F}_n$. The transformation properties of wave function under Σ action are now the controlling factor in a statistics definition when particles dynamics is described by H_{new} . The functions of the types

$$\begin{pmatrix} \psi_n \\ \hat{\sigma}^{c*}\psi_n \end{pmatrix} \text{ and } \begin{pmatrix} \psi_n \\ -\hat{\sigma}^{c*}\psi_n \end{pmatrix}$$

give the orthogonal subspaces of respectively Σ -symmetric and Σ -antisymmetric functions into $\mathfrak{F}_{-n} \oplus \mathfrak{F}_n$.

4 Multiparticle systems on tori from the topological standpoint

In the current section the productive model of one dimensional interacting particles will be developed. As above, we don't restrict the problem framework to the trivial bundle over d -particles dynamical space T^d . To carry out consequently our intents the Hamiltonian formalism should be realized from the start point as (19) to a quantization procedure. We will now put in the forefront the topological aspects and obviate the detailed discussion of dynamics.

As has already been intimated, the set of unequivalent $U(1)$ -bundles over the torus T^d and the cohomology group $H^2(T^d, \mathcal{Z})$ are in one-to-one correspondence. The elements of

$$H^2(T^d, \mathcal{Z}) = \mathcal{Z} \oplus \cdots \oplus \mathcal{Z} = \mathcal{Z}^{\frac{d(d-1)}{2}}$$

is conveniently represented by the 2-forms with the integer coefficients or, what is the same — by the antisymmetric matrices with integer-valued elements. The set $H^2(T^d, \mathcal{Z})$ falls into subsets $B_d(h_p)$ such that

$$B_d(h_p) = \{h \mid h = qh_p, q \in \mathcal{Z}\},$$

where the elements of respective matrix h_p are mutually coprime. In the matrices space, the subset $B_d(h_p)$ is the straight line in the integer lattice, connecting zero matrix and some matrix h_p :

$$H^2(T^d, \mathcal{Z}) = \bigcup_{h_p} B_d(h_p),$$

as presented by Figure 2.

It is clear that the bundle to be related to some $h = nh_p$ may be obtained from the total bundle space P_{h_p} as a quotient under discrete group $\mathcal{Z}_n$ action. In combination with the results of previous section, a space of n -equivariant functions on the bundle P_{h_p} is also a space of functions on the bundle P_h . Hence every element h on the line $B_d(h_p)$ can be thought as to correspond to the space of functions defined on the proper bundle P_{h_p} , and satisfying the equivariance condition with a number n to be equal to the greatest common divisor of the elements of h . There exists the following mapping

$$\nu : H^2(T^d, \mathcal{Z}) \rightarrow RU(1)/\sim$$

(considered as a mapping of sets) of cohomology group into the set of irreducible group $U(1)$ representations, where any two representations to be related to the characters of opposite sign are identified. This means, we don't pick out a orientation and the orientations of total space and fibres are not fixed. The choice of sign of n is only a question of convention for an individual bundle covered by $B_d(h_p)$. No self-consistent orientation may be chosen for the whole set of lines $\bigcup_{h_p} B_d$ by invariant manner (see Fig.2).

For illustrative purpose let us return to the former example of Section 2. Dynamics of a charge on the bounded region of plane in magnetic field turned out may be determined on the total space of non-trivial bundle P_n (6) when magnetic flux through the region is integer (up to 2π). Then the interrelation between the set of different bundles P_n and elements of $H^2(T^2, \mathcal{Z})$ to be described above has the form

$$P_n \leftrightarrow c_2 = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix}.$$

Such matrices c_d may be identified with elements of cohomology group and 2-forms with integer coefficients $\sum_{i<j} c_d^{ij} dy^i \wedge dy^j$. No wonder that the dynamical space of two planar charged particles in the present models to be the non-trivial bundles over T^4 , which is characterized by matrix

$$\begin{bmatrix} 0 & n & & 0 \\ -n & 0 & & \\ & & 0 & n \\ 0 & & -n & 0 \end{bmatrix}. \quad (33)$$

Intuitively, this standpoint corresponds with the usual treatment of wave functions multiplication.

We shall study the general case of d particles now and prove the last statement. Using the the base T^d diffeomorphisms f , a given matrix $h \in H^2(T^d, \mathcal{Z})$ may be transformed into the most simple form. Let us require that these maps be lifted up to transformations of $U(1)$ -bundles over T^d . The integer $n = \nu(h)$ is conserved when lifted a map for f

$$\nu(f^*h) = \nu(h), \quad (34)$$

where $f^* : H^2(T^d, \mathcal{Z}) \rightarrow H^2(T^d, \mathcal{Z})$ is homomorphism induced by the map $f \in Diff(T^d)$. Since homomorphism f^* depends on homotopy class of f ,

only. Choosing one diffeomorphism from each class of the set $[T^d, T^d]$, we cover the set of all homomorphisms f^* .

To characterize the set $[T^d, T^d]$, it is sufficient to restrict ourselves to linear transformations of the universal covering space $\mathcal{R}^d$ [33, page 164]. Namely, if $\mathbf{y}$ denotes a point of space $\mathcal{R}^d$ and torus T^d is thought as a quotient space $\mathcal{R}^d/\mathcal{Z}^d$ then the linear transformation of $\mathcal{R}^d$ to be defined by the square matrix A of order d with integer coefficients and unity determinant ($A \in SL(d, \mathcal{Z})$) specifies some torus diffeomorphism according with commutative diagram:

$$\begin{array}{ccc} \mathcal{R}^d & \xrightarrow{A_f} & \mathcal{R}^d \\ p \downarrow & & p \downarrow, \end{array} \quad (35)$$

$$T^d \xrightarrow{f} T^d$$

where p is a covering map and the map $A_f : \mathbf{y} \rightarrow A_f \mathbf{y}$. The induced cohomology group transformation is given by

$$f^* h = A_f^T h A_f. \quad (36)$$

Condition $\det A_f = 1$ ensures that the degeneracy be absent and f^* be isomorphism.

It can be easily shown that:

1. condition (34) is valid for transformations (35)–(36);
2. acting by transformations of the form (36) (Eq. 34 holds), an arbitrary matrix $h \in H^2(T^d, \mathcal{Z})$ can be reduced to the canonical type

$$c_d = \begin{bmatrix} 0 & n_1 & & & 0 \\ -n_1 & 0 & & & \\ & & 0 & n_2 & \\ & & -n_2 & 0 & \\ & & & \ddots & \\ & 0 & & & 0 & n_k \\ & & & & -n_k & 0 \end{bmatrix}, \quad (37)$$

whenever $d = 2k$ is even, or else to the type $c_{d-1} \oplus 0$, whenever d is odd.

The representation (37) is very useful to build functions on bundles over T^d . Thus, the fixed bundle to be related originally to a matrix of arbitrary type may be characterized by the canonical matrix (37) after the appropriate transformation of torus coordinates. Torus diffeomorphism f gives such new coordinatization quite as above.

New coordinates of multiparticles dynamical space are collective coordinates. It allows us to choose the natural basis in the total space of non-trivial bundle for dynamics investigation. It is wonderful that related to the new basis the system excitations are interpreted to be quasi-particles. Moreover, matrix c_d is box-type, consequently, the dynamics of every pair of new quasi-particles may be understood independently, as below is shown.

The canonical form (37) allows to construct n -equivariant functions on the bundles over T^d . The bundle P_{c_d} is constructed from the Cartesian product of k copies of bundle P_1 with the help of identification procedure according with the following equivalence relation

$$(u_1, u_2, \dots, u_k) \sim (u_1 g_1, u_2 g_2, \dots, u_k g_k), \quad (38)$$

for

$$g_1^{n_1} \cdot g_2^{n_2} \cdot \dots \cdot g_k^{n_k} = 1, \quad (39)$$

where $(u_1, u_2, \dots, u_k) \in P_1 \times \dots \times P_1$, $g_\alpha \in U(1)$. In other words, space P_{c_d} is a result of k times using of associative operation $\circ$:

$$P_{c_d} = P_{n_1} \circ P_{n_2} \circ \dots \circ P_{n_k},$$

where $P_{n_1} \circ P_{n_2} = P \times P / \sim$.

A product of functions

$$\psi_{n_1 n_2 \dots n_k} = \psi_{n_1}(u_1) \psi_{n_2}(u_2) \dots \psi_{n_k}(u_k) \quad (40)$$

to be determined on the Cartesian product $\prod_j P_{(j)}$ is constant along the parallel hypersurfaces specified by (39). Consider $\prod_j P_{(j)}$ as new bundle with the base $\prod_j T_{(j)}^2$ and fibre $\prod_j U(1)$. The direction indicated by vector of minimal length $(n_1/\nu, n_2/\nu, \dots, n_k/\nu)$ with integer coordinates (where ν denotes the greatest common divisor of n_j) to exist in every fibre is transversal to these hypersurfaces. Then a function of the form (40) is identified by us with a ν -equivariant function on the bundle $P_{c_d/\nu}$.

Multiparticles states arising on the bundle $P_{h_p} = P_{h/\nu(h)}$ can be described with the help of transforming of matrix h into the canonical form c_d . Strictly speaking this procedure is described by the bundles morphism

$$P_{c_d/\nu(h)} \xrightarrow{\hat{f}} P_{h/\nu(h)}.$$

One is induced by base diffeomorphism f such that the condition (34) is valid and $c_d = f^*h$. Let ψ_h be a function on $P_{h/\nu(h)}$ with equivariance number $\nu(h)$. There is a unique function ψ_{c_d} on $P_{c_d/\nu(h)}$ with the same equivariance number such that $\psi_{c_d} = \hat{f}^*\psi_h$. Therefore the initial function can be represented by the sum of products of the basic functions (27):

$$\psi_{n_j N_j s_j}(u_j); j = 1, \dots, k; N_j = 0, 1, \dots; s_j = 0, 1, \dots, |n| - 1;$$

$$\hat{f}^*\psi_h = \psi_{c_d} = \sum_{N_1 s_1 \dots N_k s_k} C_{N_1 s_1 \dots N_k s_k} \psi_{n_1 N_1 s_1}(u_1) \dots \psi_{n_k N_k s_k}(u_k).$$

We can write an explicit form for inner product of functions ψ_{h_1} and ψ_{h_2} whenever h_1, h_2 to lie along a one line $B_d(h_p)$ ($h_p = h/\nu(h)$):

$$\langle \psi_{h_1}, \psi_{h_2} \rangle = \int_{\hat{f}^*P_{h_p}} dM \hat{f}^*\psi_{h_1}^* \hat{f}^*\psi_{h_2},$$

where $dM = \int d\mu_1 d\mu_2 \dots d\mu_k$ denotes the partial integral through a hyper-surface to have been specified by (38)–(39).

Let us return to the problem of discrete symmetries in part to the particles transposition operator existence. Consider the d -particles dynamics on the $U(1)$ -bundle P_h over the torus T^d , which is specified by the element $h \in H^2(T^d, \mathcal{Z})$. A permutation of two particles a and b defines the torus diffeomorphism $f^{(a,b)}$ with the help of a square matrix $A_{f^{(a,b)}}$ of order d . A permutation group can't be lifted as whole if and only if every torus diffeomorphism $f^{(a,b)}$ satisfies

$$f^{(a,b)*}h = A_{f^{(a,b)}}^T h A_{f^{(a,b)}} = h. \quad (41)$$

This fact is clear in the case of trivial bundle then $h = 0$ and every element of permutation group S^d may be lifted to the total space of P_h . Whenever $h \neq 0$, no lifting exists for whole permutation group. It is impossible to test

the wave function to be given on P_h under the permutation group action. To investigate the collective quasi-particle excitations we transform the matrix h to the canonical type (37) by substituting the new coordinates. Such approach allows to decompose a complete topological interaction on P_h into a pair interaction of quasi-particles describing the involved degrees of freedom.

Taking into account the last result we are all set for the discussion of planar systems. Now d -particles configuration space is assumed to be the $U(1)$ -bundle over T^{2d} . The matrix A to be specified by the particles permutation has the box-type form. For example (33), the group $\mathcal{S}^2$ contains two elements — identity and $f^{(1,2)}$, where

$$A_{f^{(1,2)}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The property (41) is valid for the matrix h of type (33). There is not obstructions for the lifting of $\mathcal{S}^2$ action to the P_h . In general case, there is the following

Theorem 1 *Permutation group $\mathcal{S}^d$ action on a torus T^{2d} is lifted to an action in the total space of the $U(1)$ -bundle P_h over T^{2d} if and only if matrix h of order $2d$ has the box-type form with nonzeros 2×2 boxes standing only on the main diagonal*

$$\begin{bmatrix} 0 & n_1 & & & & \\ -n_1 & 0 & & & & \\ & & 0 & n_2 & & \\ & & -n_2 & 0 & & \\ & & & & \ddots & \\ & 0 & & & & 0 & n_d \\ & & & & & -n_d & 0 \end{bmatrix}, \quad (42)$$

and all n_i are equal.

This conclusion well correlates with the usual treatment of d charges in magnetic field on a plane basing on the symmetrization (or antisymmetrization)

of a product of single-particle wave functions. Hence an existence of magnetic field is manifested as appearance of the nonzeros diagonal boxes in the last formula.

To modify such picture we consider the bundle, which is specified by matrix h possessing the unique nonzeros 2×2 box outside the diagonal. There is not the lifting of transposition operator on the bundle P_h . For instance, let choose T^4 and h be

$$\begin{bmatrix} 0 & q & 0 & -q \\ -q & 0 & q & 0 \\ 0 & -q & 0 & q \\ q & 0 & -q & 0 \end{bmatrix}.$$

Using the transition to coordinates relative to the center of mass according with (12), we have the 4-fold covering

$$g: T^4(R_1, R_2, r_1, r_1) \rightarrow T^4(y_1^a, y_2^a, y_1^b, y_1^b),$$

where $\mathbf{y}^a = \mathbf{R} + \mathbf{r}$ and $\mathbf{y}^b = \mathbf{R} - \mathbf{r}$. The induced bundle g^*P_h over $T^4(R_1, R_2, r_1, r_1)$ is specified by the matrix

$$g^*h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4q \\ 0 & 0 & -4q & 0 \end{bmatrix}.$$

But such bundle is familiar to us, one similiary is $P_{4q}(r_1, r_2, \rho) \times T^2(R_1, R_2)$. Under the condition $n = 4q$ this is coincident with the bundle arising at the end of Section 2. Now then nonexistence of transposition operator on g^*P_h have been shown, one may correctly speaks about “anyons” from the point of view of the remarkable behavior of wave functions under the action of permutation group, determined on the base of bundle only.

In conclusion we will show that the quasi-particles with fractional charge may exist in the context of our approach. Consider d charged particles in external magnetic field on T^2 as before. Then resulting configuration space is the bundle to be specified by matrix (42). One may speculate that some another pertubating interaction is occurred also. We don't concretize its nature, but it is significant for us that the configuration space is changed.

Such bundle modification can be called a “topology perturbation”. The minimal one is as follows

$$\begin{bmatrix} 0 & n & & & \\ -n & 0 & & & \\ & & 0 & n & \\ & & -n & 0 & \\ & & & \ddots & \\ & 0 & & & 0 & n \\ & & & & -n & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & n & 0 & -1 & & \\ -n & 0 & 1 & 0 & & \\ 0 & -1 & 0 & n & & \\ 1 & 0 & -n & 0 & & \\ & & & \ddots & & \\ & 0 & & & 0 & n \\ & & & & -n & 0 \end{bmatrix}.$$

Under transition $h \rightarrow h_{new}$ and $P_h \rightarrow P_{h_{new}}$, the dynamics is expected to modify by minimal way.

To study new dynamics properties, a transition to quasi-particles coordinates should be done. To transform the matrix h_{new} to the canonical box-type form (37), we use some linear transformation to be specified as before by the matrix A ($\det A = 1$) with integer elements. New collective coordinates are given by $\mathbf{y}_{new} = A\mathbf{y}$. Thus in the canonical form, matrix h_{new} contains the box with $n_1 = 1$, because of the greatest common divisor of matrix elements is conserved under transorfation A . We have started with magnetic field with the flux $\Phi_0 = n$ and new quasi-particles occur in this field also. This means that charge e_{new} of the quasi-particle which is associated with coordinate $\mathbf{y}_{new}^1 = A^{1i}\mathbf{y}^i$ is equal to

$$e_{new} = e \frac{\Phi_{new}}{\Phi_0} = \frac{e}{n}, \text{ and } \Phi_{new} = n_1 = 1,$$

where e is an electron charge.

The last statement has been derived from the topological properties of space P_h only, hence one may be applied to arbitrary dynamical system possessing the similar topology.

Although the two-dimensional Schrödinger operator H in periodic magnetic field and electric field with periodic potential is familiar object, but one special property of its solutions is less well known. If the flux of magnetic field through an elementary site of the lattice is integer number n then the operator H is represented in the form of a direct integral over two-dimensional torus of the reciprocal lattice of elliptic self-adjoint operators H_{p_1, p_2} with discrete spectrum $E_N(p_1, p_2)$, $N = 0, 1, \dots$. The corresponding eigenfunctions

$\psi_{nN}(p_1, p_2)$ are the local sections of the complex-line bundle over $T^2(p_1, p_2)$ associated with the principal $U(1)$ -bundle P_n [27, 34]. It allows to extend the range of the use of topological methods and results to the quasi-momentum space $T^2(p_1, p_2)$. All constructions and conclusions are reserved for this case also.

Finally, it should be pointed out that we have tried first to introduce the “topology perturbation” before the resulting dynamics have been formulated. Such approach is inverse to the usual one when new topology occurrence is obtained from given interaction.

5 Seifert fibre spaces and rational charge

Up to this point we have deal only with topological objects to be classified by integer invariant $\Phi = n$. It is not surprising because the integer cohomology group $H^2(M, \mathcal{Z})$ is determining. However such reserves of spaces is expected to be not sufficient to describe arbitrary anyon system. One will be supplemented by new images such as orbifolds and Seifert fibre spaces in the present Section. By doing so, some generalizing spaces to possess the fractional (rational) topological invariants may be constructed and generalization of n may be obtained.

We begin with some geometrical pictures and the basic definitions⁸. Define a trivial fibred solid torus (TFST) as $S^1 \times D^2$ with the product foliation by circles. Thus the fibres are the circles $S^1 \times \{z\}$, where $z \in D^2$. A fibred solid torus (FST) is an object, which can be constructed from a trivial fibred solid torus by cutting it open along $\{g\} \times D^2$ ($g \in S^1$), rotating one of the discs so obtained through q/p of a full turn and glueing back together. Such FST will be p -fold covered by a trivial fibred solid torus. Figure 3 illustrates the former procedure.

In other terms, we recall a flow μ on a smooth manifold M , possibly with boundary, given by a smooth action $\mu : M \times \mathcal{R} \rightarrow M$ of the additive reals on M . Then FST $S^1 \times D^2$ is specified by

$$\mu(\exp 2\pi i\theta, z; t) = (\exp 2\pi i(\theta + t), z \exp 2\pi it(q/p)),$$

where p and q are coprime integers.

⁸Some of them may be not quite rigorous in a mathematical sense, but quite enough for our purposes

Seifert fibre space is defined to be a 3-manifolds M with a decomposition of M into disjoint circles, called fibres, such that each circle has a neighbourhood in M which is isomorphic to a FST. Clearly every $U(1)$ -bundle over a surface is a Seifert fibre space. A fibre is called regular if it has a neighbourhood isomorphic to a TFST and critical otherwise. A FST has only one critical fibre $O - O$ (see Fig. 3). Thus a FST and a TFST are the fundamental constituents of each Seifert fibre space.

Moreover, a Seifert fibre space turns out to possess the $U(1)$ -bundle structure we are interested in. To construct the base space B of such bundle one defines B by identifying each circle to a point. If M is a TFST, then B is simply a 2-disc and projection $M \rightarrow B$ is a bundle map. If M is a FST $T(p, q)$ then M is a quotient space of TFST $S^1 \times D^2$ by group $\mathbb{Z}_p$. This is generated by a homeomorphism to be product of rotation through $2\pi/p$ on the S^1 -factor with a rotation through $2\pi q/p$ on the D^2 -factor. Hence the space B is the quotient of D^2 by this action of $\mathbb{Z}_p$. Alternatively, one is orbifold with cone angle $2\pi/p$ ⁹(see Fig. 2).

One must distinguish between a 2-dimensional orbifold and its underlying surface, because any orbifold is homeomorphic to a manifold in dimension two. The quotient space B of a Seifert fibre space M to be obtained by identifying each fibre to a point of B is (in a topological sense) a surface and has an orbifold structure. Then each cone point corresponds to orientation preserving critical fibre in a Seifert bundle M . It is essential that the regular fibres of M are simply the orbits of the $U(1)$ -action.

Summarizing, we have

- Seifert fibre spaces M are the generalization of usual $U(1)$ -bundle over surfaces. The critical fibres to impair a bundle structure are classified by rational invariant q/p ;
- the exact $U(1)$ -bundle structure is restored by a finite covering of M ;
- functions on a Seifert fibre space to be thought as a single-valued functions on the covering space may be obtain by natural way (the most

⁹ n -dimensional orbifold is a Hausdorff, paracompact space to be locally homeomorphic to the quotient space of $\mathbb{R}^n$ by a finite group action. The orbifold's space is equipped with a covering by open sets U_i and to each U_i is associated a finite group Γ_i and an action of Γ_i on an open subset $\tilde{U}_i$ of $\mathbb{R}^n$ and a homeomorphism $\tilde{U}_i/\Gamma_i \rightarrow U_i$. For detail see [17, 35]. It is clear that orbifolds are the suitable spaces to study multivalued functions.

interesting case is that the covering space be a non-trivial $U(1)$ -bundle);

- complete picture and classification of Seifert fibre spaces can be used for the topological invariants extracting. It is unexpected but they are classified with the help of geometrical methods. Namely, Seifert fibre spaces are the remarkable spaces in dimension three, because there exist eight geometrical structures here and, for six of them, all the closed manifolds to admit the given geometry are Seifert fibre spaces.

Now we introduce the desired generalization of topological charge n . Consider Seifert bundle M over orbifold. The obvious invariant of it is the base orbifold B . Let B' denotes the underlying surface (the critical points are removed). Then the $U(1)$ -bundle P over B' is classified by integer $n = b(P)$, which is the obstruction to the existence of a section of the bundle P . The obstruction $b(P)$ lies in $H^2(B', \mathbb{Z})$ as mentioned above. Finally, given Seifert bundle over B is completed by glueing the FST corresponding to the cone points of B . Now, *Euler number $e(M)$ of a Seifert bundle M* is defined as follows

$$e(M) = - \left(b(P) + \sum_{i=1}^r \frac{\beta_i}{\alpha_i} \right), \quad (43)$$

where M has r critical fibres and $\alpha_i = p_i$, β_i is given by $\beta_i q_i \equiv 1 \pmod{p_i}$. The natural properties of $b(P)$ in the case of usual bundle take place also for $e(M)$. For example,

Theorem 2 *Let P be a Seifert bundle over a closed orbifold B with total space M . Let $\tilde{M}$ be a finite covering of M of degree d , so that $\tilde{M}$ is the total space of a Seifert bundle $\tilde{P}$ over an orbifold $\tilde{B}$. Let the orbifold covering $\tilde{B} \rightarrow B$ have degree l and let m be the degree with which regular fibres of $\tilde{P}$ cover regular fibres of P , so that $lm = d$. Then*

$$e(\tilde{P}) = e(P) \frac{l}{m}.$$

Now then all necessary tools are prepared let apply them to an appropriate systems investigation. First trivial example is provided by two anyons on a plane. Let statistics parameter Φ/π is equal to rational number q/p . Excluding the coordinate $\mathbf{R}$ of the center of mass, we have to formulate the following condition

$$\psi(\theta + 2\pi) = e^{-2i\Phi} \psi(\theta),$$

where θ denotes the relative polar angle of the two particles. This equality exactly means that the wave function $\psi(\mathbf{r})$ may be determined on the Seifert fibre space M_{pl} over orbifold with one cone point (p, q) which has the plane as an underlying space. Here Euler number is $e(M_{pl}) = -\beta/p$, where $\beta \equiv 1 \pmod{p}$.

But the more interesting dynamical space is realized when an underlying space is a closed 2-dimensional manifold and, hence one admits non-trivial $U(1)$ -bundle. The more complicated situation is explained by the fact that some anyon features are combined with a non-trivial bundle structure. To separate such objects we use the following

Theorem 3 *If a Seifert bundle M has compact total space, then one possesses a finite covering which is a trivial circle bundle over a surface, with the induced Seifert bundle structure, if and only if $e(M) = 0$.*

It is clear that the allowed spectrum of statistics for identical spinless particles on an arbitrary closed two-dimensional manifold to have been investigated by T.Imbo and J.March-Russell [23] does not contain the total dynamical spaces for the case of $e(M) \neq 0$.

We classify all Seifert bundles by their universal covering spaces and, as indicated in Appendix, by the appropriate geometrical structures. The geometry and universal covering space can be determined from the two invariants $\chi(B)$ and $e(M)$, where $\chi(B)$ is the Euler number of the base orbifold ¹⁰ and $e(M)$ is the Euler number of the Seifert bundle, according to the following table:

	$\chi(B) > 0$	$\chi(B) = 0$	$\chi(B) < 0$
$e(M) = 0$	$S^2 \times \mathcal{R}$	E^3	$H^2 \times \mathcal{R}$
$e(M) \neq 0$	S^3	Nil	$PSL(2, \mathcal{R})$

Table 1: Thurston's classification of the appropriate geometries for Seifert bundles with closed total space

According with the result of Theorem 3, the second row of the table is interesting for us. All non-trivial low-dimensional dynamical spaces, which

¹⁰If B is a "good" compact 2-dimensional orbifold with underlying surface B' [35] and r cone points (p_i, q_i) then $\chi(B) = \chi(B') - \sum_{i=1}^r \left(1 - \frac{1}{p_i}\right)$, where $\chi(B')$ is a usual Euler number of closed surface.

combines the position and gauge coordinates, are presented here. There is the well-known Hopf bundle over S^2 to be essential to the description of charge-Dirac monopole system [14, 36, 37]. The bundle P_n over T^2 to has been detailed in Section 3 is the particular example of *Nil*-geometry. As for the third column, the description of the quantum mechanics of a charged particle moving on a two-dimensional manifold of constant negative Gaussian curvature in an external magnetic field is based on the geometrical structure modelled on $P\tilde{S}L(2, \mathcal{R})$.

As a result, dynamics analysis on a given total space may be transformed to analysis on the universal covering space at the appropriate choice of discrete group of isometries. The content of Section 3 gives one of a number of examples.

Appendix

We shall now describe briefly the eight 3-dimensional homogeneous spaces, or geometries, which are needed to give geometric structures for 3-manifolds, which are dynamical spaces. These were discussed by Thurston [17] and by Scott [35] in detail. We give in each case a description of a simply-connected 3-manifolds X , and a group G of orientation-preserving diffeomorphism of X with compact stabilizers G_x . For the first three geometries, G_x is $SO(3)$.

1. Spherical geometry. X is the 3-sphere S^3 and G is $SO(4)$. All 3-dimensional spherical manifolds have been classified for a complete enumeration. The three-sphere S^3 has the structure of a group: the group of unit quaternions. Therefore $S^3 \times S^3$ acts on S^3 by the formula $x \rightarrow gxh^{-1}$. The kernel of this action is $\mathbb{Z}_2$, generated by $\{-1, 1\}$ in quaternionic notation; and this produces an incredible isomorphism

$$S^3 \times S^3 / \mathbb{Z}_2 = SO(4).$$

2. Euclidean geometry. $X = E^3$, $G = \mathcal{R}^3 \times SO(3)$, the group of Euclidean isometries. There are only 10 nonhomeomorphic 3-manifolds. Each one is finitely covered by a 3-torus.

3. Hyperbolic geometry. One good picture of hyperbolic space is the Poincare upper half space, $X = \{x, y, z | z > 0\}$. The group G is the group

$PGL(2, C)$ of nonsingular 2×2 complex matrices of determinant 1, up to multiplication by ± 1 scalars. The action of G on X may be defined as fractional linear transformation using quaternionic notation, $q = x + yi + zj$. The formula is

$$q \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \frac{aq + b}{cq + d}$$

For the next four cases, the identity components of G_x is $SO(2)$.

4. $X = S^2 \times E^1$, G consists of (spherical isometries) $\times$ (isometries of the Euclidean line). Note that an isometry which reverses the orientation of both factor preserves the total orientation. There are only two nonhomeomorphic examples of compact manifolds with this geometry.

5. $X = H^2 \times E^1$, G consists of (isometries of H^2) $\times$ (isometries of Euclidean line). Every manifolds modelled on this geometry is finitely covered by the product of a surface and a circle.

6. The space X is the universal covering space set of unit length vectors in the hyperbolic plane, $T_1(H^2)$, and the group G is $\mathcal{R} \times$ (the universal covering of the group of isometries of H^2). The isometries of H^2 act by their derivatives, and $\mathcal{R}$ acts as simultaneous rotations of all vectors, keeping their based points fixed. Even orientation-reversing isometries of H^2 preserve the orientation of $T_1(H^2)$.

7. *Nil*-geometry. X is a twisted product of E^1 with E^2 , and G is an extension of the group of isometries E^2 by $\mathcal{R}$

$$0 \rightarrow \mathcal{R} \rightarrow G \rightarrow Isom(E^2) \rightarrow 1.$$

An alternative description of this geometry is to define X as the Heisenberg group, the group of the square upper triangular matrices of order 3, and G as the semidirect product of H by S^1 acting as a group of automorphism of H which rotate the $y^a - y^b$ plane.

8. *Sol*-geometry. the identity components of G_x is trivial. Here, X is a Lie group, the solvable Lie group

$$0 \rightarrow \mathcal{R}^2 \rightarrow X \rightarrow \mathcal{R} \rightarrow 0,$$

where $\mathcal{R}$ acts on $\mathcal{R}^2$ (by conjugation) with the 0 formula

$$(x_1, x_2) \rightarrow (e^t x_1, e^{-t} x_2).$$

The group G is an extension of X by $(\mathbb{Z}_2)^2$ acting as a group of automorphisms. Whose 3 nontrivial elements are the 180° rotations

$$(x_1, x_2, t) \rightarrow (-x_1, -x_2, t), (x_1, x_2, t) \rightarrow (x_1, x_2, -t) \text{ and}$$

$$(x_1, x_2, t) \rightarrow (-x_1, -x_2, -t).$$

Any torus bundle over S^1 whose monodromy is a linear map with distinct real eigenvalues has a geometric structure of this form.

Now we turn to the classification of the 3-dimensional geometries and respective manifolds. Recall that M has a *geometric structure* if it admits a complete locally homogeneous metric. Hence the universal covering space X of M has a complete homogeneous metric, so that the isometry group of X acts transitively. Thus we have a geometry (X, G) with G being the isometry group of X . By a *geometry*, we mean a pair (X, G) where X is a manifold and G acts transitively on X with compact point stabilisers. Two geometries (X, G) and (X', G') are *equivalent* if there is a diffeomorphism of X with X' which throws the action of G onto the action of G' . Finally, one imposes restriction that there exist a subgroup Γ of G which acts on X as covering group and has compact quotient. One says that the geometry *admits a compact quotient*.

Theorem 4 (Thurston) *Any maximal, simply connected, 3-dimensional geometry which admits a compact quotient is equivalent to one of the geometries $(X, \text{Isom}(X))$ where X is one of E^3 , H^3 , S^3 , $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\tilde{PSL}(2, \mathbb{R})$, Nil or Sol as above.*

All eight above mentioned geometries are maximal and no two are equivalent because

Theorem 5 *If M is a closed 3-manifold which admits a geometric structure modelled on one of the eight geometries, then the geometry involved is unique.*

Here the geometry maximization means that the group of isometries G is maximal.

Theorem 6 *Let M be a closed 3-manifold. M possesses a geometric structure modelled on one of E^3 , S^3 , $S^2 \times \mathcal{R}$, $H^2 \times \mathcal{R}$, $P\tilde{S}L(2, \mathcal{R})$ or Nil if and only if M is a Seifert fibre space. Further if M has the structure of a Seifert bundle η over an orbifold B , then the appropriate geometry for M is determined by two invariants — Euler number $\chi(B)$ and $e(\eta)$ according to the following table:*

	$\chi(B) > 0$	$\chi(B) = 0$	$\chi(B) < 0$
$e(M) = 0$	$S^2 \times \mathcal{R}$	E^3	$H^2 \times \mathcal{R}$
$e(M) \neq 0$	S^3	Nil	$P\tilde{S}L(2, \mathcal{R})$

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Figure 1: Torus as a quotient space and transition functions of the bundle P_n

Figure 2: A set of classes (black circles) of unequivalent $U(1)$ -bundles over T^4 presented in the canonical form, and some line subsets $B_d(h_p)$. The grey square denotes the trivial bundle $(0,0)$.

Figure 3: Trivial fibred solid torus (a) and fibred solid torus $T(p, q)$ with one critical fibre $O - O$ (b)

Figure 4: The base of trivial Seifert bundle TFST (a) and the base orbifold of FST $T(p, q)$ (b)